J. OPERATOR THEORY
35(1996), 317–335

A SIMPLE PROOF OF A THEOREM OF KIRCHBERG AND RELATED RESULTS ON C^*-NORMS

GILLES PISIER

Communicated by William B. Arveson

ABSTRACT. Let F be a free group and let $C^*(F)$ be the (full) C^*-algebra of F. We give a simple proof of Kirchberg's theorem that there is only one C^*-norm on the algebraic tensor product $C^*(F)\otimes B(H)$, or equivalently that $C^*(F)\otimes_{\min} B(H) = C^*(F)\otimes_{\max} B(H)$. More generally, let A be the (unital) free product of a family $(A_i)_{i\in I}$ of (unital) C^*-algebras. We show that if $A_i \otimes_{\min} B(H) = A_i \otimes_{\max} B(H)$ holds for all i in I, then $A\otimes_{\min} B(H) = A\otimes_{\max} B(H)$.

KEYWORDS: *C^*-algebra, unicity of C^*-norms, minimal and maximal tensor product.*

AMS SUBJECT CLASSIFICATION: Primary 46L05; Secondary 46M05,47A20.

0. INTRODUCTION. MAIN RESULTS

Recently, E. Kirchberg ([14], [15]) revived the study of pairs of C^*-algebras A, B such that there is only one C^*-norm on the algebraic tensor product $A\otimes B$, or equivalently such that $A\otimes_{\min} B = A\otimes_{\max} B$. Recall that a C^*-algebra is called nuclear, cf. [17], [6] if this happens for any C^*-algebra B. Kirchberg ([14]) constructed the first example of a non-nuclear C^*-algebra such that $A\otimes_{\min} A^{\mathrm{op}} = A\otimes_{\max} A^{\mathrm{op}}$. He also proved the following striking result [15] for which we give a very simple proof and which we extend.

THEOREM 0.1. (Kirchberg [15]) *Let F be any free group and let $C^*(F)$ be the (full) C^*-algebra of F, then*

$$C^*(F)\otimes_{\min} B(H) = C^*(F)\otimes_{\max} B(H).$$

Here, and throughout the paper (unless specified otherwise) H is a separable infinite dimensional Hilbert space and $B(H)$ is the space of all bounded operators on H.

More generally, we will prove:

THEOREM 0.2. *Let $(A_i)_{i\in I}$ be a family of unital C^*-algebras such that $\forall i \in I$*

$$A_i \otimes_{\min} B(H) = A_i \otimes_{\max} B(H).$$

Then the free product $A = \underset{i\in I}{*} A_i$ (in the category of unital C^*-algebras) satisfies

$$A \otimes_{\min} B(H) = A \otimes_{\max} B(H).$$

COROLLARY 0.3. *Let $(G_i)_{i\in I}$ be a family of discrete amenable groups and let $G = \underset{i\in I}{*} G_i$ be their free product. Then*

$$C^*(G) \otimes_{\min} B(H) = C^*(G) \otimes_{\max} B(H).$$

Our method strongly uses the theory of operator spaces, recently developed in a series of papers by Effros-Ruan ([7], [8]) and Blecher-Paulsen ([1]). A key observation is that when a C^*-algebra is generated by a finite set of unitaries, then the operator space structure of the linear span of these generators (up to complete isometry) is enough to encode some important properties of the C^*-algebra they generate.

NOTATION. Our notation is standard, except perhaps that we denote by $E_1 \otimes E_2$ the algebraic tensor product of two vector spaces.

1. PROOFS

The main idea of our proof of Theorem 0.1 is that if E is the linear span of 1 and the free unitary generators of $C^*(F)$, then it suffices to check that the min- and max-norms coincide on $E \otimes B(H)$. More generally, we will prove

THEOREM 1.1. *Let A_1, A_2 be unital C^*-algebras. Let $(u_i)_{i\in I}$ (resp. $(v_j)_{j\in J}$) be a family of unitary operators which generate A_1 (resp. A_2). Let E_1 (resp. E_2) be the closed span of $(u_i)_{i\in I}$ (resp. $(v_j)_{j\in J}$). Assume $1 \in E_1$ and $1 \in E_2$. Then the following assertions are equivalent:*

(i) *The inclusion map $E_1 \otimes_{\min} E_2 \to A_1 \otimes_{\max} A_2$ is completely isometric;*

(ii) $A_1 \otimes_{\min} A_2 = A_1 \otimes_{\max} A_2$.

We will use several elementary facts, as follows. The first one is a well known property of unitary dilations.

LEMMA 1.2. *Let $u \in B(\mathcal{H}), \widehat{u} \in B(\widehat{H})$ be unitaries and let $S : \mathcal{H} \to \widehat{H}$ be an isometry with range $K \subset \widehat{H}$, such that*

$$u = S^* \widehat{u} S.$$

Then $K = S(\mathcal{H})$ is invariant under $\widehat{u}$ and $\widehat{u}^*$, so that $\widehat{u}$ commutes with P_K.

Proof. Note that $P_K = SS^*$. We have $\|P_K \widehat{u} S h\| = \|S^* \widehat{u} S h\|$, $\forall h \in \mathcal{H}$, hence

$$\|\widehat{u} S(h)\|^2 = \|S^* \widehat{u} S h\|^2 + \|(1 - P_K)\widehat{u} S h\|^2$$

hence

$$\|(1 - P_K)\widehat{u} S h\|^2 = \|\widehat{u} S h\|^2 - \|S^* \widehat{u} S h\|^2$$

so that if $S^* \widehat{u} S$ is isometric this is $= 0$. Taking adjoints we obtain the same for $\widehat{u}^*$. ∎

LEMMA 1.3. *Let $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ be finitely supported families of operators in $B(\mathcal{H})$. We have*

$$\Big\|\sum_{i \in I} a_i b_i\Big\| \leqslant \Big\|\sum a_i a_i^*\Big\|^{\frac{1}{2}} \Big\|\sum b_i^* b_i\Big\|^{\frac{1}{2}}. \tag{1.1}$$

Proof. This is an easy consequence of the Cauchy-Schwarz inequality and is left to the reader. ∎

REMARK 1.4. For any unitary u_i, the norm $\|\sum a_i u_i b_i\|$ is $\leqslant$ the right side of (1.1). However, the norm of $\sum_{i \in I} a_i b_i u_i$ can be larger in general.

The next result is known, I personally learned this kind of result from Paulsen (see e.g. [20]).

LEMMA 1.5. *Let F be a free group. Let $(U_i)_{i \in I}$ be the free unitary generators of $C^*(F)$ (i.e. these are the unitaries corresponding to the free generators of F in the universal unitary representation of F). Let $(x_i)_{i \in I}$ be a finitely supported family in $B(H)$. Then the following are equivalent:*

(i) *The linear map $T : \ell_\infty(I) \to B(H)$ defined by $T((\alpha_i)_{i \in I}) = \sum_{i \in I} \alpha_i x_i$ satisfies*

$$\|T\|_{\mathrm{cb}} < 1.$$

(ii) *We have*

$$\Big\|\sum_{i \in I} U_i \otimes x_i\Big\|_{C^*(F) \otimes_{\min} B(H)} < 1.$$

(iii) *There are operators a_i, b_i in $B(H)$ such that $x_i = a_i b_i$ and*

$$\Big\|\sum a_i a_i^*\Big\|^{\frac{1}{2}} \Big\|\sum b_i^* b_i\Big\|^{\frac{1}{2}} < 1.$$

Moreover, the same remains true if we add the unit element to the family $(U_i)_{i \in I}$.

Proof. It is easy to check, going back to the definitions of both sides, that

$$\|T\|_{\mathrm{cb}} = \left\| \sum U_i \otimes x_i \right\|_{\min}.$$

We leave this as an exercise for the reader. This shows the equivalence of (i) and (ii).

Now assume (i). By the factorization of c.b. maps we can write $T(\alpha) = V^*\pi(\alpha)W$ where $\pi : \ell_\infty(I) \to B(\widehat{H})$ is a representation and where V, W are in $B(H, \widehat{H})$ with $\|V\|\,\|W\| = \|T\|_{\mathrm{cb}}$. We can assume I finite and $\widehat{H} = H$. Let $(e_i)_{i\in I}$ be the canonical basis of $\ell_\infty(I)$, we set

$$a_i = V^*\pi(e_i) \quad \text{and} \quad b_i = \pi(e_i)W.$$

It is then easy to check (iii). Finally, the implication (iii) $\Rightarrow$ (ii) follows from Lemma 1.3 (applied to $U_i \otimes a_i$ and $\mathbf{1} \otimes b_i$). The last assertion follows from the next remark. ∎

REMARK 1.6. Let A be a C^*-algebra and let $(a_i)_{i\in I}$ be a finitely supported family of elements of A. Let F be a free group freely generated by a family $(g_i)_{i\in I}$ and let U_i be the associated unitaries in $C^*(F)$. Fix $i_0 \in I$ and let $I' = I \setminus \{i_0\}$. We wish to verify here that if α is either the minimal or the maximal C^*-norm on $C^*(F) \otimes A$ we have

$$\alpha\left(\mathbf{1} \otimes a_{i_0} + \sum_{i\in I'} U_i \otimes a_i\right) = \alpha\left(\sum_{i\in I} U_i \otimes a_i\right). \tag{1.2}$$

This can be justified as follows. (This kind of result was also pointed out to me by Vern Paulsen.)

Consider the family $(\gamma_i)_{i\in I}$ in F defined as follows

$$\gamma_i = g_{i_0}^{-1} g_i, \ \forall i \in I' \quad \text{and} \quad \gamma_{i_0} = g_{i_0}.$$

We claim that $(\gamma_i)_{i\in I}$ are again free in F and generate F. This is easy and left to the reader. It follows that the map which takes each g_i to γ_i extends to an automorphism $h : F \to F$. This automorphism h induces an isometric unital representation $\pi : C^*(F) \to C^*(F)$ which takes U_i to $U_{i_0}^* U_i$ for all i in I'. Now let $L : C^*(F) \to C^*(F)$ be the operation of left multiplication by U_{i_0}. Then the composition $L\pi \otimes I_A$ clearly is isometric with respect to the minimal or the maximal C^*-norm but it preserves $U_i \otimes a_i$ for all i in I' and takes $\mathbf{1} \otimes a_{i_0}$ to $U_{i_0} \otimes a_{i_0}$. This yields (1.2). ∎

The following simple fact is essential in our argument.

PROPOSITION 1.7. *Let A, B be two unital C^*-algebras. Let $(u_i)_{i\in I}$ be a family of unitary elements of A generating A as a unital C^*-algebra (i.e. the smallest unital C^*-subalgebra of A containing them is A itself). Let $E \subset A$ be the linear span of $(u_i)_{i\in I}$ and $\mathbf{1}_A$. Let $T : E \to B$ be a linear operator such that $T(\mathbf{1}_A) = \mathbf{1}_B$ and taking each u_i to a unitary in B. Then, $\|T\|_{\mathrm{cb}} \leqslant 1$ suffices to ensure that T extends to a (completely) contractive representation (= $*$-homomorphism) from A to B.*

Proof. Consider B as embedded into $B(\mathcal{H})$. Then, it clearly suffices to prove this statement for $B = B(\mathcal{H})$, which we now assume. By the Arveson-Wittstock extension theorem (cf. [18], p. 100), T extends to a complete contraction $\widehat{T} : A \to B(\mathcal{H})$. Since T is assumed unital, $\widehat{T}$ is unital, hence by a well known result (cf. e.g. [18]) $\widehat{T}$ must be completely positive, and more precisely (cf. [18]) we can write

$$\widehat{T}(x) = S^*\widehat{\pi}(x)S$$

where $\widehat{\pi} : A \to B(\widehat{H})$ is a unital representation (= $*$-homomorphism) and $S : \mathcal{H} \to \widehat{H}$ is an isometry. Now for any unitary U in the family $(u_i)_{i\in I}$, we have $T(U) = \widehat{T}(U) = S^*\widehat{\pi}(U)S$, hence by Lemma 1.2, since $T(U)$ is unitary by assumption, if $K = S(\mathcal{H})$ then P_K commutes with $\widehat{\pi}(U)$. Now since these operators U generate A, this implies that P_K commutes with $\widehat{\pi}(A)$, so that $\widehat{T}$ is actually a $*$-homomorphism. Thus, $\widehat{T}$ is an extension of T and a (contractive) $*$-homomorphism. This completes the proof. [Note that, a posteriori, the Arveson-Wittstock completely contractive extension $\widehat{T}$ is unique. In short, the proof reduces to this: the multiplicative domain of $\widehat{T}$ is a unital C^*-algebra (cf. [2]) and contains $(u_i)_{i\in I}$, hence it is equal to A.] ∎

REMARK 1.8. We will apply Proposition 1.7 in the following particular situation. Let $\mathcal{A} \subset A$ be the (dense) unital $*$-algebra generated by E. Consider a unital $*$-homomorphism $u : \mathcal{A} \to B$. Then $\|u|E\|_{\mathrm{cb}} \leqslant 1$ suffices to ensure that u extends to a (completely) contractive representation (= $*$-homomorphism) from the whole of A to B.

REMARK 1.9. In the same situation as in Proposition 1.7, note that, if T is a complete isometry, then $\widehat{T}$ is a faithful representation onto the C^*-algebra B_1 generated by the range of T. Indeed, by Proposition 1.7 applied to T^{-1}, $\widehat{T} : A \to B_1$ is left invertible. This can be used to give a very simple proof of the fact due to Choi ([3]) that the full C^*-algebra of any free group admits a faithful representation into a direct sum of matrix algebras. By Proposition 1.7, it suffices to check this on the free generators and this is quite easy.

Proof of Theorem 1.1. The implication (ii) $\Rightarrow$ (i) is trivial, so we prove only the converse. Assume (i). Let $E = E_1 \otimes_{\min} E_2$. We view E as a subspace of $A = A_1 \otimes_{\min} A_2$. By (i), we have an inclusion map $T : E_1 \otimes_{\min} E_2 \to A_1 \otimes_{\max} A_2$ with $\|T\|_{cb} \leqslant 1$. By Proposition 1.7, T extends to a (contractive) representation $\widehat{T}$ from $A_1 \otimes_{\min} A_2$ to $A_1 \otimes_{\max} A_2$. Clearly $\widehat{T}$ must preserve the algebraic tensor products $A_1 \otimes \mathbf{1}$ and $\mathbf{1} \otimes A_2$, hence also $A_1 \otimes A_2$. Thus we obtain (ii). ∎

REMARK 1.10. Let us denote by $E_1 \otimes \mathbf{1} + \mathbf{1} \otimes E_2$ the linear subspace spanned by elements of $A_1 \otimes A_2$ of the form $\{a_1 \otimes \mathbf{1} + \mathbf{1} \otimes a_2\}$. Then, in the situation of Theorem 1.1, $E_1 \otimes \mathbf{1} + \mathbf{1} \otimes E_2$ generates $A_1 \otimes_{\min} A_2$, so that it suffices for the conclusion of Theorem 1.1 to assume that the operator space structures induced on $E_1 \otimes \mathbf{1} + \mathbf{1} \otimes E_2$ by the min and max norms coincide.

Proof of Kirchberg's Theorem 0.1. Let $A_1 = C^*(F)$, $A_2 = B(H)$. We take $E_2 = B(H)$ and let E_1 be the linear span of the unit and the free unitary generators $(U_i \mid i \in I)$ of $C^*(F)$ (i.e. associated to the free generators of F).

Consider $x \in E_1 \otimes E_2$, with $\|x\|_{\min} < 1$. By Lemma 1.5 we can write $x = \sum_{i \in I} U_i \otimes x_i$ with $x_i \in B(H)$, $(x_i)_{i \in I}$ finitely supported, admitting a decomposition as $x_i = a_i b_i$ with $\|\sum a_i a_i^*\| < 1$, $\|\sum b_i^* b_i\| < 1$, $a_i, b_i \in B(H)$. Now, let $\pi : A_1 \otimes_{\max} A_2 \to B(\mathcal{H})$ be any faithful representation. Let $\pi_1 = \pi|A_1 \otimes \mathbf{1}$ and $\pi_2 = \pi|\mathbf{1} \otimes A_2$. We have

$$\pi(x) = \sum_{i \in I} \pi_1(U_i)\pi_2(x_i) = \sum_{i \in I} \pi_1(U_i)\pi_2(a_i)\pi_2(b_i)$$

hence, since π_1 and π_2 have commuting ranges we have $\pi(x) = y$ with

$$y = \sum_{i \in I} \pi_2(a_i)\pi_1(U_i)\pi_2(b_i).$$

Now by Lemma 1.3 (and the remark following it) we have

$$\|y\| \leqslant \Big\|\sum_{i \in I} \pi_2(a_i)\pi_2(a_i)^*\Big\|^{\frac{1}{2}} \Big\|\sum \pi_2(b_i)^*\pi_2(b_i)\Big\|^{\frac{1}{2}} < 1.$$

Hence we conclude that

$$\|x\|_{\max} = \|\pi(x)\| < 1.$$

This shows that the min and max norms coincide on $E_1 \otimes B(H)$, but since $M_n(B(H)) \approx B(H)$ for any n, this implies "automatically" that the inclusion

$$E_1 \otimes_{\min} B(H) \to A_1 \otimes_{\max} B(H)$$

is completely isometric. In other words, the operator space structures associated to the min and max norms coincide. Thus, we conclude by Theorem 1.1. ∎

We can prove the following extension of Kirchberg's theorem.

THEOREM 1.11. *Let $(A_i)_{i\in I}$ be a family of C^*-algebras (resp. unital C^*-algebras). Assume that for each i in I*

$$A_i \otimes_{\min} B(H) = A_i \otimes_{\max} B(H). \tag{1.3}$$

We will denote by $\dot{*}_{i\in I} A_i$ (resp. by $*_{i\in I} A_i$) their free product in the category of C^*-algebras (resp. in the category of unital C^*-algebras). Then we have

$$\left(\dot{*}_{i\in I} A_i\right) \otimes_{\min} B(H) = \left(\dot{*}_{i\in I} A_i\right) \otimes_{\max} B(H), \tag{1.4}$$

and in the unital case

$$\left(*_{i\in I} A_i\right) \otimes_{\min} B(H) = \left(*_{i\in I} A_i\right) \otimes_{\max} B(H). \tag{1.4}$$

REMARK 1.12. Kirchberg's theorem for $F = F_I$ corresponds to $A_i = C^*(\mathbf{Z})$ for all i in I (in the unital case).

The next result is well known, by now. It is a corollary of the Paulsen-Smith extension of the Christensen-Sinclair factorization of bilinear maps. We only sketch the standard argument. We denote by $(y_1, y_2) \to y_1 \odot y_2$ the natural bilinear map from $A_1 \otimes B(H) \times A_2 \otimes B(H)$ to $(A_1 \otimes A_2) \otimes B(H)$ which takes

$$(a_1 \otimes b_1, a_2 \otimes b_2) \quad \text{to} \quad a_1 \otimes a_2 \otimes b_1 b_2.$$

LEMMA 1.13. *Let A_1, A_2 be two operator spaces and let $y \in A_1 \otimes A_2 \otimes B(H)$, with H infinite dimensional. Then $\|y\|_{(A_1\otimes_h A_2)\otimes_{\min} B(H)} < 1$ iff there is a factorization*

$$y = y_1 \odot y_2$$

with $y_i \in A_i \otimes B(H)$ such that $\|y_i\|_{\min} < 1$ for $i = 1, 2$.

Proof. (Sketch) We may assume A_1, A_2 both finite dimensional. Then, by the self duality (and other properties) of the Haagerup tensor product (cf. [8] and [1]), y can be identified with a linear map $\tilde{y} : A_1^* \otimes_h A_2^* \to B(H)$ with $\|\tilde{y}\|_{\text{cb}} < 1$.

By the factorization theorem (cf. [21]) we have maps $\sigma_i : A_i^* \to B(\widehat{H})$ with $\|\sigma_i\|_{\text{cb}} < 1$ and operators V, W in the unit ball of $B(H, \widehat{H})$ such that $\tilde{y}(\xi_1 \otimes \xi_2) = W^* \sigma_1(\xi_1)\sigma_2(\xi_2)V$. But since H is infinite dimensional, we can assume $\widehat{H} = H$ and (incorporating W^* and V into σ_1 and σ_2) we can get rid of W^* and V. Thus we obtain $\tilde{y}_i : A_i^* \to B(H)$ with $\|\tilde{y}_i\|_{\text{cb}} < 1$ such that

$$\tilde{y}(\xi_1 \otimes \xi_2) = \tilde{y}_1(\xi_1)\tilde{y}_2(\xi_2), \quad \forall \xi_i \in A_i^*. \tag{1.5}$$

Returning to the tensor products, $\tilde{y}_i$ corresponds to $y_i \in A_i \otimes B(H)$ with $\|y_i\|_{\min} < 1$ and (1.5) means that

$$y = y_1 \odot y_2.$$

This yields the desired factorization. ∎

The key point is the following important observation concerning the Haagerup tensor product (part (ii) in Lemma 1.14 is perhaps the main new idea of this paper).

LEMMA 1.14. *Let A_1, A_2 be two C^*-algebras (resp. unital) satisfying* (1.3). *Let $A_1\dot{*}A_2$ (resp. $A_1 * A_2$) be their free product (resp. free product as unital C^*-algebras) and let $E \subset A_1\dot{*}A_2$ (resp. $E \subset A_1 * A_2$) be the linear span in $A_1\dot{*}A_2$ (resp. $A_1 * A_2$) of all elements of the form a_1a_2 with $a_i \in A_i$. Then:*

(i) *The mapping $p : a_1 \otimes a_2 \to a_1a_2$ is a complete isometry of $A_1 \otimes_h A_2$ onto the closure of E in $A_1\dot{*}A_2$ (resp. $A_1 * A_2$);*

(ii) *The min and max norms of $(A_1\dot{*}A_2) \otimes B(H)$ (resp. $(A_1 * A_2) \otimes B(H)$) coincide on $E \otimes B(H)$.*

Proof. The assertion (i) is essentially known. It is proved in [4] for the non-unital free product, and nothing is said there about the unital case. However, when A_1, A_2 are unital, the argument of [4] can be pursued to yield (i) as stated above. A similar argument is used in [13]. As far as we know, this question is nowhere considered (except for [13]). Therefore, we decided to include the details: we will now verify (i) in the unital case, starting from the non-unital case, which is treated in [4].

Consider $x = \sum a_i^1 \otimes a_i^2$ in $A_1 \otimes A_2$. By [4] we have

$$\|x\|_h = \sup\left\{\left\|\sum \sigma_1(a_i^1)\sigma_2(a_i^2)\right\|\right\} \tag{1.6}$$

where the supremum runs over all pairs $\sigma_i : A_i \to B(\widehat{H})$ of (not necessarily unital) $*$-homomorphisms, with $\widehat{H}$ an arbitrary Hilbert space. Now assume A_1, A_2 unital. Note that (by considering e.g. $A_1 \otimes_{\min} A_2$) we know that there exists a pair $\pi_i : A_i \to B(\mathcal{H})$ of unital faithful representations on the same Hilbert space $\mathcal{H}$.

Consider σ_1, σ_2 and x as above. We need to show that (1.6) can be rewritten with unital representations. Let $p = \sigma_1(1)$ and $q = \sigma_2(1)$. Note that by augmenting $\widehat{H}$ if necessary we can assume that $(1-p)(\widehat{H})$ and $(1-q)(\widehat{H})$ are of the same Hilbertian dimension, and that they are both isometric to some direct sum of copies of $\mathcal{H}$.

This allows us to define (using π_1 and π_2) $*$-homomorphisms $\widehat{\pi}_i : A_i \to B(\widehat{H})$ such that

$$\widehat{\pi}_1(1) = 1 - p \quad \text{and} \quad \widehat{\pi}_2(1) = 1 - q.$$

Then we define for $a_i \in A_i$

$$\widehat{\sigma}_1(a_1) = p\sigma_1(a_1)p + (1-p)\widehat{\pi}_1(a_1)(1-p)$$
$$\widehat{\sigma}_2(a_2) = q\sigma_2(a_2)q + (1-q)\widehat{\pi}_2(a_2)(1-q).$$

We have (note $\sigma_1(a_1)p = p\sigma_1(a_1) = \sigma_1(a_1)\ldots$)

$$\sum \sigma_1(a_i^1)\sigma_2(a_i^2) = p\sum \widehat{\sigma}_1(a_i^1)\widehat{\sigma}_2(a_i^2)q$$

hence

$$\left\|\sum \sigma_1(a_i^1)\sigma_2(a_i^2)\right\| \leqslant \left\|\sum \widehat{\sigma}_1(a_i^1)\widehat{\sigma}_2(a_i^2)\right\|$$

but now $\widehat{\sigma}_1, \widehat{\sigma}_2$ are unital representations (= $*$-homomorphisms), hence this yields

$$\|x\|_h \leqslant \|x\|_{A_1 * A_2}.$$

Since the converse is clear (using (1.6)), this shows that $p : E_1 \otimes_h E_2 \to A_1 * A_2$ is isometric.

The proof that it is completely isometric is the same with "operator coefficients" instead of scalars, we leave this to the reader.

We now turn to part (ii). Consider x in $E \otimes B(H)$ with $\|x\|_{\min} < 1$. By (i), x corresponds (via p) to an element y in $A_1 \otimes A_2$ with $\|y\|_{(A_1 \otimes_h A_2)\otimes_{\min} B(H)} < 1$. By Lemma 1.13, we have $y = y_1 \odot y_2$ with

$$y_i \in A_i \otimes B(H) \quad \text{and} \quad \|y_i\|_{\min} < 1.$$

We can write

$$y_i = \sum_k a_i^k \otimes b_i^k$$

with $a_i^k \in A_i, b_i^k \in B(H)$ and

$$x = \sum_{k,\ell} a_1^k a_2^\ell \otimes b_1^k b_2^\ell.$$

Now consider an isometric representation

$$\pi : (A_1 * A_2) \otimes_{\max} B(H) \to B(\mathcal{H})$$

and let σ_1, σ_2 and ρ be its restrictions respectively to $A_1 \otimes \mathbf{1}$, $A_2 \otimes \mathbf{1}$ and $\mathbf{1} \otimes B(H)$. We have (since the ranges of ρ and σ_2 commute)

$$\begin{aligned}\pi(x) &= \sum_{k,\ell} \sigma_1(a_1^k)\sigma_2(a_2^\ell)\rho(b_1^k)\rho(b_2^\ell)\\ &= \left(\sum_k \sigma_1(a_1^k)\rho(b_1^k)\right)\left(\sum_\ell \sigma_2(a_2^\ell)\rho(b_2^\ell)\right)\\ &= \pi(y_1)\pi(y_2).\end{aligned}$$

Hence we conclude that

$$\begin{aligned}\|x\|_{\max} = \|\pi(x)\| &\leqslant \|\pi(y_1)\|\,\|\pi(y_2)\| \\ &\leqslant \|y_1\|_{\max}\|y_2\|_{\max}\end{aligned}$$

hence by our assumption on A_1 and A_2

$$\leqslant \|y_1\|_{\min}\|y_2\|_{\min} < 1.$$

This shows by homogeneity that $\|x\|_{\max} \leqslant \|x\|_{\min}$.

Finally, to prove (ii) in the non-unital case, we simply replace A_1, A_2 by their unitizations, which clearly still satisfy (1.3). Then, by the unital case, $A_1 \dot{*} A_2$ appears (see the next remark) as an ideal in a unital C^*-algebra A such that $A \otimes_{\min} B(H) = A \otimes_{\max} B(H)$. But, as is classical, this property is inherited by closed ideals (since if I is an ideal in A and B is any other C^*-algebra, then the inclusion $I \otimes_{\max} B \subset A \otimes_{\max} B$ is isometric). ∎

REMARK 1.15. Let us denote by $\tilde{A}$ the unitization of a C^*-algebra A. Let $(A_i)_{i\in I}$ be a family of C^*-algebras. Then it is easy to check that the unitization of $\dot{*}_{i\in I} A_i$ can be identified canonically with $*_{i\in I} \tilde{A}_i$, in short we have

$$\Big(\dot{*}_{i\in I} A_i\Big)^{\sim} \simeq *_{i\in I} \tilde{A}_i.$$

Nevertheless, we do not see how to deduce from this the passage from the non-unital case (treated in [4]) to the unital one, in the first part of the preceding lemma.

Proof of Theorem 1.11. It clearly suffices to prove (1.4) in case I is finite, hence by iteration we may as well assume that $I = \{1, 2\}$. Let E be as in Lemma 1.14. We will apply Theorem 1.1 to the subspace $E \otimes B(H) \subset (A_1 * A_2) \otimes B(H)$. By Lemma 1.14, the assertion (i) in Theorem 1.1 is satisfied in this case (with E_1, E_2 now replaced by $E, B(H)$). Hence, by Theorem 1.1, we have (1.4). ∎

Let C, A be C^*-algebras. We will denote by $\mathrm{CP}(C, A)$ the set of all completely positive (in short c.p.) maps from C to A. A linear map $u : C \mapsto A$ is called decomposable if it is a linear combination of completely positive maps, i.e. it can be written as $u = u_1 - u_2 + i(u_3 - u_4)$ with all u_i's completely positive. We denote by $\mathrm{D}(C, A)$ the set of all such maps. This set could be normed by defining for instance $\|u\| = \inf\Big\{\sum_{j=1}^{4} \|u_j\|\Big\}$ but such a definition is not consistent with the algebraic context. Instead, we will use the following definition due to Haagerup:

we define $\|u\|_{\rm dec}$ as the smallest $\lambda \geqslant 0$ such that there exist S_1, S_2 in $\mathrm{CP}(C, A)$ such that $\|S_i\| \leqslant \lambda$, $i = 1, 2$, and such that

$$x \to \begin{pmatrix} S_1(x) & u(x^*)^* \\ u(x) & S_2(x) \end{pmatrix}$$

is a completely positive map from C to $M_2(A)$. If u is not decomposable, we set $\|u\|_{\rm dec} = \infty$.

Haagerup ([12]) proved that, equipped with this norm, $\mathrm{D}(C, A)$ becomes a Banach space. Moreover, he proved $\|u\|_{\rm cb} \leqslant \|u\|_{\rm dec}$ with equality when u is c.p. Also, if u is self-adjoint, then

$$\|u\|_{\rm dec} = \inf\{\|u_1 + u_2\| \,|\, u = u_1 - u_2,\ u_i \in \mathrm{CP}(C, A)\}.$$

The reader should recall (cf. [18]) that $\|u\| = \|u\|_{\rm cb}$ (resp. $= \|u(1)\|$) for any c.p. map $u : C \to A$ (resp. when C is assumed unital), and also that, if C is abelian, then a map $u : C \to A$ is c.p. iff it is *positive* in the usual sense (= positivity preserving).

Curiously, the dec norm admits several slightly different descriptions. We start with the most convenient one.

LEMMA 1.16. *Let $x_1, \ldots, x_n$ be elements in a C^*-algebra A and let $u : \ell_\infty^n \to A$ be the linear map defined by $u((\alpha_i)) = \sum \alpha_i x_i$. Then*

$$\|u\|_{\rm dec} = \inf\left\{\left\|\sum a_i a_i^*\right\|^{\frac12} \left\|\sum b_i^* b_i\right\|^{\frac12}\right\} \tag{1.7}$$

where the infimum runs over all the decompositions $x_i = a_i b_i$ with $a_i \in A$ and $b_i \in A$ $(i = 1, \ldots, n)$.

Proof. If u is positive, i.e. if $x_i \geqslant 0$ for all i, this is very easy: the optimal decomposition is simply $x_i = x_i^{1/2} x_i^{1/2}$.

Let us denote temporarily by $|||(x_i)|||$ the right side of (1.7). Assume first that $\|u\|_{\rm dec} < 1$. Then going back to the definition of the dec-norm, we can find $y_i, z_i \geqslant 0$ in A with $y_i, z_i \geqslant 0$, $\|\sum y_i\| < 1$, $\|\sum z_i\| < 1$ and such that

$$t_i = \begin{pmatrix} y_i & x_i^* \\ x_i & z_i \end{pmatrix} \geqslant 0 \quad \text{for all } i.$$

Then we have (rectangular matrix product)

$$x_i = (0, 1)\, t_i \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Let $\gamma_i = (0,1)t_i^{1/2} \in M_{1,2}(A)$ and $\delta_i = t_i^{1/2}\begin{pmatrix}1\\0\end{pmatrix} \in M_{2,1}(A)$. We have $x_i = \gamma_i\delta_i$ and

$$\left\|\sum \gamma_i\gamma_i^*\right\| = \left\|\sum z_i\right\| < 1, \quad \left\|\sum \delta_i^*\delta_i\right\| = \left\|\sum y_i\right\| < 1.$$

Let $\gamma_i = (c_i, d_i)$ and $\delta_i = \begin{pmatrix}r_i\\s_i\end{pmatrix}$ so that $x_i = c_ir_i + d_is_i$ with

$$\left\|\sum c_ic_i^* + d_id_i^*\right\| < 1 \quad \text{and} \quad \left\|\sum r_i^*r_i + s_i^*s_i\right\| < 1.$$

Assume A unital for simplicity. Let $\varepsilon > 0$ and let

$$a_i = (c_ic_i^* + d_id_i^* + \varepsilon 1)^{\frac{1}{2}} \quad \text{and} \quad b_i = a_i^{-1}x_i.$$

We can choose $\varepsilon > 0$ small enough so that

$$\left\|\sum a_ia_i^*\right\| < 1.$$

We have then $b_i = (a_i^{-1}c_i, a_i^{-1}d_i)\delta_i$, hence $b_i^*b_i = \delta_i^*\omega_i^*\omega_i\delta_i$, where $\omega_i = (a_i^{-1}c_i, a_i^{-1}d_i)$. Note that

$$\omega_i\omega_i^* = a_i^{-1}(c_ic_i^* + d_id_i^*)a_i^{-1} \leqslant a_i^{-1}(a_i^2)a_i^{-1} = 1$$

hence

$$b_i^*b_i \leqslant \delta_i^*\delta_i = r_i^*r_i + s_i^*s_i,$$

hence $\|\sum b_i^*b_i\| < 1$, so that we conclude $|||(x_i)||| < 1$. By homogeneity, this completes the proof. ∎

As a consequence, we easily derive the following (known) fact.

LEMMA 1.17. *Let $n \geqslant 1$. Let A be a C^*-algebra. Then we have an isometric identity*

$$\text{(1.8)} \qquad \mathrm{D}(\ell_\infty^n, A)^{**} = \mathrm{D}(\ell_\infty^n, A^{**}).$$

REMARK 1.18. The reader should be warned that the analogous identity $\mathrm{cb}(\ell_\infty^n, A)^{**} = \mathrm{cb}(\ell_\infty^n, A^{**})$ fails to be isometric in general when $n > 2$.

REMARK 1.19. Concerning the max-norm, we will use several times the following two known basic facts (see [17], [6]):

(i) For any C^*-algebras A, C, we have a natural isometric embedding $C \otimes_{\max} A \to C \otimes_{\max} A^{**}$ (cf. e.g. [24], p. 13).

(ii) Let C, B, A be C^*-algebras and let $\varphi : A \to B$ be a completely positive contraction. Then $I_C \otimes \varphi : C \otimes_{\max} A \to C \otimes_{\max} B$ is a completely positive contraction (cf. e.g. [24], p. 11).

Proof of Lemma 1.17. This statement reduces to the following fact: given an n-tuple (x_i) in A^{**} we have $|||(x_i)||| \leqslant 1$ iff there is a net (x_i^α) of n-tuples in A with $|||(x_i^\alpha)||| \leqslant 1$ such that $x_i^\alpha \to x_i$ $\sigma(A^{**}, A^*)$ for each i. The if part is easy and left to the reader. To prove the only if part assume without loss of generality that $|||(x_i)||| < 1$ so that $x_i = a_i b_i$ with a_i, b_i in A^{**} such that $\|\sum a_i a_i^*\| < 1$, $\|\sum b_i^* b_i\| < 1$. Let $a \in M_n(A^{**})$ (resp. $b \in M_n(A^{**})$) be the matrix admitting (a_i) (resp. (b_i)) as its first row (resp. column) and zero elsewhere. Let a^α (resp. b^α) be a net in the unit ball of $M_n(A)$ tending $\sigma(M_n(A)^{**}, M_n(A)^*)$ to be a (resp. b). By Kaplansky's theorem we can even find a net for which the convergence is in the strong sense. We define $x_i^\alpha = a^\alpha(1, i) b^\alpha(i, 1)$. Clearly $|||(x_i^\alpha)||| \leqslant 1$ and $x_i^\alpha \to x_i$ in the $\sigma(A^{**}, A^*)$-sense. (The strong convergence of a^α, b^α implies the weak convergence of x^α; moreover on bounded sets weak and σ-weak convergences coincide.) ∎

We include here the following simple observation, which is implicit in [12], Lemma 3.5.

LEMMA 1.20. *Let F be a free group and let $(U_i)_{i\in I}$ be the family of free unitary operators in $C^*(F)$ associated to the generators, to which we add the unit element. Let $(a_i)_{i\in I}$ be a finitely supported family in a C^*-algebra A and let $a : \ell_\infty(I) \to A$ be the mapping defined by $a((\alpha_i)_{i\in I}) = \sum_{i\in I} \alpha_i a_i$. Then we have*

$$\left\| \sum_{i\in I} U_i \otimes a_i \right\|_{C^*(F)\otimes_{\max} A} = \|a\|_{\mathrm{dec}}. \tag{1.9}$$

Proof. Let A be a σ-finite (= countably decomposable) von Neumann algebra. By classical facts (cf. e.g. [9]) we may assume A realized as a concrete subalgebra $N \subset B(H)$ and admitting a (cyclic and) separating vector. Then by [12], Lemma 3.5 we have

$$\|a\|_{\mathrm{dec}} = \sup\left\{ \left\| \sum v_i a_i \right\| \right\} \tag{1.10}$$

where the supremum runs over all unitaries $v_1, \ldots, v_n$ in N'. Equivalently, if we introduce the representation

$$\pi : C^*(F) \otimes N \to N' \otimes N \subset B(H \otimes H)$$

defined by $\pi(U_i \otimes a) = v_i a$, $\forall a \in N$ then we have

$$\|a\|_{\mathrm{dec}} = \left\|\pi\left(\sum U_i \otimes a_i\right)\right\|. \tag{1.11}$$

This immediately implies

$$\|a\|_{\mathrm{dec}} \leqslant \left\|\sum U_i \otimes a_i\right\|_{\max}. \tag{1.12}$$

This proves (1.12) in the σ-finite von Neumann case. By a standard direct sum argument, it is easy to extend (1.12) to the case of general von Neumann algebras. But, since the inclusion $C \otimes_{\max} A \subset C \otimes_{\max} A^{**}$ is isometric (see Remark 1.19) and since (1.8) holds we obtain (1.12) for a general C^*-algebra as a consequence of Lemma 1.17. The converse inequality

$$\left\|\sum U_i \otimes a_i\right\|_{\max} \leqslant \|a\|_{\mathrm{dec}}, \tag{1.13}$$

is actually proved in [12], Lemma 3.5. Indeed, the same argument used there shows (without any restriction on A) that for any representation $\rho : A \to B(\mathcal{H})$ and for any v_i in $\rho(A)'$ with $\|v_i\| \leqslant 1$ we have

$$\left\|\sum v_i \rho(a_i)\right\| \leqslant \|a\|_{\mathrm{dec}}.$$

This implies in particular (1.13). This completes the proof. ∎

An alternate proof of (1.13) can also be deduced from (1.7), as in the above proof of Theorem 0.1.

The following fact is due to Kirchberg in [15] (and I believe the simple proof which follows was known to Kirchberg).

LEMMA 1.21. *Let A be a C^*-algebra. Assume that*

$$C^*(F_\infty) \otimes_{\min} A = C^*(F_\infty) \otimes_{\max} A. \tag{1.14}$$

Then A is WEP.

Proof. By Lance's results ([17]), we know that A has WEP iff for any embedding $A \to B$ of A into a C^*-algebra B and for any C^*-algebra C we have an injective (= isometric) morphism

$$C \otimes_{\max} A \to C \otimes_{\max} B.$$

Note that this is obvious for the min-norms, therefore if $C \otimes_{\max} A = C \otimes_{\min} A$, this is certainly true. Hence, by assumption, this holds whenever $C = C^*(F)$ with F a free (discrete) group.

Now consider C arbitrary. Let F be a free group large enough so that there is a surjective representation $q : C^*(F) \to C$. Let J be the kernel of q. Then by the exactness properties of the max-tensor product (see e.g. [24]), we have exact sequences

$$0 \longrightarrow J \otimes_{\max} A \longrightarrow C^*(F) \otimes_{\max} A \longrightarrow C \otimes_{\max} A \longrightarrow 0$$

$$0 \longrightarrow J \otimes_{\max} B \longrightarrow C^*(F) \otimes_{\max} B \longrightarrow C \otimes_{\max} B \longrightarrow 0 .$$

By the first part of the proof we have an injective (= isometric) inclusion

$$\varphi : C^*(F) \otimes_{\max} A \to C^*(F) \otimes_{\max} B.$$

Moreover, using an approximate unit in J, it is rather easy to check that if we view $J \otimes_{\max} A$ (resp. $J \otimes_{\max} B$) as included in $C^*(F) \otimes_{\max} A$ (resp. $C^*(F) \otimes_{\max} B$) then we have

$$\varphi^{-1}(J \otimes_{\max} B) = J \otimes_{\max} A. \tag{1.15}$$

Now using (1.15) and chasing diagrams it is easy to see that the injectivity of φ implies that of the natural map $C \otimes_{\max} A \to C \otimes_{\max} B$. Thus, by Lance's criterion (as mentioned above) A is WEP. ∎

LEMMA 1.22. *Let C be a C^*-algebra. If*

$$C \otimes_{\min} B(H) = C \otimes_{\max} B(H)$$

then for any WEP *C^*-algebra A*

$$C \otimes_{\min} A = C \otimes_{\max} A. \tag{1.16}$$

Proof. If A is WEP, by definition we have a factorization $A \xrightarrow{\varphi} B(H) \xrightarrow{\psi} A^{**}$ of the canonical inclusion map, with completely positive contractions φ, ψ. By Theorem 0.1, we have a contraction

$$I_C \otimes \varphi : C \otimes_{\min} A \to C \otimes_{\min} B(H) = C \otimes_{\max} B(H).$$

We follow this by $I_C \otimes \psi$ which is contractive from $C \otimes_{\max} B(H)$ to $C \otimes_{\max} A^{**}$ by Remark 1.19 (ii). Thus we have a contractive inclusion $C \otimes_{\min} A \to C \otimes_{\max} A^{**}$ which (using Remark 1.19 (i)) implies (1.16). This shows that if (1.16) holds with $A = B(H)$, then it holds whenever A is WEP. ∎

REMARK 1.23. In his paper [15], Kirchberg shows that

$$C^*(F_\infty) \otimes_{\min} A = C^*(F_\infty) \otimes_{\max} A$$

iff A is WEP. It also follows from Theorem 0.1 and the last two lemmas.

In [15], Kirchberg also proves a general result on the tensor products $C \otimes N$ when N is an *arbitrary* von Neumann algebra. In that case, (but with C an arbitrary C^*-algebra) we can define a C^*-norm $\|\cdot\|_{\text{nor}}$ on $C \otimes N$ as follows

$$\left\|\sum a_i \otimes b_i\right\|_{\text{nor}} = \sup\left\{\left\|\sum \sigma(a_i)\pi(b_i)\right\|\right\}$$

where the supremum runs over all pairs of representations $\sigma : C \to B(\mathcal{H})$, $\pi : N \to B(\mathcal{H})$ with commuting ranges and with π *normal*. We denote by $C \otimes_{\text{nor}} N$ the completion of $C \otimes N$ for this norm. (See [6] for more information.)

Our method also allows to prove Kirchberg's theorem on this tensor norm.

THEOREM 1.24. *Let F be any free group and let $C = C^*(F)$. Let N be any von Neumann algebra. Then*

$$C \otimes_{\text{nor}} N = C \otimes_{\max} N.$$

Proof. Replacing N by $M_n(N)$ and using Theorem 1.1, it clearly suffices to prove that the norms $\|\cdot\|_{\text{nor}}$ and $\|\cdot\|_{\max}$ are equal on $E \otimes N$ when E is the linear span of the unit and the free unitary generators of C. Then, we argue as in Lemma 1.20 (first assuming N σ-finite, then passing to the general case): so that by [12], Lemma 3.5 we find, by (1.10) and (1.11), that for any t in $E \otimes N$ with associated linear map $T : E^* \to N$, we have

$$\|T\|_{\text{dec}} \leqslant \|t\|_{\text{nor}}$$

hence (see Lemma 1.20), $\|t\|_{\max} \leqslant \|t\|_{\text{nor}}$. ∎

We conclude this paper with an application to the notion of exactness for C^*-algebras. Recall that a C^*-algebra (or more generally an operator space) A is called exact (see [16]) if for any (closed 2-sided) ideal $I \subset B$ in C^*-algebra B, we have an isomorphism

$$B/I \otimes_{\min} A \approx \frac{B \otimes_{\min} A}{I \otimes_{\min} A}.$$

In [22], some of Kirchberg's results on exactness are transferred to the operator space setting. Let E be a finite dimensional operator space, and let

$$u_E : B/I \otimes_{\min} E \to \frac{B \otimes_{\min} E}{I \otimes_{\min} E}$$

be the canonical isomorphism.

Let F be another finite dimensional operator space. Recall the notation

$$d_{\mathrm{cb}}(E,F)=\inf\{\|u\|_{\mathrm{cb}}\|u^{-1}\|_{\mathrm{cb}}\}$$

where the infimum runs over all possible isomorphisms $u: E \to F$. By convention we set $d_{\mathrm{cb}}(E,F)=\infty$ if E,F are not isomorphic.

We denote

$$d_{SK}(E)=\inf\{d_{\mathrm{cb}}(E,F)\,|\,F\subset K(\ell_2)\}.$$

(Here $K(\ell_2)$ denotes the algebra of all compact operators on ℓ_2.)

When $\widehat{E}$ is an infinite dimensional operator space, we define

$$d_{SK}(\widehat{E})=\sup\{d_{SK}(E)\,|\,E\subset\widehat{E},\ \dim(E)<\infty\}.$$

In [22], by a simple adaptation of an argument of Kirchberg in [16], we show that for any exact operator space E

$$d_{SK}(E)=\sup\{\|u_E\|\}=\sup\{\|u_E\|_{\mathrm{cb}}\} \tag{1.17}$$

where the supremum runs over all possible pairs (I,B) with $I\subset B$. (Actually, it suffices to consider $I=K(\ell_2)$ and $B=B(\ell_2)$). In [16], Kirchberg showed that a C^*-algebra A is exact iff $d_{SK}(A)=1$. The point of the next result is that it suffices for the exactness of A to be able to embed (almost completely isometrically) the linear span of the unitary generators of A and the unit into $K(\ell_2)$ (or into a nuclear C^*-algebra).

THEOREM 1.25. *Let $E\subset A$ be a closed subspace of a unital C^*-algebra A. We assume that $1_A\in E$ and that E is the closed linear span of a family of unitary elements of A. Moreover, we assume that E generates A (i.e. that the smallest C^*-subalgebra of A containing E is A itself). Then, if $d_{SK}(E)=1$, A is exact.*

Proof. Let (I,B) be as above with B unital. By (1.17), if $d_{SK}(E)=1$, the unital $*$-homomorphism

$$\pi: B/I\otimes A\to\frac{B\otimes_{\min}A}{I\otimes_{\min}A}$$

becomes completely contractive when restricted to $(B/I)\otimes_{\min}E$. By Proposition 1.7, π extends to a continuous (= contractive) $*$-homomorphism on $(B/I)\otimes_{\min}A$. Hence A is exact. ∎

Acknowledgements. I am grateful to Uffe Haagerup and to Eberhard Kirchberg for useful conversations concerning Proposition 1.7, during two different visits at the Fields Institute in Waterloo in the spring of 1995. Thanks also to F. Boca for a reference, to S. Wassermann for a correction and to S. Popa for a useful comment.

Partially supported by the NSF.

REFERENCES

1. D. BLECHER, V. PAULSEN, Tensor products of operator spaces, *J. Funct. Anal.* **99**(1991), 262–292.
2. M. D. CHOI, A Schwarz inequality for positive linear maps on C^*-algebras, *Illinois J. Math.* **18**(1974), 565–574.
3. M. D. CHOI, A simple C^*-algebra generated by two finite order unitaries, *Canad. J. Math.* **31**(1979), 887–890.
4. E. CHRISTENSEN, E. EFFROS, A. SINCLAIR, Completely bounded multilinear maps and C^*-algebraic cohomology, *Invent. Math.* **90**(1987), 279–296.
5. E. CHRISTENSEN, A. SINCLAIR, A survey of completely bounded operators, *Bull. London Math. Soc.* **21**(1989), 417–448.
6. E. EFFROS, C. LANCE, Tensor products of operator algebras, *Adv. Math.* **25**(1977), 1–33.
7. E. EFFROS, Z.J. RUAN, On matricially normed spaces, *Pacific J. Math.* **132**(1988), 243–264.
8. E. EFFROS, Z.J. RUAN, Self duality for the Haagerup tensor product and Hilbert space factorization, *J. Funct. Anal.* **100**(1991), 257–284.
9. U. HAAGERUP, The standard form of a von Neumann algebra, *Math. Scand.* **37**(1975).
10. U. HAAGERUP, An example of a non-nuclear C^*-algebra which has the metric approximation property, *Invent. Math.* **50**(1979), 279–293.
11. U. HAAGERUP, Decomposition of completely bounded maps on operator algebras, unpublished manuscript, 1980.
12. U. HAAGERUP, Injectivity and decomposition of completely bounded maps, in *Operator algebras and their connection with Topology and Ergodic Theory*, Lecture Notes in Math. **1132**(1985), pp. 91–116.
13. A. HARCHARRAS, On some stability properties of the full C^*-algebra associated to the free group F_∞, to appear.
14. E. KIRCHBERG, On non-semisplit extensions, tensor products and exactness of group C^*-algebras, *Invent. Math.* **112**(1993), 449–489.
15. E. KIRCHBERG, Commutants of unitaries in UHF algebras and functorial properties of exactness, *J. Reine Angew. Math.* **452**(1994), 39–77.
16. E. KIRCHBERG, On subalgebras of the CAR-algebra, *J. Funct. Anal.* **129**(1995), 35–63.
17. C. LANCE, On nuclear C^*-algebras, *J. Funct. Anal.* **12**(1973), 157–176.
18. V. PAULSEN, *Completely Bounded Maps and Dilations*, Pitman Res. Notes Math. Ser., vol. 146, Pitman Longman (Wiley), 1986.
19. V. PAULSEN, Representation of function algebras, abstract operator spaces and Banach space geometry, *J. Funct. Anal.* **109**(1992), 113–129.
20. V. PAULSEN, The maximal operator space of a normed space, preprint, to appear.

21. V. PAULSEN, R. SMITH, Multilinear maps and tensor norms on operator systems, *J. Funct. Anal.* **73**(1987), 258–276.
22. G. PISIER, Exact operator spaces, in *Recent Advances in Operator Algebras* (Orléans 1992) Astérisque, **232**(1995), pp. 159–187, Soc. Math. France.
23. M. RIEFFEL, Induced representations of C^*-algebras, *Adv. Math.* **13**(1974), 176–257.
24. S. WASSERMANN, Exact C^*-algebras and related topics, Lecture Notes Ser., Seoul Nat. Univ., 1994.

GILLES PISIER
Texas A & M University
College Station, TX 77843
U.S.A.

and

Université Paris VI
Equipe d'Analyse, Case 186
75252 Paris Cedex 05
FRANCE

Received September 20, 1995; revised September 29, 1995 and December 7, 1995.