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# FACTORIZATION OF SELFADJOINT OPERATOR POLYNOMIALS

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ABSTRACT. Factorization theorems are obtained for selfadjoint operator polynomials $L(\lambda) := \sum_{j=0}^{n} \lambda^j A_j$ where $A_0, A_1, \ldots, A_n$ are selfadjoint bounded linear operators on a Hilbert space $\mathcal{H}$. The essential hypotheses concern the real spectrum of $L(\lambda)$ and, in particular, ensure the existence of spectral subspaces associated with the real line for the (companion) linearization. Under suitable additional conditions, the main results assert the existence of polynomial factors (a) of degrees $[\frac{1}{2}n]$ and $[\frac{1}{2}(n+1)]$ when the leading coefficient $A_n$ is strictly positive and (b) of degree $\frac{1}{2}n$ (when $n$ is even) when $A_n$ is invertible and the spectrum of $L(\lambda)$ is real. Consequences for the factorization of regular operator polynomials (when $L(\alpha)$ is invertible for some real $\alpha$) are also discussed.

KEYWORDS: *Selfadjoint operator polynomial, spectral factorization, companion operator, supporting subspace.*

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## 1. INTRODUCTION

This paper concerns new results on the factorization of selfadjoint operator polynomials, i.e. operator valued functions of the form

$$L(\lambda) = \sum_{j=0}^{n} \lambda^j A_j, \quad \lambda \in \mathbf{C},$$

where $A_0, A_1, \ldots, A_n$ are bounded selfadjoint operators on a Hilbert space $\mathcal{H}$, and the leading coefficient $A_n$ is invertible.

As with many other papers on problems of this kind we rely on properties of the "linearization" $\lambda I - C_L$ and "symmetrizer", $G_L$, of $L(\lambda)$ defined on $\mathcal{H}^n$ by

$$(1.1)\quad G_L = \begin{bmatrix} A_1 & A_2 & \dots & A_n \\ A_2 & & & 0 \\ \vdots & \iddots & & \vdots \\ A_n & 0 & \dots & 0 \end{bmatrix}, \quad C_L = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & & 0 \\ & & & \ddots & \\ & & & & I \\ -\widehat{A}_0 & -\widehat{A}_1 & \dots & & -\widehat{A}_{n-1} \end{bmatrix},$$

and $\widehat{A}_j = A_n^{-1}A_j$, $j = 0, 1, \dots, n-1$. It is easily verified that $G_L C_L = C_L^* G_L$, i.e. $C_L$ is selfadjoint in the Krein space $(\mathcal{H}^n, [\cdot,\cdot])$ generated by the inner product

$$[x, y] = (G_L x, y),$$

$x, y \in \mathcal{H}^n$. Consequently, the spectrum of $L(\lambda)$, $\sigma(L(\lambda))$ $(= \sigma(C_L))$ is symmetric with respect to the real line.

To describe the hypotheses under which our results are obtained we must first present the notion of points of spectrum of $L(\lambda)$ of "determinate type", which was introduced and developed by the authors in [8]. It can be formulated in terms of either the pair $C_L$, $G_L$, or the polynomial $L(\lambda)$ itself (see Lemma 7 of [8]). First, a point $\lambda \in \sigma(C_L)$ is in the *approximate spectrum* of $C_L$, $\sigma_{\text{ap}}(C_L)$, if there is a sequence $\{f_n\} \subset \mathcal{H}^n$ such that

$$(1.2)\qquad \|f_n\| = 1, \quad \|C_L f_n - \lambda f_n\| \to 0$$

as $n \to \infty$. Then a point of $\sigma_{\text{ap}}(C_L)$ is said to have *determinate type* if, for any sequence $\{f_n\}$ satisfying (1.2) we have either $\underline{\lim}\, [f_n, f_n] > 0$ (when $\lambda$ is said to have *plus type*), or $\overline{\lim}[f_n, f_n] < 0$ (when $\lambda$ is said to have *minus type*). It is easy to see that points of $\sigma(C_L)$ of determinate type are necessarily real.

In terms of $L(\lambda)$ itself, a real number $\lambda_0 \in \sigma(L(\lambda))$ is an *approximate eigenvalue of determinate type* if there is a sequence $\{f_n\}$ in $H$ such that $\|f_n\| = 1$ and $L(\lambda_0)f_n \to 0$ and, furthermore, for any such sequence, either $\underline{\lim}(L'(\lambda_0)f_n, f_n) > 0$ or $\overline{\lim}(L'(\lambda_0)f_n, f_n) < 0$. These two inequalities determine points of *plus* and *minus* type, respectively. We write $\sigma_+(L(\lambda))$ and $\sigma_-(L(\lambda))$ for the (real) subsets of $\sigma(L(\lambda))$ consisting of points of plus, and minus types, respectively.

The paper is devoted to the study of the factorizations of $L(\lambda)$ when *all* real points of spectrum have determinate type, i.e. when

$$(1.3)\qquad \sigma(L(\lambda)) \cap \mathbf{R} = \sigma_+(L(\lambda)) \cup \sigma_-(L(\lambda)).$$

The first result (Theorem 3.1) is obtained under the additional assumption that $A_n \gg 0$. A special case states that, with these assumptions, $L(\lambda)$ has a monic, spectral right divisor $N(\lambda)$ such that $\sigma(N(\lambda))$ is the union of $\sigma_+(L(\lambda))$ with the spectrum of $L(\lambda)$ in the open upper half of the complex plane.

When $\mathcal{H}$ is finite dimensional such a result, even without the restriction on the real spectrum, is well-known (see [11] and [4], for example). For spaces $\mathcal{H}$ of infinite dimension Theorem 3.1 is a generalization of Theorem 9 of [8] (in which the non-real spectrum is assumed to be empty), and provides a partial answer to problems posed by Rodman (see p. 192 of [13]).

The second major result (Theorem 4.1) generalizes Theorem 9 of [8] in a different direction. As in that theorem, it is assumed that all of $\sigma(L(\lambda))$ is real and of determinate type, but the condition of definite leading coefficient is weakened, and we admit an invertible indefinite leading coefficient $A_n$. This result is part of a continuing effort to obtain factorization results for polynomials with indefinite, or not invertible leading coefficient (see [6], and [7]) and, in particular, it extends the result of Theorem 3.1 in [7].

Examples are included in Section 5 to show that theorems of similar generality which include real mixed points of spectrum, or only non-real spectrum (and no real spectrum) are unlikely.

To put these results in a historical perspective, we note the early result of Krein and Langer on factorization of quadratic operator polynomials (see [5] and also Theorem 5.9.1 of [13]). This theorem involves a compactness condition on the operator coefficients and is of quite a different character. Subsequently, major contributions were made by Langer ([11]) who, in particular, related the existence of right divisors of $L(\lambda)$ (with $A_n \gg 0$) to the existence of maximal $G_L$-nonnegative and $C_L$-invariant subspaces. Gohberg, Lancaster, and Rodman ([3], [4]), developed the "supporting subspace" idea (which also appears in Langer's work) and focused attention on spectral factorization. Our analysis depends on the major results of the last three quoted papers.

The fundamental idea behind the new results is that, when the real spectrum consists of points of determinate type, then $\sigma_+(L(\lambda))$, $\sigma_-(L(\lambda))$ and the non-real spectrum are closed sets. This is discussed in Section 2. The main results are established in Sections 3 and 4 and, in Section 5 extensions of the main theorems are made to operator polynomials for which the leading coefficient may not be invertible, but there is a real $\alpha$ for which $L(\alpha)$ is either invertible, or $L(\alpha) \gg 0$.

## 2. PRELIMINARIES

The main results of this paper depend on properties of the spectrum of a selfadjoint operator polynomial $L(\lambda)$ which follow from the condition that all points of $\sigma(L(\lambda)) \cap \mathbf{R}$ are of determinate type. This section is devoted to some development of these properties.

The first lemma appears in [2] but, for completeness, the proof is repeated here. It is stated for an operator $A$ on a Krein space $\mathcal{K}$, but for our applications the space $\mathcal{K}$ is $\mathcal{H}^n$ with the inner product generated by $G_L$ of (1.2). The lemma will be applied to the operator $C_L$ on $\mathcal{K}$ (ref. Lemma 7 of [8]). Here we denote by $\rho(A)$ the resolvent set of $A$, and by $\sigma_+(A)$ the set of all its points of plus type.

LEMMA 2.1. *Let $A$ be a bounded selfadjoint operator on a Krein space. Let $a, b \in \rho(A)$ and assume that $(a,b) \subset \rho(A) \cup \sigma_+(A)$. Then there is a complex neighbourhood $U$ of $[a,b]$ such that $U \backslash [a,b] \subset \rho(A)$.*

*Proof.* Suppose the assertion is false. Then there is a sequence $\{\lambda_n\} \subset \partial\sigma(A)$ (and hence in $\sigma_{\rm ap}(A)$), and a point $\lambda_0 \in \sigma_+(A) \cap (a,b)$ such that $\lambda_n \to \lambda_0$. Hence, for each $n$ there is a vector $f_n$ such that $\|f_n\| = 1$ and

$$\|Af_n - \lambda_n f_n\| < \frac{|\mathrm{Im}\,\lambda_n|}{n}. \tag{2.1}$$

Write $\gamma_n = [Af_n - \lambda_n f_n, f_n]$ and we have

$$[Af_n, f_n] = \lambda_n [f_n, f_n] + \gamma_n.$$

Equate imaginary parts to obtain

$$[f_n, f_n] = \frac{-\mathrm{Im}\,\gamma_n}{\mathrm{Im}\,\lambda_n}$$

and, using (2.1), we find that $[f_n, f_n] \to 0$. But $\|f_n\| = 1$ and $\|Af_n - \lambda_0 f_n\| \to 0$ as well, so we contradict the assumption that $\lambda_0 \in \sigma_+(A)$. ∎

If it is assumed that $\sigma(L(\lambda)) \cap \mathbf{R}$ consists of points of determinate type then it follows from the lemma that we have a disjoint union of closed sets:

$$\sigma(L(\lambda)) = \sigma_+(L(\lambda)) \cup \sigma_-(L(\lambda)) \cup \sigma_0(L(\lambda)), \tag{2.2}$$

where $\sigma_0(L(\lambda))$ consists of non-real points.

Since $\sigma(L(\lambda))$ is symmetric with respect to the real line we can make a further sub-division into closed sets:

$$\sigma_0(L(\lambda)) = \Lambda \cup \overline{\Lambda} \tag{2.3}$$

where $\overline{\Lambda} = \{\overline{\lambda} : \lambda \in \Lambda\}$ and $\Lambda \cap \overline{\Lambda} = \emptyset$. For example, one may choose $\operatorname{Im}\lambda > 0$ and $\operatorname{Im}\lambda < 0$ for all $\lambda$ in $\Lambda$ and $\overline{\Lambda}$, respectively, and

$$\mathcal{H}^n = \mathcal{N}_+ \dot{+} \mathcal{N}_- \dot{+} \mathcal{M} \dot{+} \mathcal{M}^*, \tag{2.4}$$

where $\mathcal{N}_+$, $\mathcal{N}_-$, $\mathcal{M}$, and $\mathcal{M}^*$ are the spectral subspace of $C_L$ corresponding to $\sigma_+(L(\lambda))$, $\sigma_-(L(\lambda))$, $\Lambda$, and $\overline{\Lambda}$, respectively. Furthermore, the subspaces $\mathcal{N}_+$ and $\mathcal{N}_-$ will be uniformly $G_L$-positive and uniformly $G_L$-negative, respectively (see Lemma 5 of [8]).

## 3. THE CASE OF A DEFINITE LEADING COEFFICIENT

THEOREM 3.1. *Let $L(\lambda) = \sum_{j=0}^{n} \lambda^j A_j$ be a selfadjoint operator polynomial with $A_n \gg 0$, and assume that all points of $\sigma(L(\lambda)) \cap \mathbf{R}$ have determinate type. Then $L(\lambda)$ admits a spectral factorization*

$$L(\lambda) = M(\lambda) A_n N(\lambda), \tag{3.1}$$

*where $M(\lambda), N(\lambda)$ are monic polynomials of degrees $[\frac{1}{2}n]$ and $[\frac{1}{2}(n+1)]$, respectively, and*

$$\sigma(M(\lambda)) = \sigma_-(L(\lambda)) \cup \overline{\Lambda}, \quad \sigma(N(\lambda)) = \sigma_+(L(\lambda)) \cup \Lambda.$$

Note that $\Lambda$ and $\overline{\Lambda}$ are non-real subsets of $\sigma(L(\lambda))$ defined in Section 2. Before starting the proof it will be convenient to formulate two lemmas. For any set $\sigma \subset \mathbf{C}$ we denote by $\overline{\sigma}$ the set symmetric to $\sigma$ with respect to $\mathbf{R}$.

LEMMA 3.2. *Let $A$ be a $G$-selfadjoint operator on a Hilbert space $\mathcal{H}$ and $\sigma_1$, $\sigma_2$ be two isolated parts of $\sigma(A)$ with corresponding Riesz projectors $R_1$ and $R_2$. If $\sigma_1 \cap \overline{\sigma}_2 = \emptyset$ then $\operatorname{Im} R_1$ and $\operatorname{Im} R_2$ are $G$-orthogonal.*

*Proof.* It is easily seen that $R_2^{[*]}$ (the adjoint of $R_2$ in the $G$-inner product) is the Riesz projector corresponding to $\overline{\sigma}_2$. Then $\sigma_1 \cap \overline{\sigma}_2 = \emptyset$ implies $R_2^{[*]} R_1 = 0$ and for any $f \in \operatorname{Im} R_1$, $g \in \operatorname{Im} R_2$,

$$[f, g] = [R_1 f, R_2 g] = [0, g] = 0. \quad \blacksquare$$

Recall that subspaces $\mathcal{R}$ and $\mathcal{S}$ of a Krein space are called a "dual pair" if $\mathcal{R} \cap \mathcal{S}^{[\perp]} = \{0\}$ and $\mathcal{R}^{[\perp]} \cap \mathcal{S} = \{0\}$ (and $\mathcal{R}^{[\perp]}$, $\mathcal{S}^{[\perp]}$ are the orthogonal companions of $\mathcal{R}$ and $\mathcal{S}$, respectively, in the sense of [1]). Also, if $\mathcal{R}$ and $\mathcal{S}$ are both neutral they form a dual pair if and only if $\mathcal{R} \dot{+} \mathcal{S}$ is non-degenerate (see Section 1.10 of [1], for example).

LEMMA 3.3. *Let $A$ be $G$-selfadjoint and $\sigma_0$ be an isolated part of the spectrum for which $\sigma_0 \cap \mathbf{R} = \emptyset$ and $\sigma_0 \cap \overline{\sigma}_0 = \emptyset$. If $\mathcal{S}$ and $\mathcal{S}^*$ denote the spectral subspaces of $A$ defined by $\sigma_0$ and $\overline{\sigma}_0$, respectively, then $\mathcal{S}$ and $\mathcal{S}^*$ are a dual pair.*

*Proof.* It follows from Lemma 3.2 that $\mathcal{S}$ and $\mathcal{S}^*$ are both $G$-neutral. Then, because $\sigma_0$ (and hence $\sigma_0 \cup \overline{\sigma}_0$) is isolated, the Riesz subspace, $\mathcal{S} \dot{+} \mathcal{S}^*$, associated with $\sigma_0 \cup \overline{\sigma}_0$ is ortho-complemented (i.e. the span of $\mathcal{S} \dot{+} \mathcal{S}^*$ and $(\mathcal{S} \dot{+} \mathcal{S}^*)^{[\perp]}$ is $\mathcal{H}$). It follows that $\mathcal{S} \dot{+} \mathcal{S}^*$ is a Krein space (see Theorem V.3.4 of [1]). Finally, by Lemma I.10.2 of [1], it follows that $\mathcal{S}$ and $\mathcal{S}^*$ form a dual pair. ∎

*Proof of Theorem* 3.1. It follows from a theorem of Langer (Theorem 3 of [11]) that, if $\mathcal{N}_+ \dot{+} \mathcal{M}$ (as constructed in Section 2) is a maximal $G_L$-nonnegative subspace, then $L(\lambda)$ has a monic right divisor $N(\lambda)$ such that $\sigma(N(\lambda)) = \sigma_+(L(\lambda)) \cup \Lambda$. But $\mathcal{N}_+ \dot{+} \mathcal{M}$ is a Riesz subspace and, by Theorem 19 of [3] we know, in addition, that the spectrum of the left divisor $M(\lambda)$ is equal to $\sigma_-(L(\lambda)) \cup \overline{\Lambda}$.

As $\mathcal{N}_+$ is uniformly $G_L$-positive, $\mathcal{M}$ is $G_L$-neutral and $[\mathcal{N}_+, \mathcal{M}] = \{0\}$, it follows that $\mathcal{N}_+ \dot{+} \mathcal{M}$ is $G_L$-nonnegative.

Suppose that $\mathcal{N}_+ \dot{+} \mathcal{M}$ is not a maximal $G_L$-nonnegative, $C_L$-invariant subspace, i.e. there is a $G_L$-nonnegative $C_L$-invariant subspace $\mathcal{L}$ such that $\mathcal{L} \supset \mathcal{N}_+ \dot{+} \mathcal{M}$ properly. Equation (2.4) gives

$$(\mathcal{N}_+ \dot{+} \mathcal{M}) \dot{+} (\mathcal{N}_- \dot{+} \mathcal{M}^*) = \mathcal{H}^n,$$

so there is a nonzero $f \in (\mathcal{N}_- \dot{+} \mathcal{M}^*) \cap \mathcal{L}$.

Write $f = f_1 + f_2$ where $f_1 \in \mathcal{N}_-$, $f_2 \in \mathcal{M}^*$. Then, as $[\mathcal{N}_-, \mathcal{M}^*] = 0$ and $\mathcal{M}^*$ is neutral,

$$0 \leqslant [f, f] = [f_1, f_1] + [f_2, f_2] = [f_1, f_1] \leqslant 0.$$

Thus, $[f_1, f_1] = 0$ and, as $f_1 \in \mathcal{N}_-$, $f_1 = 0$ and $f = f_2 \in \mathcal{M}^*$.

Now for nonzero $f \in \mathcal{M}^*$ it follows from Lemma 3.3 that there is an $h \in \mathcal{M}$ such that $[f, h] \neq 0$ (otherwise $f \in \mathcal{M}^{[\perp]}$) and, by multiplying $h$ by a unimodular number, if necessary, there is an $h \in \mathcal{M}$ such that $[f, h] < 0$. Then

$$[f + h, f + h] = 2[f, h] < 0.$$

However, by definition, $f$ is also in the nonnegative subspace $\mathcal{L}$, so that $f + h \in \mathcal{L}$ as well. This is a contradiction and shows that $\mathcal{N}_+ \dot{+} \mathcal{M}$ is maximal. ∎

REMARK 3.4. Under the hypotheses of Theorem 3.1 the polynomial $L(\lambda)$ has another factorization $L(\lambda) = N_1(\lambda) A_n M_1(\lambda)$, where the polynomials $M_1(\lambda)$ and $N_1(\lambda)$ have spectral properties analogous to those of $M(\lambda)$ and $N(\lambda)$, respectively.

There are also similar dual formulations for subsequent factorization theorems.

REMARK 3.5. When $n = 2$ interesting alternatives to the hypothesis $A_n \gg 0$ are possible. These include:

(i) there is a $\lambda_0 \in \mathbf{C}$ and a $\delta > 0$ such that

$$|(L(\lambda_0)f, f)| \geqslant \delta \|f\|^2$$

for all $f \in \mathcal{H}$. (When $\lambda_0 \in \mathbf{R}$ such a condition is discussed in Section 5.)

(ii) there is a $\lambda_0 \in \mathbf{C}$ and a $\delta > 0$ such that

$$|(L'(\lambda_0)f, f)| \geqslant \delta \|f\|^2 .$$

These cases will be discussed in a future publication, in which it will be shown that the hypothesis (i) admits factorization of some rational operator functions which are selfadjoint on the unit circle.

## 4. OPERATOR POLYNOMIALS WITH REAL SPECTRUM

The basic assumptions of the next theorem are that $n$ is even, that $A_n$ is invertible (weakening a hypothesis of Theorem 3.1) and $\sigma(L(\lambda)) \subseteq \mathbf{R}$ and has determinate type (strengthening another hypothesis of Theorem 3.1). In the terminology of the paper [9], $L(\lambda)$ is a *quasihyperbolic* operator polynomial.

THEOREM 4.1. *Let* $L(\lambda) = \sum_{k=0}^{2p} \lambda^k A_k$ *be a selfadjoint operator polynomial with* $A_{2p}$ *invertible. Assume that* $\sigma(L(\lambda)) \subset \mathbf{R}$ *and* $\sigma(L(\lambda)) = \sigma_+(L(\lambda)) \cup \sigma_-(L(\lambda))$. *Then* $L(\lambda)$ *admits a spectral factorization with respect to* $\sigma_+(L(\lambda))$ *and* $\sigma_-(L(\lambda))$, *i.e. there exist monic operator polynomials* $M(\lambda)$, $N(\lambda)$ *of degrees* $p$ *such that*

$$L(\lambda) = M(\lambda) A_{2p} N(\lambda)$$

*for all* $\lambda \in \mathbf{C}$, *and* $\sigma(N(\lambda)) = \sigma_+(L(\lambda))$, $\sigma(M(\lambda)) = \sigma_-(L(\lambda))$.

It will be convenient to formulate another lemma before proving the theorem. We introduce subspaces $\mathcal{S}_p$ and $\mathcal{S}^p$ of $\mathcal{H}^{2p}$. By definition, $\mathcal{S}_p$ and $\mathcal{S}^p$ have the first $p$ components, and the last $p$ components equal to zero, respectively.

LEMMA 4.2. *If* $A_{2p}$ *is invertible then* $\mathcal{S}_p^{[\perp]} = \mathcal{S}_p$.

*Proof.* Let $y \in \mathcal{S}_p$. Then for any $f \in \mathcal{S}_p$ we easily verify that $[f, y] = (G_L f, y) = 0$, i.e. $\mathcal{S}_p \subseteq \mathcal{S}_p^{[\perp]}$.

On the other hand, if $y \in \mathcal{S}_p^{[\perp]}$ and we write

$$G_L = \begin{bmatrix} G_{11} & G_{12} \\ G_{12} & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 \\ \widehat{f} \end{bmatrix}, \quad y = \begin{bmatrix} \widehat{y}_1 \\ \widehat{y}_2 \end{bmatrix}$$

with respect to $\mathcal{H}^{2p} = \mathcal{H}^p \oplus \mathcal{H}^p$, then

$$(G_L f, y)_{2p} = (G_{12}\widehat{f}, \widehat{y}_1)_p$$

for all $\widehat{f} \in H^p$. But

$$G_{12} = \begin{bmatrix} A_{p+1} & A_{p+2} & \dots & A_{2p} \\ A_{p+2} & & & \\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ A_{2p} & & \dots & 0 \end{bmatrix}$$

and is therefore selfadjoint and invertible. Thus, $(\widehat{f}, G_{12}\widehat{y}_1) = 0$ for all $\widehat{f} \in H^p$ implies $\widehat{y}_1 = 0$, i.e. $y \in \mathcal{S}_p$. ∎

*Proof of Theorem* 4.1. We use Theorems 16 and 19 of [3] (see also Theorem 1 of [11]) which ensure that a factorization of the required form exists provided that there is a direct sum decomposition

$$\mathcal{N}_+ \dot{+} \mathcal{S}_p = \mathcal{H}^{2p}. \tag{4.1}$$

Since $\sigma_+(L(\lambda)) \cup \sigma_-(L(\lambda)) = \sigma(L(\lambda)) = \sigma(C_L)$ we have $\mathcal{N}_+ \dot{+} \mathcal{N}_- = \mathcal{H}^{2p}$ and, furthermore, the sum is $G_L$-orthogonal.

Let us first prove that $\mathcal{N}_+ \cap \mathcal{S}_p = \{0\}$ and the direct sum $\mathcal{N}_+ \dot{+} \mathcal{S}_p$ is closed. If not, then there are two sequences $\{f_n\} \subset \mathcal{S}_p$, $\{g_n\} \subset \mathcal{N}_+$ such that $\|f_n\| = 1$ and $\|f_n - g_n\| \to 0$ as $n \to \infty$. Let

$$f_n = \begin{bmatrix} 0 \\ \widehat{f}_n \end{bmatrix}$$

in $H^p \oplus H^p$. Then, denoting possibly nonzero components by $*$, we have

$$G_L f_n = \begin{bmatrix} * \\ 0 \end{bmatrix}, \quad (G_L f_n, f_n) = 0.$$

But $\|f_n - g_n\| \to 0$ then implies that $(G_L g_n, g_n) \to 0$ as $n \to \infty$, as well. Since $g_n \in \mathcal{N}_+$ it follows that $g_n \to 0$ and this contradicts the hypothesis that $\|f_n\| = 1$ and $\|f_n - g_n\| \to 0$.

Now let us prove that $\mathcal{N}_+ \dot{+} \mathcal{S}_p$ is the whole of $\mathcal{H}^{2p}$. If not, there is a nonzero $h \in (\mathcal{N}_+ \dot{+} \mathcal{S}_p)^{[\perp]}$. But $\mathcal{N}_+^{[\perp]} = \mathcal{N}_-$ and, by Lemma 4.2, $\mathcal{S}_p^{[\perp]} = \mathcal{S}_p$. Thus, $h \in \mathcal{N}_- \cap \mathcal{S}_p$. However, as in the first part of the proof, we see that $\mathcal{N}_- \cap \mathcal{S}_p = \{0\}$, and we have our contradiction. Equation (4.1) is established. ∎

REMARK 4.3. When $L(\lambda)$ is quadratic ($p = 1$), then $N(\lambda)$ and $M(\lambda)$ take the form $N(\lambda) = \lambda I - Z$, $M(\lambda) = \lambda I - Y$, and the operators $Z$, $Y$ are similar to selfadjoint operators on $\mathcal{H}$. To see this, observe that $Z$ is similar to the restriction $C_L \mid \mathcal{N}_+$ (see Theorem 28.2 of [12], for example). Then it follows from Lemma 5 of [8] that $C_L \mid \mathcal{N}_+$ is similar to a selfadjoint operator.

Note that it has been shown in [7] that, even in the finite dimensional case, there is no analogue for Theorem 4.1 for selfadjoint operator polynomials with *odd degree* and invertible leading coefficient $A_{2p+1}$, unless $A_{2p+1}$ is definite. In the latter case, Theorem 3.1 applies.

It can be argued that Theorem 4.1 is the best of its kind in the sense that, if the restrictions on $\sigma(L(\lambda))$ are removed, then there may be no spectral factorization. For example, one might conjecture that with no real spectrum a factorization would be possible. Example 4.5 below shows that this is not the case. Example 4.4 is well-known and demonstrates that, with mixed real points of spectrum, there may be no factorization.

EXAMPLE 4.4. Let $L(\lambda) = \begin{bmatrix} 0 & \lambda^2 \\ \lambda^2 & 1 \end{bmatrix}$. Here, $\lambda = 0$ is an eigenvalue of mixed type and there is no right divisor of the form $\lambda I - Z$.

EXAMPLE 4.5. Let

$$L(\lambda) = \begin{bmatrix} 0 & 0 & 0 & (\lambda + i)^2 \\ 0 & 0 & (\lambda + i)^2 & 1 \\ 0 & (\lambda - i)^2 & 0 & 0 \\ (\lambda - i)^2 & 1 & 0 & 0 \end{bmatrix}.$$

Here, $\sigma(L(\lambda)) = \{i, -i\}$ and each eigenvalue has algebraic multiplicity 4 and geometric multiplicity 1. To show that there is no right divisor of the form $\lambda I - Z$ we consider chains of generalized eigenvectors for the two distinct eigenvalues.

It is found that for the eigenvalue i there is a chain of the form $\{e_1, e_1, *, *\}$, where $e_1$ denotes the first unit coordinate vector in $\mathbb{C}^4$ and $*$ denotes a vector of no immediate concern. Similarly, for the eigenvalue $-i$ there is a chain $\{e_3, e_3, *, *\}$.

A theorem of Langer ([10]) states that, for the existence of the right divisor $\lambda I - Z$, it must be possible to select a basis for $\mathbb{C}^4$ from the *leading* vectors of the chains. Clearly, this is not possible for this example. ∎

Observe that in Examples 4.4 and 4.5 all values of $L(\lambda)$, $\lambda \in \mathbb{R}$, are indefinite (and not only the leading coefficient). This is necessary because, otherwise, a factorization is known to exist (see Remark 3.5 (i) and Theorem 5.2).

5. REGULAR QUASIHYPERBOLIC OPERATOR POLYNOMIALS

In this section we make some remarks on the extension of Theorems 3.1 and 4.1 to quasihyperbolic polynomials (i.e. with only real and determinate spectrum including the point at infinity when appropriate) for which the leading coefficient is not invertible, but for which there is an $\alpha \in \mathbf{R}$ such that $L(\alpha)$ is invertible. Such a polynomial is said to be *regular*.

The technique applied here is to "shift and invert" the eigenvalue parameter so that earlier theorems can be applied and then transform back to the parameter $\lambda$. Some of the necessary preliminaries have been developed in Section 7 of [9].

Let us first consider the case when the degree of $L(\lambda)$ is even, say $n = 2p$. We set $\mu = (\lambda - \alpha)^{-1}$ and

$$M(\mu) := \mu^{2p} L(\alpha + \mu^{-1}) = \sum_{j=0}^{2p} \frac{\mu^{2p-j}}{j!} L^{(j)}(\alpha) =: \sum_{j=0}^{2p} \mu^j M_j.$$

Thus, $M_{2p} = L(\alpha)$ and is invertible. In this case, finite points of $\sigma(L(\lambda))$ of determinate type also have determinate type for $M(\mu)$ but the signs are reversed "plus" $\leftrightarrow$ "minus" (Lemma 5 of [9]). Also (when $A_{2p}$ is not invertible) $0 \in \sigma(M(\mu))$ and has the *same* type as $\infty \in \sigma(L(\lambda))$. (We recall that $\infty \in \sigma(L(\lambda))$ has plus type if, as $n \to \infty$,

$$\underline{\lim}(A_{2p-1} f_n, f_n) > 0$$

whenever $\|f_n\| = 1$ and $\|A_{2p} f_n\| \to 0$, and similarly for the case when $\infty$ has minus type. See Section 7 of [9].)

The following result is easily obtained from Theorem 4.1.

THEOREM 5.1. *Let $L(\lambda)$ be a quasihyperbolic operator polynomial of degree $2p$ with $A_{2p}$ not invertible, and assume there is an $\alpha \in \mathbf{R}$ such that $L(\alpha)$ is invertible. Then there are operator polynomials $L_1(\lambda)$ and $L_2(\lambda)$ of degree $p$ such that*

$$L(\lambda) = L_2(\lambda) L(\alpha) L_1(\lambda)$$

*and $\sigma(L_1(\lambda)) = \sigma_+(L(\lambda))$, $\sigma(L_2(\lambda)) = \sigma_-(L(\lambda))$.*

*Furthermore, if $L_k(\lambda) = \sum_{j=0}^{p} \lambda^j L_{k,j}$, $k = 1, 2$, then $L_{1,p}$ and $L_{2,p}$ are invertible according as $\infty$ has plus or minus type, respectively, as a point of $\sigma(L(\lambda))$.*

When $L(\lambda)$ has odd degree, say $n = 2p + 1$, one might try the device of defining

$$\widehat{L}(\lambda) = \lambda^{2p+2} 0 + L(\lambda)$$

and apply Theorem 5.1 to $\widehat{L}(\lambda)$. However, it is easily seen that $\infty$ is not a point of determinate type for $\widehat{L}(\lambda)$, so Theorem 5.1 does not apply.

If $L(\lambda)$ is a QHP of odd degree, $2p+1$, and there is an $\alpha > 0$ such that $L(\alpha) \gg 0$ then Theorem 3.1 can be used to obtain a factorization, but the conditions $\sigma(L_1(\lambda)) = \sigma_+(L(\lambda))$, $\sigma(L_2(\lambda)) = \sigma_-(L(\lambda))$ are not satisfied. Points of spectrum of plus and minus types will appear in both factors. Indeed, when the degree is odd it is known that, in general, factorizations of the kind described in Theorem 5.1 do not exist (see Theorem 3.6 of [7]).

To obtain a "mixed" factorization in the case of odd degree, shift and invert the parameter as above to obtain

$$M(\mu) := \mu^{2p+1}L(\alpha+\mu^{-1}) = \sum_{j=0}^{2p+1} \frac{\mu^{2p+1-j}}{j!}L^{(j)}(\alpha) =: \sum_{j=0}^{2p+1} \mu^j M_j,$$

where $M_{2p+1}$ is invertible. The "mixing" of points of spectrum occurs because, if $\lambda_0 \in \sigma(L(\lambda))$ has plus (minus) type, then $\mu_0 = (\lambda_0 - \alpha)^{-1}$ has the type reversed when $\lambda_0 > \alpha$ and preserved when $\lambda_0 < \alpha$ (Lemma 5 of [9]). Also, if $\infty$ has plus or minus type for $L(\lambda)$ then this type is preserved for the point $0 \in \sigma(M(\mu))$. It is found from Theorem 3.1 that:

THEOREM 5.2. *Let $L(\lambda)$ be a quasihyperbolic operator polynomial of odd degree, $2p+1$, with $L(\alpha) \gg 0$ for some $\alpha \in \mathbf{R}$. Then there are operator polynomials $L_1(\lambda)$ and $L_2(\lambda)$ of degrees $p+1$ and $p$, respectively, such that*

$$L(\lambda) = L_2(\lambda)L(\alpha)L_1(\lambda)$$

*and $\sigma(L_1(\lambda))$ is the union of*

$$\sigma_+(L(\lambda)) \cap \{\lambda \mid \lambda > \alpha\}, \quad \sigma_-(L(\lambda)) \cap \{\lambda \mid \lambda < \alpha\},$$

*and $\{\infty\}$ if $\infty$ has plus type for $L(\lambda)$. Also, $\sigma(L_2(\lambda))$ is the union of*

$$\sigma_+(L(\lambda)) \cap \{\lambda \mid \lambda < \alpha\}, \quad \sigma_-(L(\lambda)) \cap \{\lambda \mid \lambda > \alpha\}$$

*and $\{\infty\}$ if $\infty$ has minus type for $L(\lambda)$.*

We note that the leading coefficient of $L_1(\lambda)$ (of $L_2(\lambda)$) is invertible if $\infty$ has plus type for $L(\lambda)$ (has minus type for $L(\lambda)$).

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