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BOOK REVIEW:

Deformation Quantization and Index Theory, **Boris Fedosov**, Akademie Verlag, Berlin 1995, 325 pag., ISBN 3-05-501716-1.

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AMS SUBJECT CLASSIFICATION: Primary 81S10; Secondary 35J10,35S05.

The problem of defining a precise and efficient procedure of transferring the classical image of the evolution of a system into its quantum one and back is a rather old problem that has not yet got a satisfactory answer but has continuously generated a lot of very important concepts and techniques (Weyl calculus, semiclassical analysis, geometric quantization, ...). One of the newest theories elaborated in connection with this problem is the so called *Deformation Quantization*.

The Volume 9 in the *"Mathematical Topics"* series of Akademie Verlag, contains the monograph *"Deformation Quantization and Index Theory"* by Boris Fedosov, dedicated to a pedagogical survey of the author's results (published in the period 1986-1995), concerning the definition of the index in the study of the deformation quantization and some of its important applications. Deformation quantization, a structure first introduced in the 1978 paper by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer, consists in defining the algebra of quantum observables as an algebra of formal power series with respect to a formal parameter h. More precisely, given a symplectic manifold M, one considers the linear space $Z = C^{\infty}(M)\,[[h]]$ of formal power series in h with coefficients smooth functions on M and defines on it an associative product (the star-product), satisfying a locality condition and a correspondence principle, meaning that the leading term of the star-product is just the usual product of the leading terms of the factors and the commutator for the star-product has a leading term linear in h and given by the usual Poisson bracket. The advantage of this construction is that it provides a good candidate for generalizing the usual Weyl Calculus from the case of a symplectic space to any symplectic manifold. The problem that one has to solve after rigorously defining the algebra of observables is to provide a Hilbert space representation for it, in the spirit of quantum theory and this task proves to be a rather difficult one. The author defines a kind of symbolic calculus by associating the so called *Formal Weyl Algebras Bundle* to the algebra Z previously defined and generalizes the notion of topological index for this case. Then one can show that this index is in fact a formal Laurent series in the parameter h. The important fact now is that if a Hilbert space representation is to be found then the associated analytical index being an integer one is obliged to consider the previously formal parameter h as taking only those values that give integer values for the topological index too. This observation leads the author to elaborate the so called *Asymptotic Operator Representation* inspired from the 1984 paper by Karasev and Maslov.

The book by Boris Fedosov also contains a rather detailed exposition of the main definitions and results from symplectic geometry, pseudodifferential calculus and index theory, that are needed for the main developments concerning the Formal Weyl Algebras Bundle and the Asymptotic Operator Representation. Moreover the book ends with some interesting examples (the flag manifold and the projective space) and with some developments of the AOR technique concerned with symplectic reduction and trace formula for Schrödinger operators.

The first chapter entitled "*Elements of Differential Geometry*" is a short but self-contained survey of the main facts concerning connections on vector bundles and some important topological invariants. The Chern class and the Chern character are defined and some of their basic properties are discussed. The Grothendieck K-functor and the K-functor with compact supports are defined too and the Thom isomorphism theorems are proved.

The second chapter "*Symplectic Manifolds*" introduces the main notions concerning symplectic manifolds and symplectic bundles and the framework of Hamiltonian mechanics and symplectic transformations. The notion of generating function of a symplectomorphism is discussed. The chapter ends with a brief study of the important example of the coadjoint orbits of the group U(n).

The third chapter "*The Weyl Quantization*" gives a brief exposition of the theory of global h-pseudodifferential operators emphasizing mainly on the composition formula, the L^2 boundedness and the explicit form of the unitary representation of the linear symplectic isomorphisms. The symbols are considered as asymptotic series in the real parameter h. A paragraph is devoted to the definition of the Gaussian symbols and the wave packets, providing some efficient tools for localizing in phase-space.

Chapter 4, "*Introduction to Index Theory*" is dedicated to the introduction of the notions of Fredholm elements and the associated index and to prove the Atiyah-Singer index theorem for elliptic operators in $\mathbf{R}^n$ as a model for more complicated index theorems discussed in further chapters. One starts from the classical definition of a Fredholm operator, puts into evidence its relation with the K-theory with compact supports and defines the notion of a Fredholm element and the Hilbert bundle that it generates. The chapter closes with two paragraphs defining the Weyl Algebras Bundle, the Fock Bundle (providing a representation for the Weyl Algebras Bundle) and the Bott generator.

Chapter 5, *"Deformation Quantization"* introduces the main algebra, the algebra of quantum observables and its associated calculus. Starting from an arbitrary symplectic manifold one defines its Formal Weyl Algebras Bundle (FWAB) by considering formal power series in the formal parameter h with coefficients in the tangent bundle of the symplectic manifold and with a special composition law inspired from the classical Weyl calculus. On this bundle one considers connections induced by symplectic connections on the symplectic manifold, defines the notion of an Abelian connection and shows that such an Abelian connection always exists and is even unique (if one also imposes some constraints on its form). The subalgebra of flat sections of the FWAB is defined as the algebra of quantum observables ant it is shown to be in a bijective correspondence with the linear space Z; moreover the star-product on Z is defined to be the product induced by the composition law of the FWAB. One can then prove existence and uniqueness of the solution of the Heisenberg Equation, a theorem stating that any quantum algebra is locally trivial and the existence and uniqueness of a trace on any quantum algebra. The chapter ends with the study of the case of regular Poisson manifolds and of the case of symplectic manifolds with an action of a Lie group, case of interest for the geometric quantization.

Chapter 6, *"The Index Theorem for Quantum Algebras"*, introduces the notion of *elliptic elements* of a quantum algebra and is dedicated to a detailed proof of the following theorem

Theorem: *For any Symplectic manifold M and for any virtual bundle with compact support on it ξ (as defined in section 1.4.1.), there exists an elliptic element Ξ in the associated quantum algebra having ξ as leading term. The index of Ξ depends only on the class $ch\xi$ and one has the formula:*

$$ind{:}\,\Xi = \int_M ch\xi \, \exp\left(\frac{-\Omega}{2\pi h}\right) \hat{A}(M)$$

where A is the operator of the elliptic quadruple and Ω is the curvature of D (the connection of the quantum algebra).

Chapter 7, *"The Asymptotic Operator Representation"* gives the details of an explicit construction of an asymptotic Hilbert space representation for the quantum algebra associated to a compact symplectic manifold and discusses the particular cases of the flag manifold and the projective space. The main conclusion of this chapter is that the property of the topological index of being an integer imposes some conditions on the parameter h, that has to belong to an admissible set, conditions known as the quantization conditions.

Chapter 8, *"Symplectic Reduction"* presents the Marsden-Weinstein reduction theorem for the case of one integral of motion and then formulates and proves a reduction theorem for the deformation quantization.

Chapter 9, *"A Trace Formula for the Schrödinger Operator"* discusses the problem of solving the Schrödinger equation. Formally one would have to define a star-exponential, but unfortunately this thing is not possible. Using the Asymptotic Operator Representation, the author can define the trace of the exponential applied to an element with compact support and studies its behavior for $h \to 0$ over an admissible set. The important conclusion is that this asymptotic behavior is governed by the fixed point set of the classical Hamiltonian flow.

The topics discussed are of very much interest for many domains but especially for Schrödinger operators, Hamiltonian systems, quantization and semiclassical limit and the theory of pseudodifferential operators. In the same time the presentation being self-contained and very detailed, the book is very useful also for students preparing to work in the above fields.

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