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PREANNIHILATORS, THE OPERATOR APPROXIMATION PROPERTY AND DUAL PRODUCTS

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ABSTRACT. It was shown by Effros, Kraus and Ruan that an ultraweakly closed subspace of $B(H)$ has the weak*-operator approximation property if and only if its predual has the operator approximation property. We show that the preannihilator of such a subspace S has the operator approximation property if and only if any ultraweakly closed subspace T of $B(K)$ the dual product of S and T is the ultraweakly closed linear span of the algebraic tensor product of S and $B(K)$ and the algebraic tensor product of $B(H)$ and T. Using this we also show that there are reflexive subspaces S and T for which the dual product is strictly larger than this ultraweakly closed linear span, answering a question of the second author.

KEYWORDS: *Operator space, tensor product, dual product, hyperreflexive, operator approximation property, preannihilator.*

AMS SUBJECT CLASSIFICATION: 46L05, 47A15, 47D25, 47A45

0. INTRODUCTION

It was shown by the second author in [19] (and independently by Tsuji in [?]) that if $\mathcal{H}$ is a separable Hilbert space, then a σ-weakly (i.e., ultraweakly, weak-$*$) closed subalgebra A of $B(\mathcal{H})$ (the algebra of bounded operators on $\mathcal{H}$) is reflexive (i.e., $A = \operatorname{Alg}\operatorname{Lat} A$) if and only if the preannihilator $A_\perp$ of A in the predual $B(\mathcal{H})_*$ of $B(\mathcal{H})$ is generated (as a closed linear space) by rank $\leqslant 1$ operators (where we identify $B(\mathcal{H})_*$ with the trace-class operators in the usual way). This result was extended to reflexive linear subspaces in [14]. Arveson observed in [1], Chapter 7 that those reflexive algebras for which a distance formula holds satisfy an equivalent condition which he called hyperreflexivity. Hyperreflexivity also has characterizations in terms of the preannihilator (one characterization was obtained in [19], and another in [1]), and can be defined for subspaces (see [14], [15]). The first example of a nonhyperreflexive reflexive algebra was constructed in [14]. The first example of a nonhyperreflexive CSL algebra (reflexive algebra with commutative subspace lattice) was constructed by Davidson and Power in [5]. Motivated by [5], the second author introduced the notion of *dual product* in [20]. It is shown in [20] that if S and T are reflexive subspaces of $B(\mathcal{H})$, then the dual product $S * T$ of S and T is the smallest reflexive subspace containing $S \otimes B(\mathcal{H}) + B(\mathcal{H}) \otimes T$, and the question was raised of whether $S * T$ is always the σ-weak closure of $S \otimes B(\mathcal{H}) + B(\mathcal{H}) \otimes T$. Using recent results from the theory of operator spaces, we show in this paper that the answer to this question is no. We also give a characterization of those σ-weakly closed subspaces S for which $S * T$ is the σ-weak closure of $S \otimes B(\mathcal{H}) + B(\mathcal{H}) \otimes T$ for all σ-weakly closed subspaces T. These turn out to be precisely those subspaces whose preannihilators have the operator approximation property.

The operator approximation property (the OAP), introduced by Effros and Ruan in [8], is the natural analogue for operator spaces of Grothendieck's approximation property for Banach spaces. It is shown in [6] that an operator space V has the OAP if and only if V^* has the weak*-OAP, which is the dual version of the OAP. (Precise definitions of the OAP and the weak*-OAP, as well as a discussion of basic facts about operator spaces, can be found in Section 1 below.) In particular, if S is a σ-weakly closed subspace of $B(\mathcal{H})$ (where $\mathcal{H}$ is not necessarily separable), then S has the weak*-OAP if and only if its operator predual S_* (i.e. $B(\mathcal{H})_*/S_\perp$ with the quotient operator space structure) has the OAP.

The preannihilator $S_\perp$ of S also has a natural operator space structure, namely the one it inherits as a subspace of the operator predual $B(\mathcal{H})_*$ of $B(\mathcal{H})$. Thus two natural questions are:

(1) If S has the weak*-OAP, what properties does $S_\perp$ have?

(2) If $S_\perp$ has the OAP, what properties does S have?

We give answers to both of these questions in this paper.

Before starting our main results, we need a few definitions. If $S \subset B(\mathcal{H})$ and $T \subset B(\mathcal{K})$ are σ-weakly closed subspaces, the *dual product* $S * T$ of S and T is defined by $S * T = (S_\perp \otimes T_\perp)^\perp$ [20] (where $S_\perp \otimes T_\perp$ denotes the closed linear span of $\{x \otimes y : x \in S_\perp \text{ and } y \in T_\perp\}$ in $B(\mathcal{H} \otimes \mathcal{K})_*$). We say that S has the *dual product density property* (the DDP) if $S * T$ is the σ-weak closure of $S \overline{\otimes} B(\mathcal{K}) + B(\mathcal{H}) \overline{\otimes} T$ whenever T is a σ-weakly closed subspace of $B(\mathcal{K})$ for some $\mathcal{K}$. If $V \subset B(\mathcal{H})_*$ and $W \subset B(\mathcal{K})_*$ are closed subspaces, the *predual product* $V * W$ of V and W is defined by $V * W = (V^\perp \overline{\otimes} W^\perp)_\perp$ (where $V^\perp \overline{\otimes} W^\perp$ denotes the σ-weakly closed linear span of $\{a \otimes b : a \in V^{\perp}$ and $b \in W^{\perp}\}$ in $B(\mathcal{H}) \overline{\otimes} B(\mathcal{K}) = B(\mathcal{H} \otimes \mathcal{K})$). A closed subspace $V \subset B(\mathcal{H})_*$ has the *predual product density property* (the PDP) if $V * W$ is the norm closure of $V \otimes B(\mathcal{K})_* + B(\mathcal{H})_* \otimes W$ whenever W is a closed subspace of $B(\mathcal{K})_*$ for some $\mathcal{K}$.

Our main results are:

(1) S has the weak*-OAP $\Leftrightarrow S_{\perp}$ has the PDP (Theorem 2.6), and

(2) $S_{\perp}$ has the OAP $\Leftrightarrow S$ has the DDP (Theorem 3.2).

Suppose $\mathcal{H}$ is a separable infinite-dimensional Hilbert space. Since there are closed subspaces of $B(\mathcal{H})_*$ which do not have the OAP, it follows from Theorem 3.2 that the σ-weak closure of $S \overline{\otimes} B(\mathcal{H}) + B(\mathcal{H}) \overline{\otimes} T$ can be strictly smaller than $S * T$. As noted above, it was asked in [20] whether this can happen when S and T are reflexive subspaces. We show that this can happen, and, in fact, there are reflexive subalgebras A and B of $B(\mathcal{H})$ for which $A * B$ is not the σ-weak closure of $A \overline{\otimes} B(\mathcal{H}) + B(\mathcal{H}) \overline{\otimes} B$ (Theorem 5.10).

For a discussion of reflexivity and hyperreflexivity properties of operator algebras and subspaces of operators we refer the reader to the articles [4], [14], [15], [19], [20], [21].

1. PRELIMINARIES AND NOTATIONS

For a Hilbert space $\mathcal{H}$, let $\mathcal{H}^{(n)}$ denote the direct sum of n copies of $\mathcal{H}$, $n = 1, 2, \ldots$. If V is a subspace of $B(\mathcal{H})$, then $M_n(V)$, the space of $n \times n$ matrices with entries in V, can be viewed as a subspace of $B(\mathcal{H}^{(n)})$ (where we make the usual identification of $M_n(B(\mathcal{H}))$ with $B(\mathcal{H}^{(n)})$). Let $\|\cdot\|_n$ denote the restriction of the norm of $B(\mathcal{H}^{(n)})$ to $M_n(V)$, $n = 1, 2, \ldots$. A (concrete) *operator space* is such a subspace V, together with the sequence of norms $\|\cdot\|_n$ on $M_n(V)$. In this paper we will always assume our operator spaces are closed, and so are Banach spaces. A (complete) L^{∞}-*matricially normed space* is a Banach space V over $\mathbb{C}$, together with a sequence of norms $\|\cdot\|_n$ on the spaces $M_n(V)$ satisfying:

(i) $\|\alpha v \beta\|_n \leqslant \|\alpha\| \, \|v\|_n \|\beta\|, \quad \alpha, \beta \in M_n(\mathbb{C}), v \in M_n(V)$;

(ii) $\|v_1 \oplus v_2\|_{n+m} = \max\{\|v_1\|_n, \|v_2\|_m\}, \quad v_1 \in M_n(V), v_2 \in M_m(V)$.

If V and W are L^{∞}-matricially normed spaces, and if $\varphi \in B(V, W)$, we can define a map $\varphi_n \in B(M_n(V), M_n(W))$ ($n = 1, 2, \ldots$) by $\varphi_n([v_{ij}]) = [\varphi(v_{ij})]$ ($[v_{ij}] \in M_n(V)$). Such a map φ is *completely bounded* if $\sup_n \|\varphi_n\| < \infty$, and a *complete isometry* if each φ_n is an isometry. If φ is completely bounded, the completely bounded norm of φ is defined by $\|\varphi\|_{\mathrm{cb}} = \sup_n \|\varphi_n\|$. The space of completely bounded maps from V to W is denoted by $\mathrm{CB}(V, W)$.

Concrete operators spaces are L^∞-matricially normed spaces, and Ruan proved in [24] that any L^∞-matricially normed space is completely isometric to a concrete operator space (see [10] for a short proof of Ruan's theorem). This abstract characterization of operator spaces is very useful. For example, if V and W are operator spaces, and if we define a norm $\|\cdot\|_n$ on $M_n(\mathrm{CB}(V,W))$ by identifying $M_n(\mathrm{CB}(V,W))$ in the usual way with $\mathrm{CB}(V,M_n(W))$ (i.e., associate the map $v \to [\varphi_{ij}(v)]$ in $\mathrm{CB}(V,M_n(W))$ to the element $[\varphi_{ij}]$ of $M_n(\mathrm{CB}(V,W))$), then it is easily checked that $\{\mathrm{CB}(V,W), \|\cdot\|_n\}$ is an L^∞-matricially normed space, and so is an operator space.

If V is an operator space, then V^* together with the norms $\|\cdot\|_n$ on $M_n(V^*)$ obtained by identifying $M_n(V^*)$ with $\mathrm{CB}(V, M_n(\mathbf{C}))$ is called the *operator dual* [25] (or *standard dual* [3]) of V. (If $\varphi \in V^*$, then $\|\varphi\| = \|\varphi\|_{\mathrm{cb}}$, so $\|\varphi\|_1 = \|\varphi\|$.) The canonical map from V to V^{**} is a complete isometry (see [3] or [9]). If $S = V^*$ for some operator space V, then S is said to be a *dual operator space* and V is an *operator predual* of S. If $\mathcal{H}$ is a Hilbert space, we will always assume that $B(\mathcal{H})_*$ has the operator space structure it inherits as a subspace of $B(\mathcal{H})^*$. Then $B(\mathcal{H})$ (with its usual operator space structure) is the operator dual of $B(\mathcal{H})_*$ ([2], Theorem 2.9). If $V \subset B(\mathcal{H})_*$ is a subspace, we will always assume its operator space structure is the one it inherits from $B(\mathcal{H})_*$.

If $S \subset B(\mathcal{H})$ is a σ-weakly closed subspace, then we will always assume that $S_* = B(\mathcal{H})_*/S_\perp$ has the quotient operator space structure (i.e. we identify $M_n(S_*)$ with $M_n(B(\mathcal{H})_*)/M_n(S_\perp)$, and put the quotient norm on $M_n(S_*)$ for all n). Then S is the operator dual of S_* ([2], Corollary 2.4). If $S \subset B(\mathcal{H})$ and $T \subset B(\mathcal{K})$ are σ-weakly closed subspaces, we denote by $CB_\sigma(S,T)$ the space of normal ($=\sigma$-weakly continuous) completely bounded maps from S to T.

Let K_∞ denote the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space $\mathcal{H}_0$, and let T_∞ denote the space of trace class operators, identified as an operator space with $B(\mathcal{H}_0)_*$. If $V \subset B(\mathcal{H})$ is an operator space, we denote the spatial tensor product of V and K_∞ by $K_\infty(V)$; i.e., $K_\infty(V)$ is the closed linear span of $\{x \otimes y : x \in V \text{ and } y \in K_\infty\}$ in $B(\mathcal{H} \otimes \mathcal{H}_0)$. If V and W are operator spaces, and if $\varphi \in \mathrm{CB}(V,W)$, there is a unique map $\varphi_\infty \in \mathrm{CB}(K_\infty(V), K_\infty(W))$ such that $\varphi_\infty(x \otimes y) = \varphi(x) \otimes y$ ($x \in V$, $y \in K_\infty$). If φ is a complete isometry, so is φ_∞, and thus $K_\infty(V)$ does not depend on how V is concretely represented as an operator space. If we view the elements of $K_\infty(V)$ as $\infty \times \infty$-matrices with entries in V in the obvious way, then $\varphi_\infty([v_{ij}]) = [\varphi(v_{ij})]$. (See [8] for a detailed discussion of $K_\infty(V)$.)

If V and W are operator spaces, a net $\{\varphi_\lambda\}$ in $\mathrm{CB}(V,W)$ is said to converge to $\varphi \in \mathrm{CB}(V,W)$ in the *stable point norm topology* if $(\varphi_\lambda)_\infty(v) \to (\varphi)_\infty(v)$ in norm for all $v \in K_\infty(V)$. Let $F(V)$ denote the space of bounded (and hence completely bounded ([7], Corollary 3.4)) finite rank operators from V to V. A Banach space V has the *approximation property* (AP) of Grothendieck ([12]) if there is a net $\{\varphi_\lambda\}$ in $F(V)$ such that φ_λ converges to the identity operator id_V from V to V uniformily on compact sets. If V is also an operator space, then V has the *operator approximation property* (OAP) of Effros and Ruan ([8]) if there is a net $\{\varphi_\lambda\}$ in $F(V)$ such that φ_λ converges to id_V in the stable point norm topology. If V has the OAP, then V has the AP. (This follows easily from the equivalence of (i) and (ii) on p. 185 of [8]. See also Remark 5.9 in [18].) It seems likely that there are operator spaces which have the AP and don't have the OAP, but no examples of such spaces are known.

Now suppose S is a dual operator space. Then there is a weak*-σ-weak homeomorphism and complete isometry of S onto a σ-weakly closed subspace of some $B(\mathcal{H})$. (See [2] or [8].) In this paper, since we are interested in preannihilators, we will always assume dual operator spaces are concretely represented as σ-weakly closed subspaces of some $B(\mathcal{H})$. If $S \subset B(\mathcal{H})$ is a σ-weakly closed subspace, we set $M_\infty(S) = S \overline{\otimes} B(\mathcal{H}_0)$, where $\mathcal{H}_0$ is a separable infinite-dimensional Hilbert space. If $S \subset B(\mathcal{H})$ and $T \subset B(\mathcal{K})$ are σ-weakly closed subspaces, then for each $\varphi \in \mathrm{CB}_\sigma(S,T)$ there is a unique $\varphi_\infty \in \mathrm{CB}_\sigma(M_\infty(S), M_\infty(T))$ such that $\varphi_\infty(a \otimes b) = \varphi(a) \otimes b$ ($a \in S$, $b \in B(\mathcal{H}_0)$). If we view the elements of $M_\infty(S)$ as $\infty \times \infty$ matrices with entries in S, then $\varphi_\infty([a_{ij}]) = [\varphi(a_{ij})]$ ($[a_{ij}] \in M_\infty(S)$). A net $\{\varphi_\lambda\}$ in $\mathrm{CB}_\sigma(S,T)$ is said to converge to $\varphi \in \mathrm{CB}_\sigma(S,T)$ in the *stable point-weak* topology* if $(\varphi_\lambda)_\infty(a) \to \varphi_\infty(a)$ σ-weakly for every $a \in M_\infty(S)$. We say that S has the *weak*-operator approximation property* (weak*-OAP) if there is a net $\{\varphi_\lambda\}$ in $F_\sigma(S)$ (the space of normal maps in $F(S)$) such that $\{\varphi_\lambda\}$ converges to id_S in the stable point weak*-topology.

There is a close connection between approximation properties and slice maps. If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, and if $x \in B(\mathcal{H})_*$, the *right slice map* R_x associated with x is the unique normal map from $B(\mathcal{H}) \overline{\otimes} B(\mathcal{K})$ to $B(\mathcal{K})$ such that

$$R_x(a \otimes b) = \langle x, a\rangle b, \quad a \in B(\mathcal{H}),\ b \in B(\mathcal{K}). \tag{1.1}$$

The left slice maps $L_y : B(\mathcal{H}) \overline{\otimes} B(\mathcal{K}) \to B(\mathcal{H})$ ($y \in B(\mathcal{K})_*$) are similarly defined. (If $x \in B(\mathcal{H})_*$, the slice map R_x depends on $\mathcal{K}$ as well as x. However, it should always be clear from context what the domain and range of R_x are. A similar remark applies to left slice maps.) It is easily checked that

$$\langle x \otimes y, c\rangle = \langle y, R_x(c)\rangle = \langle x, L_y(c)\rangle \tag{1.2}$$

whenever $x \in B(\mathcal{H})_*$, $y \in B(\mathcal{K})_*$ and $c \in B(\mathcal{H}) \overline{\otimes} B(\mathcal{K})$.

If $S \subset B(\mathcal{H})$ and $T \subset B(\mathcal{K})$ are σ-weakly closed subspaces, the *Fubini product* $S \overline{\otimes}_F T$ of S and T is $\{c \in B(\mathcal{H} \otimes \mathcal{K}) : R_x(c) \in T \ \forall x \in B(\mathcal{H})_* \text{ and } L_y(c) \in S \ \forall y \in B(\mathcal{K})_*\}$. A useful fact is that $S \overline{\otimes}_F T = \{c \in S \overline{\otimes} B(\mathcal{K}) : R_x(c) \in T \ \forall x \in B(\mathcal{H})_*\} = \{c \in B(\mathcal{H}) \overline{\otimes} T : L_y(c) \in S \ \forall y \in B(\mathcal{K})_*\}$ (see Remark 1.5 in [16]).

We always have $S \overline{\otimes} T \subset S \overline{\otimes}_F T$. We say that S has *Property* S_σ if $S \overline{\otimes} T = S \overline{\otimes}_F T$ whenever T is a σ-weakly closed subspace of $B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$. It is shown in [18] that S has Property $S_\sigma \Leftrightarrow S$ has the weak*-OAP. (The weak*-OAP is referred to as the σ-weak approximation property in [18].) It is also shown in [18] that there are von Neumann algebras without the weak*-OAP. This is relevant to reflexivity theory: If $\mathcal{L}_1$ is a subspace lattice, then $\operatorname{Alg}(\mathcal{L}_1 \overline{\otimes} \mathcal{L}_2) = \operatorname{Alg} \mathcal{L}_1 \overline{\otimes} \operatorname{Alg} \mathcal{L}_2$ for all subspace lattices $\mathcal{L}_2$ if and only if $\operatorname{Alg} \mathcal{L}_1$ has Property S_σ (see [18], Remark 1.1).

If V and W are operator spaces, and if $u \in M_n(V \odot W)$ (where $V \odot W$ denotes the algebraic tensor product of V and W), let

$$\|u\|_n^{\wedge} = \inf\{\|\alpha\| \, \|v\| \, \|w\| \, \|\beta\|\},$$

where the infimum is taken over all possible representations $u = \alpha(v \otimes w)\beta$ with $v \in M_k(V)$ for some k, $w \in M_\ell(W)$ for some ℓ, $\alpha \in M_{n,k\ell}(\mathbf{C})$ and $\beta \in M_{k\ell,n}(\mathbf{C})$. (If $v = [v_{ij}] \in M_k(V)$ and $w = [w_{pq}] \in M_\ell(W)$, then $v \otimes w$ denotes the $k\ell \times k\ell$ matrix whose $(i,p) \times (j,q)$ entry is $v_{ij} \otimes w_{pq}$, $1 \leqslant i,j \leqslant k$, $1 \leqslant p,q \leqslant \ell$.) It is shown in [9] that $\|\cdot\|_n^{\wedge}$ is a norm for each n, and that the completion $V \widehat{\otimes} W$ of $V \odot W$ with respect to $\|\cdot\|_1^{\wedge}$ is an L^∞-matricially normed space (for the norms $\|\cdot\|_n^{\wedge}$), and so is an operator space, which is called the operator space projective tensor product of V and W. (The operator space projective tensor product was introduced independently in [9] and [3]. See [3] for a different, equivalent, definition of $\|\cdot\|_n^{\wedge}$.)

If V and W are operator spaces, and if $\varphi \in \mathrm{CB}(V, W^*)$, we can define a linear map $\widetilde{\varphi} : V \odot W \to \mathbf{C}$ by $\widetilde{\varphi}(v \otimes w) = \langle \varphi(v), w \rangle$. It is shown in [9] and [3] that $\widetilde{\varphi}$ extends to a map from $V \widehat{\otimes} W$ to $\mathbf{C}$, and that the map $\varphi \to \widetilde{\varphi}$ is a complete isometry from $\mathrm{CB}(V, W^*)$ onto $(V \widehat{\otimes} W)^*$. Another fact that we will need is that if V_i and W_i are operator spaces, $i = 1,2$, and if $\varphi_i \in \mathrm{CB}(V_i, W_i)$, $i = 1,2$, then there is a unique map $\varphi_1 \widehat{\otimes} \varphi_2$ in $\mathrm{CB}(V_1 \widehat{\otimes} V_2, W_1 \widehat{\otimes} W_2)$ such that $(\varphi_1 \widehat{\otimes} \varphi_2)(v_1 \otimes v_2) = \varphi_1(v_1) \otimes \varphi_2(v_2)$, and $\|\varphi_1 \widehat{\otimes} \varphi_2\|_{cb} \leqslant \|\varphi_1\|_{cb} \|\varphi_2\|_{cb}$ ([3], Proposition 5.11).

It is shown in [8] that if $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, then the bilinear map $(x,y) \to x \otimes y$ from $B(\mathcal{H})_* \times B(\mathcal{K})_*$ to $B(\mathcal{H} \otimes \mathcal{K})_*$ determines a complete isometry θ from $B(\mathcal{H})_* \widehat{\otimes} B(\mathcal{K})_*$ onto $B(\mathcal{H} \otimes \mathcal{K})_*$. (More generally, if $S \subset B(\mathcal{H})$ and $T \subset B(\mathcal{K})$ are σ-weakly closed subspaces, and if S has the weak*-OAP, then $(x, y) \to x \otimes y$ $(x \in S_*,\ y \in T_*)$ determines a complete isometry from $S_* \widehat{\otimes} T_*$ onto $(S \overline{\otimes} T)_*$, ([25], Corollary 3.7).) Since θ is a complete isometry, so is its adjoint $\theta^* : B(\mathcal{H}) \overline{\otimes} B(\mathcal{K}) \to (B(\mathcal{H})_* \widehat{\otimes} B(\mathcal{K})_*)^*$. If we make the canonical identifications of $(B(\mathcal{H})_* \widehat{\otimes} B(\mathcal{K})_*)^*$ with $\mathrm{CB}(B(\mathcal{H})_*, B(\mathcal{K}))$ and of $M_n(\mathrm{CB}(B(\mathcal{H})_*, B(\mathcal{K})))$ with $\mathrm{CB}(B(\mathcal{H})_*, M_n(B(\mathcal{K})))$, then it is easily checked that for $n = 1, 2, \ldots$ and for any $x \in B(\mathcal{H})_*$,

$$(R_x)_n(c) = [(\theta^*)_n(c)](x), \quad c \in M_n(B(\mathcal{H}) \overline{\otimes} B(\mathcal{K})).$$

Hence $\|R_x\|_{\mathrm{cb}} = \|x\|\ \ \forall x \in B(\mathcal{H})_*$. Similarly, $\|L_y\|_{\mathrm{cb}} = \|y\|\ \ \forall y \in B(\mathcal{K})_*$. In what follows we will often use θ to identify $B(\mathcal{H})_* \widehat{\otimes} B(\mathcal{K})_*$ with $B(\mathcal{H} \otimes \mathcal{K})_*$, and write $B(\mathcal{H})_* \widehat{\otimes} B(\mathcal{K})_* = B(\mathcal{H} \otimes \mathcal{K})_*$.

If $a \in B(\mathcal{H})$, then $a \in \mathrm{CB}(B(\mathcal{H})_*, \mathbb{C})$, and so there is a map $a \widehat{\otimes} \mathrm{id}_{B(\mathcal{K})_*} \in \mathrm{CB}(B(\mathcal{H})_* \widehat{\otimes} B(\mathcal{K})_*,\ \mathbb{C} \widehat{\otimes} B(\mathcal{K})_*)$ for any Hilbert space $\mathcal{K}$. Let R_a denote the composition of $a \widehat{\otimes} \mathrm{id}_{B(\mathcal{K})_*}$ with the canonical complete isometry (extending $\lambda \otimes y \to \lambda y$) from $\mathbb{C} \widehat{\otimes} B(\mathcal{K})_*$ to $B(\mathcal{K})_*$. Then $R_a \in \mathrm{CB}(B(\mathcal{H} \otimes \mathcal{K})_*,\ B(\mathcal{K})_*)$. It can be easily shown that $\|R_a\|_{\mathrm{cb}} = \|a\|$, and

$$(1.3) \qquad R_a(x \otimes y) = \langle x, a \rangle y, \quad x \in B(\mathcal{H})_*,\ y \in B(\mathcal{K})_*.$$

We call R_a the *right slice map associated with* a. The domain and range of R_a are not indicated by the notation, but these spaces should always be clear from context. The left slice maps $L_b \in \mathrm{CB}(B(\mathcal{H} \otimes \mathcal{K})_*, B(\mathcal{H})_*)$ $(b \in B(\mathcal{K}))$ are similarly defined. A straightforward calculation shows

$$(1.4) \quad \langle z, a \otimes b \rangle = \langle R_a(z), b \rangle = \langle L_b(z), a \rangle, \quad z \in B(\mathcal{H} \otimes \mathcal{K})_*,\ a \in B(\mathcal{H}),\ b \in B(\mathcal{K}).$$

Let $V \subset B(\mathcal{H})_*$ and $W \subset B(\mathcal{K})_*$ be closed subspaces. Then, as noted in the introduction, we use $V \otimes W$ to denote the closed linear span of $\{x \otimes y : x \in V$ and $y \in W\}$ in $B(\mathcal{H})_* \widehat{\otimes} B(\mathcal{K})_* = B(\mathcal{H} \otimes \mathcal{K})_*$. The *Fubini product* $V \otimes_F W$ is defined by $V \otimes_F W = \{z \in B(\mathcal{H} \otimes \mathcal{K})_* : R_a(z) \in W\ \ \forall a \in B(\mathcal{H})$ and $L_b(z) \in V$ $\forall b \in B(\mathcal{K})\}$. We always have $V \otimes W \subset V \otimes_F W$, as is easily checked. We show in Section 3 (Theorem 3.1) that $V \otimes W = V \otimes_F W$ for every $W \subset B(\mathcal{K})_*$, $\mathcal{K}$ any Hilbert space, if and only if V has the OAP.

The next result follows easily from (1.2) and (1.4), and the proof is left to the reader.

PROPOSITION 1.1. *Let $S \subset B(\mathcal{H})$ and $T \subset B(\mathcal{K})$ be σ-weakly closed subspaces, and let $V \subset B(\mathcal{H})_*$ and $W \subset B(\mathcal{K})_*$ be closed subspaces.*

(i)
$$\begin{aligned}(S\overline{\otimes}T)_\perp &= \{z \in B(\mathcal{H}\otimes\mathcal{K})_* : R_a(z) \in T_\perp \ \forall a \in S\} \\ &= \{z \in B(\mathcal{H}\otimes\mathcal{K})_* : L_b(z) \in S_\perp \ \forall b \in T\}.\end{aligned}$$

(ii)
$$\begin{aligned}(V \otimes W)^\perp &= \{c \in B(\mathcal{H}\otimes\mathcal{K}) : R_x(c) \in W^\perp \ \forall x \in V\} \\ &= \{c \in B(\mathcal{H}\otimes\mathcal{K}) : L_y(c) \in V^\perp \ \forall y \in W\}.\end{aligned}$$

PROPOSITION 1.2. *Let $S \subset B(\mathcal{H})$ and $T \subset B(\mathcal{K})$ be σ-weakly closed subspaces, and let $V \subset B(\mathcal{H})_*$ and $W \subset B(\mathcal{K})_*$ be norm closed subspaces. Then:*

(i) *$(S\overline{\otimes}_F T)_\perp$ is the norm closure of $S_\perp \otimes B(\mathcal{K})_* + B(\mathcal{H})_* \otimes T_\perp$.*

(ii) *$(V \otimes_F W)^\perp$ is the σ-weak closure of $V^\perp \overline{\otimes} B(\mathcal{K}) + B(\mathcal{H})\overline{\otimes} W^\perp$.*

Proof. Using Proposition 1.1 (ii), we have that

$$\begin{aligned}&(S_\perp \otimes B(\mathcal{K})_* + B(\mathcal{H})_* \otimes T_\perp)^\perp \\ &\quad = (S_\perp \otimes B(\mathcal{K})_*)^\perp \cap (B(\mathcal{H})_* \otimes T_\perp)^\perp \\ &\quad = \{c \in B(\mathcal{H}\otimes\mathcal{K}) : L_y(c) \in (S_\perp)^\perp = S \quad \forall y \in B(\mathcal{K})_*\} \\ &\qquad\quad \cap \{c \in B(\mathcal{H}\otimes\mathcal{K}) : R_x(c) \in (T_\perp)^\perp = T \quad \forall x \in B(\mathcal{H})_*\} \\ &\quad = S\overline{\otimes}_F T,\end{aligned}$$

from which (i) follows immediately. A similar application of Proposition 1.1 (i) yields (ii). ∎

A subspace $S \subset B(\mathcal{H})$ is called *n-reflexive* if the n-fold ampliation $S^{(n)} = \{s \oplus \cdots \oplus s : s \in S\}$ is reflexive in $B(\mathcal{H}^{(n)})$.

PROPOSITION 1.3. *Suppose $S \subset B(\mathcal{H})$ and $T \subset B(\mathcal{K})$ are σ-weakly closed subspaces, with $\mathcal{H}$ and $\mathcal{K}$ separable, and that S and T are both reflexive (resp. n-reflexive, weakly closed). Then $S\overline{\otimes}_F T$ is reflexive (resp. n-reflexive, weakly closed.)*

Proof. From [19], a σ-weakly closed subspace is reflexive (resp. n-reflexive, weakly closed) if and only if its preannihilator is generated by operators of rank $\leqslant 1$ (resp. rank $\leqslant n$, finite-rank). So an application of Proposition 1.2 (1) yields the result. ∎

The next result will prove useful in what follows, and should be compared with the analogous results for σ-weakly closed subspaces (with $B(\mathcal{K})_*$ replaced by $B(\mathcal{K})$).

PROPOSITION 1.4. *Let $V \subset B(\mathcal{H})_*$ and $W \subset B(\mathcal{K})_*$ be closed subspaces.*

(i) $V \otimes_F B(\mathcal{K})_* = V \otimes B(\mathcal{K})_*$.

(ii) $V \otimes_F W = \{z \in V \otimes B(\mathcal{K})_* : R_a(z) \in W \quad \forall a \in B(\mathcal{H})\}$.

Proof. (i) Since $(B(\mathcal{K})_*)^\perp = \{0\}$, it follows from Proposition 1.2 (ii) that $(V \otimes_F B(\mathcal{K})_*)^\perp = V^\perp \overline{\otimes} B(\mathcal{K})$. Since $B(\mathcal{K})$ has Property S_σ ([16], Theorem 1.9), $V^\perp \overline{\otimes} B(\mathcal{K}) = V^\perp \overline{\otimes}_F B(\mathcal{K})$. Since $B(\mathcal{K})_\perp = \{0\}$, it follows from Proposition 1.2 (i) that $(V^\perp \overline{\otimes}_F B(\mathcal{K}))_\perp = (V^\perp)_\perp \otimes B(\mathcal{K})_* = V \otimes B(\mathcal{K})_*$. Hence $V \otimes_F B(\mathcal{K})_* = ((V \otimes_F B(\mathcal{K})_*)^\perp)_\perp = V \otimes B(\mathcal{K})_*$.

(ii) By (i), $V \otimes_F W \subset V \otimes_F B(\mathcal{K})_* = V \otimes B(\mathcal{K})_*$, so we get $\subset$. On the other hand, if $z \in V \otimes B(\mathcal{K})_*$, then $L_b(z) \in V$ for all $b \in B(\mathcal{K})$, and so we also get $\supset$. ∎

2. ANNIHILATORS WITH THE WEAK*-OAP

For σ-weakly closed subspaces $S \subset B(\mathcal{H})$ and $T \subset B(\mathcal{K})$ we define the *dual Fubini product* $S *_F T$ by

$$S *_F T = (S_\perp \otimes_F T_\perp)^\perp.$$

For closed subspaces $V \subset B(\mathcal{H})_*$ and $W \subset B(\mathcal{K})_*$, we define the *predual Fubini product* $V *_F W$ by

$$V *_F W = (V^\perp \overline{\otimes}_F W^\perp)_\perp.$$

The next result follows immediately from the definitions and Proposition 1.2.

PROPOSITION 2.1. (i) *If $S \subset B(\mathcal{H})$ is a σ-weakly closed subspace, then S has the* DDP $\Leftrightarrow$ *$S * T = S *_F T$ whenever T is a σ-weakly closed subspace of $B(\mathcal{K})$ for some Hilbert space $\mathcal{K}$.*

(ii) *If $V \subset B(\mathcal{H})_*$ is a closed subspace, then V has the* PDP $\Leftrightarrow$ *$V * W = V *_F W$ whenever W is a closed subspace of $B(\mathcal{K})_*$ for some Hilbert space $\mathcal{K}$.*

The next proposition, whose routine proof is left to the reader, will be useful in Sections 4 and 5.

PROPOSITION 2.2. (i) *Suppose $S \subset B(\mathcal{H})$ and $T \subset B(\mathcal{K})$ are σ-weakly closed subspaces. Then $S * T = S *_F T \Leftrightarrow T * S = T *_F S$.*

(ii) *Suppose $V \subset B(\mathcal{H})_*$ and $W \subset B(\mathcal{K})_*$ are closed subspaces. Then $V * W = V *_F W \Leftrightarrow W * V = W *_F V$.*

THEOREM 2.3. *Let $V \subset B(\mathcal{H})_*$ be a closed subspace, let $S = V^\perp$, and let $\mathcal{H}_0$ be a separable infinite-dimensional Hilbert space. The following conditions are equivalent:*

(i) *V has the* PDP;

(ii) $V * W = V *_F W$ *whenever* $W \subset B(\mathcal{H}_0)_*$ *is a closed subspace;*

(iii) S *has the weak**-OAP.

Proof. (i) $\Rightarrow$ (ii) follows immediately from Proposition 2.1 (ii).

(ii) $\Rightarrow$ (iii). Let $T \subset B(\mathcal{H}_0)$ be a σ-weakly closed subspace. Then $S \overline{\otimes} T = (S_\perp * T_\perp)^\perp = (V * T_\perp)^\perp = (V *_F T_\perp)^\perp = S \overline{\otimes}_F (T_\perp)^\perp = S \overline{\otimes}_F T$. Hence S has the weak*-OAP by Theorem 2.6 in [18].

(iii) $\Rightarrow$ (i). Let $\mathcal{K}$ be a Hilbert space, and let $W \subset B(\mathcal{K})_*$ be a closed subspace. Let $T = W^\perp$. Since S has the weak*-OAP, it has Property S_σ, and so $V * W = (S \overline{\otimes} T)_\perp = (S \overline{\otimes}_F T)_\perp = S_\perp *_F T_\perp = V *_F W$. Hence V has the PDP by Proposition 2.1.(ii). ∎

REMARK 2.4. If S is a dual operator space, then, as noted above, S can be realized concretely as a σ-weakly closed subspace of $B(\mathcal{H})$ for some $\mathcal{H}$. Theorem 2.3 implies that although $S_\perp$ depends on the concrete realization of S, if S has the weak*-OAP, then $S_\perp$ has the PDP for any such realization.

3. PREANNIHILATORS WITH THE OAP

THEOREM 3.1. *Let $S \subset B(\mathcal{H})$ be a σ-weakly closed subspace, let $V = S_\perp$, and let $\mathcal{H}_0$ be a separable infinite-dimensional Hilbert space. The following conditions are equivalent:*

(i) S *has the* DDP*;*

(ii) $S * T = S *_F T$ *whenever* $T \subset B(\mathcal{H}_0)$ *is a σ-weakly closed subspace;*

(iii) $V \otimes W = V \otimes_F W$ *whenever* W *is a closed subspace of* $B(\mathcal{K})_*$ *for some Hilbert space* $\mathcal{K}$*;*

(iv) $V \otimes W = V \otimes_F W$ *whenever* $W \subset B(\mathcal{H}_0)_*$ *is a closed subspace;*

(v) V *has the* OAP.

Before proving Theorem 3.1, we need a few lemmas. The first lemma is a special case of Proposition 3.3 in [9].

LEMMA 3.2. *Suppose V is a closed subspace of an operator space W, let $\mathcal{K}$ be a Hilbert space, and let $i_V : V \to W$ be the inclusion map. Then $i_V \widehat{\otimes} \mathrm{id}_{B(\mathcal{K})_*}$: $V \widehat{\otimes} B(\mathcal{K})_* \to W \widehat{\otimes} B(\mathcal{K})_*$ is a complete isometry. If $W = B(\mathcal{H})_*$ for some Hilbert space $\mathcal{H}$, then $i_V \widehat{\otimes} \mathrm{id}_{B(\mathcal{K})_*}$ maps $V \widehat{\otimes} B(\mathcal{K})_*$ onto $V \otimes B(\mathcal{K})_*$.*

LEMMA 3.3. *Let $V \subset B(\mathcal{H})_*$ be a closed subspace, let $i_V : V \to B(\mathcal{H})_*$ be the inclusion map, let $\mathcal{K}$ be a Hilbert space and let $\varphi = i_V \widehat{\otimes} \mathrm{id}_{B(\mathcal{K})_*}$. For*

$z \in V \otimes B(\mathcal{K})_*$, let $W_z = \{R_a(z) : a \in B(\mathcal{H})\}$. Then $V \otimes W_z$ is the norm closure of $\{\varphi((\psi \widehat{\otimes} \mathrm{id}_{B(\mathcal{K})_*})(\varphi^{-1}(z)) : \psi \in F(V)\}$.

Proof. Let $\psi \in F(V)$. Since $V \subset B(\mathcal{H})_*$, we can choose $a_1, \ldots, a_n$ in $B(\mathcal{H})$ and $x_1, \ldots, x_n$ in V for some n so that

$$\psi(x) = \sum_{i=1}^{n} \langle x, a_i \rangle x_i, \quad x \in V. \tag{3.1}$$

Then

$$\varphi((\psi \widehat{\otimes} \mathrm{id}_{B(\mathcal{K})_*})(\varphi^{-1}(z))) = \sum_{i=1}^{n} x_i \otimes R_{a_i}(z) \tag{3.2}$$

when $z = x \otimes y$ for some $x \in V$, $y \in B(\mathcal{K})_*$ (since LHS $= \varphi((\psi \widehat{\otimes} \mathrm{id}_{B(\mathcal{K})_*})(x \otimes y)) = \varphi(\psi(x) \otimes y) = \varphi\Big(\sum_{i=1}^{n} \langle x, a_i \rangle x_i \otimes y) = \sum_{i=1}^{n} x_i \otimes \langle x, a_i \rangle y = \sum_{i=1}^{n} x_i \otimes R_{a_i}(x \otimes y)\Big)$, from which it follows easily that (3.2) is valid for any z in $V \otimes B(\mathcal{K})_*$. Since the elements of $F(V)$ are precisely the maps defined by (3.1) for some choice of $a_1, \ldots, a_n$ in $B(\mathcal{H})$ and $x_1, \ldots, x_n$ in V, it follows immediately from (3.2) that the norm closure of $\{\varphi((\psi \widehat{\otimes} \mathrm{id}_{B(\mathcal{K})_*})(\varphi^{-1}(z)) : \psi \in F(V)\}$ is $V \otimes W_z$. ∎

Proof of Theorem 3.1. (v) $\Rightarrow$ (iii). By Theorem 2.4 in [6], there is a net $\{\psi_\lambda\}$ in $F(V)$ such that $\psi_\lambda \widehat{\otimes} \mathrm{id}_{B(\mathcal{K})_*} : V \widehat{\otimes} B(\mathcal{K})_* \to V \widehat{\otimes} B(\mathcal{K})_*$ converges in the point-norm topology to the identity map of $V \widehat{\otimes} B(\mathcal{K})_*$. Now let $W \subset B(\mathcal{K})_*$ be a closed subspace, and let $z \in V \otimes_F W$. Then $z \in V \otimes B(\mathcal{K})_*$ by Proposition 1.4 (ii), and so $\varphi(\varphi^{-1}(z)) = z$ (where, as in Lemma 3.3, $\varphi = i_V \widehat{\otimes} \mathrm{id}_{B(\mathcal{K})_*}$). On the other hand, $(\psi_\lambda \widehat{\otimes} \mathrm{id}_{B(\mathcal{K})_*})(\varphi^{-1}(z)) \to \varphi^{-1}(z)$ in norm, and φ is an isometry by Lemma 3.2, so $\varphi((\psi_\lambda \widehat{\otimes} \mathrm{id}_{B(\mathcal{K})_*})(\varphi^{-1}(z)) \to z$ in norm. Since $W_z = \{R_a(z) : a \in B(\mathcal{H})\} \subset W$, it follows from Lemma 3.3 that $z \in V \otimes W$. Hence $V \otimes_F W \subset V \otimes W$. But the reverse inclusion always holds, so $V \otimes_F W = V \otimes W$.

(iii) $\Rightarrow$ (iv) and (ii) $\Leftrightarrow$ (iv) are trivial, and (i) $\Rightarrow$ (ii) and (i) $\Leftrightarrow$ (iii) follow from Proposition 2.1 (i).

(iv) $\Rightarrow$ (v). By Theorem 2.4 in [6], to show V has the OAP it suffices to show that V^* has the weak*-OAP. By Propositions 2.2 and 2.3 in [18], to show V^* has the weak*-OAP it suffices to show that for any $z \in (V^* \overline{\otimes} B(\mathcal{H}_0))_*$, z is in the norm closure of $\{\widetilde{\psi}(z) : \psi \in F(V)\}$, where $\widetilde{\psi} = ((\psi^*)_\infty)_*$. Since $B(\mathcal{H}_0)$ has Property S_σ, there is a complete isometry θ from $V \widehat{\otimes} B(\mathcal{H}_0)_*$ onto $(V^* \overline{\otimes} B(\mathcal{H}_0))_*$

such that $\theta(x\otimes y) = x\otimes y$ ([25], Corollary 3.7). If we use θ to identify $V\widehat{\otimes}B(\mathcal{H}_0)_*$ with $(V^*\overline{\otimes}B(\mathcal{H}_0))_*$, then $\widetilde{\psi} = \psi\widehat{\otimes}\mathrm{id}_{B(\mathcal{H}_0)_*}$. Hence it suffices to show that if $z\in V\widehat{\otimes}B(\mathcal{H}_0)_*$, then z is in the norm closure of $\{(\psi\widehat{\otimes}\mathrm{id}_{B(\mathcal{H}_0)_*})(z) : \psi\in F(V)\}$.

So let $z \in V\widehat{\otimes}B(\mathcal{H}_0)_*$, and let $\varphi = i_V\widehat{\otimes}\mathrm{id}_{B(\mathcal{H}_0)_*}$. Then $\varphi(z) \in V\otimes B(\mathcal{K})_*$, and so if we set $W = \{R_a(\varphi(z)) : a\in B(\mathcal{H})_*\}$, then $\varphi(z)\in V\otimes_F W$ by Proposition 1.4 (ii). Since W is a closed subspace of $B(H_0)_*$, $\varphi(z)\in V\otimes W$. Hence by Lemma 3.3, $\varphi(z)$ is in the norm closure of $\{\varphi((\psi\widehat{\otimes}\mathrm{id}_{B(\mathcal{H}_0)_*})(\varphi^{-1}(\varphi(z)))) : \psi\in F(V)\}$, and so, since φ is an isometry by Lemma 3.2, z is in the norm closure of $\{(\psi\widehat{\otimes}\mathrm{id}_{B(\mathcal{H}_0)_*})(z) : \psi\in F(V)\}$, as required. ∎

4. SUBSPACES WITH THE DDP

PROPOSITION 4.1. *Let $S\subset B(\mathcal{H})$ be a σ-weakly closed subspace. If $S_\perp$ is finite dimensional, then S has the* DDP.

Proof. Since $S_\perp$ is finite dimensional, it has the OAP, and so S has the DDP by Theorem 3.1. ∎

Note that if $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional Hilbert spaces, then for any subspaces $S\subset B(\mathcal{H})$ and $T\subset B(\mathcal{K})$ we have that $S*T = S\otimes B(\mathcal{K}) + B(\mathcal{H})\otimes T$, since S has the DDP by Proposition 4.1, and $S\otimes B(\mathcal{K}) + B(\mathcal{H})\otimes T$ is finite dimensional and so σ-weakly closed. This result is proved more directly in [20].

THEOREM 4.2. (i) *Let $S_i\subset B(\mathcal{H}_i)$ be nonzero σ-weakly closed subspaces, $i=1,2$. Then $S_1 * S_2$ has the* DDP $\Leftrightarrow$ *S_1 and S_2 have the* DDP.

(ii) *Let $V_i\subset B(\mathcal{H}_i)_*$ be nonzero closed subspaces. Then $V_1\otimes V_2$ has the* OAP $\Leftrightarrow$ *V_1 and V_2 have the* OAP.

Proof. By Theorem 3.1, it suffices to prove (ii). So first assume that V_1 and V_2 have the OAP, and let $S_i = V_i^\perp$, $i=1,2$. Let $\mathcal{K}$ be a Hilbert space, and let T be a σ-weakly closed subspace of $B(\mathcal{K})$. Then $(S_1 * S_2) * T = ((S_1 * S_2)_\perp\otimes T_\perp)^\perp$ $= ((V_1\otimes V_2)\otimes T_\perp)^\perp = (V_1\otimes(V_2\otimes T_\perp))^\perp = S_1*(S_2*T)$. Moreover, it follows easily from Proposition 1.2 (ii) that $(S_1 *_F S_2)*_F T = S_1 *_F (S_2 *_F T)$. Since V_1 and V_2 have the OAP, it follows from Theorem 3.1 that $(S_1*S_2)*T = S_1*(S_2*T) = S_1*_F(S_2*T)$ $= S_1 *_F (S_2 *_F T) = (S_1 *_F S_2) *_F T = (S_1 * S_2) *_F T$. Hence $S_1 * S_2$ has the DDP and so $V_1\otimes V_2$ has the OAP.

Now assume that $V_1\otimes V_2$ has the OAP. Then $V_2\otimes V_1$ also has the OAP (since the map $x\otimes y\to y\otimes x$ extends to a complete isometry from $B(\mathcal{H}_1)_*\widehat{\otimes}B(\mathcal{H}_2)_*$ onto

$B(\mathcal{H}_2)_* \widehat{\otimes} B(\mathcal{H}_1)_*)$, and so by symmetry it suffices to show that V_1 has the OAP. Choose a fixed unit vector y_0 in V_2, and choose $b_0 \in B(\mathcal{H}_2)$ such that $\langle y_0, b_0\rangle = 1$. Let $\varphi(x) = x \otimes y_0$, $x \in V_1$. Then φ is a complete isometry when viewed as a map from V_1 into $V_1 \widehat{\otimes} B(\mathcal{H}_2)_*$ (since the projective operator space tensor norm is an operator space cross norm ([3], Theorem 5.5)), and so, identifying $V_1 \widehat{\otimes} B(\mathcal{H}_2)_*$ with $V_1 \otimes B(\mathcal{H}_2)_*$ via Lemma 3.2, we get that φ is a complete isometry from V_1 into $V_1 \otimes B(\mathcal{H}_2)_*$ (and, of course, $\varphi(V_1) \subset V_1 \otimes V_2$). Moreover, $L_{b_0}(\varphi(x)) = \langle y_0, b_0\rangle x = x$ for all x in V_1. Set $V = V_1 \otimes V_2$. Since V has the OAP, there is a net $\{\psi_\lambda\}$ in $F(V)$ such that $(\psi_\lambda)_\infty(z) \to z$ in norm for all $z \in K_\infty(V)$. In particular, since $\varphi_\infty : K_\infty(V_1) \to K_\infty(V)$, $(\psi_\lambda)_\infty(\varphi_\infty(y)) \to \varphi_\infty(y)$ in norm for all y in $K_\infty(V_1)$. Let $\varphi_\lambda = L_{b_0} \circ \psi_\lambda \circ \varphi$ for all λ. Then $\varphi_\lambda \in F(V_1)$ for all λ, and if $y \in K_\infty(V_1)$, then

$$\begin{aligned}(\varphi_\lambda)_\infty(y) &= (L_{b_0})_\infty((\psi_\lambda)_\infty(\varphi_\infty(y))) \to (L_{b_0})_\infty(\varphi_\infty(y))\\ &= (L_{b_0} \circ \varphi)_\infty(y) = (\mathrm{id}_{V_1})_\infty(y) = y\end{aligned}$$

in norm. Hence V_1 has the OAP. ∎

It was shown in [18] that if L is a completely distributive commutative subspace lattice (a CDCSL) on a separable Hilbert space $\mathcal{H}$, and if $A = \mathrm{Alg}\ L$ is the associated CDCSL-algebra, then any σ-weakly closed A-bimodule has Property S_σ. The proof used the fact that there is a net $\{a_\lambda\}$ of finite rank operators in A such that $a_\lambda \to 1$ σ-weakly, which follows from Laurie and Longstaff's result ([22]) that the set of finite rank operators in A is σ-weakly dense in A. We next show that every σ-weakly closed A-bimodule has the DDP. We need some preliminary results. (See [13] or [4] for exposition on CDCSL-algebras.)

For $a \in B(\mathcal{H})$, let ℓ_a and r_a denote the maps $\ell_a(b) = ab$ and $r_a(b) = ba$ ($b \in B(\mathcal{H})$). Then $\ell_a, r_a \in \mathrm{CB}_\sigma(B(\mathcal{H}), B(\mathcal{H}))$, and so $(\ell_a)_*$ and $(r_a)_*$ are in $\mathrm{CB}(B(\mathcal{H})_*, B(\mathcal{H})_*)$. For $x \in B(\mathcal{H})_*$, set $ax = (r_a)_*(x)$ and $xa = (\ell_a)_*(x)$. Then

$$(4.1)\qquad \langle ax, b\rangle = \langle x, ba\rangle,\ \langle xa, b\rangle = \langle x, ab\rangle, \quad a, b \in B(\mathcal{H}),\ x \in B(\mathcal{H})_*.$$

It follows immediately from (4.1) that if $S \subset B(\mathcal{H})$ is a σ-weakly closed subspace, and if $a \in B(\mathcal{H})$, then $aS \subset S \Rightarrow S_\perp a \subset S_\perp$ and $Sa \subset S \Rightarrow aS_\perp \subset S_\perp$.

PROPOSITION 4.3. *Let $S \subset B(\mathcal{H})$ be σ-weakly closed, and suppose there are nets $\{a_\lambda\}$ and $\{b_\gamma\}$ of finite rank operators in $B(\mathcal{H})$ such that $a_\lambda \to 1$ and $b_\gamma \to 1$ σ-weakly, and such that $Sa_\lambda \subset S\ \ \forall \lambda \in \Lambda$ and $b_\gamma S \subset S\ \ \forall \gamma \in \Gamma$. Then S has the* DDP.

Proof. Let $V = S_\perp$, and let $\mathcal{K}$ be a Hilbert space. Let $\varphi_\lambda = (r_{a_\lambda})_* \widehat{\otimes} \mathrm{id}_{B(\mathcal{K})_*}$ ($\lambda \in \Lambda$) and $\psi_\gamma = (\ell_{b_\gamma})_* \widehat{\otimes} \mathrm{id}_{B(\mathcal{K})_*}$ ($\gamma \in \Gamma$). Then if $a \in B(\mathcal{H})$, $b \in B(\mathcal{K})$, and

$z \in B(\mathcal{H} \otimes \mathcal{K})_*$, $\langle \varphi_\lambda(z), a \otimes b\rangle = \langle z, aa_\lambda \otimes b\rangle = \langle z, (a \otimes b)(a_\lambda \otimes \mathbf{1})\rangle$ for all $\lambda \in \Lambda$, so $\langle \varphi_\lambda(z), c\rangle = \langle z, c(a_\lambda \otimes \mathbf{1})\rangle, c \in B(\mathcal{H} \otimes \mathcal{K})$. Since $c(a_\lambda \otimes \mathbf{1}) \to c$ σ-weakly, $\langle \varphi_\lambda(z), c\rangle \to \langle z, c\rangle$. Similarly, $\langle \psi_\gamma(z), c\rangle \to \langle z, c\rangle$ for all $z \in B(\mathcal{H} \otimes \mathcal{K})_*$ and $c \in B(\mathcal{H} \otimes \mathcal{K})$. Hence

$$(4.2) \qquad \varphi_\lambda(z) \to z \text{ weakly and } \psi_\gamma(z) \to z \text{ weakly} \qquad (z \in B(\mathcal{H} \otimes \mathcal{K})_*).$$

If $b \in B(\mathcal{K})$, then $\langle R_a(\varphi_\lambda(z)), b\rangle = \langle \varphi_\lambda(z), a \otimes b\rangle = \langle z, aa_\lambda \otimes b\rangle = \langle R_{aa_\lambda}(z), b\rangle$, so $R_a(\varphi_\lambda(z)) = R_{aa_\lambda}(z)$ $(a \in B(\mathcal{H}),\ \lambda \in \Lambda,\ z \in B(\mathcal{H}\otimes\mathcal{K})_*)$. Similarly, $R_a(\psi_\gamma(z)) = R_{b_\gamma a}(z)$ $(a \in B(\mathcal{H}), \gamma \in \Gamma, z \in B(\mathcal{H}\otimes\mathcal{K})_*)$. If $z = x \otimes y$, $(x \in V, y \in B(\mathcal{K})_*)$ then $\varphi_\lambda(\psi_\gamma(z)) = a_\lambda x b_\gamma \otimes y \in a_\lambda V b_\gamma \otimes B(\mathcal{K})_*$ $(\gamma \in \Gamma, \lambda \in \Lambda)$, and so $\varphi_\lambda(\psi_\gamma(z)) \in a_\lambda V b_\gamma \otimes B(\mathcal{K})_*$ if $z \in V \otimes B(\mathcal{K})_*$.

Now let $W \subset B(\mathcal{K})_*$ be a closed subspace, and let $z \in V \otimes_F W$. Let $\lambda \in \Lambda, \gamma \in \Gamma$. Since $z \in V \otimes B(\mathcal{K})_*$, $\varphi_\lambda(\psi_\gamma(z)) \in a_\lambda V b_\gamma \otimes B(\mathcal{K})_*$. Moreover, if $a \in B(\mathcal{H})$, then $R_a(\varphi_\lambda(\psi_\gamma(z))) = R_{b_\gamma a a_\lambda}(z) \in W$, since $z \in V \otimes_F W$. Hence $\varphi_\lambda(\psi_\gamma(z)) \in a_\lambda V b_\gamma \otimes_F W$ by Proposition 1.4 (ii). Since $a_\lambda V b_\gamma$ is finite dimensional, it has the OAP, and so $a_\lambda V b_\gamma \otimes_F W = a_\lambda V b_\gamma \otimes W$ by Theorem 3.1. Since $b_\gamma S a_\lambda \subset S$, $a_\lambda V b_\gamma \subset V$, and so $\varphi_\lambda(\psi_\gamma(z)) \in V \otimes W$ $(\lambda \in \Lambda, \gamma \in \Gamma)$. Hence it follows from (4.2) that $\psi_\gamma(z) \in V \otimes W \quad \forall \gamma \in \Gamma$, and another application of (4.6) shows that $z \in V \otimes W$. Thus $V \otimes_F W = V \otimes W$, and so $S = V^\perp$ has the DDP by Theorem 3.1. ∎

COROLLARY 4.4. *Suppose $\mathcal{H}$ is a separable Hilbert space, and $A_i \subset B(\mathcal{H})$, $i = 1, 2$, are* CDCSL*-algebras. Let $S \subset B(\mathcal{H})$ be a σ-weakly closed A_1–A_2 bimodule. Then S has the* DDP. *In particular, all* CDCSL*-algebras have the* DDP.

Let $\mathcal{H}_0$ be a separable infinite-dimensional Hilbert space, and let $S \subset B(\mathcal{H}_0)$ be a σ-weakly closed subspace. Then S is *compactly dense* (or *local*) if $S \cap K_\infty$ is σ-weakly dense in S, S is *finitely dense* if $S \cap F(B(\mathcal{H}_0))$ is σ-weakly dense in S, and S is *rank one dense* if the linear span of the rank one operators in S is σ-weakly dense in S. The term *local subspace* was first used by Fall, Arveson and Muhly in their work on compact perturbations of operator algebras ([11]), and more recently the term *compactly dense* was used by Ruan in [25] for the algebra case.

The next result follows easily from the definition of the DDP and the fact that $B(\mathcal{H}_0)$ is rank one dense.

PROPOSITION 4.5. *Suppose $\mathcal{H}_0$ is a separable infinite-dimensional Hilbert space, and that $S_i \subset B(\mathcal{H}_0)$, $i = 1, 2$, are σ-weakly closed subspaces, at least one of which has the* DDP. *Then if both S_1 and S_2 are compactly dense (resp. finitely dense, rank one dense), $S_1 * S_2$ is also compactly dense (resp. finitely dense, rank one dense).*

The following is related to a result in [20].

COROLLARY 4.6. *Let $\mathcal{H}_0$ be a separable infinite-dimensional Hilbert space, and let $A_i \subset B(\mathcal{H}_0)$, $i = 1, \ldots, n$ be* CDCSL-*algebras. Let*

$$A = \begin{bmatrix} A_1 \otimes \cdots \otimes A_n & A_1 * \cdots * A_n \\ 0 & A_1 \otimes \cdots \otimes A_n \end{bmatrix}.$$

Then A is a CDCSL-*algebra.*

Proof. Let $L_i = \operatorname{Lat} A_i$, $i = 1, \ldots, n$. Then $A_1 \otimes \cdots \otimes A_n = \operatorname{Alg}(L_1 \otimes \cdots \otimes L_n)$ ([17]), so $A_1 \otimes \cdots \otimes A_n$ is a CDCSL-algebra. It is easily checked that $A_1 * \cdots * A_n$ is an $A_1 \otimes \cdots \otimes A_n$ bimodule, and hence A is a CSL-algebra. Indeed, A is a σ-weakly closed algebra, and we have

$$A_\perp = \begin{bmatrix} (A_1 \otimes \cdots \otimes A_n)_\perp & 0 \\ (A_1 * \cdots * A_n)_\perp & (A_1 \otimes \cdots \otimes A_n)_\perp \end{bmatrix}.$$

Since $A_1 \otimes \cdots \otimes A_n$ and $A_1 * \cdots * A_n$ are reflexive their preannihilators are generated by rank $\leqslant 1$ operators. So $A_\perp$ is gene[rated by rank $\leqslant 1$ operators, and hence A is reflexive. Since $A_1 \otimes \cdots \otimes A_n$ is a CSL-algebra it contains a m.a.s.a., and hence A is a reflexive algebra which contains a m.a.s.a., so A is a CSL-algebra. Moreover, since each A_i has the DDP by Corollary 4.4, it follows from Proposition 4.5 and induction that $A_1 * \cdots * A_n$ is rank one dense, and hence A is finitely dense. Since A is a finitely dense CSL-algebra, it is a CDCSL-algebra ([22]). ∎

5. SUBSPACES WITHOUT THE DDP

It follows from Proposition 4.1 that if $\mathcal{H}$ is a finite dimensional Hilbert space, then every (σ-weakly closed) subspace of $B(\mathcal{H})$ has the DDP. The next result implies that the converse is true.

THEOREM 5.1. *Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. Then there are σ-weakly closed subspaces of $B(\mathcal{H})$ which do not have the* DDP. *Moreover, there are σ-weakly closed subspaces S and T of $B(\mathcal{H})$ such that $S * T \neq S *_F T$.*

Proof. Since $\mathcal{H}$ is infinite-dimensional, $\ell^1(\mathbb{N})$ can be isometrically embedded in $B(\mathcal{H})_*$. Hence, since $\ell^1(\mathbb{N})$ has subspaces without the AP ([26]), so does $B(\mathcal{H})_*$. Let $V \subset B(\mathcal{H})_*$ be a subspace without the AP. Then V doesn't have the OAP, and so $S = V^\perp$ doesn't have the DDP, by Theorem 3.1. Since $\mathcal{H}$ contains a separable infinite-dimensional subspace $\mathcal{H}_0$, another application of Theorem 3.1 yields a σ-weakly closed subspace T of $B(\mathcal{H})$ such that $S * T \neq S *_F T$. ∎

If S doesn't have the DDP, then $S_\perp$ doesn't have the OAP, and so $(S_\perp)^*$ doesn't have the weak*-OAP ([6]). Hence if $\mathcal{K}$ is any infinite-dimensional Hilbert space, there is a σ-weakly closed subspace $T \subset B(\mathcal{K})$ such that $(S_\perp)^* \overline{\otimes} T \neq (S_\perp)^* \overline{\otimes}_F T$. It turns out that T also satisfies $S * T \neq S *_F T$.

PROPOSITION 5.2. *Suppose $S \subset B(\mathcal{H})$ is a σ-weakly closed subspace, and let $V = S_\perp$. Suppose $T \subset B(\mathcal{K})$ is a σ-weakly closed subspace such that $V^* \overline{\otimes} T \neq V^* \overline{\otimes}_F T$. Then $S * T \neq S *_F T$.*

Proof. Let $\varphi = i_V \widehat{\otimes} \mathrm{id}_{B(\mathcal{K})_*}$, where i_V is the inclusion map from V to $B(\mathcal{H})_*$. Then, since $\varphi : V \widehat{\otimes} B(\mathcal{K})_* \to B(\mathcal{H} \otimes \mathcal{K})_*$ is a complete isometry (Lemma 3.2), φ^* is a complete quotient map from $B(\mathcal{H} \otimes \mathcal{K})$ onto $(V \widehat{\otimes} B(\mathcal{K})_*)^*$ ([2], Proposition 2.3). Since $B(\mathcal{K})$ has Property S_σ, the inclusion map from $V^* \odot B(\mathcal{K})$ into $(V \widehat{\otimes} B(\mathcal{K})_*)^*$ extends to a σ-weak-weak* homeomorphic complete isometry of $V^* \overline{\otimes} B(\mathcal{K})$ onto $(V \widehat{\otimes} B(\mathcal{K})_*)^*$ ([25], Corollary 3.7). Hence we can view φ^* as a map from $B(\mathcal{H} \otimes \mathcal{K})$ onto $V^* \overline{\otimes} B(\mathcal{K})$. Moreover, for any $c \in B(\mathcal{H} \otimes \mathcal{K})$

$$\langle R_x(\varphi^*(c)), y\rangle = \langle \varphi^*(c), x \otimes y\rangle = \langle c, i_V(x) \otimes y\rangle = \langle c, x \otimes y\rangle, \tag{5.1}$$

$x \in V$, $y \in B(\mathcal{K})_*$ and so (since $\varphi^*(c) \in V^* \overline{\otimes} B(\mathcal{K})$), $\varphi^*(c) \in V^* \overline{\otimes}_F T \Leftrightarrow R_x(\varphi^*(c)) \in T\ \forall x \in (V^*)_* = V = S_\perp$ ([16], Remark 1.5) $\Leftrightarrow \langle c, x \otimes y\rangle = 0$ when $x \in S_\perp$ and $y \in T_\perp \Leftrightarrow c \in (S_\perp \otimes T_\perp)^\perp = S * T$. Thus, since φ^* is onto,

$$V^* \overline{\otimes}_F T = \varphi^*(S * T). \tag{5.2}$$

It also follows easily from (5.1) that $\varphi^*(S \overline{\otimes} B(\mathcal{K})) = 0$. Moreover, since $(i_V)^*$ maps $B(\mathcal{H})$ onto V^* and since $\varphi^*(a \otimes b) = (i_V)^*(a) \otimes b$ $(a \in B(\mathcal{H}), b \in B(\mathcal{K}))$, we also have that $\varphi^*(B(\mathcal{H}) \overline{\otimes} T) = V^* \overline{\otimes} T$. Hence it follows from Proposition 1.2.(2) that

$$\varphi^*(S *_F T) = V^* \overline{\otimes} T. \tag{5.3}$$

Since $V^* \overline{\otimes} T \neq V^* \overline{\otimes}_F T$, (5.2) and (5.3) together imply that $S * T \neq S *_F T$. ∎

The main result in this section is that if $\mathcal{H}_0$ is a separable infinite-dimensional Hilbert space, then there are reflexive subalgebras $A_i \subset B(\mathcal{H}_0)$, $i = 1,2$, such that $A_1 * A_2 \neq A_1 *_F A_2$. The proof has several steps. We first show that if $S_i \subset B(\mathcal{H}_i)$ are σ-weakly closed subspaces such that $S_1 * S_2 \neq S_1 *_F S_2$, then $A(S_1) * A(S_2) \neq A(S_1) *_F A(S_2)$, where $A(S_i)$ is the σ-weakly closed subalgebra of $M_2(B(\mathcal{H}_i)) = B(\mathcal{H}_i^{(2)})$ consisting of all matrices of the form

$$\begin{bmatrix} \lambda 1 & s \\ 0 & \mu 1 \end{bmatrix}, \quad \lambda, \mu \in \mathbf{C}, \ s \in S_i.$$

We next prove that if $S_i \subset B(\mathcal{H}_i)$, $i = 1,2$, and $T_i \subset B(\mathcal{K}_i)$, $i = 1,2$, are σ-weakly closed subspaces, then $S_1 * S_2 \neq S_1 *_F S_2 \Rightarrow (S_1 \overline{\otimes} T_1) * (S_2 \overline{\otimes} T_2) \neq (S_1 \overline{\otimes} T_1) *_F (S_2 \overline{\otimes} T_2)$. Combining these two results we get that if $S_i \subset B(\mathcal{H}_0)$ $(i = 1,2)$ are σ-weakly closed subspaces such that $S_1 * S_2 \neq S_1 *_F S_2$ (such subspaces exist by Theorem 5.1) then

$$(A(S_1) \overline{\otimes} \mathbf{C}1) * (A(S_2) \overline{\otimes} \mathbf{C}1) \neq (A(S_1) \overline{\otimes} \mathbf{C}1) *_F (A(S_2) \overline{\otimes} \mathbf{C}1),$$

where 1 is the identity operator in $B(\mathcal{H}_0)$. Since each of the σ-weakly closed algebras $A(S_i) \overline{\otimes} \mathbf{C}1$ is reflexive ([23], Theorem 3.5), and since $\mathcal{H}_0^{(2)} \otimes \mathcal{H}_0$ is separable, we get our example.

PROPOSITION 5.3. *Let S_{ij}, $1 \leqslant i,j \leqslant 2$, be σ-weakly closed subspaces of $B(\mathcal{H})$, and let S be the σ-weakly closed subspace of $M_2(B(\mathcal{H})) = B(\mathcal{H}^{(2)})$ defined by*

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \left\{ \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} : s_{ij} \in S_{ij} \right\}.$$

Then for any σ-weakly closed subspace $T \subset B(\mathcal{K})$ we have that

$$S * T = \begin{bmatrix} S_{11} * T & S_{12} * T \\ S_{21} * T & S_{22} * T \end{bmatrix} \quad \text{and} \quad S *_F T = \begin{bmatrix} S_{11} *_F T & S_{12} *_F T \\ S_{21} *_F T & S_{22} *_F T \end{bmatrix} \tag{5.4}$$

(where we identify $M_2(B(\mathcal{H} \otimes \mathcal{K}))$ with $M_2(B(\mathcal{H})) \overline{\otimes} B(\mathcal{K})$). Moreover, S has the DDP $\Leftrightarrow$ *S_{ij} have the* DDP, $1 \leqslant i,j \leqslant 2$.

Proof. By Proposition 2.1 (i), (5.4) implies that S has the DDP $\Leftrightarrow$ S_{ij} have the DDP, $1 \leqslant i,j \leqslant 2$, so it suffices to prove (5.4).

Let $T \subset B(\mathcal{K})$ be a σ-weakly closed subspace. Simple duality computations show that

$$S_\perp = \begin{bmatrix} (S_{11})_\perp & (S_{21})_\perp \\ (S_{12})_\perp & (S_{22})_\perp \end{bmatrix},$$

and that

$$S * T = (S_\perp \otimes T_\perp)^\perp = \left(\begin{bmatrix} (S_{11})_\perp \otimes T_\perp & (S_{21})_\perp \otimes T_\perp \\ (S_{12})_\perp \otimes T_\perp & (S_{22})_\perp \otimes T_\perp \end{bmatrix}\right)^\perp$$
$$= \begin{bmatrix} ((S_{11})_\perp \otimes T_\perp)^\perp & ((S_{12})_\perp \otimes T_\perp)^\perp \\ ((S_{21})_\perp \otimes T_\perp)^\perp & ((S_{22})_\perp \otimes T_\perp)^\perp \end{bmatrix} = \begin{bmatrix} S_{11} * T & S_{12} * T \\ S_{21} * T & S_{22} * T \end{bmatrix}.$$

Let $b \in B(\mathcal{K})$. Then for any $[z_{ij}] \in M_2(B(\mathcal{H})_* \widehat{\otimes} B(\mathcal{K})_*) = M_2(B(\mathcal{H})_*) \widehat{\otimes} B(\mathcal{K})_*$ we have

$$L_b([z_{ij}]) = [L_b(z_{ij})]. \tag{5.5}$$

Indeed, if $[z_{ij}] = [x_{ij} \otimes y_{ij}]$, then

$$L_b([z_{ij}]) = L_b\left(\begin{bmatrix} x_{11} & 0 \\ 0 & 0 \end{bmatrix} \otimes y_{11} + \begin{bmatrix} 0 & x_{12} \\ 0 & 0 \end{bmatrix} \otimes y_{12}\right.$$
$$\left. + \begin{bmatrix} 0 & 0 \\ x_{22} & 0 \end{bmatrix} \otimes y_{21} + \begin{bmatrix} 0 & 0 \\ 0 & x_{22} \end{bmatrix} \otimes y_{22}\right)$$
$$= \sum_{1 \leqslant i,j \leqslant 2} L_b([\delta_{ij} x_{ij}] \otimes y_{ij}) = \sum_{1 \leqslant i,j \leqslant 2} \langle y_{ij}, b\rangle [\delta_{ij} x_{ij}]$$
$$= \begin{bmatrix} \langle y_{11}, b\rangle x_{11} & \langle y_{12}, b\rangle x_{12} \\ \langle y_{21}, b\rangle x_{21} & \langle y_{22}, b\rangle x_{22} \end{bmatrix} = \begin{bmatrix} L_b(z_{11}) & L_b(z_{12}) \\ L_b(z_{21}) & L_b(z_{22}) \end{bmatrix}.$$

Hence

$$L_b([z_{ij}]) \in S_\perp \Leftrightarrow L_b(z_{ij}) \in (S_{ji})_\perp, \quad 1 \leqslant i,j \leqslant 2. \tag{5.6}$$

Moreover, by Proposition 1.4 (ii) (reversing the roles of $\mathcal{H}$ and $\mathcal{K}$ in the proof)

$$S_\perp \otimes_F T_\perp = \{[z_{ij}] \in B(\mathcal{H}^{(2)})_* \widehat{\otimes} T_\perp : L_b([z_{ij}]) \in S_\perp \quad \forall b \in B(\mathcal{K})\} \tag{5.7}$$

and for $1 \leqslant i,j \leqslant 2$,

$$(S_{ij})_\perp \otimes_F T_\perp = \{z \in B(\mathcal{H})_* \widehat{\otimes} T_\perp : L_b(z) \in (S_{ij})_\perp \quad \forall b \in B(\mathcal{K})\}. \tag{5.8}$$

Combining (5.6)–(5.8) we get

$$S_\perp \otimes_F T_\perp = \begin{bmatrix} (S_{11})_\perp \otimes_F T_\perp & (S_{21})_\perp \otimes_F T_\perp \\ (S_{12})_\perp \otimes_F T_\perp & (S_{22})_\perp \otimes_F T_\perp \end{bmatrix},$$

from which the second equality in (5.4) follows easily. ∎

COROLLARY 5.4. *Let $S_i \subset B(\mathcal{H}_i)$, $i = 1, 2$, be σ-weakly closed subspaces, and let*

$$A(S_i) = \begin{bmatrix} \mathbf{C}1 & S_i \\ 0 & \mathbf{C}1 \end{bmatrix}.$$

Then $S_1 * S_2 \neq S_1 *_F S_2 \Rightarrow A(S_1) * A(S_2) \neq A(S_1) *_F A(S_2)$.

Proof. Since $S_1 * S_2 \neq S_1 *_F S_2$, Proposition 5.3 implies that $A(S_1) * S_2 \neq A(S_1) *_F S_2$, and so, by Proposition 2.2 (i), $S_2 * A(S_1) \neq S_2 *_F A(S_1)$. Hence $A(S_2) * A(S_1) \neq A(S_2) *_F A(S_1)$ and so $A(S_1) * A(S_2) \neq A(S_1) *_F A(S_2)$. ∎

THEOREM 5.5. *Suppose that $S_i \subset B(\mathcal{H}_i)$ and $T_i \subset B(\mathcal{K}_i)$, $i = 1, 2$, are nonzero σ-weakly closed subspaces. If $S_1 * S_2 \neq S_1 *_F S_2$, then $(S_1 \overline{\otimes} T_1) * S_2 \neq (S_1 \overline{\otimes} T_1) *_F S_2$ and $(S_1 \overline{\otimes} T_1) * (S_2 \overline{\otimes} T_2) \neq (S_1 \overline{\otimes} T_1) *_F (S_2 \overline{\otimes} T_2)$.*

Proof. It follows easily from Proposition 2.2 (i) that it suffices to show that $S_1 * S_2 \neq S_1 *_F S_2 \Rightarrow S_1 * (S_2 \overline{\otimes} T_2) \neq S_1 *_F (S_2 \overline{\otimes} T_2)$.

Let $\mathcal{H} = \mathcal{H}_2 \otimes \mathcal{K}_2$, $V_i = (S_i)_\perp$, $i = 1, 2$, and let $W = (S_2 \overline{\otimes} T_2)_\perp$. Fix a unit vector b_0 in T_2, and choose a unit vector y_0 in $B(\mathcal{K}_2)_*$ such that $\langle y_0, b_0 \rangle = 1$. Define a map $\varphi \in \mathrm{CB}(B(\mathcal{H}_2)_*, B(\mathcal{H})_*)$ by $\varphi(x) = x \otimes y_0$ ($x \in B(\mathcal{H}_2)_*$). Then it follows from Proposition 1.1 (i) that $\varphi : V_2 = (S_2)_\perp \to (S_2 \overline{\otimes} T_2)_\perp = W$.

Let $\psi = \mathrm{id}_{B(\mathcal{H}_1)_*} \widehat{\otimes} \varphi \in \mathrm{CB}(B(\mathcal{H}_1)_* \widehat{\otimes} B(\mathcal{H}_2)_*, B(\mathcal{H}_1)_* \widehat{\otimes} B(\mathcal{H})_*)$. Then

$$\psi(B(\mathcal{H}_1)_* \otimes V_2) \subset B(\mathcal{H}_1)_* \otimes W$$

and

$$\psi(V_1 \otimes B(\mathcal{H}_2)_*) \subset V_1 \otimes B(\mathcal{H})_*.$$

Now let $z \in V_1 \otimes_F V_2$. Then $z \in V_1 \otimes_F B(\mathcal{H}_2)_* = V_1 \otimes B(\mathcal{H}_2)_*$, so $\psi(z) \in V_1 \otimes B(\mathcal{H})_*$. Moreover, if $a \in B(\mathcal{H}_1)$ and $b \in B(\mathcal{H})$, then $\langle R_a(\psi(z)), b \rangle = \langle \psi(z), a \otimes b \rangle = \langle z, a \otimes \varphi^*(b) \rangle = \langle R_a(z), \varphi^*(b) \rangle = \langle \varphi(R_a(z)), b \rangle$. Hence

$$R_a(\psi(z)) = \varphi(R_a(z)), \quad a \in B(\mathcal{H}_1).$$

But $z \in V_1 \otimes_F V_2 \Rightarrow R_a(z) \in V_2 \ \ \forall a \in B(\mathcal{H}_1)$, so $R_a(\psi(z)) \in W \ \ \forall a \in B(\mathcal{H}_1)$. Hence by Proposition 1.4 (ii), $\psi(z) \in V_1 \otimes_F W$.

By assumption, there is an element $z_0 \in V_1 \otimes_F V_2$ such that $z_0 \notin V_1 \otimes V_2$. Choose $c_0 \in B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that

$$\langle z_0, c_0 \rangle \neq 0, \quad \langle z, c_0 \rangle = 0, \quad z \in V_1 \otimes V_2. \tag{5.9}$$

Then $\langle\psi(x_1 \otimes x_2), c_0 \otimes b_0\rangle = \langle x_1 \otimes \varphi(x_2), c_0 \otimes b_0\rangle = \langle x_1 \otimes x_2 \otimes y_0, c_0 \otimes b_0\rangle = \langle x_1 \otimes x_2, c_0\rangle\langle y_0, b_0\rangle = \langle x_1 \otimes x_2, c_0\rangle$ ($x_1 \in B(\mathcal{H}_1)_*$, $x_2 \in B(\mathcal{H}_2)_*$). Hence

$$\langle\psi(z), c_0 \otimes b_0\rangle = \langle z, c_0\rangle, \quad z \in B(\mathcal{H}_1)_* \widehat{\otimes} B(\mathcal{H}_2)_*.$$

In particular,

$$\langle\psi(z_0), c_0 \otimes b_0\rangle \neq 0. \tag{5.10}$$

It follows from (5.10) that in order to show that $V_1 \otimes_F W \neq V_1 \otimes W$ (and so $S_1 *_F (S_2 \overline{\otimes} T_2) \neq S_1 * (S_2 \overline{\otimes} T_2)$) it suffices to show that

$$0 = \langle u, c_0 \otimes b_0\rangle = \langle L_{b_0}(u), c_0\rangle \quad \text{if } u \in V_1 \otimes W, \tag{5.11}$$

where L_{b_0} is the left slice map from $B(\mathcal{H}_1 \otimes \mathcal{H}_2)_* \widehat{\otimes} B(\mathcal{K}_2)_* \to B(\mathcal{H}_1 \otimes \mathcal{H}_2)_*$ associated with b_0. If we let $\widetilde{L}_{b_0}$ denote the left slice map from $B(\mathcal{H}_2)_* \widehat{\otimes} B(\mathcal{K}_2)_* \to B(\mathcal{H}_2)_*$ associated with b_0, then $L_{b_0}(x_1 \otimes (x_2 \otimes y_2)) = \langle y_2, b_0\rangle(x_1 \otimes x_2) = x_1 \otimes (\langle y_2, b_0\rangle x_2) = x_1 \otimes \widetilde{L}_{b_0}(x_2 \otimes y_2)$ ($x_1 \in B(\mathcal{H}_1)_*$, $x_2 \in B(\mathcal{H}_2)_*$, $y_2 \in B(\mathcal{K}_2)_*$), and so

$$L_{b_0}(x_1 \otimes y) = x_1 \otimes \widetilde{L}_{b_0}(y), \quad x_1 \in B(\mathcal{H}_1)_*, \quad y \in B(\mathcal{H})_*. \tag{5.12}$$

Now let $u = x_1 \otimes y$ ($x_1 \in V_1, y \in W$). Then it follows from Proposition 1.1 (i) that $\widetilde{L}_{b_0}(y) \in (S_2)_\perp = V_2$. Hence it follows from (5.12) that $L_{b_0}(u) \in V_1 \otimes V_2$, and hence $\langle L_{b_0}(u), c_0\rangle = 0$ by (5.9). Since the map $u \to \langle L_{b_0}(u), c_0\rangle$ is linear and continuous, (5.11) holds, as required. ∎

The next two corollaries follow immediately from Theorem 5.5 and Theorem 3.1.

COROLLARY 5.6. *Suppose that $S_i \subset B(\mathcal{H}_i)$, $i = 1, 2$, are nonzero σ-weakly closed subspaces, and $V_i \subset B(\mathcal{H}_i)_*$, $i = 1, 2$, are nonzero closed subspaces.*

(i) *If $S_1 \overline{\otimes} S_2$ has the* DDP, *then S_1 and S_2 have the* DDP.

(ii) *If $V_1 * V_2$ has the* OAP, *then V_1 and V_2 have the* OAP.

COROLLARY 5.7. *Suppose $S \subset B(\mathcal{H})$ is a σ-weakly closed subspace and let* **1** *denote the unit of $B(\mathcal{H})$. If S does not have the* DDP, *then neither does the reflexive subspace $S \overline{\otimes} \mathbf{C1}$ of $B(\mathcal{H} \otimes \mathcal{H})$.*

Let $\mathcal{H}_0$ be a separable infinite-dimensional Hilbert space, and suppose S is a σ-weakly closed subspace of $B(\mathcal{H}_0)$ without the DDP (which exists by Theorem 5.1). Then since $S \overline{\otimes} \mathbf{C1}$ is reflexive, $(S \overline{\otimes} \mathbf{C1})_\perp \subset B(\mathcal{H}_0 \otimes \mathcal{H}_0)_* \cong T_\infty$ is generated by rank one operators. Since $S \overline{\otimes} \mathbf{C1}$ doesn't have the DDP, $(S \otimes \mathbf{C1})_\perp$ is an example of a subspace of T_∞ which is generated by rank ones and fails to have the OAP. As noted above, the OAP implies the AP, but there may be operator spaces which have the AP but don't have the OAP. For this reason, the following result is of interest.

PROPOSITION 5.8. *Suppose* $S_i \subset B(\mathcal{H}_i)$, $i = 1,2$, *are nonzero* σ-*weakly closed subspaces. If* $(S_1 \overline{\otimes} S_2)_\perp$ *has the AP, then* $(S_1)_\perp$ *and* $(S_2)_\perp$ *have the AP.*

Proof. Let $V_i = (S_i)_\perp$, $i = 1,2$. It suffices to show that V_1 has the AP. Let $V = (S_1 \overline{\otimes} S_2)_\perp$. Choose unit vectors $b_0 \in S_2$ and $y_0 \in B(\mathcal{H}_2)_*$ such that $\langle y_0, b_0\rangle = 1$, and let $\varphi(x) = x \otimes y_0$, $x \in V_1$. Then $\varphi : V_1 \to V$ by Proposition 1.1 (i). Let $K \subset V_1$ be compact, and let $\varepsilon > 0$. Since φ is continuous, $\varphi(K)$ is a compact subset of V, and so, since V has the AP, there is a $\psi \in F(V)$ such that $\|\psi(\varphi(x)) - \varphi(x)\| < \varepsilon \ \ \forall x \in K$. By Proposition 1.1 (i), $L_{b_0}(V) \subset V_1$, so $L_{b_0} \circ \psi \circ \varphi \in F(V_1)$. Moreover, $\|L_{b_0}\| = \|b_0\| = 1$, so $\|(L_{b_0} \circ \psi \circ \varphi)(x) - L_{b_0}(\varphi(x))\| < \varepsilon \ \ \forall x \in K$. But if $x \in V_1$, $L_{b_0}(\varphi(x)) = L_{b_0}(x \otimes y_0) = \langle y_0, b_0\rangle x = x$, so $\|(L_b \circ \psi \circ \varphi)(x) - x\| < \varepsilon$ $\forall x \in K$. Hence V_1 has the AP. ∎

COROLLARY 5.9. *There is a subspace of* T_∞ *which is generated by rank one operators but fails to have the* AP.

Proof. Let $\mathcal{H}_0$ be a separable infinite-dimensional Hilbert space. Then we can identify T_∞ with $B(\mathcal{H}_0 \otimes \mathcal{H}_0)_*$. By the proof of Theorem 5.1, there is a subspace $V \subset B(\mathcal{H}_0)_*$ without the AP. Let $S = V^\perp$. Then it follows from Proposition 5.8 that $(S \overline{\otimes} \mathbf{C}\mathbf{1})_\perp$ doesn't have the AP, where $\mathbf{1}$ is the unit of $B(\mathcal{H}_0)$. As noted above, $(S \overline{\otimes} \mathbf{C}\mathbf{1})_\perp$ is generated by rank one operators. ∎

THEOREM 5.10. *Let* $\mathcal{H}_0$ *be an infinite-dimensional separable Hilbert space. Then there are reflexive algebras* $A_i \subset B(\mathcal{H}_0)$, $i = 1,2$, *such that* $A_1 * A_2 \neq A_1 *_F A_2$.

Proof. Choose a Hilbert space $\mathcal{H}_1$ such that $\mathcal{H}_1^{(2)} \otimes \mathcal{H}_1^{(2)} = \mathcal{H}_0$. By Theorem 5.1, we can find a subspace $S_1 \subset B(\mathcal{H}_1)$ that doesn't have the DDP, and by Theorem 3.1 we can find $S_2 \subset B(\mathcal{H}_1)$ such that $S_1 * S_2 \neq S_1 *_F S_2$. Let

$$A(S_i) = \begin{bmatrix} \mathbf{C}\mathbf{1} & S_i \\ 0 & \mathbf{C}\mathbf{1} \end{bmatrix}, \quad i = 1,2,$$

(where $\mathbf{1}$ is the identity of $B(\mathcal{H}_1)$). Then by Corollary 5.4, $A(S_1) * A(S_2) \neq A(S_1) *_F A(S_2)$. Finally, let $A_i = A(S_i) \otimes \mathbf{C} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $i = 1,2$. Then $A_i \subset B(\mathcal{H}_0)$ is a reflexive algebra, $i = 1,2$, and, by Theorem 5.5, $A_1 * A_2 \neq A_1 *_F A_2$. ∎

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