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THE DERIVATION PROBLEM AND THE SIMILARITY PROBLEM ARE EQUIVALENT

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ABSTRACT. There is proved a proposition which implies that the similarity problem and the derivation problem for C^*-algebras are equivalent, and also that they are equivalent to a bicommutation problem.

KEYWORDS: *C^*-algebra, derivation, $*$-representation.*

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Let $A \subseteq B(H)$ be a C^*-algebra. The similarity problem and the derivation problem for A are the following open questions respectively:

Is every bounded (possibly non-selfadjoint) representation r of A on a Hilbert space K equivalent to a $*$-representation of A on K?

Can we find for every $*$-representation $D : A \to B(K)$ on a Hilbert space K and every bounded derivation $\delta : A \to B(K)$ with respect to D an operator $t \in B(K)$ such that $\delta = [t, D(\cdot)]$?

Let L be a bounded linear map from A into $B(K)$ then by $\|L\|$ respectively $\|L\|_{\mathrm{cb}}$ we denote the norm of L in $B(A, B(K))$ and the supremum over $n \in N$ of $\|L \otimes \mathrm{id}_n\|$ in $B(M_n(A), M_n(B(K)))$ respectively.

From works of Christensen ([1], [2]) and Haagerup ([4]) we know that:

(i) the derivation problem for A has a positive answer if and only if there exists a (best = minimal) constant $\chi < \infty$, such that $\|\delta\|_{\mathrm{cb}} \leqslant \chi\|\delta\|$ for all $*$-homomorphisms $D : A \to B(K)$ and all derivations δ with respect to D;

(ii) moreover it suffices to consider in (i) derivations of form $\delta(\cdot) = [t, D(\cdot)]$ to check the existence of the (same best) constant $\chi < \infty$;

(iii) the similarity problem for A has a positive answer if and only if there exists a continuous function F on R_+ with

$$\|r\|_{\mathrm{cb}} \leqslant f(\|r\|) \tag{1}$$

for every completely bounded (possibly non-selfadjoint) representation r from A into $B(K)$.

With help of ultrapower technics one can easily obtain from [4] that moreover;

(iv) it suffices to consider bounded representations $r : A \to B(K)$ of the form $r(a) = \mathrm{e}^{-h} D(a) \mathrm{e}^{h}$ for a $*$-representation $D : A \to B(K)$ and $h = h^*$ in $B(K)$ to show (1) in full generality with same f.

We are now in position to give the results. Remark that the corollaries are obvious consequences of (i)–(iv), Proposition 1 and results of [4].

PROPOSITION 1. *Let $A \subset B(H)$ be a C^*-algebra and assume that there exists a constant $k < \infty$ such that $\|\delta_t\|_{\mathrm{cb}} \leqslant k\|\delta_t\|$ for every selfadjoint $t \in B(H)$, where $\delta_t(a) := [t, a]$ and δ_t is considered as a map from A into $B(H)$. Then for $h = h^* \in B(H)$ and the representation*

$$r_h : a \in A \mapsto r_h(a) := \mathrm{e}^{-h} a \mathrm{e}^{h} \in B(H)$$

we have that $\|r_h\|_{\mathrm{cb}} \leqslant \exp(k(1 + \|r\|))$.

COROLLARY 1. *The similarity problem for a C^*-algebra A and the derivation problem for A are equivalent.*

COROLLARY 2. *The similarity problem is equivalent to the following bicommutation problem: Let $M \subset B(H)$ be a von Neumann algebra and ω a free ultrafilter. Is $(M_\omega)' \cap (B(H))_\omega = (M')_\omega$?*

Moreover: To study (fixed) A it suffices to consider the finite part M of A^{**}.

Proof of Proposition 1. Let $t = t^* \in B(H)$. Fix $a \in A$ and put $f(z) = \mathrm{e}^{-zt} a \mathrm{e}^{zt} - a$, $z \in \mathbb{C}$. Then $f'(0) = [t, a]$, $f(0) = 0$ and $|f(z)| \leqslant \|a\| + \max\{\|\mathrm{e}^{-t} a \mathrm{e}^{t}\|, \|\mathrm{e}^{t} a \mathrm{e}^{-t}\|\}$ for $-1 \leqslant \mathrm{Re}(z) \leqslant 1$.

We apply Schwarz-lemma to the unit disc around $z = 0$ and use $\mathrm{e}^{t} a \mathrm{e}^{-t} = (\mathrm{e}^{-t} a^* \mathrm{e}^{t})^*$ to get $\|[t, a]\| \leqslant \|a\| + \max\{\|\mathrm{e}^{-t} a \mathrm{e}^{t}\|, \|\mathrm{e}^{-t} a^* \mathrm{e}^{t}\|\}$.

Thus, for $\delta_t(a) = [t, a]$ it follows that $\|\delta_t\| \leqslant 1 + \|\mathrm{e}^{-t}(\cdot)\mathrm{e}^{t}\|$ (with norms in $B(A, B(H))$) and $\|\delta_t\|_{\mathrm{cb}} \leqslant k(1 + \|\mathrm{e}^{-t}(\cdot)\mathrm{e}^{t}\|_{B(A,B(H))})$ by assumption (of Proposition 1).

Now let be given $h = h^* \in B(H)$ and $r_h : a \in A \mapsto r_h(a) = \mathrm{e}^{-h} a \mathrm{e}^{h} \in B(H)$.

Then $\|r_h\|_{\mathrm{cb}} = \|\mathrm{e}^{-h\otimes 1}(\cdot)\mathrm{e}^{h\otimes 1}\|$ where the operator norm on the right hand side means the norm in $B(A\otimes\mathcal{K}, B(H)\otimes\mathcal{K})$ and $\mathcal{K}$ denotes the compact operators on l_2. It follows $\|r_h\|_{\mathrm{cb}} < \infty$.

By [4] there exists an invertible $T \in B(H)$ with $\|r_h\|_{\mathrm{cb}} = \|T\|\,\|T^{-1}\|$ and $r_h(a) = T^{-1}aT$ for every $a \in A$.

We have $T = \mathrm{e}^t \cdot U$ for some $t = t^* \in B(H)$ and unitary $U \in B(H)$. We get $\|r_h\|_{B(A,B(H))} = \|\mathrm{e}^{-t}(\cdot)\mathrm{e}^{t}\|_{B(A,B(H))}$. Thus

$$\|\delta_t\|_{\mathrm{cb}} \leqslant k(1 + \|r_h\|). \tag{2}$$

On the other hand $\|r_h\|_{\mathrm{cb}} = \|T\|\,\|T^{-1}\| = \|\mathrm{e}^{-t}\|\,\|\mathrm{e}^{t}\| = \|\mathrm{e}^{-S}\|\,\|\mathrm{e}^{S}\|$, where we take $S^* = S := t \otimes \mathbf{1}$ in $B(H) \otimes B(l_2) \subset B(H \otimes l_2)$.

Using spectral calculus for $\mathrm{e}^{\tau S}$ on commutative C^*-algebras we get $\|\mathrm{e}^{\tau S}\| = \|\mathrm{e}^{S}\|^{\tau}$ for $\tau \geqslant 0$, $\|\mathrm{e}^{-\tau S}\|\,\|\mathrm{e}^{\tau S}\| = (\|\mathrm{e}^{-S}\|\,\|\mathrm{e}^{S}\|)^{\tau}$ and

$$\int_0^1 (\|\mathrm{e}^{-\tau S}\|\,\|\mathrm{e}^{\tau S}\|)\,\mathrm{d}\tau = \int_0^1 \|r_h\|_{\mathrm{cb}}^{\tau}\,\mathrm{d}\tau = \frac{\|r_h\|_{\mathrm{cb}} - 1}{\log\|r_h\|_{\mathrm{cb}}}$$

if $\|r_h\|_{\mathrm{cb}} > 1$. The latter right identity comes from

$$\int_0^1 \mathrm{e}^{\tau\sigma}\,\mathrm{d}\tau = \frac{\mathrm{e}^{\sigma} - 1}{\sigma}$$

for $\delta > 0$.

Now let be $a \in A \otimes \mathcal{K}$ (and $S = t \otimes \mathbf{1}$ as above). We obtain

$$\begin{aligned}
\|r_h \otimes \mathrm{id}(a)\| = \|\mathrm{e}^{-S} a \mathrm{e}^{S}\| &\leqslant \|a\| + \left\| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\tau}(\mathrm{e}^{-\tau S} a \mathrm{e}^{\tau S})\,\mathrm{d}\tau \right\| \\
&\leqslant \|a\| + \int_0^1 \|\mathrm{e}^{-\tau S}([S,a])\mathrm{e}^{\tau S}\|\,\mathrm{d}\tau \\
&\leqslant \|a\| + \|[S,a]\| \int_0^1 (\|\mathrm{e}^{-\tau S}\|\,\|\mathrm{e}^{\tau S}\|)\,\mathrm{d}\tau \\
&= \|a\| + \|(\delta_t \otimes \mathrm{id})(a)\| \left(\frac{\|r_h\|_{\mathrm{cb}} - 1}{\log\|r_h\|_{\mathrm{cb}}} \right)
\end{aligned}$$

if $\|r_h\| > 1$, because $\delta_t \otimes \mathrm{id}(a) = [S, a]$.

Now we take in the above inequality the supremum on both sides for $a \in A\otimes\mathcal{K}$ and $\|a\| \leqslant 1$. We get $\|r_h\|_{\mathrm{cb}} \leqslant 1 + \|\delta_t\|_{\mathrm{cb}} \left(\frac{\|r_h\|_{\mathrm{cb}}-1}{\log\|r_h\|_{\mathrm{cb}}}\right)$ if $\|r_h\|_{\mathrm{cb}} > 1$. Thus $\log(\|r_h\|_{\mathrm{cb}}) \leqslant \|\delta_t\|_{\mathrm{cb}}$ again because $\|r_h\|_{\mathrm{cb}} > 1$.

It follows (from (2)) that $\|r_h\|_{\mathrm{cb}} \leqslant \exp(k(1 + \|r_h\|))$ for $r_h : A \to B(H)$. ∎

REMARK. It is unknown if for $C^*_{\rm red}(SL_n(Z))$ the derivation problem has a positive answer for $n = 2, 3, \ldots$.

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