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PROJECTIONS OF INVARIANT SUBSPACES AND TOEPLITZ OPERATORS

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ABSTRACT. Let K be a compact abelian group dual to a discrete abelian group which possesses an archimedean linear order. Let $W = EH^2(K)$ be a Beurling subspace of $L^2(K)$, where $H^2(K)$ is the space of analytic functions and E is a unimodular function on K. We show that if E satisfies an approximation condition, then there is a standard invariant subspace H so that the orthogonal projection $\mathrm{pr} : W \to H$ is injective and has dense range. We explain that this kind of consideration can be regarded as a generalization of the study of Toeplitz operators.

KEYWORDS: *Orthogonal projections, Beurling subspaces, compact abelian groups.*

AMS SUBJECT CLASSIFICATION: Primary 47A15; Secondary 47B35.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let K be a compact abelian group dual to a discrete abelian group Γ. Suppose Γ possesses an archimedean linear order, so we may regard Γ as a subgroup of $\mathbf{R}$. For $\gamma \in \Gamma$, let $\mathcal{X}_\gamma$ be the character on K given by $\mathcal{X}_\gamma(x) = x(\gamma)$ for each $x \in K$. The set $\{\mathcal{X}_\gamma \mid \gamma \in \Gamma\}$ is called the *standard orthonormal basis* of L^2. We shall call an invariant subspace a *standard invariant subspace* if it is spanned by a subset of the standard orthonormal basis. In particular, the space $H^2(K)$ of all analytic functions in $L^2(K)$ is a standard invariant subspace. Let W be any shift invariant subspace of $L^2(K)$. The orthogonal projection $\mathrm{pr} : W \to H$, where H is a standard invariant subspace, will be called a *standard projection*. If K is the circle $\mathbf{S}^1$, and therefore Γ is the group of integers $\mathbf{Z}$, then every standard invariant subspace is of the form $\mathcal{X}_n H^2(\mathbf{S}^1)$, where $n \in \mathbf{Z}$. A celebrated result of Beurling ([2]) tells us that every simply invariant subspace has the form $EH^2(\mathbf{S}^1)$, where E is a unimodular function on $\mathbf{S}^1$. Thus the consideration of the standard projection $\mathrm{pr}_n : W \to \mathcal{X}_n H^2(\mathbf{S}^1)$ is the same as the consideration of the Toeplitz operator $T(\mathcal{X}_{-n}E) = \mathrm{pr}_0 \circ \mathcal{X}_{-n}E$. By rephrasing some of the well known results from the theory of Toeplitz operators ([1]), we are able to say much about pr_n. In particular, if $\mathcal{X}_{-n}E$ can be approximated by a bounded analytic function under the L^∞ norm, then pr_n is injective. On the other hand, if $\mathcal{X}_{-n}E$ can be approximated by the complex conjugate of a bounded analytic function under the L^∞ norm, then pr_n is surjective. Of course, if there are integers n_1 and n_2 so that pr_{n_1} is injective while pr_{n_2} is surjective, then there must exist an integer n_3 so that pr_{n_3} is bijective.

The situation becomes considerably more complicated when K is not the circle. First of all, there are more standard invariant subspaces than those of the form $\mathcal{X}_\gamma H^2(K)$. One of them is the space $H_0^2(K)$ of all analytic functions in $L^2(K)$ with zero mean. Secondly, not every simply invariant subspace of $L^2(K)$ is a Beurling subspace ([2]). In this paper, we shall only prove results about the case when W is a Beurling subspace. Even with this restriction, we are still facing a situation more general than that of Toeplitz operators. Furthermore, studies of Toeplitz operators defined on $H^2(K)$, which have appeared recently, have been limited to Toeplitz operators with continuous symbols ([3]). Nevertheless, we are able to show in this paper that essentially, the above results about the projections persist. Precisely, we shall prove the following theorems.

THEOREM 1.1. *Let K be a compact abelian group dual to a discrete abelian group which possesses an archimedean linear order. Let $W = EH^2(K)$ be a Beurling subspace of $L^2(K)$, where E is a unimodular function on K. Suppose* $\mathrm{dist}_{L^\infty}(E, H^\infty(K)) < 1$. *Then the orthogonal projection* $\mathrm{pr}_0 : W \to H^2(K)$ *is injective.*

The notation $H^\infty(K)$ stands for the set of all bounded analytic functions on K, while $\mathrm{dist}_{L^\infty}(E, H^\infty(K))$ stands for $\inf\{\|E - h\|_\infty \mid h \in H^\infty(K)\}$.

THEOREM 1.2. *Let K, W and E be as in Theorem* 1.1.

(i) *Suppose* $\mathrm{dist}_{L^\infty}(E, H^\infty(K)) < 1$. *Then the orthogonal projection* $\mathrm{pr}_0 : W \to H^2(K)$ *has dense range.*

(ii) *Suppose* $\mathrm{dist}_{L^\infty}(E, H^\infty(K)) < 1/5$. *Then the orthogonal projection* $\mathrm{pr}_0 : W \to H^2(K)$ *is surjective.*

THEOREM 1.3. *Let K, W and E be as in Theorem* 1.1. *Suppose there are standard projections* $\mathrm{pr}_1 : W \to H_1$ *and* $\mathrm{pr}_2 : W \to H_2$ *so that* pr_1 *is injective* *while* pr_2 *has dense range. Then there exists a standard projection* $\mathrm{pr}_3 : W \to H_3$ *which is injective and has dense range.*

The proofs of the theorems will be given in Section 3. The proofs employ elementary Hilbert space techniques and they do not rely on results from the theory of Toeplitz operators. In fact, they provide the proofs of the corresponding statements about Toeplitz operators, which we shall present as corollaries to the theorems, together with other concluding remarks, in Section 4.

2. PRELIMINARIES

In this section, we shall briefly recall some information about Fourier analysis on groups and the function theory of invariant subspaces. References for a more detailed discussion will be supplied as we go along.

We shall drop the letter K from the notations of the function spaces when ambiguity does not arise. For example, we shall simply write L^2 for $L^2(K)$. The norm on L^2 shall be denoted by $\|\cdot\|$. Every vector v in L^2 has a Fourier expansion

$$\sum_{\gamma\in\Gamma} a_\gamma \mathcal{X}_\gamma,$$

where a_γ is called the Fourier coefficient for the frequency (or exponent) γ. The subspace H^2 of L^2 consists of the vectors whose Fourier coefficients for the negative frequencies all vanish. A theorem of Helson and Lowdenslager ([2]) says that a function in H^2 that vanishes on a set of positive measure is null almost everywhere. The space of all bounded linear operators on L^2 shall be denoted by $\mathcal{B}$, and the norm on $\mathcal{B}$ shall be denoted by $\|\cdot\|_{\mathcal{B}}$. Let GL denote the general linear group of all bounded linear operators possessing bounded inverses.

References on invariant subspaces can be found in the masterly written article of Helson ([2]). A closed subspace W of L^2 is invariant if $\mathcal{X}_\gamma W \subseteq W$ for all $\gamma \geqslant 0$. W is called doubly invariant if the inclusion holds for $\gamma < 0$ as well. Otherwise it is called simply invariant. The doubly invariant subspaces are classified by Wiener's theorem which says that a doubly invariant subspace of L^2 consists exactly of the functions which vanish on some fixed subset of K. In particular, if W is a doubly invariant subspace that contains a function which is non-null almost everywhere, then W is L^2 itself. On the other hand, the theory of simply invariant subspaces turns out differently depending on whether K is a circle or not. If K is $\mathbb{S}^1$, then as noted in the previous section, every simply invariant subspace is a Beurling subspace. If K is not $\mathbb{S}^1$, the theory is highly complex and many questions remain open. Since our focus is on the case where K is not $\mathbf{S}^1$, we shall simply assume that this is the case and we shall be explicit otherwise.

Given a simply invariant subspace W of L^2, let $W_+ = \bigcap_{\gamma<0} \mathcal{X}_\gamma W$ and $W_- = \sum_{\gamma>0} \mathcal{X}_\gamma W$ which is the closure of $\bigcup_{\gamma>0} \mathcal{X}_\gamma W$. Then we have the relation $W_- \subseteq W \subseteq W_+$ and W_- has codimension at most 1 in W_+. So W has to coincide with one of them. W_+ and W_- are called the left continuous and right continuous versions of W, respectively. The distinction between W_+ and W_- is not interesting. What is interesting is the question of whether W_+ and W_- are actually different! If W is left continuous, meaning $W = W_+$, then W is a Beurling subspace if and only if $W \ominus W_-$ is non-trivial. There are left continuous simply invariant subspaces which are not Beurling subspaces.

The standard invariant subspaces spanned by the sets $\{\mathcal{X}_\gamma \mid \gamma \in \Gamma, \gamma \geqslant r\}$ and $\{\mathcal{X}_\gamma \mid \gamma \in \Gamma, \gamma > r\}$, where $r \in \mathbf{R}$, will be denoted by $H(r)$ and $H_0(r)$ respectively, while the corresponding standard projections will be denoted respectively by $\mathrm{pr}(r)$ and $\mathrm{pr}'(r)$. Notice that if r is not an element of Γ, then $H(r) = H_0(r)$. The set of all standard invariant subspaces is linearly ordered by inclusion. Therefore, if there exists $r_1 \in \mathbf{R}$ so that $\mathrm{pr}(r_1)$ is injective, then either every standard projection is injective or there exists $\alpha \in \mathbf{R}$ so that $\mathrm{pr}(r)$ is injective if and only if $r \leqslant \alpha$. We shall call α the *injectivity least upper bound*. Similarly, if there exists $r_2 \in \mathbf{R}$ so that $\mathrm{pr}'(r_2)$ has dense range, then either every standard projection has dense range or there exists $\beta \in \mathbf{R}$ so that $\mathrm{pr}'(r)$ has dense range if and only if $r \geqslant \beta$. We shall call β the *density greatest lower bound*.

3. PROOFS OF THE THEOREMS

Proof of Theorem 1.1. Let $\widetilde{E}$ be an analytic function in L^2 so that $\|E - \widetilde{E}\|_\infty < 1$. Thus $\widetilde{E}$ has a bounded inverse and it defines an element of GL by multiplication. Let P be the operator $\widetilde{E}E^{-1}$. P is in GL and

$$\|P - I\|_B = \|\widetilde{E}E^{-1} - EE^{-1}\|_B \leqslant \|\widetilde{E} - E\|_B \|E^{-1}\|_B < 1,$$

where I is the identity operator in B. Let $\widetilde{W}$ be $P(W) = \widetilde{E}H^2$. Hence $\widetilde{W}$ is contained in H^2 and this of course means that the orthogonal projection of $\widetilde{W}$ into H^2 is injective. The claim is that the orthogonal projection $\mathrm{pr}_0 : W \to H^2$ is also injective. Suppose the contrary. Then we can find a vector $w \in W$ so that $\mathrm{pr}_0(w) = 0$. The kernel of pr_0 lies in the orthogonal complement of H^2 in L^2 and so w is orthogonal to $\widetilde{W} \subseteq H^2$. In particular, w is orthogonal to $\widetilde{w} = P(w) \in \widetilde{W}$. Therefore,

$$\|\widetilde{w} - w\| = \left(\|\widetilde{w}\|^2 + \|w\|^2\right)^{\frac{1}{2}},$$

which is at least $\|w\|$. However,

$$\|\widetilde{w} - w\| = \|P(w) - w\| \leqslant \|P - I\|_{\mathcal{B}}\|w\| < \|w\|.$$

We have a contradiction. Hence pr_0 is injective. ∎

Proof of Theorem 1.2. Part (i) follows from Theorem 1.1 by considering the orthogonal complement of W. We shall now proof part (ii). Let $\widetilde{E}$ be an element of $\overline{H^2}$ so that $\|E - \widetilde{E}\|_\infty < 1/5$. As before, the operator $P = \widetilde{E}E^{-1}$ is in GL and $\|P - I\|_{\mathcal{B}} < 1/5$. Furthermore, a direct calculation gives $\|P^{-1} - I\|_{\mathcal{B}} < 1/4$. Let $\widetilde{W} = P(W) = \widetilde{E}H^2$ and $\widetilde{W}' = \widetilde{E}\overline{H_0^2}$. Note that $\widetilde{W}'$ is contained in $\overline{H_0^2} = (H^2)^\perp$. Now, since $P \in \mathrm{GL}$, L^2 can be written as a direct sum $\widetilde{W} + \widetilde{W}'$. So if we take any $v \in H^2$, then $v = \widetilde{w} + \widetilde{w}'$ where $\widetilde{w} \in \widetilde{W}$ and $\widetilde{w}' \in \widetilde{W}'$. Hence $\widetilde{w} = v - \widetilde{w}'$ is projected to v by the orthogonal projection $\widetilde{\mathrm{pr}}_0 : \widetilde{W} \to H^2$. Therefore $\widetilde{\mathrm{pr}}_0$ is surjective. We claim that the orthogonal projection $\mathrm{pr}_0 : W \to H^2$ is also surjective.

In order to see that the claim is true, we consider the following linear maps:

$$H^2 \xrightarrow{\sigma} \widetilde{V} \xrightarrow{P^{-1}} V \xrightarrow{\mathrm{pr}_0} H^2.$$

The map σ with image $\widetilde{V} \subseteq \widetilde{W}$ is defined by nothing but the same procedure described above of finding a pre-image $\widetilde{w}$ of $v \in H^2$ under $\widetilde{\mathrm{pr}}_0$. Since the sum $L^2 = \widetilde{W} + \widetilde{W}'$ is direct, the choice of $\widetilde{w} \in \widetilde{W}$, and also $\widetilde{w}' \in \widetilde{W}'$, for the expression $v = \widetilde{w} + \widetilde{w}'$ is unique. We let $\sigma(v) = \widetilde{w}$. The space V is the image of $\widetilde{V}$ under P^{-1}, therefore V lies in W. We want to show that the composition $\mathrm{pr}_0 P^{-1}\sigma$ is invertible. Of course, if that is true, then for any $v \in H^2$, we will be able to find a $v' \in H^2$, giving $w = P^{-1}\sigma(v') \in W$, so that

$$\mathrm{pr}_0(w) = \mathrm{pr}_0 P^{-1}\sigma(v') = v,$$

which means that pr_0 is surjective.

We can achieve our goal by checking that the operator norm of $\mathrm{pr}_0 P^{-1}\sigma - \mathrm{id}$ is less than 1, where id is the identity operator on H^2. We take any unit vector $v \in H^2$, let $\widetilde{w} = \sigma(v) \in \widetilde{W}$ and let $\widetilde{w}'$ be the element of $\widetilde{W}'$ satisfying the equation $\widetilde{w} = v - \widetilde{w}'$. Then

$$\begin{aligned}\|\mathrm{pr}_0 P^{-1}\sigma(v) - v\| = \|\mathrm{pr}_0 P^{-1}(\widetilde{w}) - v\| &= \|\mathrm{pr}_0 P^{-1}(v - \widetilde{w}') - \mathrm{pr}_0(v)\| \\ &\leqslant \|\mathrm{pr}_0(P^{-1}(v) - v)\| + \|\mathrm{pr}_0 P^{-1}(\widetilde{w}')\| \\ &\leqslant \|(P^{-1} - I)(v)\| + \|\mathrm{pr}_0 P^{-1}(\widetilde{w}')\| \\ &< \frac{1}{4} + \|\mathrm{pr}_0 P^{-1}(\widetilde{w}')\|.\end{aligned}$$

Since $\mathrm{pr}_0(\widetilde{w}') = 0$, we have

$$\begin{aligned}\|\mathrm{pr}_0 P^{-1}(\widetilde{w}')\| &= \|\mathrm{pr}_0 P^{-1}(\widetilde{w}') - \mathrm{pr}_0(\widetilde{w}')\| \\ &\leqslant \|(P^{-1} - I)(\widetilde{w}')\| \\ &\leqslant \|P^{-1} - I\|_B \|\widetilde{w}'\| \\ &< \frac{1}{4}\|\widetilde{w}'\|.\end{aligned}$$

We are able to obtain what we need because $\|\widetilde{w}'\|$ is smaller than 1 for the following reason. Roughly, by the way everything has been defined, $\widetilde{w}'$ is almost perpendicular to $\widetilde{w}$. Since $\widetilde{w}'$ and $\widetilde{w}$ add up to a unit vector v, we are able to conclude that $\widetilde{w}'$ is of length less than 1. Precisely, notice that if the absolute value of the inner product

$$A = \left\langle \frac{\widetilde{w}}{\|\widetilde{w}\|}, \frac{\widetilde{w}'}{\|\widetilde{w}'\|} \right\rangle = \left\langle \frac{v - \widetilde{w}'}{\sqrt{1 + \|\widetilde{w}'\|^2}}, \frac{\widetilde{w}'}{\|\widetilde{w}'\|} \right\rangle$$

of unit vectors can be made smaller than $1/\sqrt{2}$, then

$$\frac{\|\widetilde{w}'\|}{\sqrt{1 + \|\widetilde{w}'\|^2}} = |A| < \frac{1}{\sqrt{2}}$$

which gives $\|\widetilde{w}'\| < 1$. So it remains to show that A has size less than $1/\sqrt{2}$.

Let $\widetilde{u}$ and $\widetilde{u}'$ be any two unit vectors in $\widetilde{W}$ and $\widetilde{W}'$ respectively. Then $P^{-1}(\widetilde{u}) \in W$ and $P^{-1}(\widetilde{u}') \in W^{\perp}$. So

$$\begin{aligned}|\langle \widetilde{u}, \widetilde{u}' \rangle| &= \left|\langle \widetilde{u} - P^{-1}(\widetilde{u}) + P^{-1}(\widetilde{u}), \widetilde{u}' - P^{-1}(\widetilde{u}') + P^{-1}(\widetilde{u}')\rangle\right| \\ &\leqslant \|(I - P^{-1})\widetilde{u}\| \, \|(I - P^{-1})\widetilde{u}'\| \\ &\qquad + \|(I - P^{-1})\widetilde{u}\| \, \|P^{-1}\widetilde{u}'\| + \|P^{-1}\widetilde{u}\| \, \|(I - P^{-1})\widetilde{u}'\| + 0 \\ &\leqslant \|I - P^{-1}\|_B^2 + 2\|I - P^{-1}\|_B \|P^{-1}\|_B \\ &< \left(\frac{1}{4}\right)^2 + 2\left(\frac{1}{4}\right)\left(\frac{5}{4}\right) \\ &< \frac{1}{\sqrt{2}}.\end{aligned}$$

Thus the proof of the theorem is complete. ∎

Proof of Theorem 1.3. Suppose the theorem is false. Then the hypothesis of the theorem implies that the injectivity least upper bound α and the density greatest lower bound β exist. Clearly, α cannot be larger than β. If $\alpha = \beta$, there are two possibilities. If α is not an element of Γ, then the two subspaces $H(\alpha)$

and $H'(\beta)$ are the same and so the two projections $\mathrm{pr}(\alpha)$ and $\mathrm{pr}'(\beta)$ are the same. This means that $\mathrm{pr}(\alpha)$ is injective and has dense range, which is contrary to our assumption that the theorem is false. If $\alpha \in \Gamma$, then the fact that $\mathrm{pr}'(\alpha)$ is not injective implies that there is a vector in W so that its Fourier expansion is of the form

$$\sum_{\gamma \leqslant \alpha} a_\gamma \mathcal{X}_\gamma,$$

with $a_\alpha \neq 0$. Therefore $\mathcal{X}_\alpha \in H(\alpha) = \mathcal{X}_\alpha H^2$ is an image of $\mathrm{pr}(\alpha)$. We already know that $\mathrm{pr}'(\alpha)$ has dense range. Putting all this information together, we conclude, contrary to our assumption, that $\mathrm{pr}(\alpha)$ is injective and has dense range. Hence we must have $\alpha < \beta$.

By the definition of α and β, there is a vector $w \in W$ such that w is orthogonal to $H(r_{1/2})$ where $\alpha < r_{1/2} < \beta$. The theorem of Helson and Lowdenslager mentioned in the preliminaries tells us that as a function in L^2, w is non-null almost everywhere. Now Wiener's theorem implies that the closed subspace of L^2 generated by the set $\{\mathcal{X}_\gamma w \mid \gamma \in \Gamma\}$ is L^2 itself. Let V and V' be the closed subspaces spanned by $\{\mathcal{X}_\gamma w \mid \gamma \in \Gamma, \gamma \geqslant 0\}$ and $\{\mathcal{X}_\gamma w \mid \gamma \in \Gamma, \gamma < 0\}$ respectively. Then the vector space sum $V + V'$ (not necessarily direct) is dense in L^2 and notice that $V \subseteq W$ while $V' \subseteq H(r_{1/2})^\perp$. Now take any vector $u \in H(r_{1/2})$. Then there is a vector $\widetilde{u} \in L^2$ of arbitrarily small norm so that $u = v + v' + \widetilde{u}$, where $v \in V$ and $v' \in V'$. Therefore $v = u - v' - \widetilde{u}$. It follows that u differs from an image of the projection $\mathrm{pr}(r_{1/2}) : W \to H(r_{1/2})$ by the vector $\mathrm{pr}(r_{1/2})(\widetilde{u})$ whose norm is arbitrarily small. This means that $\mathrm{pr}(r_{1/2})$ has dense range. But this is absurd because $r_{1/2} < \beta$. Thus we have arrived at a contradiction and the proof of the theorem is complete. ∎

4. SOME OBSERVATIONS

The study of the standard projection of a Beurling subspace EH^2 into H^2 is the same as the study of the Toeplitz operator $T(E)$. Theorems 1.1 and 1.2 give the following immediately.

COROLLARY 4.1. *Let K be a compact abelian group dual to a discrete abelian group which possesses an archimedean linear order, and let E be a unimodular function on K.*

(i) *Suppose* $\mathrm{dist}_{L^\infty}(E, H^\infty(K)) < 1/5$. *Then $T(E)$ is left invertible.*

(ii) *Suppose* $\mathrm{dist}_{L^\infty}(E, \overline{H^\infty(K)}) < 1/5$. *Then $T(E)$ is right invertible.*

Proof. Part (ii) follows from Theorem 1.2, while part (i) follows from part (ii) because $T(E)^* = T(\overline{E})$. ∎

By allowing all possible standard projections and by considering invariant subspaces that are not Beurling, the study of standard projections can be regarded as a generalization of the study of Toeplitz operators. A typical result about a Toeplitz operator relates the invertibility or the Fredholm property of the operator to the nature of its symbol. Similarly, a result about a standard projection of an invariant subspace should establish a relationship between the properties of the projection and the nature of the generators of the invariant subspace. The properties of the projection can be bijectivity or injectivity and having dense range. However, we do have the following proposition.

PROPOSITION 4.2. *Let W be an invariant subspace of $L^2(K)$, where K is a compact abelian group dual to a discrete abelian group Γ which has an archimedean linear order. Suppose there exists $\gamma_0 \in \Gamma$ such that the standard projection* $\mathrm{pr}(\gamma_0)$: $W \to \mathcal{X}_{\gamma_0}H^2(K)$ *is bijective. Then $W = EH^2(K)$ where E is a unimodular function on K.*

Proof. We need only to find a vector in W that is not in $\sum_{\gamma>0} \mathcal{X}_\gamma W$. Let $w_0 \in W$ be the unique pre-image of $\mathcal{X}_{\gamma_0}$ under $\mathrm{pr}(\gamma_0)$. So w_0 is orthogonal to $\mathcal{X}_{\gamma_0}H_0^2$. Since $\mathrm{pr}(\gamma_0)$ is bijective, the orthogonal projection of $\sum_{\gamma>0} \mathcal{X}_\gamma W$ onto $\mathcal{X}_{\gamma_0}H_0^2$ is also bijective. Thus $w_0 \in \sum_{\gamma>0} \mathcal{X}_\gamma W$ would mean that $w_0 = 0$, which is not true. Hence w_0 is what we need and hence the proposition. ∎

Given an invariant subspace W, the problem of finding a standard projection of W which is injective and has dense range probes deeply into the structure of W. We expect that what this problem will reveal is quite different from the side of the story told by results of the Szegö-Beurling type. Even among the Beurling subspaces, there are opposite responses to the problem. For example, suppose $W = EH^2$ and E is continuous. Since E can be approximated by polynomials, our theorems show that there exists a standard projection of W which is injective and has dense range. In fact, results from the theory of Toeplitz operators with continuous symbols ([3]) imply that there is a $\gamma \in \Gamma$ so that $\mathrm{pr}(\gamma) : W \to \mathcal{X}_\gamma H^2$ is bijective. On the other hand, we can find an inner function $E \in H^2(\mathbf{S}^1)$ so that $EH^2(\mathbf{S}^1)$ has infinite codimension in $H^2(\mathbf{S}^1)$ ([4]). So in this case, there does not exist a standard projection which is injective and has dense range.

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