J. OPERATOR THEORY
36(1996), 157–177

REMARKS ON BRAIDED C^*-CATEGORIES AND ENDOMORPHISMS OF C^*-ALGEBRAS

ANNA PAOLUCCI

Communicated by Norberto Salinas

ABSTRACT. The duality theory for compact groups of Doplicher and Roberts deals with the category of finite dimensional continuous representations as an abstract C^*-category. We study braided C^*-categories for a compact matrix quantum group to model the non commutativity of the tensor product. Let $\mathrm{F_d}$ be the category of corepresentations of the quantum group $U_q(d)$. We associate to the Yang-Baxter category $\mathrm{YB(F_d)}$ a C^*-algebra, $(O_d)^{U_q(d)}$.

We give conditions for an endomorphism of a unital C^*-algebra $\mathcal{A}$ to determine an action of the strict braided tensor C^*-category of corepresentations of $U_q(d)$ on $\mathcal{A}$. Such actions correspond to $*$-monomorphism of the fixed point subalgebra $(O_d)^{SU_q(d)}$ into A with natural intertwining properties.

KEYWORDS: *Braided C^*-categories, compact quantum group, endomorphisms, C^*-algebras.*

AMS SUBJECT CLASSIFICATION: Primary 46Lxx, 81R50, 46Mxx; Secondary 16W30, 16W, 17B37.

1. INTRODUCTION

The results in this paper were motivated by those of [6], [7], [8] on the abstract duality theory for compact groups where the basic results rest on endomorphisms of C^*-algebras. Specifically, the authors gave a characterization of compact group duals as abstract categories. The categories that Doplicher and Roberts studied arise as dual objects of a compact group. As categories they have objects which are abstract elements, and the set of arrows are linear spaces which are not given a priori as spaces of linear operators on finite dimensional Hilbert spaces. They characterize such categories among the full subcategories of $\mathrm{End}(\mathcal{A})$, where $\mathcal{A}$ is a

C^*-algebra. The class of categories they studied are C^*-categories which are the categorical analogues of C^*-algebras, equipped with additional structures. There is a monoidal structure modelling the tensor product and it is symmetric, which is equivalent to the fact that the tensor product is commutative.

Each object has a conjugate which is the analogue of conjugate representations for a compact group. Doplicher and Roberts' main results characterize the abstract category of finite-dimensional continuous unitary representations of a compact group: every strict symmetric monoidal tensor C^*-category with conjugates which has subobjects and direct sums and for which the C^*-algebra of endomorphisms of the monoidal unit reduces to the complex numbers is isomorphic to a category of finite-dimensional continuous unitary representations of a compact group, unique up to isomorphism. Instead the theory of Tannaka-Krein characterized the category of continuous unitary finite-dimensional representations of a compact group G within the category $\mathrm{Vect}_{\mathbb{C}}$ of finite-dimensional vector Hilbert spaces.

To cover the duality theory of compact Lie groups, it was sufficient to deal with minimal duals T, where the objects were all powers of a given endomorphism ρ of a C^*-algebra $\mathcal{A}$. If $\mathcal{A}$ is minimal in the sense that the arrows of the category T generate $\mathcal{A}$ as C^*-algebra, then the crossed product C^*-algebra $\mathcal{A} \times \mathrm{T}$ which contains $\mathcal{A}$ as a subalgebra is the Cuntz algebra $\mathcal{O}_d$ ([2]), where d is the dimension of ρ.

The compact group G appears to be the group of all automorphisms of $\mathcal{A} \times \mathrm{T}$ leaving $\mathcal{A}$ pointwise fixed and $\mathcal{A}$ is the set of all G-fixed points in $\mathcal{A} \times \mathrm{T}$. The action of G is a canonical action, so that the pair $\{\mathcal{A}, \rho\}$ is identified with the universal model $\{\mathcal{O}_G, \sigma_G\}$, where $\mathcal{O}_G$ is the fixed point subalgebra of $\mathcal{O}_d$ under the canonical action of G and σ_G is the restriction of the canonical endomorphism of $\mathcal{O}_d$.

One natural question which will then arise would be how to generalize this construction to an abstract duality theory for compact quantum groups. Thus the first step should be to study model coaction of Hopf algebras on Cuntz algebras. Model actions of finite-dimensional Hopf algebra on Cuntz algebra have been investigated in [3], [16], [18]. Longo in [16] has developed a different approach to the construction of a crossed product related to the work of Cuntz and based on the theory of inclusions of subfactors.

There is a history of generalizations of the Tannaka-Krein theory for locally compact groups by Ernest, Stinespring, Tatsuuma, Takesaki. Generalizations to compact quantum group have been given by Woronowicz ([23]), and in the setting of category theory by Yetter ([25]). A unified treatment covering all those cases has been given by Baaj and Skandalis ([1]), based only on multiplicative unitaries.

We should emphasize that all these developments deal with concrete duals in the spirit of the Tannaka-Krein duality. As shown in [10], model theories on a 2-dimensional space time give rise to superselection structures described by a braided monoidal subcategory ([12]) of End($\mathcal{A}$). In [15] properties of conjugation are developed for finite-dimensional objects in a strict tensor C^*-category and a definition of dimension. It is shown that the braiding gives rise to a central element and that it is compatible with conjugates and model endomorphisms canonically associated with an object in a strict tensor C^*-category.

In this paper we deal with compact matrix quantum groups which we denote by $G = (\mathcal{A}, u)$ as in [23] and the representation category of G will be a C^*-category, i.e. the categorial analogue of C^*-algebras. These C^*-categories are equipped with a tensor product structure and they will be braided to model the non commutativity of the tensor product. The objects of our category which we denote by $\mathrm{F_d}$, are the tensor power of the fundamental corepresentation u and the arrows are the intertwiners. Out of the braiding of the C^*-category $\mathrm{F_d}$ we consider a new category $\mathrm{YB(F_d)}$. The objects of $\mathrm{YB(F_d)}$ are pairs $(V, c_{V,V})$ where V is an object of $\mathrm{F_d}$ and $c_{V,V}$ is the Yang-Baxter operator given by the commutativity constraint, i.e. by the braiding. Let $\mathrm{F_d}$ be the category of corepresentations of the quantum $U_q(d)$. To this category we associate in a natural way a C^*-algebra $(O_d)^{U_q(d)}$ which turns out to be the fixed point subalgebra of the Cuntz algebra under a natural coaction Γ of $U_q(d)$ on O_d.

The paper is organized as follows. Section 2 contains main definitions and results needed about the fixed point algebra under a natural coaction of the compact quantum group $U_q(d)$ on O_d. In Section 3 we give the definition of a strict braided tensor C^*-category and we construct the Yang-Baxter category. The representation category of the quantum compact matrix group $U_q(d)$ is considered. It is a strict tensor C^*- category and it has a natural braiding given by the R-matrix associated to $U_q(d)$. We associate to the strict tensor braided C^*-category $\mathrm{YB(F_d)}$ the C^*-algebra $(O_d)^{U_q(d)}$. This C^*-algebra can be identified as the fixed points denoted $(O_d)^{U_q(d)}$ of the Cuntz algebra O_d under a natural coaction of $U_q(d)$. It carries a canonical endomorphism denoted by $\sigma_{U_q(d)}$ which is the restriction of the canonical endomorphism $\widehat{\sigma}$ of O_d. The strict braided tensor C^*-category $\mathrm{YB(F_d)}$ can be reconstructed from $(O_d)^{U_q(d)}$ and $\sigma_{U_q(d)}$. This C^*-algebra has been studied in [17] where also the coaction of $SU_q(d)$ has been considered.

In Section 4 we give conditions for an endomorphism ρ of a unital C^*-algebra $\mathcal{A}$ to determine an action of $\mathrm{YB}(\mathrm{F_d})$ on $\mathcal{A}$. Such actions will correspond to $*$-monomorphisms of the fixed point algebra $(O_d)^{SU_q(d)}$ into $\mathcal{A}$ with natural intertwining properties.

Our main result, Theorem 4.3, is about endomorphisms ρ of a C^*-algebra $\mathcal{A}$ with braiding of dimension d. Specifically, we specify $\mathcal{A}$ with a representation ε of the braid group B_∞. Our result gives conditions for the existence and uniqueness of monomorphisms $\mu : (O_d)^{U_q(d)} \to \mathcal{A}$, $\mu(\theta) = \varepsilon$ defining an action of $(O_d)^{U_q(d)}$ on $\mathcal{A}$.

2. BASIC DEFINITIONS

In this section we give the definition of compact quantum group and in particular we consider the quantum $U_q(d)$ and $SU_q(d)$. We state some results about the fixed point algebra under a natural coaction of a compact quantum group on the Cuntz algebra. As in [24] we denote by $(\mathcal{A}, \Delta)$ a compact quantum group where $\mathcal{A}$ is a unital C^*-algebra and Δ is the comultiplication. Let $u \in M_n \otimes \mathcal{A}$ be a unitary corepresentation of $\mathcal{A}$, i.e. u is unitary and it satisfies:

$$(i \otimes \Delta)u = u_{12}u_{13}.$$

Write $u = \sum e_{ij} \otimes u_{ij}$. Since u is unitary we have: $\sum\limits_p u^*_{pr} u_{pq} = \delta_{rq}\mathbf{1}$ and $\sum\limits_p u_{rp} u^*_{qp} = \delta_{rq}\mathbf{1}$. From $(\mathbf{1} \otimes \Delta)u = u_{12}u_{13}$ we obtain the formula for the comultiplication: $\Delta(u_{ij}) = \sum\limits_p u_{ip} \otimes u_{pj}$.

Cuntz ([2]) defined the C^*-algebra O_d generated by the isometries $S_1, S_2, \ldots, S_d$ so that:

$$\sum S_i S_i^* = \mathbf{1} \quad \text{and} \quad S_i^* S_j = \delta_{ij}\mathbf{1} \tag{1.1}$$

It has a canonical endomorphism $\widehat{\sigma}$ such that $\widehat{\sigma}(X) = \sum\limits_i S_i X S_i^*$.

DEFINITION 2.1. Let $\mathcal{B}$ be a C^*-algebra and π be a $*$-homomorphism from $\mathcal{B}$ to $\mathcal{B} \otimes \mathcal{A}$. We say that π is a *coaction* of a compact quantum group $G = (\mathcal{A}, \Delta)$ on $\mathcal{B}$ if

$$(\pi \otimes \mathrm{id}_{\mathcal{A}})\pi = (\mathrm{id} \otimes \Delta)\pi$$

where Δ is the comultiplication. This is equivalent to saying commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{B} \otimes \mathcal{A} & \xrightarrow{\pi \otimes \mathrm{id}_{\mathcal{A}}} & \mathcal{B} \otimes \mathcal{A} \otimes \mathcal{A} \\ \pi \uparrow & & \uparrow \mathrm{id}_{\mathcal{B}} \otimes \Delta \\ \mathcal{B} & \xrightarrow[\pi]{} & \mathcal{B} \otimes \mathcal{A}. \end{array}$$

The $*$-homomorphism $\Gamma : O_d \to O_d \otimes \mathcal{A}$, defined by $\Gamma(S_i) = \sum_p S_p \otimes u_{pi}$, is a coaction. It is the natural analogue of the action of the matrix group $U(d)$ on the Cuntz algebra.

Recall the definition of the quantum $U_q(d)$ as in [14]. It is the algebra generated by u_{pq} satisfying the relations

$$\sum_{k,l} R_{ij}^{kl} u_{km} u_{lp} = \sum_{k,l} R_{kl}^{mp} u_{ik} u_{jl}$$

for all i, j, m, p where

$$R_{ii}^{ii} = q^{-1},\ R_{ij}^{ji} = \mathbf{1}, \quad \text{for} \quad i \neq j;$$
$$R_{ij}^{ij} = q^{-1} - q, \quad \text{for } i > j, \quad R_{mp}^{ij} = 0 \text{ otherwise.}$$

It is known that R satisfies the Yang-Baxter equations.

DEFINITION 2.2. Let O_d be the Cuntz algebra and let Γ be a coaction of the quantum $U_q(d)$ on O_d. We define the fixed point subalgebra $(O_d)^{U_q(d)}$ of O_d by Γ as follows

$$(O_d)^{U_q(d)} = \{x \in O_d : \Gamma(x) = x \otimes \mathbf{1}\}.$$

Denote by M_d^k the k-times tensor product of the $d \times d$ matrix algebra M_d and define a canonical embedding $\eta : M_d^k \to O_d$ by

$$\eta(e_{i_1 j_1} \otimes \cdots \otimes e_{i_k j_k}) = S_{i_1} \cdots S_{i_k} S_{j_k}^* \cdots S_{j_1}^*$$

where $\{e_{ij}\}_{i,j=1}^d$ is a system of matrix units of M_d. This embedding is compatible with the canonical inclusion of M_d^k into M_d^{k+1}.

We denote by M_d^∞ the UHF algebra which is the inductive limit of $\{M_d^k\}_{k=1}^\infty$. Observe that the UHF algebra M_d^∞ can be considered as a C^*-subalgebra of O_d through the embedding η.

Define $u^k = u \otimes \cdots \otimes u$ to be the tensor product of the unitary corepresentation u with itself k times. Clearly u^k is a unitary corepresentation if u is. The restriction of the coaction Γ to the UHF algebra M_d^∞ is also a coaction of the compact quantum group G on M_d^∞.

Set $\varphi = \Gamma|_{\mathrm{UHF}}$. Then it satisfies

$$\varphi(e_{i_1 j_1} \otimes \cdots \otimes e_{i_k j_k}) = \sum_{a_1,\ldots,a_k,b_1,\ldots,b_k} e_{a_1 b_1} \otimes \cdots \otimes e_{a_k b_k} \otimes u_{a_1 i_1} \cdots u_{a_k i_k} u_{a_k j_k}^* \cdots u_{a_1 j_1}^*,$$

for every k positive integer. Therefore φ can be represented in the following form

$$\varphi(x) = u^k (x \otimes \mathbf{1}_{\mathcal{A}})(u^k)^*,$$

for every $x \in M_d^k$.

In this algebra we have the quantum determinant

$$D = \sum_{\sigma \in S(n)} (-q)^{l(\sigma)} u_{\sigma_1 1} \cdots u_{\sigma_n n}.$$

This D is in the centre. We have $\Delta(D) = \Delta \otimes D$, where $\Delta(u_{ij}) = \sum u_{ip} \otimes u_{pj}$. We obtain $U_q(d)$ if we require D to be invertible. Then we get a Hopf $*$-algebra. The antipode can be expressed in terms of the u's and D^{-1}.

As [19] and [23] we define the compact quantum group $SU_q(d)$ to be the algebra generated by elements (u_{pq}) such that (u_{pq}) is a unitary matrix and

$$\sum E(k_0, k_1, \ldots, k_n)\ u_{l_0 k_0} u_{l_1 k_1} \cdots u_{l_n k_n} = E(l_0, l_1, \ldots, l_n)$$

where $E(k_0, k_1, \ldots, k_n) = 0$ if two indices are equal and otherwise

$$E(k_0, k_1, \ldots, k_n) = (-q)^{I(k_0, k_1, \ldots, k_n)}$$

where $I(k_0, k_1, \ldots, k_n)$ is the number of inverted pairs in the permutation $(k_0, k_1, \ldots, k_n)$.

Set $S_q = \sum E(k_0, k_1, \ldots, k_n) S_{k_0} S_{k_1} \cdots S_{k_d}$. It is easily seen that it is a fixed point of Γ.

Note that the canonical endomorphism $\widehat{\sigma}$ of the Cuntz algebra O_d leaves the fixed point subalgebra in O_d invariant under Γ.

The map

$$\Gamma(S_i) = \sum S_p \otimes u_{pi}$$

is a map from H to $H \otimes \mathcal{A}$. This is a corepresentation of $\mathcal{A}$ on H. The trivial corepresentation is given by $\lambda \in \mathbf{C} \to \lambda \otimes 1 \in \mathbf{C} \otimes \mathcal{A}$.

Observe that a fixed point $x \in H^k$ can be considered as an intertwiner by the following diagram:

$$\begin{array}{ccc} \mathbf{C} & \longrightarrow & H^k \\ \Gamma \downarrow & & \downarrow \Gamma \\ \mathbf{C} \otimes \mathcal{A} & \longrightarrow & H^k \otimes \mathcal{A}. \end{array}$$

Let (H^r, H^s) be an intertwiner from $H^r \mapsto H^s$. Hence an intertwiner (H, H) is given by a map

$$T(S^i) = \sum a_i^j S_j$$

and satisfies

$$\Gamma(T(S_i)) = \sum a_i^j \Gamma(S_j) = \sum a_i^j S_p \otimes u_{pj},$$

$$(T \otimes 1)\Gamma(S_j) = (T \otimes 1) \sum S_p \otimes u_{pi} = \sum a_q^p S_p \otimes u_{qi}.$$

This implies that for all i and p, we have

$$\sum_q a_q^p u_{qi} = \sum_j a_q^j u_{pj}$$

or

$$\sum_i a_i^p u_{iq} = \sum_j a_q^j u_{pj}.$$

Similarly, an intertwining between H^2 and itself gives relations between the products $u_{ij}u_{pq}$. Hence the operator R defined above is an intertwiner between H^2 and H^2.

Let us recall the following theorem from [17]:

THEOREM 2.3. *The fixed point subalgebra $(O_d)^{U_q(d)}$ of O_d by the coaction Γ coincides with the fixed point subalgebra $(M_d^\infty)^{SU_q(d)}$ of the* UHF *algebra M_d^∞ by the coaction of $SU_q(d)$. In particular*

$$(O_d)^{U_q(d)} = (M_d^\infty)^{U_q(d)} = (M_d^\infty)^{SU_q(d)}$$

with an explicit formula for the isomorphism given in [17].

3. BRAIDED STRICT TENSOR C^*-CATEGORY

In this section we give the definition of braided strict tensor C^*-categories and of the Yang-Baxter category built from the braiding. We assume that the tensor category is generated by tensor powers of a single object and we show that there exists a unique representation of B_∞ in O_d. We then specialize to the category generated by the tensor powers of corepresentations of the compact quantum group $U_q(d)$ and we show that θ of Corollary 7 of [17] gives a representation of B_∞ in O_d.

Let us now recall a few facts about tensor categories. Let C be a category and we denote by $\otimes : \mathrm{C} \times \mathrm{C} \to \mathrm{C}$ a functor called a tensor product that associates to any pair of objects (V, W) of C the object $V \otimes W$. Thus we say that a tensor category $(\mathrm{C}, \otimes, I, a, l, r)$ is a category which is equipped with a tensor product, with an object I, called the unit of the tensor category, with an associativity constraint a, a left unit constraint l and a right unit constraint r with respect to I. The following law relating the tensor product and the composition of arrows holds in the tensor category:

$$(T \circ T') \otimes (S \circ S') = (T \otimes S) \circ (T' \otimes S').$$

The tensor category is said to be strict if the associativity and the unit constraints a, l, r are all identities of the category. Hence in strict tensor categories we are not concerned about keeping track of parenthesis. We say that a category is braided if there exists a commutativity constraint c such that for each pair of objects of the category we have an isomorphism $c_{U,V} : U \otimes V \to V \otimes U$ such that if $T \in (U, V)$, $T' \in (U', V')$ we have

$$c_{V,V'} \circ T \otimes T' = T' \otimes T \circ c_{U,U'},$$

i.e. the constraint c is natural on both V, U. In a strict braided tensor category the commutativity constraint and the associativity constraint satisfy a dodecagon diagram which can be seen as the categorial version of the Yang-Baxter equation ([13]). A strict tensor C^*-category is a strict tensor category with the associativity constraint and the unit commuting with the $*$-operation and where each arrow is a complex Banach space.

Let H be a Hilbert space of dimension $d < \infty$. From now on we let $\mathrm{F_d}$ to be the category whose objects are the tensor powers of H, denoted by

$$H^r = H \otimes \cdots \otimes H \quad (r \text{ times}), \quad r \in N_0.$$

The set of arrows from H^s to H^r is denoted by (H^s, H^r) and they are linear mappings. The tensor product is defined within the category $\mathrm{F_d}$ since the objects are the tensor powers of the given Hilbert space H.

The tensor product is strictly associative. Hence $\mathrm{F_d}$ is a strict tensor category, also called strict monoidal category, where the monoidal product is the tensor product. We adopt the definition of tensor category. Such category $\mathrm{F_d}$ has a space of arrows which is a complex Banach space. The composition of arrows denoted by $\circ$ gives a bilinear map: $A, B \to A \circ B$ (with $\|A \circ B\| \leqslant \|A\| \, \|B\|$). There is also the adjoint denoted by $*$, which is an involutive contravariant functor acting as an identity on objects, i.e. $* : \mathrm{F_d} \to \mathrm{F_d}$. If $A \in (H^s, H^r)$ then $A^* \in (H^r, H^s)$. The identity is $H^0 = \mathbf{C}$. The norm satisfies the C^*-property: $\|A^* \circ A\| = \|A\|^2$, which makes $\mathrm{F_d}$ a C^*-category.

On arrows the tensor product is defined by $A \times A' \in (H^r \otimes H^s, H^{r'} \otimes H^{s'})$, if $A \in (H^r, H^{r'})$, $A' \in (H^s, H^{s'})$, $A \times A' = A \otimes A'$. We write $H^r H^s$ to mean: $H^r \otimes H^s = H^{r+s}$.

In our C^*-category we have one more element of structure, a commutativity constraint, i.e. the braiding, which models the non symmetry of the strict tensor C^*-category $\mathrm{F_d}$ opposed to the Doplicher-Roberts strict symmetric C^*-category in which the symmetry models the commutativity of the tensor product. Let R

be any operator from the two fold tensor product of H into itself that satisfies the Yang-Baxter equation and let τ be the flip operator. Later we will consider R to be the solution of the Yang-Baxter equation associated to $U_q(d)$. Define the commutativity constraint $c_{H,H} : H\otimes H \to H\otimes H$ by $c_{H,H}(h_1\otimes h_2) = \tau_{H,H}(R(h_1\otimes h_2))$, where $h_1,\ h_2 \in H$. It is shown in [13] that the operator $c_{H,H}$ so defined gives a braiding in the C^*-category $\mathrm{F_d}$. Let us define $c_{H^r,H^s} : H^r \otimes H^s \to H^s \otimes H^r$ inductively for any r and s as a linear mapping such that $c_{H^r,H^s} \circ c_{H^s,H^r} \neq \mathrm{id}_{H^r,H^s}$ and which permutes the order of the factors in the tensor product. We define c_{H^r,H^s} for any $r,\ s$, by the following equation

$$
\begin{aligned}
&c_{H^r,H^s} = \Pi_{s-1}\Pi_{s-2}\cdots\Pi_r,\\
&\Pi_{s-1} = (\tau_{H^r,H^r}(R)\circ\tau_{H^{r+1},H^{r+1}}(R)\circ\cdots\circ\tau_{H^{s-1},H^{s-1}}(R)),\\
&\Pi_{s-2} = (\tau_{H^r,H^r}(R)\circ\cdots\circ\tau_{H^{s-2},H^{s-2}}(R)),\ \ldots\\
&\Pi_r = \tau_{H^r,H^r}(R),
\end{aligned}
\tag{3.1}
$$

where $\tau_{H^r,H^r}(R)$ stands for a copy of R, after the flipping in the $r, r+1$ position. Hence we define a new category $\mathrm{YB(F_d)}$ from the commutativity constraint $c_{H,H}$ on the tensor category $\mathrm{F_d}$. It is called the Yang-Baxter category and denoted by $\mathrm{YB(F_d)} = (\mathrm{F_d}, c_{H,H})$. It is a strict braided tensor C^*-category, where the objects of the category are pairs (H^r, c_{H^r,H^r}), with H^r objects of $\mathrm{F_d}$ and c_{H^r,H^r} the *Yang-Baxter operator* on $H^r\otimes H^r$ above defined.

The braid category B is a strict braided tensor category, whose set of objects is the set $\mathbf{N}$ and such that there exists a family of isomorphisms $c_{n,m} : n\otimes m \to m\otimes n$ defined as follows. For any pair of nonnegative integers (n,m), $c_{0,n} = \mathrm{id} = c_{n,0}$; then

$$c_{n,m} = (g_m g_{m-1}\cdots g_1)(g_{m+1}g_m\cdots g_2)(g_{m+n+1}\cdots g_n)$$

where $g_1,\ldots,g_{m+n+1}$ are the generators of the braid group B_{n+m-1}.

The family $\{c_{n,m}\}$ is a braiding for B, see [13]. Hence there is a functor $\mathcal{F} : \mathrm{B} \to \mathrm{F_d}$, uniquely defined on objects, i.e. it must be $\mathcal{F}(0) = \mathrm{id}$, $\mathcal{F}(n) = H^n$, $\mathcal{F}(I) = I$ and there exists an isomorphism

$$\varphi_2(H^r,H^s) : \mathcal{F}(r)\otimes\mathcal{F}(s) \to \mathcal{F}(r\otimes s),$$

which is actually an identity since this tensor category is strict.

Also the $c_{1,1} : \mathbf{1}\otimes\mathbf{1} \to \mathbf{1}\otimes\mathbf{1}$ automorphism is a Yang-Baxter operator ([13]), Theorem 1.3.21, on the object $\mathbf{1}$ of B. Hence the automorphism $\varphi_2^{-1}\mathcal{F}(c_{1,1})\varphi_2$ is a Yang-Baxter operator on $\mathcal{F}(\mathbf{1})$ in the category $\mathrm{F_d}$. This defines an object

$(\mathcal{F}(1), \varphi_2^{-1}\mathcal{F}(c_{1,1})\varphi_2) = \Theta(\mathcal{F})$ in $\mathrm{YB}(\mathrm{F_d})$. Therefore by Theorem XIII.3.3 of [13], Θ extends to a functor

$$\mathrm{Tens}(\mathrm{B}, \mathrm{F_d}) \to \mathrm{YB}(\mathrm{F_d})$$

and Θ gives an equivalence of categories.

Let B_∞ be the infinite braid group. One of the usual models for B_n is as follows. Fix on each of two parallel lines in 3 space an "upper" and a "lower" one, n points labelled by the numbers $1, 2, \ldots, n$. We require that this labelling respects the natural ordering of our index set. A braid is obtained by connecting each of the upper points with a point of the lower line by a curve going only downwards. Multiplication of two braids is defined by connecting the lower points of the first braid with the upper points of the second braid with matching labels.

Similarly B_∞ can be realized by countably many points such that all but finitely many of them go straight downwards. Hence we see that we can realize B_∞ as inductive limit of the groups B_n, by the inclusion map

$$B_n \subset B_{n+1} \text{ which sends } g_n \to g_n \otimes \mathbf{1}$$

where by the tensor we mean to add an extra label on the "upper" and "lower" line connected by a straight line.

The shift $\sigma : B_n \to B_{n+1}$ which sends $g_n \mapsto \mathbf{1} \otimes g_n = \sigma(g_n) = g_{n+1}$, by the compatibility with the inclusion, then extends to an endomorphism which we denote by the same letter $\sigma : B_\infty \to B_\infty$.

Consider the subgroup B_n of B_∞ generated by $g_1, \ldots, g_n$. As a consequence of the construction of the category $\mathrm{YB}(\mathrm{F_d})$ we will see explicitly that the braiding in the category gives rise to a representation of B_n on $H^n = H^{\otimes n}$. Now let us introduce the shorthand notation for the braiding defined above

$$c_i \equiv \mathrm{id}_{H^{\otimes(i-1)}} \otimes c_{H,H} \otimes \mathrm{id}_{H^{\otimes(n-i-1)}}. \tag{3.2}$$

Observe that these are automorphisms of H^n, for every $i = 1, \ldots, n-1$, in $\mathrm{YB}(\mathrm{F_d})$.

The following result is Lemma XIII.3.5 of [13].

LEMMA 3.1. *Let* C *be a strict tensor category and* (V, σ) *an object of* YB(C). *Then there exists a unique strict tensor functor* $F : \mathrm{B} \to \mathrm{C}$ *such that* $F(1) = V$ *and* $F(c_{1,1}) = \sigma$.

Proof. If such a functor F exists, then it implies $F(n) = V^{\otimes n}$ and

$$F(g_i) = F(\mathrm{id}_1^{\otimes(i-1)} \otimes c_{1,1} \otimes \mathrm{id}_1^{\otimes(n-i-1)}) = \mathrm{id}_V^{\otimes(i-1)} \otimes \sigma \otimes \mathrm{id}_V^{\otimes(n-i-1)}$$

for $1 \leqslant i \leqslant n-1$. This proves the uniqueness of F in view of the fact that $g_1, \ldots, g_{n-1}$ generate B_n as a group. For the existence of F we set $F(n) = V^{\otimes n}$. Define as above automorphisms $c_1, \ldots, c_{n-1}$ of $F(n)$ by

$$c_i = \mathrm{id}^{\otimes(i-1)} \otimes \sigma \otimes \mathrm{id}^{\otimes(n-i-1)}$$

when $1 \leqslant i \leqslant n-1$. Since the automorphism σ is a Yang Baxter operator, the c_i satisfy the braid group relations. Then from Theorem X.6.5 of [13] there exists a unique morphism of groups F from the braid group B_n to $\mathrm{Aut}(F(n))$ such that $F(\sigma_i) = c_i$ for all i. The functor F is a strict tensor functor from B to C and $F(c_{1,1}) = c_1 = \sigma$. ∎

From this result, Theorem XIII.1.3 and Corollary X.6.3 of [13] there is a unique group morphism

$$\varepsilon : B_n \to \mathrm{Aut}(H^n)$$

sending the generator g_i of B_n to c_i, $\varepsilon(g_i) = c_i$, for every $i = 1, \ldots, n-1$. The representation ε is called the ***braid group representation*** associated to the braided tensor category $\mathrm{YB}(\mathrm{F_d})$.

The braid group B_∞ is the inductive limit of the subgroups B_n's by the inclusion $B_n \subset B_{n+1}$, thus the representation ε extends to a representation of B_∞.

Our next step will be to identify the strict braided C^*-category $\mathrm{YB}(\mathrm{F_d}) = (\mathrm{F_d}, c_{H,H})$ with $\mathrm{End}(O_d)$, where O_d is the Cuntz algebra.

The $*$-algebra over the complex numbers with unit $\mathbf{1}$ generated by the S_i with the relations (1.1) will be denoted by ${}^oO_d$ the algebraic part of O_d. The linear span of the S_i 's for $i = 1, \ldots, d$ will be denoted by H and it is the canonical Hilbert space. The scalar product on H is defined by $\langle S, S' \rangle \mathbf{1} = S^* S'$. Consequently the linear subspace generated by $S_{i_1} \cdots S_{i_r}$ will be denoted by H^r and the linear span of terms of the form $S_{i_1} \cdots S_{i_r} S_{j_s}^* \ldots S_{j_1}^*$ by (H^s, H^r).

Fix an object in the category $\mathrm{F_d}$. Following [7] we give an alternative construction of the Cuntz algebra ${}^oO_d$. Now ${}^oO_d$ and hence O_d can be derived from the category F_d from the following construction. We consider the embeddings $(H^r, H^{r+k}) \to (H^{r+1}, H^{r+k+1})$ given by the operation of tensoring on the right by the identity , i.e. $X \to X \otimes \mathbf{1}_H$ which is injective. The $*$-algebra structure is given as follows. The product of $A \in {}^oO_d^j$, $B \in {}^oO_d^k$ can always be defined to be an element of ${}^oO_d^{j+k}$ since for r sufficiently large $B \in (H^r, H^{r+k})$, $A \in (H^{r+k}, H^{r+k+j})$ and $(A \otimes \mathbf{1}) \circ (B \otimes \mathbf{1}) = (A \circ B) \otimes \mathbf{1}$. Also

B^* is well defined in ${}^\circ O_d^k$, by $(B \otimes \mathbf{1})^* = B^* \otimes \mathbf{1}$. Hence we can form the graded $*$-algebra

$$ {}^\circ O_d = \bigoplus_{k \in \mathbf{Z}} {}^\circ O_d^k \quad \text{with} \quad {}^\circ O_d^k = \lim_{\to r}(H^r, H^{r+k}). $$

Thus the operation of tensoring on the right by the identity identifies $\mathrm{F_d}$ with ${}^\circ O_d$ and induces the identity automorphism on ${}^\circ O_d$. The operation of tensoring on the left by the identity on $\mathrm{F_d}$ induces the non trivial endomorphism $\widehat{\sigma}$ on ${}^\circ O_d$. In other words, letting $i : X \to i(X)$ be the embedding of (H^r, H^s) into O_d we have

$$ i(X \times \mathbf{1}_H) = i(X), \quad i(\mathbf{1}_H \times X) = \widehat{\sigma} \circ i(X) $$

which defines the endomorphism $\widehat{\sigma}$. Observe that the operation of tensoring on the left by $\mathbf{1}$ on $\mathrm{F_d}$ and the interchange law:

$$ (X \otimes \mathbf{1}) \circ (\mathbf{1} \otimes S) = X \otimes S = (\mathbf{1} \otimes S) \circ (X \otimes \mathbf{1}) $$

induce in ${}^\circ O_d$ a natural endomorphism $\widehat{\sigma}$ such that

(3.3) $$ \Psi C = \widehat{\sigma}(C)\Psi, \quad \Psi \in (\mathbf{C}, H), \quad C \in {}^\circ O_d. $$

This rewritten in terms of the generators of ${}^\circ O_d$ by using relations (1.1) gives

$$ \widehat{\sigma}(C) = \sum_{i=1}^{d} S_i C S_i^*, \quad C \in {}^\circ O_d, $$

which is the canonical endomorphism of the Cuntz algebra. Pick now $S_i \in (\mathbf{C}, H)$ such that $S_i\, \mathbf{1}$, $i = 1, \ldots, d$ is an orthonormal basis of H. It is easy to see that S_i generate ${}^\circ O_d$ as a $*$-algebra. Let us now consider ${}^\circ O_d$ equipped with the natural endomorphism $\widehat{\sigma}(X) = \mathbf{1}_H \otimes X$, with $X \in (H^r, H^s)$, $r, s \in \mathbf{N}_0$, where we identify arrows between powers of H with their images in O_d. Denote by $\widehat{\sigma}^r$ with $r \in \mathbf{N}_0$ the composition $\widehat{\sigma} \circ \cdots \circ \widehat{\sigma}$ (r times). Hence under this identification we have

$$ (H^r, H^s) \subset (\widehat{\sigma}^r, \widehat{\sigma}^s), \quad r, s \in \mathbf{N}_0. $$

The C^*-algebra O_d is defined as the completion of ${}^\circ O_d$.

Let us summarize our construction. We have associated to the category $\mathcal{F}_d$ a C*algebra O_d as follows. To each $T \in (H^r, H^{r+1})$ of $\mathrm{F_d}$ there corresponds an element $i(T)$ of O_d. The map $T \to i(T)$ is linear and preserves composition and the adjoint operation. Regarding O_d as a C*-category with a single object then $i : \mathrm{F_d} \to O_d$ is a $*$-functor. The image of the arrows of $\mathrm{F_d}$ generates O_d as a C*-algebra. Also

$$ i(T \times \mathbf{1}_H) = i(T), \quad i(\mathbf{1}_H \times T) = \widehat{\sigma}(i(T)) $$

where $\widehat{\sigma}$ represents the canonical endomorphism of O_d. Now, by looking at this from the strict tensor category of $\mathrm{End}(O_d)$ we have the following embedding theorem for $\mathrm{F_d}$ ([5]).

THEOREM 3.2. *There is a C^*-algebra and a strict tensor $*$-functor $i : \mathcal{F}_d \to$* End(O_d).

Proof. Since we have

$$i(\mathbf{1}_H) = \mathbf{1}, \quad i(\mathbf{1}_H \times T) = \widehat{\sigma} i(T), \quad i(T \times \mathbf{1}_H) = i(T)$$

we can define i on objects by $i(\mathbf{C}) = \mathbf{id}$, $i(H^m) = \widehat{\sigma}^m$. We need only to check that if $T \in (H^r, H^{r+k})$ then for $A \in O_d$,

$$i(T)\widehat{\sigma}^r(A) = \widehat{\sigma}^{r+k}(A)i(T).$$

It suffices to take $A = i(S)$ for $S \in (H^q, H^{j+q})$. Then

$$\begin{aligned} i(T)\widehat{\sigma}^r(A) &= i(T)\widehat{\sigma}^r(i(S)) = i(T \times \mathbf{1}_{H^{j+q}})i(\mathbf{1}_H \times S) \\ &= i(\mathbf{1}_{H^{k+r}} \times S \circ T \times \mathbf{1}_{H^q}) = \widehat{\sigma}^{k+r}(i(S))i(T) \end{aligned}$$

since in our category $\mathrm{F_d}$ the interchange law holds and gives

$$T \times \mathbf{1}_{H^{j+q}} \circ \mathbf{1}_{H^r} \times S = \mathbf{1}_{H^{j+q}} \times S \circ T \times \mathbf{1}_{H^q}. \quad \blacksquare$$

Observe that End(O_d) arises from the category $\mathrm{F_d}$ as an abstract tensor C^*-category.

We consider a strict braided C^*-category $\mathrm{F_d}$ generated by tensor powers of a single object $\rho = H$ and with braiding ε_ρ. Then $\varepsilon_\rho(\rho^r, \rho^s) \in (\rho^{r+s}, \rho^{r+s})$. Hence the Yang-Baxter category is generated by the category $\mathrm{F_d}$ and the braiding ε_ρ. In the next result we use the strict tensor functor defined in Lemma 3.1. Hence we have

THEOREM 3.3. *Let $(\mathrm{F_d}, \varepsilon_\rho)$ be the braided strict tensor C^*-category. Then there exists a unique representation $\varepsilon : B_\infty \to O_d$ of the infinite braid group B_∞ in the Cuntz algebra O_d such that $\varepsilon(g_0) = \varepsilon_\rho$, $\varepsilon(\sigma g) = \widehat{\sigma}(\varepsilon(g))$ and ε implements the shift for every $g \in B_\infty$.*

Proof. If ε exists, it is unique since the infinite braid group is generated by the element g_0 corresponding to the transposition $(1, 2)$ and by the shift σ. Given $(\mathrm{F_d}, \varepsilon_\rho)$ we can form the Yang-Baxter category YB$(\mathrm{F_d})$ consisting of pairs (ρ, ε_ρ) where ρ is an object of $\mathrm{F_d}$ and ε_ρ is the Yang Baxter operator coming from the commutativity constraint. By Lemma 3.1 we have that there exists a unique tensor functor $F_\rho : \mathrm{B} \to \mathrm{F_d}$ such that $F_\rho(\mathbf{1}) = \rho$, $F_\rho(c_{1,1}) = \varepsilon_\rho$. Hence F_ρ is defined inductively on the braid groups $B_1, B_2, \ldots, B_n$ since the set of arrows of B are braid group generators for every n. We denote them by g_i for convenience so each $g_i \in B_\infty$ is sent into the corresponding braiding in $\mathrm{F_d}$ i.e. $F_\rho(g_i) = \varepsilon_{\rho^i}$, where we denote by $\varepsilon_{\rho^i} = \varepsilon_\rho(\rho^i, \rho^i)$ and they are computed by using the formula (3.1). In particular the ε_{ρ^i}'s are solution of a Yang-Baxter equation. By Theorem 3.2 there exists a functor

$$i : \mathrm{F_d} \to O_d.$$

To prove the existence, let us set: $\varepsilon(g) = i(F_\rho(g))$ for every $g \in B_n$ for some n. Hence $\varepsilon(g) \in (\widehat{\sigma}^n, \widehat{\sigma}^n)$ by Theorem 3.2. Since $\varepsilon(g \otimes 1) = i(F_\rho(g \otimes 1)) = i(F_\rho(g) \otimes 1) = \varepsilon(g)$ then ε is compatible with the inclusion $B_n \subset B_{n+1}$. Thus it gives a representation of B_∞ in O_d.

Now , $\varepsilon(\sigma(g)) = i(1 \otimes F_\rho(g)) = \widehat{\sigma}(i(F_\rho(g)) = \widehat{\sigma}(\varepsilon(g))$. ∎

From now on $\mathrm{F_d}$ will be the representation category of $U_q(d)$ generated by tensor powers of the corepresentation u. Any intertwiner in the category $\mathrm{F_d}$ can be realized as an intertwiner between tensor powers of a Hilbert space H of finite dimension d. Let R be the R-matrix associated to the compact matrix quantum group $U_q(d)$ and let $c_{H,H}$ be the operator given by the composition of the flip τ with R. Let us denote by c_{H^r,H^s} the operator as in (3.1). We denote by $c_{i_1 \ldots\ldots i_r}^{j_1 \ldots\ldots j_s}$ the matrix elements of c_{H^r,H^s} with respect to the basis on $H^r \otimes H^s$. Let us remark that S_q, defined in Section 2 by $S_q = \sum E(k_0, k_1, \ldots, k_n) S_{k_0} S_{k_1} \cdots S_{k_d}$, belongs to $(i, \widehat{\sigma}^d)$ and it is actually a q-determinant.

Let us assume that the self intertwiners of the tensor unit reduce to $\mathbf{C}$. Then Theorem 3.3 takes the following form.

COROLLARY 3.4. *Suppose that R is an operator on H^2, where H is an Hilbert space of dimension $d < \infty$ satisfying the Yang-Baxter equation. Define θ as*

$$\theta(g) = \sum_{i_1 \cdots i_k} c_{i_1 \ldots\ldots\ldots i_k}^{p(i_k) \ldots\ldots p(i_1)} S_{i_1} \cdots S_{i_k} S^*_{p(i_k)} \cdots S^*_{p(i_1)}.$$

Then θ is a representation of B_∞ in O_d. Moreover θ is the unique representation satisfying $\theta(\sigma(g)) = \widehat{\sigma}(\theta(g))$ for every $g \in B_\infty$.

Proof. Let $\mathrm{F_d}$ be the strict tensor C^*-category whose objects are the tensor powers of the Hilbert space H. Then R gives a commutativity constraint for the category $\mathrm{F_d}$ and denote by $c_{H,H} = \tau(R)$ where τ is the flip. The element $c_{H,H}$ is an intertwiner from the two fold tensor product of H into itself in the C^*-category $\mathrm{F_d}$. Let e_i be the basis of H. Denote by $\tilde{R}_{i_1 i_2}^{j_1 j_2}$ the elements of the matrix R, composed with the flip. Then $c_{H,H}$ has the following form

$$c_{H,H}(e_{j_1} \otimes e_{j_2}) = \sum_{j_1, j_2} \tilde{R}_{j_1 j_2}^{j_{p(2)} j_{p(1)}} e_{j_{p(2)}} \otimes e_{j_{p(1)}}.$$

Thus

$$i(c_{H,H}(e_{j_1}\otimes e_{j_2})) = i\Big(\sum_{j_1,j_2} \tilde{R}^{j_{p(2)}j_{p(1)}}_{j_1j_2} e_{j_{p(2)}}\otimes e_{j_{p(1)}}\Big) = \sum_{j_1,j_2} \tilde{R}^{j_{p(2)}j_{p(1)}}_{j_1j_2} i(e_{j_{p(2)}}\otimes e_{j_{p(1)}}).$$

Then it follows

$$i(c_{H,H}(e_{j_1}\otimes e_{j_2})) = \sum_{j_1,j_2} \tilde{R}^{j_{p(2)}j_{p(1)}}_{j_1j_2} S_{j_{p(2)}} S_{j_{p(1)}},$$

from which

$$i(c_{H,H}) = \sum_{j_1,j_2} \tilde{R}^{j_{p(2)}j_{p(1)}}_{j_1j_2} S_{j_1} S_{j_2} S^*_{j_{p(2)}} S^*_{j_{p(1)}}.$$

Consider the following composition

$$i(\tilde{R}^{j_1j_2}_{i_1i_2}(e_{i_1}\otimes e_{i_2})^* \circ (e_{j_1}\otimes e_{j_2})) = i(\tilde{R}^{j_1j_2}_{i_1i_2}(e_{i_1}\otimes e_{i_2})^*) \circ i(\tilde{R}^{j_1j_2}_{i_1i_2}(e_{j_1}\otimes e_{j_2}))$$

where $(e_{i_1}\otimes e_{i_2})^*$ denotes the intertwiner between H^2 and $\mathbf{C}$ and $e_{j_1}\otimes e_{j_2}$ denotes the intertwiner between $\mathbf{C}$ and H^2. Then $i(\tilde{R}^{j_1j_2}_{i_1i_2}(e_{i_1}\otimes e_{i_2})^* \circ (e_{j_1}\otimes e_{j_2}))$ is an intertwiner from $\mathbf{C}$ to $\mathbf{C}$. By our assumption that $(\mathbf{C},\mathbf{C}) = \mathrm{id}$ in the C*-category $\mathrm{F_d}$ we have that

$$i(\tilde{R}^{j_1j_2}_{i_1i_2}(e_{i_1}\otimes e_{i_2})^* \circ (e_{j_1}\otimes e_{j_2})) = \lambda\mathbf{1}$$

where $\lambda \in \mathbf{C}$. Observe that the left hand side is

$$\tilde{R}^{j_1j_2}_{i_1i_2} S^*_{i_1} S^*_{i_2} S_{j_2} S_{j_1} = \lambda\mathbf{1}$$

and so λ can be defined to be

$$\lambda = \langle \tilde{R}^{j_1j_2}_{i_1i_2} S_{j_2} S_{j_1}, S_{i_1} S_{i_2}\rangle.$$

This means that $i(c_{H,H})$ is identified with an element of O_d. Let $\mathrm{YB}(\mathrm{F_d}) = (\mathrm{F_d}, c_{H,H})$.

Hence i extends to $\hat{i} : \mathrm{YB}(\mathrm{F_d}) \to O_d$. If θ exists, it is clear that it is unique since B_∞ is generated by g_0 and by the shift σ. By Theorem 3.3 there exists a unique representation θ from B_∞ to O_d such that $\theta(g) = i(F_H(g))$ for every $g \in B_n$ for some n.

Recall that the element g_0 of the braid group B_∞ corresponds to the transposition $(1,2)$ and by using the shift σ, we can write any generator of B_∞ in terms of g_0 and σ as follows: $g_1 = \sigma(g_0) = \mathbf{1}\otimes g_0$, $g_i = \sigma^{(i)}(g_0)$.

The image of the operator $\theta(g_o)$ in terms of the generators of O_d has the following form:

$$\theta(g_0) = \sum_{i_1,i_2} c_{H,H}(S_{i_1} \otimes S_{i_2})S^*_{i_{p(2)}} \otimes S^*_{i_{p(1)}}, \quad g_0 \in B_\infty \tag{3.4}$$

where we then identify $S_{i_1} \otimes S_{i_2}$ with $S_{i_1}S_{i_2}$ and p stands for the unique permutation of $(1,2)$ associated to the element $g_0 \in B_\infty$ which in turn implies $g_0 \in B_n$, for some n.

Observe that θ implements the shift σ. In fact

$$\begin{aligned}
\theta(\sigma(g_0)) = \theta(1 \otimes g_0) &= \sum_{i_1,i_2,i_3} (1 \otimes c_{H,H})(S_{i_3}S_{i_1}S_{i_2})S^*_{i_{p(2)}}S^*_{i_{p(1)}}S^*_{i_3} \\
&= \sum_{i_3,i_1,i_2} S_{i_3}(c_{H,H}(S_{i_1}S_{i_2})S^*_{i_{p(2)}}S^*_{i_{p(1)}})S^*_{i_3} \\
&= \sum_{i_3} S_{i_3}\theta(g_0)S^*_{i_3} = \widehat{\sigma}(\theta(g_0))
\end{aligned}$$

where on the right side $\widehat{\sigma}$ represents the canonical endomorphism of the Cuntz algebra O_d. Then for a generic r,

$$\begin{aligned}
\theta(\sigma g_r) = \theta(1 \otimes g_r) = &\sum_{i_{r+1},\dots,i_r} (1 \otimes c_{H^r,H^r})(S_{i_{r+1}} \cdots S_{i_r})S^*_{i_{p(r)}} \cdots S^*_{i_{p(1)}}S^*_{i_{r+1}} \\
&= \widehat{\sigma}(\theta(g_r)).
\end{aligned}$$

Now $F_H(g) = c_{H^n H^n}$ for some n, so $\theta(g) = i(c_{H^n H^n})$ and θ has the desidered form. Then $\theta(g_n) \in (\widehat{\sigma}^n, \widehat{\sigma}^n)$ by Theorem 3.2. Also

$$\theta(g_n \otimes 1) = i(c_n \otimes 1) = i(c_n) = \theta(g_n).$$

Thus θ is compatible with the inclusion $B_n \subset B_{n+1}$. We have thus defined a representation of B_∞ in O_d. ∎

4. ENDOMORPHISMS OF A C^*-ALGEBRA WITH BRAIDINGS

In this section we apply the results of the previous section for the representation category of the quantum groups $U_q(d)$ and $SU_q(d)$ respectively to give conditions for an endomorphism ρ of a C^*-algebra $\mathcal{A}$ to determine an action of the category on $\mathcal{A}$.

Let $\mathcal{A}$ be a C^*-algebra and let $\mathrm{End}(\mathcal{A})$ denote the C^*-category whose objects are the endomorphisms of $\mathcal{A}$ and the set of arrows (ρ, ρ') from ρ to ρ' are the intertwiners $T \in (\rho, \rho')$ i.e. $T\rho(A) = \rho'(A)T$, if $A \in \mathcal{A}$.

$\mathrm{End}(\mathcal{A})$ has the tensor structure defined as follows: let $T \in (\rho, \rho')$, $S \in (\sigma, \sigma')$; we define

$$T \times S := T\rho(s) = \rho'(S)T \in (\rho\sigma, \rho'\sigma').$$

Let i be the identity automorphism of $\mathcal{A}$ and $(i, i) = \mathrm{centre}(\mathcal{A})$.

Now we restrict to the case of the category generated by the powers of a single endomorphism ρ. As subsets of $\mathcal{A}$ we have

$$(\rho^r, \rho^s) \subset (\rho^{r+k}, \rho^{s+k}), \ (\rho^r, \rho^s)^* = (\rho^s, \rho^r), \ k, r, s \in \mathbf{N}_0.$$

We study the problem of the existence of an action of the fixed point algebra $(O_d)^{U_q(d)}$ on the C^*-algebra $\mathcal{A}$, i.e. the existence of a morphism $\mu : (O_d)^{U_q(d)} \to \mathcal{A}$ such that $\mu \circ \sigma = \rho \circ \mu$, where by fixed point subalgebra we understand under the coaction of $U_q(d)$ on O_d, and σ is the canonical endomorphism of O_d. This is equivalent to the problem of finding an action of $\mathrm{F_d}$ on $\mathcal{A}$ where $\mathrm{F_d}$ is the strict tensor braided C^*-category of corepresentations of the Hopf algebra associated to $U_q(d)$. Hence there must exist a functor $M : \mathrm{F_d} \to \mathrm{End}(\mathcal{A})$ such that $(\rho, \sigma) \subset (M(\rho), M(\sigma))$, the induced maps are linear, the unit of $\mathrm{F_d}$ is mapped onto the identity automorphism and

$$M(T) = M(T)^*, \ M(\rho \otimes \rho') = M(\rho)M(\rho'),$$

$$M(T \otimes T') = M(T) \times M(T').$$

We want to study conditions such that there exists such action which in turn will give a representation of the braid group B_∞ in $\mathcal{A}$.

DEFINITION 4.1. Let $\rho \in \mathrm{End}(\mathcal{A})$. We say that ρ has a braiding if there exists a representation $\varepsilon : B_\infty \hookrightarrow \mathcal{A}$ such that if we let σ be the shift operator,

(i) $\varepsilon(\sigma g) = \rho(\varepsilon(g))$, $g \in B_\infty$;

(ii) $\varepsilon(g_0) \in (\rho^2, \rho^2)$;

(iii) $\varepsilon(g_s)x = \rho(x)\varepsilon(g_r)$, if $x \in (\rho^r, \rho^s)$, where $g_s = \sigma^s(g_0)$, $g_r = \sigma^r(g_0)$.

Hence it follows that $\varepsilon(g_n) \in (\rho^n, \rho^n)$, $g_n \in B_n$.

For a given braiding we want to find the kernel of the corresponding morphism $\varepsilon : C^*(B_\infty) \to \mathcal{A}$. This means to find the quasiequivalence class of the representation of each B_n. As an example we consider $(O_d)^{U_q(d)}$, the fixed point subalgebra of O_d by the coaction Γ of the quantum group $U_q(d)$ and the restriction of the canonical endomorphism $\widehat{\sigma}$ to $(O_d)^{U_q(d)}$ i.e. $\widehat{\sigma}|_{(O_d)^{U_q(d)}}$. It is an endomorphism with braiding given by θ as in Corollary 3.5. By Corollary 6 in [17] $(O_d)^{U_q(d)}$ is generated by the representation θ. The first property is easily verified. Now $\theta(g_0)$ is an intertwiner between (H^2, H^2) and we have seen that any intertwiner $T \in (H^r, H^s)$ by using the identification with its image in O_d, arises as $T \in (\widehat{\sigma}^r, \widehat{\sigma}^s)$. Thus $\theta(g_0) \in (\widehat{\sigma}^2, \widehat{\sigma}^2)$, where $\widehat{\sigma}$ can be thought of as $\widehat{\sigma} = 1_H \otimes x$. Given $x \in (O_d)^{U_q(d)}$, since $\theta(g) \circ T \otimes T' = T' \otimes T \circ \theta(g)$ holds for every $g \in B_\infty$ and T, $T' \in (O_d)^{U_q(d)}$, then $\widehat{\sigma}(x)$ is still fixed, it follows that $\theta(g_r)x = \widehat{\sigma}(x)\theta(g_s)$. Therefore the third property is satisfied.

We need to observe that the Yang-Baxter operator R associated to the compact quantum group $U_q(d)$ satisfies the Hecke algebra relations ([11]). This means that the representation of the braid group factors through the Hecke algebra. For q not a root of unity ([11]) the representations of B_n are in one to one correspondence with Young diagrams. Their decomposition rule and their dimension are the same as for the symmetric group of order n. Let a Young diagram with one row correspond to the trivial irreducible representation of the symmetric group and a diagram with one column to the parity irreducible representation. By extending this convention to the trivial irreducible representation of B_n for $q \neq 1$ we have that it sends each g_i into $q1$ and the parity representation sends g_i into -1. The ideals in $C^*(B_\infty)$ are specified by giving minimal central projections in subalgebras $C^*(B_n)$ and they generate the ideals. The minimal projections in $C^*(B_n)$ are in one to one correspondence with the Young diagram with n squares. The Young diagram with a single column of length $d+1$ gives the fixed point algebra $(O_d)^{U_q(d)}$ as a quotient of $C^*(B_\infty)$ by some ideal I_d. The ideal I_d is generated by the q-antisymmetric projection on $C^*(B_{d+1})$ and it is the kernel of θ. Analogously to the Doplicher and Roberts terminology for permutation symmetry, we say that the endomorphism $\widehat{\sigma}|_{(O_d)^{U_q(d)}}$ has a braiding of dimension d. This motivates the following definition.

DEFINITION 4.2. An endomorphism ρ has a braiding of dimension d if the representation $\varepsilon : C^*(B_\infty) \to \mathcal{A}$ has the ideal I_d generated in $C^*(B_\infty)$ by the projection on the totally antisymmetric space in $C^*(B_{d+1})$ as its kernel.

We are now ready to state the main theorem.

THEOREM 4.3. *If ρ is an endomorphism of a C^*-algebra $\mathcal{A}$ with a braiding of "dimension d" realized by the representation $\varepsilon : g \to \varepsilon(g)$ of B_∞ in $\mathcal{A}$, then there is a unique monomorphism $\mu : (O_d)^{U_q(d)} \to \mathcal{A}$ with $\mu(\theta) = \varepsilon$ and $\mu \circ \widehat{\sigma} = \rho \circ \mu$ defining an action of $(O_d)^{U_q(d)}$ in $\mathcal{A}$. If there is a $R \in (i, \rho^d)$ with $RR^* = \sum_{p \in S(d)} \frac{1}{d!}((-q)^{\#l(p)})\varepsilon(g)$, then the action extends uniquely to $(O_d)^{SU_q(d)}$ if we require $\mu(S_q) = R$.*

Proof. By definition a braiding of dimension d yields a monomorphism $\mu : (O_d)^{U_q(d)} \to \mathcal{A}$ with $\mu(\theta(g)) = \varepsilon(g)$ which satisfies $\mu(\theta) = \varepsilon$ and $\mu \circ \widehat{\sigma} = \rho \circ \mu$. We want to prove that we have an action of $(O_d)^{U_q(d)}$ on $\mathcal{A}$. We know by Theorem 2.3 and Corollary 7 of [17] that $(O_d)^{U_q(d)} = (M_d^\infty)^{U_q(d)} = C^*(\theta(g))$. Let us recall the results about the intertwiners:

$$(M_d^k)^{U_q(d)} = \{x \in M_d^k : u^k(x \otimes 1_{\mathcal{A}}) = (x \otimes 1_{\mathcal{A}})u^k\}, \quad i = 0, ..., k-1.$$

M_d^∞ is the canonical UHF algebra inside the Cuntz algebra O_d. Also if $x \in (H^k, H^k)$ in the fixed point subalgebra, by the identification of the category F_d with $\mathrm{End}(O_d)$ we have that $x \in (\widehat{\sigma}^k, \widehat{\sigma}^k)$. Since the canonical endomorphism preserves fixed points, this implies that $x \in (\widehat{\sigma}^k, \widehat{\sigma}^k)$ is still in the fixed point subalgebra. It follows that the set $(\widehat{\sigma}^k, \widehat{\sigma}^k)$ is the linear span of $\theta(g_i)$, $i = 0, ..., k-1$, by Corollary 6 of [17]. This implies $\mu((\widehat{\sigma}^k, \widehat{\sigma}^k)) \subset (\rho^k, \rho^k)$, so that μ is an action of $(O_d)^{U_q(d)}$ on $\mathcal{A}$. We want to prove that it extends uniquely to a monomorphism of $(O_d)^{U_q(d)}$ on $\mathcal{A}$. From Lemma 7 of [17], we know that $(O_d)^{SU_q(d)}$ is constructed from $(O_d)^{U_q(d)}$ and the q-determinant called S_q. Note that $S_q \in (i, \widehat{\sigma}^d)$. Then S_q induces an isomorphism of $(O_d)^{U_q(d)}$ onto a subalgebra. In fact, we define

$$\tau(A) = S_q A S_q^*, \ A \in (O_d)^{U_q(d)},$$

$$\tau(I) = S_q S_q^* = \frac{1}{d!} \sum_{p \in S(d)} (-q)^{\#l(p)} S_{i_1} \cdots S_{i_d} S_{p(j_d)}^* \cdots S_{p(j_1)}^*,$$

where $\#l(p)$ stands for the length of the permutation p. Thus τ is an isomorphism onto the subalgebra $\tau(I)(O_d)^{U_q(d)}\tau(I)$ and $\tau^{-1}(B) = S_q^* B S_q$ for $B \in \tau(O_d)^{U_q(d)}$ since $\tau(S_q^* B S_q) = S_q S_q^* B S_q S_q^* = B$, $B \in \tau(I)(O_d)^{U_q(d)}\tau(I)$. $(O_d)^{SU_q(d)}$ is the crossed product of $(O_d)^{U_q(d)}$ by the action τ. Since any element can be written uniquely as XS_q^k with $X = X\tau^k(I)$, then the morphism μ extends to $(O_d)^{SU_q(d)}$ with $\mu(S_q) = R$.

Thus if there is an $R \in (\iota, \rho^d)$ and if $A \in (O_d)^{U_q(d)}$, then

$$R\mu(A)R^* = \rho^d \circ \mu(A)RR^* = \mu \circ \widehat{\sigma}^d(A)\mu(S_qS_q^*) = \mu \circ \tau(A)$$

where τ is the above isomorphism guaranteeing that μ extends uniquely to a monomorphism $\mu : (O_d)^{SU_q(d)} \to \mathcal{A}$ with $\mu(S_q) = R$ and $\mu(\widehat{\sigma}(S_q)) = \rho(R)$.

To check that we will still have an action by Lemma 7 of [17], we need only to show that if $X \in (\widehat{\sigma}^r, \widehat{\sigma}^{r+kd})$ then $\mu(X) \in (\rho^r, \rho^{r+kd})$. From Lemma 7 of [17], we have that any X in $(O_d)^{U_q(d)}$ consists of XS_q^{k*} and S_q^k. We have seen that $XS_q^{k*} \in (\widehat{\sigma}^r, \widehat{\sigma}^{r+kd})$ as an element of $(O_d)^{U_q(d)}$ while $\mu(S_q^k) = R^k \in (\rho^r, \rho^{r+kd})$ so this gives that $\mu(X) \in (\rho^r, \rho^{r+kd})$. ∎

Acknowledgements. The author deeply thanks Professor G. Elliott for the support received during her permanence at the Fields Institute, and to Professors A. Van Daele and P.E.T. Jorgensen for the very useful discussions on this subject.

REFERENCES

1. S. BAAJ, G. SKANDALIS, Unitaires multiplicatifs et dualité pour le produit cróises de C^*-algèbres, preprint.
2. J. CUNTZ, Simple C^*-algebras generated by isometries, *Comm. Math. Phys.* **57**(1977), 173–185.
3. J. CUNTZ, Regular actions of Hopf algebras on the C^*-algebra generated by a Hilbert space, preprint.
4. S. DOPLICHER, Abstract compact group duals, operator algebras and quantum field theory, in *Proceedings of the International Congress of Mathematicians* (Kyoto, 1990), vol. II, p. 1319–1333, Math. Soc. Japan, Tokyo 1991.
5. S. DOPLICHER, J.E. ROBERTS, A new duality theory for compact groups, *Invent. Math.* **98**(1989), 157–218.
6. S. DOPLICHER, J.E. ROBERTS, Endomorphisms of C^*-algebras, cross products and duality for compact groups, *Ann. of Math. (2)* **130**(1989), 75–119.
7. S. DOPLICHER, J.E. ROBERTS, Duals of compact Lie groups realized in the Cuntz algebras, *J. Funct. Anal.* **74**(1987), 90–120.
8. S. DOPLICHER, J.E. ROBERTS, Compact group actions on C^*-algebras, *J. Operator Theory*, **19**(1988), 283–305.
9. S. DOPLICHER, Operator algebras and abstract duals: progress and problems, to appear in the *Proceedings of the International Symposium on Non-commutative Analysis*, Publ. Res. Inst. Math. Sci., Kyoto University 1992.
10. K. FREDENHAGEN, K.H. REHREN, B. SCHROER, Superselection sectors with braid group statistics and exchange algebras I, *Comm. Math. Phys.* **125**(1989), 201–226.

11. V.F.R. JONES, Hecke algebra representations of braid groups and link polynomials, *Ann. of Math. (2)* **126**(1987), 335–388.
12. A. JOYAL, R. STREET, Braided monoidal categories, Macquarie Math. Report, n. 860081, 1986.
13. C. KASSEL, *Quantum Groups*, Springer Verlag, 1994.
14. H.T. KOELINK, On $*$-representations of the Hopf $*$-algebra associated with the quantum group $U_q(n)$, *Compositio Math.* **77**(1991), 199–231.
15. R. LONGO, J.E. ROBERTS, Theory of dimension, preprint.
16. R. LONGO, A duality for Hopf algebras and for subfactors I, *Comm. Math. Phys.* **159**(1994), 133–150
17. A. PAOLUCCI, Coactions of Hopf algebras on Cuntz algebras and their fixed point algebras, *Proc. Amer. Math. Soc.*, to appear.
18. M TAKESAKI, A characterization of group algebras as a converse of Tannaka-Stinespring-Tatsuuma duality theorem, *Amer. J. Math.* **91**(1965), 529–564.
19. A. VAN DAELE, Dual pairs of Hopf $*$-algebras, *Bull. London Math. Soc.* **25**(1993), 209–230.
20. Y. KONISHI, M. NAGISA, Y. WATATANI, Some remarks on actions of compact matrix quantum groups on C^*-algebras, *Pacific J. Math.* **153**(1992), 119–127.
21. S.L. WORONOWICZ, Compact matrix pseudogroups, *Comm. Math. Phys.* **111**(1987), 613–665.
22. S.L. WORONOWICZ, Twisted SU(2) group. An example of a non commutative differential calculus, *Publ. Res. Inst. Math. Sci.* **23**(1987), 117–181.
23. S.L. WORONOWICZ, Tannaka-Krein duality for compact matrix pseudogroups, twisted $SU(N)$ groups, *Invent. Math.* **93**(1988), 35–76.
24. S.L. WORONOWICZ, Compact Quantum Group, preprint, Warsaw, 1993.
25. D.N. YETTER, Quantum groups and representations of monoidal categories, *Math. Proc. Cambridge Philos. Soc.* **108**(1990), 261–290.

ANNA PAOLUCCI
School of Mathematics
University of Leeds
Leeds LS2 9JT, ENGLAND
UNITED KINGDOM

Received August 2, 1995; revised January 31, 1996.