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UPPER AND LOWER MULTIPLICITY FOR IRREDUCIBLE REPRESENTATIONS OF C^*-ALGEBRAS. II

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ABSTRACT. We use the notions of upper and lower multiplicity, $\mathrm{M_U}(\pi)$ and $\mathrm{M_L}(\pi)$, for an irreducible representation π of a C^*-algebra A to investigate some of the possible structure in point-strong limits of essentially irreducible representations of A on a large Hilbert space. In particular, this leads to a generalization of Gardner's theorem on 'the third definition' of the topology on the spectrum $\hat{A}$ of A and also to new characterizations of $\mathrm{M_U}(\pi)$ and $\mathrm{M_L}(\pi)$. We also investigate the possible gap between $\mathrm{M_U}(\pi)$ and $\mathrm{M_L}(\pi)$ by introducing upper and lower multiplicities for π relative to a net in $\hat{A}$.

KEYWORDS: *Multiplicity, spectrum, C^*-algebra, lower semi-continuity, irreducible representation, pure states.*

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1. INTRODUCTION

Let A be a C^*-algebra and let H be a Hilbert space that is infinite dimensional and large enough for A (in the sense that $\dim(H)$ is at least as large as the dimension of every irreducible representation of A). Following [7] and [8], we let $\mathrm{Rep}(A, H)$ be the set of all (possibly degenerate) representations of A on H, and we equip $\mathrm{Rep}(A, H)$ with the strong topology (that is, the topology of pointwise strong operator convergence on H). Let $\mathrm{Irr}(A, H)$ be the subspace of $\mathrm{Rep}(A, H)$ consisting of those non-zero representations σ for which $\mathrm{Ess}(\sigma)$ (the essential part of σ) is irreducible, and let Φ be the canonical surjection from $\mathrm{Irr}(A, H)$ onto $\widehat{A}$ which maps σ to the unitary equivalence class of $\mathrm{Ess}(\sigma)$.

In [12], Gardner showed that Φ is a continuous, open mapping (a somewhat different approach to the topology on $\widehat{A}$ was subsequently given by Ernest ([9]).

As a consequence of the open property, if $\Omega = (\pi_\alpha)_{\alpha\in\Lambda}$ is a net in $\widehat{A}$ which is convergent to π then there exists a net $\Omega_1 = (\sigma_\mu)_{\mu\in\Delta}$ in $\mathrm{Irr}(A, H)$ and a representation $\sigma \in \mathrm{Irr}(A, H)$ such that $\Phi(\mathrm{Ess}(\sigma)) = \pi$, $(\Phi(\mathrm{Ess}(\sigma_\mu)))_{\mu\in\Delta}$ is a subnet of Ω, and $\sigma_\mu \to \sigma$. That is, speaking informally, the convergence of π_α to π can be lifted to $\mathrm{Rep}(A, H)$ (at the expense of passing to a subnet and possibly introducing degeneracy into the representations).

In general, $\widehat{A}$ may fail to be Hausdorff and a convergent net Ω in $\widehat{A}$ may have a large (possibly uncountable) limit set $L(\Omega)$. If π and π' are distinct elements of $L(\Omega)$ then it is not possible to use the same net Ω_1 in the previous paragraph for both π and π', because $\mathrm{Rep}(A, H)$ is always a Hausdorff space. However, one might naturally ask whether it is possible to find a net $\Omega_1 = (\sigma_\mu)_{\mu\in\Delta}$ in $\mathrm{Irr}(A, H)$ and a representation $\sigma \in \mathrm{Rep}(A, H)$ such that $\mathrm{Ess}(\sigma) \simeq \pi \oplus \pi'$ (unitary equivalence), $(\Phi(\mathrm{Ess}(\sigma_\mu)))_{\mu\in\Delta}$ is a subnet of Ω, and $\sigma_\mu \to \sigma$.

Existing results enable this question to be answered in the affirmative: the proof of [2], Theorem 1 shows that the convergence of Ω to π and π' can be lifted to the convergence of a suitable net in $\overline{P(A)}$ (the weak*-closure of the set of pure states of A) and then the lifting to $\mathrm{Rep}(A, H)$ can be achieved by applying a theorem of Bichteler ([7]). This method can be extended to deal with a finite subset of $L(\Omega)$. However, the method of proof in [2], Theorem 1 is not applicable to infinite subsets of $L(\Omega)$, and the use of Bichteler's theorem does not permit the introduction of multiplicity in the direct sum decomposition of $\mathrm{Ess}(\sigma)$. This is because Bichteler's theorem deals with Gelfand-Naimark-Segal (GNS) representations for states and therefore cannot be used to obtain non-cyclic representations in the closure of $\mathrm{Irr}(A, H)$ (see also [1], p. 1 in this connection).

One of our main results (Theorem 4.2) enables these problems to be overcome by a different approach. This shows, in particular, that if Ω is a convergent net in $\widehat{A}$ with limit set $L(\Omega)$ (such that Ω is not frequently equal to any point in $L(\Omega)$) and if $\{m_\pi \mid \pi \in L(\Omega)\}$ is a set of non-zero cardinal numbers such that $m_\pi \leqslant \mathrm{M_L}(\pi)$ whenever the lower multiplicity $\mathrm{M_L}(\pi)$ is finite (see [3] and Section 2 below), then there exists a Hilbert space H which is infinite dimensional and large enough for A, and representations σ and $(\sigma_\mu)_{\mu\in\Delta}$ of A on H such that:

(i) $\sigma_\mu \in \mathrm{Irr}(A, H)$ for each $\mu \in \Delta$,

(ii) $(\Phi(\sigma_\mu))_{\mu\in\Delta}$ is a subnet of Ω,

(iii) $\mathrm{Ess}(\sigma) \simeq \bigoplus_{\pi\in L(\Omega)} m_\pi \cdot \pi$,

(iv) $\sigma_\mu \to \sigma$ in $\mathrm{Rep}(A, H)$.

That is, the simultaneous convergence of Ω to the members of its limit set can be lifted to $\mathrm{Rep}(A, H)$ whilst also taking account of multiplicity.

Most of the technicalities required for the proof of Theorem 4.2 are dealt with in Lemma 3.7 which is also applicable to the study of upper multiplicity for a single element of $L(\Omega)$ (see Theorem 4.1). In turn, the proof of Lemma 3.7 depends in a crucial way on two basic lemmas concerning pure states, the first of which appears to be previously unknown (see Lemma 2.5, Lemma 2.6 and the discussion which precedes them).

Theorems 4.1 and 4.2 lead easily to characterizations of $\mathrm{M_U}(\pi)$ and $\mathrm{M_L}(\pi)$ (respectively) in terms of point-strong limits of essentially irreducible representations: Theorems 4.5 and 4.7 characterize $\mathrm{M_U}(\pi)$ and $\mathrm{M_L}(\pi)$ by means of inequalities, whereas Theorems 4.9 and 4.12 give exact formulae in terms of the multiplicity of π in elements of $\overline{\mathrm{Irr}(A,H)}$. The formula for $\mathrm{M_U}(\pi)$ can easily be described as follows. Suppose that H is infinite dimensional and large enough for A. For $\sigma \in \overline{\mathrm{Irr}(A,H)}$, let $m(\pi,\sigma) \in \mathbf{N} \cup \{\infty\}$ denote the multiplicity of π in σ. (Throughout the paper we use $\mathbf{N}$ to denote the non-negative integers, and $\mathbf{P}$ to denote the positive integers.) Then

$$\mathrm{M_U}(\pi) = \sup\{m(\pi,\sigma) \mid \sigma \in \overline{\mathrm{Irr}(A,H)}\}.$$

A further consequence of these results is a generalization of Gardner's theorem that Φ is an open map. The openness of Φ is equivalent to the fact that if $\sigma \in \mathrm{Irr}(A,H)$ with $\Phi(\sigma) = \pi \in \widehat{A}$, then Φ carries the neighbourhood base of σ in $\mathrm{Rep}(A,H)$ into the neighbourhood base of π in $\widehat{A}$. In Corollary 4.8 we show that if $\{\pi\}$ is not open in $\widehat{A}$, if $n \in \mathbf{P} \cup \{\infty\}$ and if $\sigma \in \mathrm{Rep}(A,H)$ satisfies $\mathrm{Ess}(\sigma) \simeq n \cdot \pi$ for some cardinal realization of n (see Definition 3.6) then Φ carries the neighbourhood base at σ into the (possibly deleted) neighbourhood base at π if and only if $n \leqslant \mathrm{M_L}(\pi)$.

The definitions of $\mathrm{M_U}(\pi)$ and $\mathrm{M_L}(\pi)$ in [3] may be informally described as follows. Let φ be a pure state associated with $\pi \in \widehat{A}$. Both $\mathrm{M_U}(\pi)$ and $\mathrm{M_L}(\pi)$ are related to the counting of the number of mutually orthogonal pure states, associated with an element π' close to π in $\widehat{A}$, which are simultaneously close to φ. The number $\mathrm{M_U}(\pi)$ is related to a maximal count obtained by optimising the way in which π' approaches π, whereas the number $\mathrm{M_L}(\pi)$ gives a guaranteed minimum count that can be obtained when π' ($\neq \pi$) approaches π in an arbitrary way. Both $\mathrm{M_U}(\pi)$ and $\mathrm{M_L}(\pi)$ are independent of the choice of φ and take values in $\mathbf{P} \cup \{\infty\}$.

In order to investigate the possible gap between $\mathrm{M_L}(\pi)$ and $\mathrm{M_U}(\pi)$ for $\pi \in \widehat{A}$, we introduce upper and lower multiplicities $\mathrm{M_U}(\pi,\Omega)$ and $\mathrm{M_L}(\pi,\Omega)$ relative to nets Ω in $\widehat{A}$. These are defined by restricting attention to elements of $\widehat{A}$ which lie in Ω. This has the effect of increasing $\mathrm{M_L}$ and decreasing $\mathrm{M_U}$ (see, however, Proposition 2.1 and the remarks which precede it). Several of our results (in particular Theorems 4.1 and 4.2) are framed in terms of the quantities $\mathrm{M_L}(\pi,\Omega)$ and $\mathrm{M_U}(\pi,\Omega)$, thereby achieving greater precision. For example, as a consequence of Theorem 4.2, we show in Theorem 4.3 (i) that if $\Omega = (\pi_\alpha)_{\alpha\in\Lambda}$ is a net in $\widehat{A}$, then

$$\liminf \mathrm{Tr}(\pi_\alpha(a)) \geqslant \sum_{\pi\in\hat{A}} \mathrm{M_L}(\pi,\Omega)\,\mathrm{Tr}(\pi(a)) \quad (a \in A^+) \tag{1.1}$$

(where Tr is the usual trace defined on positive operators in Hilbert space). This strengthens the lower semi-continuity result of [3], Theorem 3.2 in which $\mathrm{M_L}(\pi,\Omega)$ is replaced by $\mathrm{M_L}(\pi)$, and the sum is restricted to those limits π of Ω for which eventually $\pi_\alpha \neq \pi$.

Given $\pi \in \widehat{A}$ with $\{\pi\}$ not open, there exists a net Ω in $\widehat{A}\setminus\{\pi\}$ such that $\mathrm{M_L}(\pi) = \mathrm{M_L}(\pi,\Omega)$ and $\mathrm{M_U}(\pi) = \mathrm{M_U}(\pi,\Omega)$ (Proposition 2.2). It is this fact which enables us to obtain Theorems 4.5 and 4.7 (concerning $\mathrm{M_U}(\pi)$ and $\mathrm{M_L}(\pi)$) as special cases of Theorems 4.1 and 4.2.

In general, if Ω is a net in $\widehat{A}$ and $\pi \in \widehat{A}$ then there exist subnets Ω_1 and Ω_2 such that

$$\mathrm{M_L}(\pi,\Omega_1) = \mathrm{M_U}(\pi,\Omega_1) = \mathrm{M_L}(\pi,\Omega),$$
$$\mathrm{M_L}(\pi,\Omega_2) = \mathrm{M_U}(\pi,\Omega_2) = \mathrm{M_U}(\pi,\Omega)$$

(Proposition 2.3). In particular, it is possible to achieve $\mathrm{M_L}(\pi)$ and $\mathrm{M_U}(\pi)$ in this way by taking Ω as in the previous paragraph. These techniques enable us to vary (1.1) above as follows: if π_0 is any prescribed member of $\widehat{A}$ then we can replace $\mathrm{M_L}(\pi_0,\Omega)$ by $\mathrm{M_U}(\pi_0,\Omega)$ in the righthand side of (1.1) at the expense of passing to a subnet on the lefthand side (Theorem 4.3 (ii)).

We close this section by discussing the relationship of $\mathrm{M_L}(\pi,\Omega)$ and $\mathrm{M_U}(\pi,\Omega)$ with the multiplicity numbers obtained by previous authors. With a view to applications to group C^*-algebras, Fell ([10]), Perdrizet ([18]), Milicic ([16]) and Ludwig ([15]) have variously considered situations in which

$$\mathrm{Tr}(\pi_\alpha(a)) \to \sum_{\pi\in L(\Omega)} m_\pi \mathrm{Tr}(\pi(a)) < \infty$$

for all positive elements a in a dense self-adjoint subalgebra of a C^*-algebra A, where $\Omega = (\pi_\alpha)_{\alpha\in\Lambda}$ is a convergent net in $\widehat{A}$ with limit set $L(\Omega)$ (necessarily equal to the set of cluster points of Ω in this context) and m_π is a positive integer for each $\pi \in L(\Omega)$. It necessarily follows that

$$m_\pi = \mathrm{M_L}(\pi,\Omega) = \mathrm{M_U}(\pi,\Omega) \qquad (\pi \in L(\Omega)).$$

We shall defer the details of the proof to a subsequent paper ([6]).

2. MULTIPLICITIES AND PURE STATES

Let A be a C^*-algebra and let π be an irreducible representation of A. We shall denote by H_π the Hilbert space on which $\pi(A)$ acts and we shall adopt the common practice of using the same symbol π to denote the unitary equivalence class in $\widehat{A}$ of the irreducible representation. Thus if π_1 and π_2 are equivalent irreducible representations ($\pi_1 \simeq \pi_2$) then $\pi_1 = \pi_2$ in $\widehat{A}$. Note that, in this situation, $\mathrm{Tr}(\pi_1(a)) = \mathrm{Tr}(\pi_2(a))$ for all $a \in A^+$ and so we may write unambiguously the expression $\mathrm{Tr}(\pi(a))$ whenever $\pi \in \widehat{A}$ and $a \in A^+$.

Unless stated otherwise, we shall always regard (subsets of) A^* as being equipped with the weak*-topology. We denote by $\mathcal{N}$ the weak*-neighbourhood base at zero in the Banach dual A^* consisting of all open sets of the form

$$N = \{\psi \in A^* : |\psi(a_i)| < \varepsilon, 1 \leqslant i \leqslant n\}$$

where $\varepsilon > 0$ and $a_1, a_2, \ldots, a_n \in A$. Let $P(A)$ be the set of pure states of A and let $\theta : P(A) \to \widehat{A}$ be the continuous, open mapping given by $\theta(\varphi) = \pi_\varphi$ where π_φ is (the equivalence class of) the GNS representation associated with φ ([8], 3.4.11).

Let $\pi \in \widehat{A}$. We begin by recalling descriptions of the upper and lower multiplicities $\mathrm{M_U}(\pi)$ and $\mathrm{M_L}(\pi)$. Let φ be a pure state of A associated with π and let $N \in \mathcal{N}$. Let

$$V(\varphi, N) = \theta((\varphi + N) \cap P(A)),$$

an open neighbourhood of π in $\widehat{A}$. For $\sigma \in \widehat{A}$ let

$$\mathrm{Vec}(\sigma, \varphi, N) = \{\eta \in H_\sigma : \| \eta \| = 1, \langle \sigma(\cdot)\eta, \eta\rangle \in \varphi + N\}.$$

Note that $\mathrm{Vec}(\sigma, \varphi, N)$ is nonempty if and only if $\sigma \in V(\varphi, N)$. For $\sigma \in V(\varphi, N)$ we define $d(\sigma, \varphi, N)$ to be the supremum (in $\mathbf{P} \cup \{\infty\}$) of the cardinalities of finite orthonormal subsets of $\mathrm{Vec}(\sigma, \varphi, N)$. It is convenient to define $d(\sigma, \varphi, N) = 0$ for $\sigma \in \widehat{A} \setminus V(\varphi, N)$.

From [3], Section 2 and Proposition 3.4, we have

$$\mathrm{M_U}(\pi) = \inf_{N \in \mathcal{N}} \Big(\limsup_{\sigma \to \pi} d(\sigma, \varphi, N) \Big) \in \mathbf{P} \cup \{\infty\}$$

and, if π is not open in $\widehat{A}$,

$$\mathrm{M_L}(\pi) = \inf_{N \in \mathcal{N}} \Big(\liminf_{\sigma \to \pi, \sigma \neq \pi} d(\sigma, \varphi, N) \Big) \in \mathbf{P} \cup \{\infty\}.$$

As noted in [3], Lemma 2.1, $\mathrm{M_U}(\pi)$ and $\mathrm{M_L}(\pi)$ are independent of the choice of φ. Examples which motivate these definitions and illustrate the computations are given in [3], Section 2.

Now suppose, in addition, that $\Omega = (\pi_\alpha)_{\alpha\in\Lambda}$ is a net in $\widehat{A}$. For $N \in \mathcal{N}$ let

$$\mathrm{M_U}(\varphi, N, \Omega) = \limsup_{\alpha} d(\pi_\alpha, \varphi, N) \in \mathbf{N} \cup \{\infty\}.$$

Note that if $N' \in \mathcal{N}$ and $N' \subseteq N$ then $\mathrm{M_U}(\varphi, N', \Omega) \leqslant \mathrm{M_U}(\varphi, N, \Omega)$. We define

$$\mathrm{M_U}(\pi, \Omega) = \inf_{N\in\mathcal{N}} \mathrm{M_U}(\varphi, N, \Omega) \in \mathbf{N} \cup \{\infty\}$$

(which is independent of the choice of φ by an argument similar to that used in the proof of [3], Lemma 2.1). Similarly, for $N \in \mathcal{N}$, let

$$\mathrm{M_L}(\varphi, N, \Omega) = \liminf_{\alpha} d(\pi_\alpha, \varphi, N) \in \mathbf{N} \cup \{\infty\}.$$

Then $\mathrm{M_L}(\varphi, N, \Omega)$ decreases with N and we define

$$\mathrm{M_L}(\pi, \Omega) = \inf_{N\in\mathcal{N}} \mathrm{M_L}(\varphi, N, \Omega) \in \mathbf{N} \cup \{\infty\}$$

(which is, again, independent of the choice of φ). Note that it is not required that Ω converge to π. However it follows from these definitions that $\mathrm{M_U}(\pi, \Omega) > 0$ if and only if π is a cluster point of Ω, and that $\mathrm{M_L}(\pi, \Omega) > 0$ if and only if Ω converges to π.

We remark that in the definition of $\mathrm{M_L}(\pi, \Omega)$ we have not required that eventually $\pi_\alpha \neq \pi$, as might be expected given the definition of $\mathrm{M_L}(\pi)$ (see, e.g., [3], Remark 2.2). This allows us to state our results concerning lower multiplicity without the burden of requiring that a net eventually be unequal to each of its limits. However this introduces a small peculiarity into the properties of $\mathrm{M_L}(\pi, \Omega)$ which we describe below.

Routine topological arguments (which do not require the use of the definition of $d(\sigma, \varphi, N)$) show that

$$\mathrm{M_L}(\pi, \Omega) \leqslant \mathrm{M_U}(\pi, \Omega) \leqslant \mathrm{M_U}(\pi); \tag{2.1}$$

if eventually $\pi_\alpha \neq \pi$ then

$$\mathrm{M_L}(\pi) \leqslant \mathrm{M_L}(\pi, \Omega); \tag{2.2}$$

if Ω_0 is a subnet of Ω then

$$\mathrm{M_L}(\pi, \Omega) \leqslant \mathrm{M_L}(\pi, \Omega_0) \leqslant \mathrm{M_U}(\pi, \Omega_0) \leqslant \mathrm{M_U}(\pi, \Omega). \tag{2.3}$$

However, in certain situations the inequality (2.2) fails. These are specified in the following result.

PROPOSITION 2.1. *Let $\pi \in \widehat{A}$, and let $\Omega = (\pi_\alpha)_{\alpha\in\Lambda}$ be a net in $\widehat{A}$ converging to π. Assume that $\{\pi\}$ is not open in $\widehat{A}$. Then $\mathrm{M_L}(\pi) > \mathrm{M_L}(\pi, \Omega)$ (that is, inequality (2.2) fails) if and only if the following conditions all hold:*

(i) *$\pi_\alpha = \pi$ frequently;*

(ii) *$\pi(A)$ contains the compact linear operators on H_π;*

(iii) *$\mathrm{M_L}(\pi) > 1$.*

In this case, $\mathrm{M_L}(\pi, \Omega) = 1$.

Proof. We first assume that conditions (i), (ii) and (iii) hold. Let ξ be a unit vector in H_π and put $\varphi = \langle \pi(\cdot)\xi, \xi\rangle$. By (ii) there exists $a \in A$ such that $\pi(a)$ is the rank one projection onto $\mathbb{C}\xi$. Let $N = \{\psi \in A^* \mid |\psi(a)| < 1/4\}$. If $\eta \in \mathrm{Vec}(\pi, \varphi, N)$ then $\langle \pi(\cdot)\eta, \eta\rangle \in \varphi + N$, and hence $1 - |\langle \eta, \xi\rangle|^2 < 1/4$. Then, replacing η by $e^{i\theta}\eta$ for suitable θ, we have

$$\|\eta - \xi\|^2 = 2(1 - |\langle \eta, \xi\rangle|) \leqslant 2(1 - |\langle \eta, \xi\rangle|^2) < \frac{1}{2}.$$

If $\eta_1, \eta_2 \in \mathrm{Vec}(\pi, \varphi, N)$, we make the above phase change for each, and find that

$$\|\eta_1 - \eta_2\| \leqslant \|\eta_1 - \xi\| + \|\eta_2 - \xi\| < \sqrt{2}.$$

It follows that η_1 and η_2 are not orthogonal. Hence $d(\pi, \varphi, N) = 1$. By (i) it follows that $\mathrm{M_L}(\pi, \Omega) = 1$, which is less than $\mathrm{M_L}(\pi)$ by (iii).

Conversely, we will show that $\mathrm{M_L}(\pi) \leqslant \mathrm{M_L}(\pi, \Omega)$ if any one of (i), (ii) or (iii) fails. If (i) fails, the desired inequality follows from (2.2). If (iii) fails then $\mathrm{M_L}(\pi) = 1 \leqslant \mathrm{M_L}(\pi, \Omega)$, since Ω converges to π. Finally suppose that (ii) fails. Let φ be a pure state of A associated with π. Let $k \in \mathbf{P}$ and $N \in \mathcal{N}$, and set $N_1 = \frac{1}{2}N$. From the proof of [3], Theorem 4.4 (i) it follows that $d(\pi, \varphi, N_1) = \infty$, so there exists an orthonormal set $\{\xi_1, \ldots, \xi_k\}$ in H_π with $\langle \pi(\cdot)\xi_i, \xi_i\rangle \in \varphi + N_1$ for $1 \leqslant i \leqslant k$. By [3], Lemma 3.1 (applied to N_1 with $n = 1$, $\pi_1 = \pi$, $m_1 = k$ and $d_1 = 1$) there exists an open neighbourhood W of π such that if $\sigma \in W$ then there exists an orthonormal set $\{\eta_1, \ldots, \eta_k\}$ in H_σ such that

$$\langle \sigma(\cdot)\eta_i, \eta_i\rangle \in \langle \pi(\cdot)\xi_i, \xi_i\rangle + N_1 \quad (1 \leqslant i \leqslant k).$$

Let $\beta \in \Lambda$ such that $\alpha \geqslant \beta \Rightarrow \pi_\alpha \in W$. Fix $\alpha \geqslant \beta$. Let $\{\eta_1, \ldots, \eta_k\} \in H_{\pi_\alpha}$ as above. Then

$$\langle \pi_\alpha(\cdot)\eta_i, \eta_i\rangle \in \varphi + N_1 + N_1 = \varphi + N \quad (1 \leqslant i \leqslant k)$$

and so $d(\pi_\alpha, \varphi, N) \geqslant k$. Then

$$\inf_{\alpha\geqslant\beta} d(\pi_\alpha, \varphi, N) \geqslant k,$$

and hence

$$\mathrm{M_L}(\varphi, N, \Omega) \geqslant k.$$

Since $k \in \mathbf{P}$ and $N \in \mathcal{N}$ were arbitrary, it follows that $\mathrm{M_L}(\pi, \Omega) = \infty$. (We note that in this case $\mathrm{M_L}(\pi, \Omega) = \mathrm{M_L}(\pi) = \infty$ by [3], Theorem 4.4 (ii).) ∎

PROPOSITION 2.2. *Let A be a C^*-algebra, let $\pi \in \widehat{A}$, and assume that $\{\pi\}$ is not open in $\widehat{A}$. Then there is a net Ω in $\widehat{A} \setminus \{\pi\}$ converging to π such that*

$$\mathrm{M_U}(\pi) = \mathrm{M_U}(\pi, \Omega),$$
$$\mathrm{M_L}(\pi) = \mathrm{M_L}(\pi, \Omega).$$

Proof. Fix a pure state φ of A associated with π. Let

$$\Lambda = \{(N, \sigma) \in \mathcal{N} \times (\widehat{A} \setminus \{\pi\}) \mid \sigma \in V(\varphi, N)\}$$

with direction given by

$$(N, \sigma) \geqslant (N', \sigma') \Longleftrightarrow N \subseteq N'.$$

For $\alpha = (N, \sigma) \in \Lambda$, let $\pi_\alpha = \sigma$. Then $\Omega = (\pi_\alpha)_{\alpha \in \Lambda}$ is a net in $\widehat{A} \setminus \{\pi\}$. Since $\{V(\varphi, N) \mid N \in \mathcal{N}\}$ is a neighbourhood base at π, and $V(\varphi, N)$ decreases with N, it follows that $\pi_\alpha \to \pi$. For $N \in \mathcal{N}$, note that

$$\begin{aligned}
\limsup_{\alpha} d(\pi_\alpha, \varphi, N) &= \inf_{\beta} \sup_{\alpha \geqslant \beta} d(\pi_\alpha, \varphi, N) \\
&= \inf_{P \in \mathcal{N}} \sup_{\sigma \in V(\varphi, P) \setminus \{\pi\}} d(\sigma, \varphi, N) \\
&= \inf \left\{ \sup_{\sigma \in V \setminus \{\pi\}} d(\sigma, \varphi, N) \mid V \text{ an open neighbourhood of } \pi \right\}
\end{aligned}$$

(since $\{V(\varphi, P) \mid P \in \mathcal{N}\}$ is a neighbourhood base at π)

$$= \limsup_{\sigma \to \pi, \, \sigma \neq \pi} d(\sigma, \varphi, N).$$

Thus it follows that

$$\begin{aligned}
\mathrm{M_U}(\pi) &= \inf_{N} \limsup_{\sigma \to \pi, \, \sigma \neq \pi} d(\sigma, \varphi, N) \quad \text{(cf. [3], 3.4 and 3.5)} \\
&= \inf_{N} \limsup_{\alpha} d(\pi_\alpha, \varphi, N) \\
&= \mathrm{M_U}(\pi, \Omega).
\end{aligned}$$

The proof that $\mathrm{M_L}(\pi) = \mathrm{M_L}(\pi, \Omega)$ is analogous. ∎

PROPOSITION 2.3. *Let A be a C^*-algebra, let $\pi \in \widehat{A}$, and let $\Omega = (\pi_\alpha)_{\alpha\in\Lambda}$ be a net in $\widehat{A}$. Then there exist subnets Ω_1 and Ω_2 of Ω such that*

$$\mathrm{M_L}(\pi, \Omega_1) = \mathrm{M_U}(\pi, \Omega_1) = \mathrm{M_L}(\pi, \Omega), \tag{2.4}$$

$$\mathrm{M_L}(\pi, \Omega_2) = \mathrm{M_U}(\pi, \Omega_2) = \mathrm{M_U}(\pi, \Omega). \tag{2.5}$$

Proof. Let φ be a pure state of A associated with π. To prove (2.4) it suffices to assume that $\mathrm{M_L}(\pi, \Omega) < \infty$; otherwise use $\Omega_1 = \Omega$. Moreover, by (2.3) above, it is enough to show the inequality $\mathrm{M_U}(\pi, \Omega_1) \leqslant \mathrm{M_L}(\pi, \Omega)$. Since $\mathrm{M_L}(\pi, \Omega) = \inf_{N\in\mathcal{N}} \liminf_{\alpha\in\Lambda} d(\pi_\alpha, \varphi, N)$, there is $N_0 \in \mathcal{N}$ such that

$$\liminf_{\alpha\in\Lambda} d(\pi_\alpha, \varphi, N_0) = \mathrm{M_L}(\pi, \Omega).$$

Then there is $\beta_0 \in \Lambda$ such that

$$\beta \geqslant \beta_0 \Longrightarrow \inf_{\alpha\geqslant\beta} d(\pi_\alpha, \varphi, N_0) = \mathrm{M_L}(\pi, \Omega).$$

Since $\mathrm{M_L}(\pi, \Omega)$ is finite, the infimum in the previous line is attained. Let

$$\Lambda_1 = \{\alpha \in \Lambda \mid d(\pi_\alpha, \varphi, N_0) = \mathrm{M_L}(\pi, \Omega)\}.$$

It follows from the above that Λ_1 is a cofinal directed subset of Λ. Define $\Omega_1 = (\pi_\alpha)_{\alpha\in\Lambda_1}$. Then Ω_1 is a subnet of Ω, and

$$\mathrm{M_U}(\pi, \Omega_1) = \inf_{N\subseteq N_0} \limsup_{\alpha\in\Lambda_1} d(\pi_\alpha, \varphi, N).$$

Note that for $N \subseteq N_0$, $d(\pi_\alpha, \varphi, N) \leqslant d(\pi_\alpha, \varphi, N_0)$. Hence

$$\limsup_{\alpha\in\Lambda_1} d(\pi_\alpha, \varphi, N) \leqslant \limsup_{\alpha\in\Lambda_1} d(\pi_\alpha, \varphi, N_0) = \mathrm{M_L}(\pi, \Omega).$$

Taking the infimum over $N \subseteq N_0$, we obtain $\mathrm{M_U}(\pi, \Omega_1) \leqslant \mathrm{M_L}(\pi, \Omega)$.

To prove (2.5) it suffices to show that $\mathrm{M_L}(\pi, \Omega_2) \geqslant \mathrm{M_U}(\pi, \Omega)$. In order to include the possibility that $\mathrm{M_U}(\pi, \Omega) = \infty$, it is convenient to introduce the set $R = \{k \in \mathbb{N} \mid k \leqslant \mathrm{M_U}(\pi, \Omega)\}$. Let $\Lambda_2 = \mathcal{N} \times \Lambda \times R$, with the product direction. For $\mu = (N, \beta, k) \in \Lambda_2$ we have $\limsup_{\alpha\in\Lambda} d(\pi_\alpha, \varphi, N) \geqslant k$. Hence there exists $\alpha(\mu) \in \Lambda$ such that $\alpha(\mu) \geqslant \beta$ and $d(\pi_{\alpha(\mu)}, \varphi, N) \geqslant k$. Note that $\Omega_2 = (\pi_{\alpha(\mu)})_{\mu\in\Lambda_2}$ is a subnet of Ω. (For, if $\beta_0 \in \Lambda$ let $\mu_0 = (N_0, \beta_0, k_0)$, with N_0 and k_0 chosen arbitrarily. Then $\mu = (N, \beta, k) \geqslant \mu_0$ implies that $\alpha(\mu) \geqslant \beta \geqslant \beta_0$.)

Now fix $\mu_0 = (N_0, \beta_0, k_0) \in \Lambda_2$. For $\mu = (N, \beta, k) \geqslant \mu_0$ we have

$$d(\pi_{\alpha(\mu)}, \varphi, N_0) \geqslant d(\pi_{\alpha(\mu)}, \varphi, N) \geqslant k \geqslant k_0.$$

Therefore $\liminf_{\mu\in\Lambda_2} d(\pi_{\alpha(\mu)}, \varphi, N_0) \geqslant k_0$. This is true for each $k_0 \in R$, so

$$\liminf_{\mu\in\Lambda_2} d(\pi_{\alpha(\mu)}, \varphi, N_0) \geqslant \mathrm{M_U}(\pi, \Omega).$$

Since this is true for each $N_0 \in \mathcal{N}$, it follows that $\mathrm{M_L}(\pi, \Omega_2) \geqslant \mathrm{M_U}(\pi, \Omega)$. ∎

Suppose that $\pi \in \widehat{A}$ and that $\{\pi\}$ is not open in $\widehat{A}$. We define $R(\pi)$ to be the set of those numbers $k \in \mathbf{P} \cup \{\infty\}$ for which there exists a net Ω in $\widehat{A} \setminus \{\pi\}$ such that Ω is convergent to π and $k = \mathrm{M_L}(\pi, \Omega) = \mathrm{M_U}(\pi, \Omega)$. It follows from (2.1) and (2.2) above that

$$\mathrm{M_L}(\pi) \leqslant k \leqslant \mathrm{M_U}(\pi) \quad (k \in R(\pi))$$

and from Propositions 2.2 and 2.3 that

$$\{\mathrm{M_L}(\pi), \mathrm{M_U}(\pi)\} \subseteq R(\pi).$$

Given that $\mathrm{M_L}(\pi) < \mathrm{M_U}(\pi)$, elementary examples of the type considered in [3], Section 2, show that one might have $R(\pi) = \{\mathrm{M_L}(\pi), \mathrm{M_U}(\pi)\}$ or, at the other extreme,

$$R(\pi) = \{k \in \mathbf{P} \cup \{\infty\} \mid \mathrm{M_L}(\pi) \leqslant k \leqslant \mathrm{M_U}(\pi)\}.$$

The following result is an immediate consequence of Proposition 2.3.

COROLLARY 2.4. *Let A be a C^*-algebra, let $\pi \in \widehat{A}$ and suppose that $\{\pi\}$ is not open in $\widehat{A}$. Let $k \in \mathbf{P} \cup \{\infty\}$. The following conditions are equivalent:*

(i) $k \in R(\pi)$,

(ii) *there exists a net Ω in $\widehat{A} \setminus \{\pi\}$ which is convergent to π such that either $k = \mathrm{M_L}(\pi, \Omega)$ or $k = \mathrm{M_U}(\pi, \Omega)$.*

Most of Section 3 below is devoted to the proof of Lemma 3.5, which is crucial for the main results in Section 4. The bulk of the proof of this lemma is devoted to the approximation of a representation of the form $\rho = \left(\bigoplus_{\pi \in L} m_\pi \cdot \pi\right) \oplus 0$ by an essentially irreducible representation σ. In order to achieve this we need to reproduce approximately (in the essential space for σ) the invariant subspace structure for ρ. We require, in particular, that for all $a \in A$, $\langle \sigma(a)\xi, \eta \rangle \approx 0$ when either

(i) ξ and η mimic vectors in two different copies of H_π for some $\pi \in L$

or

(ii) ξ mimics a vector from a copy of some H_π and η mimics a vector from a copy of some $H_{\pi'}$ where $\pi' \in L$ and $\pi' \neq \pi$ in $\widehat{A}$.

To deal with these two cases, we require the following lemmas concerning pure states. The first enables us to deal with case (i) above, whilst the second relates to case (ii).

LEMMA 2.5. *Let A be a C^*-algebra and let $\varphi \in P(A)$. Let (π_α) be a net of (not necessarily irreducible) representations of A. Let H_α be the Hilbert space of π_α, let $\{\xi_\alpha, \eta_\alpha\}$ be an orthonormal set in H_α, and let $\varphi_\alpha = \langle \pi_\alpha(\cdot)\xi_\alpha, \xi_\alpha\rangle$, $\psi_\alpha = \langle \pi_\alpha(\cdot)\eta_\alpha, \eta_\alpha\rangle$. Suppose that $\varphi_\alpha \to \varphi$ and $\psi_\alpha \to \varphi$. Then $\langle \pi_\alpha(a)\,\xi_\alpha, \eta_\alpha\rangle \to 0$ for each $a \in A$.*

Proof. We show first that $\langle \pi_\alpha(a)\,\xi_\alpha, \eta_\alpha\rangle \to 0$ for all $a \in \ker(\varphi)$. To see this, let $a \in \ker(\varphi)$. Since φ is pure, it follows from [14], Theorem 10.2.8 that $a = b + c$, where $b^*b, cc^* \in \ker(\varphi)$. Then

$$\|\pi_\alpha(b)\xi_\alpha\|^2 = \varphi_\alpha(b^*b) \to \varphi(b^*b) = 0,$$

hence $\langle \pi_\alpha(b)\,\xi_\alpha, \eta_\alpha\rangle \to 0$ by the Cauchy-Schwarz inequality. Moreover,

$$\begin{aligned}\left|\langle \pi_\alpha(c)\xi_\alpha, \eta_\alpha\rangle\right|^2 &\leqslant \|\pi_\alpha(c^*)\eta_\alpha\|^2 \\ &= \psi_\alpha(cc^*) \\ &\to \varphi(cc^*) = 0.\end{aligned}$$

By linearity $\langle \pi_\alpha(a)\,\xi_\alpha, \eta_\alpha\rangle \to 0$. Note that, so far, we have not needed to use the assumption that $\langle \xi_\alpha, \eta_\alpha\rangle = 0$.

Let P_α be the projection from H_α onto the essential space for π_α. Since φ is a state, $\|P_\alpha\xi_\alpha\|^2 \to 1$ and $\|P_\alpha\eta_\alpha\|^2 \to 1$.

Now let $a \in A$ be arbitrary. Suppose first that A is unital. By the first part of the proof, $\langle \pi_\alpha\,(a - \varphi(a)\mathbf{1})\,\xi_\alpha, \eta_\alpha\rangle \to 0$. On the other hand

$$\begin{aligned}|\langle \pi_\alpha(\varphi(a)\mathbf{1})\xi_\alpha, \eta_\alpha\rangle| &= |\varphi(a)|\ |\langle P_\alpha\xi_\alpha - \xi_\alpha, \eta_\alpha\rangle| \\ &\leqslant |\varphi(a)|\,\|P_\alpha\xi_\alpha - \xi_\alpha\|\end{aligned}$$

and $\|P_\alpha\xi_\alpha - \xi_\alpha\| \to 0$ since $\|P_\alpha\xi_\alpha\|^2 \to 1$. Combining these two cases, we obtain that $\langle \pi_\alpha(a)\xi_\alpha, \eta_\alpha\rangle \to 0$.

Now suppose that A is non-unital, and let $\widetilde{A} = A + \mathbf{C}\mathbf{1}$. Let $\widetilde{\varphi}$ be the unique extension of φ to a pure state of $\widetilde{A}$ and let $\widetilde{\pi}_\alpha$ be the unique representation of $\widetilde{A}$ on H_α with essential space $P_\alpha H_\alpha$ which extends π_α. Since $\|P_\alpha\xi_\alpha\|^2 \to 1$, $\langle \widetilde{\pi}_\alpha(\cdot)\xi_\alpha, \xi_\alpha\rangle \to \widetilde{\varphi}$. Similarly, $\langle \widetilde{\pi}_\alpha(\cdot)\eta_\alpha, \eta_\alpha\rangle \to \widetilde{\varphi}$. Hence, by the unital case, $\langle \widetilde{\pi}_\alpha(b)\xi_\alpha, \eta_\alpha\rangle \to 0$ for all $b \in \widetilde{A}$. Taking $b = a$ gives the required result. ∎

The next result is Lemma 2 of [2] and we recall that the proof depends on a continuity property of transition probabilities for pure states that was observed in [4]. This same continuity property may also be used to prove Lemma 2.5 in the case where each π_α is assumed to be irreducible (which holds in our subsequent application).

LEMMA 2.6. *Let A be a C^*-algebra. Let (π_α) be a net of irreducible representations of A, let H_α be the Hilbert space of π_α, let ξ_α, η_α be unit vectors in H_α, and let $\varphi_\alpha = \langle \pi_\alpha(\cdot)\xi_\alpha, \xi_\alpha\rangle$, $\psi_\alpha = \langle \pi_\alpha(\cdot)\eta_\alpha, \eta_\alpha\rangle$. Let φ, ψ be inequivalent pure states of A such that $\varphi_\alpha \to \varphi$ and $\psi_\alpha \to \psi$. Then $\langle \xi_\alpha, \eta_\alpha\rangle \to 0$ and $\langle \pi_\alpha(a)\xi_\alpha, \eta_\alpha\rangle \to 0$ for each $a \in A$.* ∎

In order to approximate a finite set of vectors (in a Hilbert space) which is *almost orthonormal* by a nearby orthonormal set, we shall require the following well-known version of the Gram-Schmidt process.

LEMMA 2.7. *Given a positive integer M and $\delta_0 > 0$, there exists $\delta_1 > 0$ (depending only on M and δ_0) such that if $1 \leqslant m \leqslant M$ and $\xi_1, \xi_2, \ldots, \xi_m$ are vectors in a Hilbert space H satisfying*

$$|\langle \xi_i, \xi_j\rangle - \delta_{(i,j)}| < \delta_1 \quad (1 \leqslant i, j \leqslant m)$$

(where the first δ in the preceding line is a Kronecker delta function) then there exists an orthonormal system $\{\eta_1, \eta_2, \ldots, \eta_m\}$ in H such that

$$\|\eta_i - \xi_i\| < \delta_0 \quad (1 \leqslant i \leqslant m).$$

This can be obtained from the version for unit vectors (see, for example, [5], Lemma 2.1) by an elementary scaling argument. Alternatively, a direct proof can be given by the operator-theoretic method cited in [5].

3. LIMITS OF ESSENTIALLY IRREDUCIBLE REPRESENTATIONS

DEFINITION 3.1. Let A be a C^*-algebra and let H be a Hilbert space. We recall from [12] that H is said to be *large enough for A* if the dimension of H is at least as large as the dimension of each irreducible representation of A.

NOTATION 3.2. If r is a cardinal number, we let r also denote a set of cardinality r.

DEFINITION 3.3. Let $\Omega = (\pi_\alpha)_{\alpha\in\Lambda}$ be a net in $\widehat{A}$, and let ρ be a non-degenerate representation of A.

(1) *Property $P(\Omega, \rho)$* states that for any cardinal number r such that $\ell^2(r) \oplus H_\rho$ is infinite dimensional and large enough for A, there is a subnet $\left(\pi_{\alpha(\mu)}\right)_{\mu\in\Delta}$ of Ω, there is a Hilbert space H, and there are representations σ, $(\sigma_\mu)_{\mu\in\Delta}$ of A on H such that:

(a) $\mathrm{Ess}(\sigma_\mu) \simeq \pi_{\alpha(\mu)}$;
(b) $\mathrm{Ess}(\sigma) \simeq \rho$;

(c) (σ_μ) converges to σ in $\mathrm{Rep}(A,H)$;

(d) $\dim\big(\sigma(A)H\big)^\perp = r$.

(2) *Property* $P'(\Omega,\rho)$ states that there is a subnet $\big(\pi_{\alpha(\mu)}\big)_{\mu\in\Delta}$ of Ω, there is a Hilbert space H, and there are representations σ, $(\sigma_\mu)_{\mu\in\Delta}$ of A on H such that:

(a) $\mathrm{Ess}(\sigma_\mu) \simeq \pi_{\alpha(\mu)}$;

(b') ρ is unitarily equivalent to a subrepresentation of $\mathrm{Ess}(\sigma)$;

(c) (σ_μ) converges to σ in $\mathrm{Rep}(A,H)$.

REMARK 3.4. The significance of the cardinal r in the definition of property $P(\Omega,\rho)$ will be discussed following Corollary 4.8.

In the following result, no assumption is made about the dimension of the Hilbert space H, but we use the notation $\mathrm{Rep}(A,H)$, $\mathrm{Irr}(A,H)$ and Φ in the same way as before (see Section 1), bearing in mind that Φ might not be a surjection. The result generalizes the fact that the map Φ is continuous. It shows also that if either $P(\Omega,\rho)$ or $P'(\Omega,\rho)$ holds then $\Phi(\sigma_\mu)\to\pi$ for any $\pi\in\widehat{A}$ that is unitarily equivalent to a subrepresentation of ρ (or σ). The proof uses a standard line of argument.

LEMMA 3.5. *Let A be a C^*-algebra, let H be a Hilbert space, and suppose that $(\sigma_\alpha)_{\alpha\in\Lambda}$ is a net in $\mathrm{Irr}(A,H)$ which is convergent to $\rho\in\mathrm{Rep}(A,H)$. Then $\Phi(\sigma_\alpha)\to\pi$ for any $\pi\in\widehat{A}$ which is unitarily equivalent to a subrepresentation of ρ.*

Proof. If such exists, let π_0 be an element of $\widehat{A}$ which is unitarily equivalent to a subrepresentation of ρ. Let V be an open neighbourhood of π_0 in $\widehat{A}$. Then there exists a closed two-sided ideal J of A such that

$$V = \big\{\pi\in\widehat{A} \mid \pi(J)\neq\{0\}\big\}.$$

Since $\pi_0(J)\neq\{0\}$ it follows that $\rho(J)\neq\{0\}$, and hence that eventually $\sigma_\alpha(J)\neq\{0\}$. Thus eventually $\Phi(\sigma_\alpha)\in V$, as required. ∎

In the next lemma, and in several later results as well, it will be necessary to make a choice of an infinite cardinal in association with a representation whose (upper or lower) multiplicity is infinite. In view of this we make the following definition.

DEFINITION 3.6. Let $m\in\mathbb{N}\cup\{\infty\}$ be fixed. By a *cardinal realization* for m we mean m itself if $m\in\mathbb{N}$, and we mean a choice of infinite cardinal if $m=\infty$. Suppose that A is a C^*-algebra, that F is a non-empty subset of $\widehat{A}$ and that, for each $\pi\in F$, $m_\pi\in\mathbb{N}\cup\{\infty\}$. By a *cardinal realization* for $\{m_\pi \mid \pi\in F\}$ we mean an assignment of a cardinal realization for each m_π (more formally, a function f

from F into the cardinal numbers such that $f(\pi)$ is a cardinal realization for m_π for each $\pi \in F$).

The next lemma is the key technical requirement for the main results in Section 4. By isolating the lemma in this way, we have avoided the need to repeat broadly similar arguments at various stages of the proofs of Theorems 4.1 and 4.2.

LEMMA 3.7. *Let $\Omega = (\pi_\alpha)_{\alpha \in \Lambda}$ be a net in $\widehat{A}$, and let F be a non-empty subset of $\widehat{A}$. For each $\pi \in F$ let $m_\pi \in \mathbf{P} \cup \{\infty\}$. The following are equivalent:*

(i) *For each finite non-empty subset $F_0 \subseteq F$, for each collection $\{\varphi_\pi \mid \pi \in F_0\}$ of pure states of A with φ_π associated to π, for each collection $\{n_\pi \mid \pi \in F_0\}$ of positive integers with $n_\pi \leqslant m_\pi$, for each $N \in \mathcal{N}$, and for each $\alpha \in \Lambda$, there exists $\beta \in \Lambda$ with $\beta \geqslant \alpha$ such that $d(\pi_\beta, \varphi_\pi, N) \geqslant n_\pi$ for all $\pi \in F_0$.*

(ii) *For every cardinal realization for $\{m_\pi \mid \pi \in F\}$, $P\Big(\Omega, \bigoplus\limits_{\pi \in F} m_\pi \cdot \pi\Big)$ holds.*

(iii) *For some cardinal realization for $\{m_\pi \mid \pi \in F\}$, $P'\Big(\Omega, \bigoplus\limits_{\pi \in F} m_\pi \cdot \pi\Big)$ holds.*

Proof. (i) $\Rightarrow$ (ii): By adjoining an identity, if necessary, we may assume that A is unital (note that if A is non-unital and if $\varphi \in P(A)$ is associated with $\pi \in \widehat{A}$, then the unique pure extension $\tilde{\varphi}$ of φ to $A + \mathbf{C}1$ is associated with the canonical image of π in the spectrum of $A + \mathbf{C}1$). Let $K_\pi = \ell^2(m_\pi)$, with orthonormal basis $\{\delta_p \mid p \in m_\pi\}$, and let $H = \Big(\bigoplus\limits_{\pi \in F} (H_\pi \otimes K_\pi)\Big) \oplus \ell^2(r)$ where r is any cardinal number such that H is infinite dimensional and large enough for A. For each $\pi \in F$, let B_π be a fixed orthonormal basis of H_π. For $\xi \in B_\pi$ and $p \in m_\pi$ let $\xi^p = \xi \otimes \delta_p$ in $H_\pi \otimes K_\pi$. Let $\widetilde{B}_\pi = \{\xi^p \mid \xi \in B_\pi, p \in m_\pi\}$. Let $B = \bigcup\limits_{\pi \in F} \widetilde{B}_\pi$, and let C be a fixed orthonormal basis for $\ell^2(r)$. Then $B \cup C$ is an orthonormal basis for H.

Let $\mathcal{E}$ be the set of all non-empty finite subsets of $B \cup C$, and let $\mathcal{S}$ be the set of all finite subsets of A containing the identity element. Let $\Delta = \Lambda \times \mathcal{S} \times \mathbf{R}^+ \times \mathcal{E}$, directed by defining:

$$(\alpha, S, \varepsilon, E) \geqslant (\alpha', S', \varepsilon', E') \iff \alpha \geqslant \alpha', S \supseteq S', \varepsilon \leqslant \varepsilon', E \supseteq E'.$$

Let $\mu = (\alpha, S, \varepsilon, E) \in \Delta$ be fixed. Since E is finite, we can find a finite non-empty subset F_0 of F, and for each $\pi \in F_0$ a positive integer $n_\pi \leqslant m_\pi$ and a finite orthonormal set $\{\xi_{\pi,1}, \ldots, \xi_{\pi,t_\pi}\}$ in H_π such that

$$E \cap B \subseteq \bigcup_{\pi \in F_0} \{\xi^p_{\pi,i} \mid 1 \leqslant i \leqslant t_\pi,\ 1 \leqslant p \leqslant n_\pi\}.$$

Define $\varphi_\pi = \langle \pi(\cdot)\xi_{\pi,1}, \xi_{\pi,1}\rangle \in P(A)$ for each $\pi \in F_0$. By Kadison's transitivity theorem there exist unitary operators $u_{\pi,i}$ in A such that

$$\pi(u_{\pi,i})\xi_{\pi,1} = \xi_{\pi,i} \quad (\pi \in F_0,\ 1 \leqslant i \leqslant t_\pi),$$

and we may assume that $u_{\pi,1} = 1$ for all $\pi \in F_0$. Let $c = \max\{\|a\| \mid a \in S\} \geqslant 1$, let $\delta_0 = \varepsilon/4c$, and let $M = \sum_{\pi \in F_0} n_\pi t_\pi$. Then let δ_1 be obtained from δ_0 and M as in Lemma 2.7. Let $\delta_2 = \min\{\delta_1, \varepsilon/2\}$.

Now let $\Gamma = \mathcal{S} \times \mathbf{R}^+$ be directed by defining:

$$(T, h) \geqslant (T', h') \Longleftrightarrow T \supseteq T',\ h \leqslant h'.$$

For $\gamma = (T, h) \in \Gamma$, define $N_\gamma = \{\psi \in A^* \mid |\psi(a)| < h \quad (a \in T)\}$. Note that $\{N_\gamma \mid \gamma \in \Gamma\}$ forms an open neighbourhood base at zero in A^*. Let

$$L_\gamma = \{\psi \in A^* \mid |\psi(u_{\pi,j}^* a u_{\pi,i})| < h \quad (\pi \in F_0,\ a \in T,\ 1 \leqslant i, j \leqslant t_\pi)\}.$$

Since $u_{\pi,1} = 1$, $L_\gamma \subseteq N_\gamma$.

Assuming (i), there exists $\beta(\gamma) \in \Lambda$ with $\beta(\gamma) \geqslant \alpha$ such that

$$d(\pi_{\beta(\gamma)}, \varphi_\pi, L_\gamma) \geqslant n_\pi \quad (\pi \in F_0).$$

Denote $\pi_{\beta(\gamma)}$ by ρ_γ and let H_γ be the Hilbert space on which ρ_γ acts.

For each $\pi \in F_0$ there exists in H_γ an orthonormal set $\{\eta_{\pi,\gamma,1}, \ldots, \eta_{\pi,\gamma,n_\pi}\}$ such that

$$\langle \rho_\gamma(\cdot)\eta_{\pi,\gamma,p}, \eta_{\pi,\gamma,p}\rangle \in \varphi_\pi + L_\gamma \subseteq \varphi_\pi + N_\gamma \quad (1 \leqslant p \leqslant n_\pi). \tag{3.1}$$

Hence

$$\langle \rho_\gamma(u_{\pi,j}^* \cdot u_{\pi,i})\eta_{\pi,\gamma,p}, \eta_{\pi,\gamma,p}\rangle \in \varphi_\pi(u_{\pi,j}^* \cdot u_{\pi,i}) + N_\gamma \tag{3.2}$$

for $\pi \in F_0$, $1 \leqslant i, j \leqslant t_\pi$, and $1 \leqslant p \leqslant n_\pi$. From (3.1),

$$\lim_\gamma \langle \rho_\gamma(\cdot)\eta_{\pi,\gamma,p}, \eta_{\pi,\gamma,p}\rangle = \varphi_\pi \quad (\pi \in F_0,\ 1 \leqslant p \leqslant n_\pi)$$

and so it follows from Lemma 2.5 that

$$\lim_\gamma \langle \rho_\gamma(\cdot)\eta_{\pi,\gamma,p}, \eta_{\pi,\gamma,q}\rangle = 0 \quad \text{in } A^* \tag{3.3}$$

whenever $\pi \in F_0$, $1 \leqslant p, q \leqslant n_\pi$, and $p \neq q$. Also, from (3.2),

$$\lim_\gamma \langle \rho_\gamma(u_{\pi,j}^* \cdot u_{\pi,i})\eta_{\pi,\gamma,p}, \eta_{\pi,\gamma,p}\rangle = \varphi_\pi(u_{\pi,j}^* \cdot u_{\pi,i}) \tag{3.4}$$

for $\pi \in F_0$, $1 \leqslant i, j \leqslant t_\pi$, and $1 \leqslant p \leqslant n_\pi$. Since F_0 and S are finite, it follows from (3.3) and (3.4) that there exists $\gamma_0 \in \Gamma$ such that for $\gamma \geqslant \gamma_0$

$$\left|\langle \rho_\gamma(u_{\pi,j}^* a u_{\pi,i})\eta_{\pi,\gamma,p}, \eta_{\pi,\gamma,q}\rangle\right| < \delta_2 \tag{3.5}$$

for all $a \in S$, $\pi \in F_0$, $1 \leqslant i, j \leqslant t_\pi$, and $1 \leqslant p, q \leqslant n_\pi$ with $p \neq q$, and

$$\left|\langle \rho_\gamma(u_{\pi,j}^* a u_{\pi,i})\eta_{\pi,\gamma,p}, \eta_{\pi,\gamma,p}\rangle - \varphi_\pi(u_{\pi,j}^* a u_{\pi,i})\right| < \delta_2 \tag{3.6}$$

for all $a \in S$, $\pi \in F_0$, $1 \leqslant i, j \leqslant t_\pi$, and $1 \leqslant p \leqslant n_\pi$.

Suppose that π and π' are distinct elements in F_0 and that $1 \leqslant i \leqslant t_\pi$ and $1 \leqslant j \leqslant t_{\pi'}$. From (3.4), for $1 \leqslant p \leqslant n_\pi$ and $1 \leqslant q \leqslant n_{\pi'}$ we have

$$\lim_\gamma \langle \rho_\gamma(u_{\pi,i}^* \cdot u_{\pi,i})\eta_{\pi,\gamma,p}, \eta_{\pi,\gamma,p}\rangle = \varphi_\pi(u_{\pi,i}^* \cdot u_{\pi,i}) \tag{3.7}$$

and

$$\lim_\gamma \langle \rho_\gamma(u_{\pi',j}^* \cdot u_{\pi',j})\eta_{\pi',\gamma,q}, \eta_{\pi',\gamma,q}\rangle = \varphi_{\pi'}(u_{\pi',j}^* \cdot u_{\pi',j}). \tag{3.8}$$

Since the limits in (3.7) and (3.8) are inequivalent pure states, it follows from Lemma 2.6 that

$$\lim_\gamma \langle \rho_\gamma(a)\rho_\gamma(u_{\pi,i})\eta_{\pi,\gamma,p}, \rho_\gamma(u_{\pi',j})\eta_{\pi',\gamma,q}\rangle = 0$$

for all $a \in A$. Since F_0 and S are finite there exists $\gamma(\mu) \geqslant \gamma_0$ such that

$$\left|\langle \rho_{\gamma(\mu)}(a)\rho_{\gamma(\mu)}(u_{\pi,i})\eta_{\pi,\gamma(\mu),p}\ ,\ \rho_{\gamma(\mu)}(u_{\pi',j})\eta_{\pi',\gamma(\mu),q}\rangle\right| < \delta_2 \tag{3.9}$$

whenever $a \in S$, π and π' are distinct elements of F_0, $1 \leqslant i \leqslant t_\pi$, $1 \leqslant j \leqslant t_{\pi'}$, $1 \leqslant p \leqslant n_\pi$, and $1 \leqslant q \leqslant n_{\pi'}$. In a minor abuse of notation we let π_μ denote $\pi_{\beta(\gamma(\mu))} = \rho_{\gamma(\mu)}$, and $\eta_{\pi,\mu,p}$ denote $\eta_{\pi,\gamma(\mu),p}$, for $\pi \in F_0$ and $1 \leqslant p \leqslant n_\pi$.

Let H_μ be the Hilbert space for π_μ. For each $\pi \in F_0$ we have the following $n_\pi \times t_\pi$ array of unit vectors in H_μ:

$$\begin{array}{l} \pi_\mu(u_{\pi,1})\eta_{\pi,\mu,1}, \ldots, \pi_\mu(u_{\pi,t_\pi})\eta_{\pi,\mu,1} \\ \cdots \\ \cdots \\ \cdots \\ \pi_\mu(u_{\pi,1})\eta_{\pi,\mu,n_\pi}, \ldots, \pi_\mu(u_{\pi,t_\pi})\eta_{\pi,\mu,n_\pi}. \end{array}$$

From (3.5) we have

$$\left|\langle \pi_\mu(u_{\pi,j}^* a u_{\pi,i})\eta_{\pi,\mu,p}, \eta_{\pi,\mu,q}\rangle\right| < \delta_2 \tag{3.10}$$

for all $a \in S$, $\pi \in F_0$, $1 \leqslant i,j \leqslant t_\pi$, and $1 \leqslant p,q \leqslant n_\pi$ with $p \neq q$. In particular, taking $a = 1$,

$$\left|\langle \pi_\mu(u_{\pi,i})\eta_{\pi,\mu,p}, \pi_\mu(u_{\pi,j})\eta_{\pi,\mu,q}\rangle\right| < \delta_2 \tag{3.11}$$

for all $\pi \in F_0$, $1 \leqslant i,j \leqslant t_\pi$, and $1 \leqslant p,q \leqslant n_\pi$ with $p \neq q$. (So any two vectors u and v in different rows of the same array satisfy $|\langle u, v\rangle| < \delta_2$.)

From (3.6) we have

$$\left|\langle \pi_\mu(u_{\pi,j}^* a u_{\pi,i})\eta_{\pi,\mu,p}, \eta_{\pi,\mu,p}\rangle - \varphi_\pi(u_{\pi,j}^* a u_{\pi,i})\right| < \delta_2 \tag{3.12}$$

for all $a \in S$, $\pi \in F_0$, $1 \leqslant i,j \leqslant t_\pi$, and $1 \leqslant p \leqslant n_\pi$. In particular, taking $a = 1$ and noting that

$$\begin{aligned}\varphi_\pi(u_{\pi,j}^* u_{\pi,i}) &= \langle \pi(u_{\pi,i})\xi_{\pi,1}, \pi(u_{\pi,j})\xi_{\pi,1}\rangle \\ &= \langle \xi_{\pi,i}, \xi_{\pi,j}\rangle \\ &= \delta_{(i,j)} \quad \text{(Kronecker delta)}\end{aligned}$$

we have that

$$\left|\langle \pi_\mu(u_{\pi,i})\eta_{\pi,\mu,p}, \pi_\mu(u_{\pi,j})\eta_{\pi,\mu,p}\rangle\right| < \delta_2 \tag{3.13}$$

for all $a \in S$, $\pi \in F_0$, $1 \leqslant p \leqslant n_\pi$, and $1 \leqslant i,j \leqslant t_\pi$ with $i \neq j$. (So any two vectors u and v occupying different positions in the same row of the same array satisfy $|\langle u, v\rangle| < \delta_2$.)

From (3.9) we have

$$\left|\langle \pi_\mu(a)\pi_\mu(u_{\pi,i})\eta_{\pi,\mu,p}, \pi_\mu(u_{\pi',j})\eta_{\pi',\mu,q}\rangle\right| < \delta_2 \tag{3.14}$$

whenever $a \in S$, π and π' are distinct elements of F_0, $1 \leqslant i \leqslant t_\pi$, $1 \leqslant j \leqslant t_{\pi'}$, $1 \leqslant p \leqslant n_\pi$, and $1 \leqslant q \leqslant n_{\pi'}$. In particular, taking $a = 1$,

$$\left|\langle \pi_\mu(u_{\pi,i})\eta_{\pi,\mu,p}, \pi_\mu(u_{\pi',j})\eta_{\pi',\mu,q}\rangle\right| < \delta_2 \tag{3.15}$$

whenever π and π' are distinct elements of F_0, $1 \leqslant i \leqslant t_\pi$, $1 \leqslant j \leqslant t_{\pi'}$, $1 \leqslant p \leqslant n_\pi$, and $1 \leqslant q \leqslant n_{\pi'}$. (So any two vectors u and v from different arrays satisfy $|\langle u, v\rangle| < \delta_2$.)

Since $\delta_2 \leqslant \delta_1$, it follows from (3.11), (3.13), (3.15) and Lemma 2.7 that there exists in H_μ an orthonormal set

$$\{\xi_{\pi,\mu,i,p} \mid \pi \in F_0,\ 1 \leqslant i \leqslant t_\pi,\ 1 \leqslant p \leqslant n_\pi\}$$

such that $\|\pi_\mu(u_{\pi,i})\eta_{\pi,\mu,p} - \xi_{\pi,\mu,i,p}\| < \delta_0$ for all $\pi \in F_0$, $1 \leqslant i \leqslant t_\pi$, and $1 \leqslant p \leqslant n_\pi$.

From (3.10) we have

$$\left|\langle \pi_\mu(a)\xi_{\pi,\mu,i,p}, \xi_{\pi,\mu,j,q}\rangle\right| < \delta_2 + 2c\delta_0 \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{3.16}$$

for all $a \in S$, $\pi \in F_0$, $1 \leqslant i, j \leqslant t_\pi$, and $1 \leqslant p, q \leqslant n_\pi$ with $p \neq q$.

From (3.12) we have

$$\left|\langle \pi_\mu(a)\xi_{\pi,\mu,i,p}, \xi_{\pi,\mu,j,p}\rangle - \langle \pi(a)\xi_{\pi,i}, \xi_{\pi,j}\rangle\right| < \delta_2 + 2c\delta_0 \leqslant \varepsilon \tag{3.17}$$

for all $a \in S$, $\pi \in F_0$, $1 \leqslant i, j \leqslant t_\pi$, and $1 \leqslant p \leqslant n_\pi$.

From (3.14) we have

$$\left|\langle \pi_\mu(a)\xi_{\pi,\mu,i,p}, \xi_{\pi',\mu,j,q}\rangle\right| < \delta_2 + 2c\delta_0 \leqslant \varepsilon \tag{3.18}$$

whenever $a \in S$, π and π' are distinct elements of F_0, $1 \leqslant i \leqslant t_\pi$, $1 \leqslant j \leqslant t_{\pi'}$, $1 \leqslant p \leqslant n_\pi$, and $1 \leqslant q \leqslant n_{\pi'}$.

We define a linear isometry $V_\mu : H_\mu \to H$ as follows. We let

$$V_\mu(\xi_{\pi,\mu,i,p}) = \xi^p_{\pi,i}$$

for $\pi \in F_0$, $1 \leqslant i \leqslant t_\pi$, and $1 \leqslant p \leqslant n_\pi$, and we let V_μ map

$$\left(\mathrm{Span}\{\xi_{\pi,\mu,i,p} \mid \pi \in F_0,\ 1 \leqslant i \leqslant t_\pi,\ 1 \leqslant p \leqslant n_\pi\}\right)^\perp$$

to any subspace of H which is orthogonal to

$$\mathrm{Span}\left\{E \cup \{\xi^p_{\pi,i} \mid \pi \in F_0,\ 1 \leqslant i \leqslant t_\pi,\ 1 \leqslant p \leqslant n_\pi\}\right\}.$$

Note that V_μ can be defined because H is infinite dimensional and large enough for A. We define a representation σ_μ of A on H by

$$\sigma_\mu(a) = V_\mu \pi_\mu(a) V_\mu^* \quad (a \in A).$$

Then

$$\mathrm{Ess}(\sigma_\mu) = \sigma_\mu \mid V_\mu H_\mu \simeq \pi_\mu.$$

Define

$$\sigma = \Big(\bigoplus_{\pi\in F}(\pi\otimes \mathbf{1}_{K_\pi})\Big)\oplus 0$$

(note that $\pi\otimes\mathbf{1}_{K_\pi} = m_\pi\cdot\pi$). It follows from (3.16) that

$$(3.19)\qquad \big|\langle\sigma_\mu(a)\xi^p_{\pi,i},\xi^q_{\pi,j}\rangle - \langle\sigma(a)\xi^p_{\pi,i},\xi^q_{\pi,j}\rangle\big| = \big|\langle\pi_\mu(a)\xi_{\pi,\mu,i,p},\xi_{\pi,\mu,j,q}\rangle - 0\big| < \varepsilon$$

for all $a\in S$, $\pi\in F_0$, $1\leqslant i,j\leqslant t_\pi$, and $1\leqslant p,q\leqslant n_\pi$ with $p\neq q$. From (3.17) we have

$$(3.20)\qquad \big|\langle\sigma_\mu(a)\xi^p_{\pi,i},\xi^p_{\pi,j}\rangle - \langle\sigma(a)\xi^p_{\pi,i},\xi^p_{\pi,j}\rangle\big| < \varepsilon$$

for all $a\in S$, $\pi\in F_0$, $1\leqslant i,j\leqslant t_\pi$, and $1\leqslant p\leqslant n_\pi$.

From (3.18) we have

$$(3.21)\qquad \big|\langle\sigma_\mu(a)\xi^p_{\pi,i},\xi^q_{\pi',j}\rangle - \langle\sigma(a)\xi^p_{\pi,i},\xi^q_{\pi',j}\rangle\big| = \big|\langle\pi_\mu(a)\xi_{\pi,\mu,i,p},\xi_{\pi',\mu,j,q}\rangle - 0\big| < \varepsilon$$

whenever $a\in S$, π and π' are distinct elements of F_0, $1\leqslant i\leqslant t_\pi$, $1\leqslant j\leqslant t_{\pi'}$, $1\leqslant p\leqslant n_\pi$, and $1\leqslant q\leqslant n_{\pi'}$.

It follows from (3.19), (3.20), (3.21), and the fact that

$$E\cap\mathcal{B}\subseteq\{\xi^p_{\pi,i}\mid \pi\in F_0,\ 1\leqslant i\leqslant t_\pi,\ 1\leqslant p\leqslant n_\pi\},$$

that

$$(3.22)\qquad \big|\langle\sigma_\mu(a)\xi,\eta\rangle - \langle\sigma(a)\xi,\eta\rangle\big| < \varepsilon$$

whenever $a\in S$ and $\xi,\eta\in E\cap\mathcal{B}$.

Now suppose that $\xi\in E\cap\mathcal{C}$ and $\eta\in H$. Then $V^*_\mu(\xi)=0$ and so $\sigma_\mu(a)\xi=0$ for all $a\in A$. Hence $\langle\sigma_\mu(a)\xi,\eta\rangle$, $\langle\sigma_\mu(a)\eta,\xi\rangle$, $\langle\sigma(a)\xi,\eta\rangle$, $\langle\sigma(a)\eta,\xi\rangle$ are zero for all $a\in A$. Combining this with (3.22) we have

$$(3.23)\qquad \big|\langle\sigma_\mu(a)\xi,\eta\rangle - \langle\sigma(a)\xi,\eta\rangle\big| < \varepsilon\quad (a\in S,\ \xi,\eta\in E).$$

Up until now, μ has been fixed in Δ. We now show that σ_μ converges to σ in the strong topology of $\mathrm{Rep}(A,H)$. Let $a\in A$ and let $\xi,\eta\in\mathcal{B}\cup\mathcal{C}$ (the fixed basis for H). Let $\varepsilon>0$. Choose any $\alpha_0\in\Lambda$ and let $S_0=\{\mathbf{1},a\}$ and $E_0=\{\xi,\eta\}$. Let $\mu_0=(\alpha_0,S_0,\varepsilon,E_0)$. Then it follows from (3.23) that, for $\mu\geqslant\mu_0$,

$$\big|\langle\sigma_\mu(a)\xi,\eta\rangle - \langle\sigma(a)\xi,\eta\rangle\big| < \varepsilon.$$

Thus $\lim_\mu\langle\sigma_\mu(a)\xi,\eta\rangle = \langle\sigma(a)\xi,\eta\rangle$.

Since $\|\sigma_\mu(a)\| \leqslant \|a\|$ for all $\mu \in \Delta$, it follows by linearity and continuity that $\sigma_\mu(a)$ converges to $\sigma(a)$ in the weak operator topology. As this holds for any $a \in A$, it follows that $\sigma_\mu(a)$ converges to $\sigma(a)$ in the strong operator topology ([8], 3.5.2).

(ii) $\Rightarrow$ (iii): immediate.

(iii) $\Rightarrow$ (i): Assuming (iii), there exists a subnet $\Omega_1 = (\pi_{\alpha(\mu)})_{\mu\in\Delta}$ of Ω, a Hilbert space H, and representations σ, $(\sigma_\mu)_{\mu\in\Delta}$ of A on H such that

$$\mathrm{Ess}(\sigma_\mu) \simeq \pi_{\alpha(\mu)} \quad (\mu \in \Delta),$$

$\bigoplus_{\pi\in F} m_\pi \cdot \pi$ is unitarily equivalent to a subrepresentation of σ, and

$$\lim_\mu \sigma_\mu(a) = \sigma(a) \quad (a \in A)$$

where the limit is taken in the strong operator topology. We may assume without loss of generality that $H = \left(\bigoplus_{\pi\in F} (H_\pi \otimes K_\pi)\right) \oplus K$ for some Hilbert space K, and that $\sigma = \left(\bigoplus_{\pi\in F} (\pi \otimes \mathbf{1}_{K_\pi})\right) \oplus \sigma_0$ (where the Hilbert spaces $\{K_\pi\}$ are as in the beginning of the proof of (i) $\Rightarrow$ (ii)). Let $F_0 \subseteq F$, $\{\varphi_\pi | \pi \in F_0\} \subseteq P(A)$, $\{n_\pi | \pi \in F_0\} \subseteq \mathbf{P}$, $N \in \mathcal{N}$, and $\alpha_0 \in \Lambda$ be given as in (i). For $\pi \in F_0$ let $\xi_\pi \in H_\pi$ be a unit vector such that $\varphi_\pi = \langle \pi(\cdot)\xi_\pi, \xi_\pi\rangle$. There are $\varepsilon \in \mathbf{R}^+$ and a finite non-empty subset S of A such that $N = \{\psi \in A^* \mid |\psi(a)| < \varepsilon \text{ for } a \in S\}$. Let $c = 1 + \max\{\|a\| \mid a \in S\} \geqslant 1$, let $\delta_0 = \varepsilon/4c$, and let $M = \sum_{\pi\in F_0} n_\pi$. From δ_0, M, and Lemma 2.7 we obtain δ_1.

For $1 \leqslant p \leqslant n_\pi$ let $\xi_\pi^p = \xi_\pi \otimes \delta_p$ in $H_\pi \otimes K_\pi$. Let P_μ be the projection of H onto $\mathrm{Ess}(\sigma_\mu)$. Write $\eta_{\pi,p}^\mu = P_\mu \xi_\pi^p$. For $\pi \in F_0$ and $1 \leqslant p \leqslant n_\pi$, we have

$$\text{(3.24)} \qquad \lim_\mu \langle \sigma_\mu(\cdot)\eta_{\pi,p}^\mu, \eta_{\pi,p}^\mu\rangle = \lim_\mu \langle \sigma_\mu(\cdot)\xi_\pi^p, \xi_\pi^p\rangle = \langle \sigma(\cdot)\xi_\pi^p, \xi_\pi^p\rangle = \varphi_\pi$$

where the limit is taken in the weak*-topology. In particular,

$$\text{(3.25)} \qquad \lim_\mu \|\eta_{\pi,p}^\mu\|^2 = 1 \quad (\pi \in F_0,\ 1 \leqslant p \leqslant n_\pi).$$

We have $1 = \|\xi_\pi^p\|^2 = \|\eta_{\pi,p}^\mu\|^2 + \|(1 - P_\mu)\xi_\pi^p\|^2$, so $\lim_\mu \|P_\mu \xi_\pi^p - \xi_\pi^p\| = 0$, for $\pi \in F_0$ and $1 \leqslant p \leqslant n_\pi$. Hence for $(\pi, p) \neq (\pi', p')$ we have

$$\text{(3.26)} \qquad \lim_\mu \langle \eta_{\pi,p}^\mu, \eta_{\pi',p'}^\mu\rangle = \lim_\mu \langle P_\mu \xi_\pi^p, \xi_{\pi'}^{p'}\rangle = \langle \xi_\pi^p, \xi_{\pi'}^{p'}\rangle = 0.$$

By (3.24), and the fact that Ω_1 is a subnet of Ω, there exists $\mu_0 \in \Delta$ such that

$$(3.27) \qquad \langle\sigma_\mu(\cdot)\eta^\mu_{\pi,p},\eta^\mu_{\pi,p}\rangle \in \varphi_\pi + \frac{1}{2}N \quad (\pi \in F_0,\ 1 \leqslant p \leqslant n_\pi,\ \mu \geqslant \mu_0 \text{ in } \Delta),$$

and such that $\alpha(\mu) \geqslant \alpha_0$ for all $\mu \geqslant \mu_0$.

By (3.25) and (3.26) there is $\mu \geqslant \mu_0$ in Δ such that for $\pi, \pi' \in F_0$, $1 \leqslant p \leqslant n_\pi$, and $1 \leqslant p' \leqslant n_{\pi'}$,

$$\left|\langle\eta^\mu_{\pi,p},\eta^\mu_{\pi',p'}\rangle - \delta_{\left((\pi,p),(\pi',p')\right)}\right| < \delta_1$$

(where the first δ in the preceding line is a Kronecker delta function). From Lemma 2.7 there exists an orthonormal set of vectors $\{\xi^\mu_{\pi,p} \mid \pi \in F_0,\ 1 \leqslant p \leqslant n_\pi\}$ in $P_\mu H$ such that

$$\|\xi^\mu_{\pi,p} - \eta^\mu_{\pi,p}\| < \delta_0 \quad (\pi \in F_0,\ 1 \leqslant p \leqslant n_\pi).$$

For $a \in S$, $\pi \in F_0$, and $1 \leqslant p \leqslant n_\pi$:

$$\left|\langle\sigma_\mu(a)\xi^\mu_{\pi,p},\xi^\mu_{\pi,p}\rangle - \langle\sigma_\mu(a)\eta^\mu_{\pi,p},\eta^\mu_{\pi,p}\rangle\right| < 2c\delta_0 = \frac{\varepsilon}{2}.$$

Hence, using (3.27), we have

$$\langle\sigma_\mu(\cdot)\xi^\mu_{\pi,p},\xi^\mu_{\pi,p}\rangle \in \langle\sigma_\mu(\cdot)\eta^\mu_{\pi,p},\eta^\mu_{\pi,p}\rangle + \frac{1}{2}N \subseteq \varphi_\pi + N.$$

Since $\mathrm{Ess}(\sigma_\mu) \simeq \pi_{\alpha(\mu)}$ it follows that $\pi_{\alpha(\mu)} \in V(\varphi_\pi, N)$, and $d(\pi_{\alpha(\mu)}, \varphi_\pi, N) \geqslant n_\pi$, for $\pi \in F_0$. Since $\mu \geqslant \mu_0$, it follows that $\alpha(\mu) \geqslant \alpha_0$, and so (i) is established. ∎

REMARK 3.8. It will be convenient to extend Definition 3.3 to allow the representation ρ to be the zero representation of A on a zero dimensional Hilbert space. In part (1) we take this to mean that the representation σ is the zero representation on H. In part (2), item (b′) is satisfied for any representation σ. Then, with this convention, $P(\Omega, 0)$ and $P'(\Omega, 0)$ hold for any net Ω in $\widehat{A}$. To see that $P(\Omega, 0)$ (and hence also $P'(\Omega, 0)$) holds, let H be a Hilbert space which is infinite dimensional and large enough for A. Let $\Omega = (\pi_\alpha)_{\alpha\in\Lambda}$ and let $\mathcal{E}$ be the set of all finite subsets of a fixed orthonormal basis for H. Let $\Delta = \Lambda \times \mathcal{E}$ with the product direction. For $\mu = (\beta, E) \in \Delta$ let $\alpha(\mu) = \beta$, and choose $\sigma_\mu \in \mathrm{Irr}(A, H)$ such that $\Phi(\sigma_\mu) = \pi_{\alpha(\mu)}$ and such that the essential space for σ_μ is orthogonal to E.

4. THE MAIN RESULTS

We begin by using Lemma 3.7 to give characterizations of $\mathrm{M_U}(\pi,\Omega)$ and $\mathrm{M_L}(\pi,\Omega)$ in terms of point-strong limits of essentially irreducible representations (Theorems 4.1 and 4.2).

THEOREM 4.1. *Let A be a C^*-algebra and let $\pi \in \widehat{A}$. Let $\Omega = (\pi_\alpha)_{\alpha\in\Lambda}$ be a net in $\widehat{A}$. Let $m \in \mathbf{N}\cup\{\infty\}$. The following are equivalent:*

(i) $\mathrm{M_U}(\pi,\Omega) \geqslant m$.

(ii) *For every cardinal realization for m, $P(\Omega, m\cdot\pi)$ holds.*

(iii) *There exists a cardinal realization for m such that $P'(\Omega, m\cdot\pi)$ holds.*

Proof. Let φ be a pure state of A associated with π. By Remark 3.8 we may assume that $m > 0$.

(i) $\Rightarrow$ (ii): Assuming (i), we proceed to verify condition (i) of Lemma 3.7 with $F = \{\pi\}$. Let $n \in \mathbf{P}$ with $n \leqslant m$, let $N \in \mathcal{N}$, and let $\alpha \in \Lambda$. Since $\mathrm{M_U}(\pi,\Omega) \geqslant m$, we have $\mathrm{M_U}(\varphi, N, \Omega) \geqslant m$. Thus there exists $\beta \geqslant \alpha$ with $d(\pi_\beta, \varphi, N) \geqslant n$. Statement (ii) now follows by Lemma 3.7 ((i) $\Rightarrow$ (ii)).

(ii) $\Rightarrow$ (iii): immediate.

(iii) $\Rightarrow$ (i): Assume (iii). Applying Lemma 3.7 ((iii) $\Rightarrow$ (i)) with $F = \{\pi\}$, we obtain that for each positive integer $n \leqslant m$ and each $N \in \mathcal{N}$, $\limsup_\alpha d(\pi_\alpha, \varphi, N) \geqslant n$. That is, $\mathrm{M_U}(\varphi, N, \Omega) \geqslant n$, and so $\mathrm{M_U}(\pi,\Omega) \geqslant n$. It follows that $\mathrm{M_U}(\pi,\Omega) \geqslant m$. ∎

THEOREM 4.2. *Let A be a C^*-algebra and let $\Omega = (\pi_\alpha)_{\alpha\in\Lambda}$ be a net in $\widehat{A}$. Let F be any subset of $\widehat{A}$. Let $m : F \to \mathbf{N}\cup\{\infty\}$ be any function. The following are equivalent:*

(i) $\mathrm{M_L}(\pi,\Omega) \geqslant m_\pi$ *for each* $\pi \in F$.

(ii) *For every subnet Ω_1 of Ω and for every cardinal realization for $\{m_\pi \mid \pi \in F\}$, $P\Big(\Omega_1, \bigoplus_{\pi\in F} m_\pi\cdot\pi\Big)$ holds.*

(iii) *For every subnet Ω_1 of Ω there exists a cardinal realization for $\{m_\pi \mid \pi \in F\}$ such that $P'\Big(\Omega_1, \bigoplus_{\pi\in F} m_\pi\cdot\pi\Big)$ holds.*

Proof. If $F = \emptyset$ or if m is identically zero, the result follows from Remark 3.8. Therefore we assume that $F \neq \emptyset$ and (without loss of generality) that $m : F \to \mathbf{P}\cup\{\infty\}$. For each $\pi \in F$, let φ_π be a pure state of A associated with π.

(i) $\Rightarrow$ (ii): Let $\Omega_1 = (\pi_{\alpha(\mu)})_{\mu\in\Delta}$ be a subnet of Ω. Assuming (i), we have $\mathrm{M_L}(\pi,\Omega_1) \geqslant \mathrm{M_L}(\pi,\Omega) \geqslant m_\pi$ for each $\pi \in F$. Let $F_0 \subseteq F$ be finite and non-empty.

Let $N \in \mathcal{N}$, let $\mu \in \Delta$, and for each $\pi \in F_0$ let $n_\pi \in \mathbf{P}$ satisfy $n_\pi \leqslant m_\pi$. We have for each $\pi \in F_0$,

$$\liminf_{\nu} d(\pi_{\alpha(\nu)}, \varphi_\pi, N) = \mathrm{M_L}(\varphi_\pi, N, \Omega_1) \geqslant \mathrm{M_L}(\pi, \Omega_1) \geqslant m_\pi.$$

Since F_0 is finite there exists $\nu \geqslant \mu$ such that $d(\pi_{\alpha(\nu)}, \varphi_\pi, N) \geqslant n_\pi$ for all $\pi \in F_0$. This verifies condition (i) of Lemma 3.7, and hence $P\Big(\Omega_1, \bigoplus_{\pi \in F} m_\pi \cdot \pi\Big)$ holds.

(ii) $\Rightarrow$ (iii): immediate.

(iii) $\Rightarrow$ (i): Fix $\pi \in F$ and suppose that $\mathrm{M_L}(\pi, \Omega) = k < m_\pi$. Then there is $N \in \mathcal{N}$ such that $\mathrm{M_L}(\varphi_\pi, N, \Omega) = k$. That is, $\liminf_{\alpha} d(\pi_\alpha, \varphi_\pi, N) = k$. Therefore there is a subnet $\Omega_1 = (\pi_{\alpha(\mu)})_{\mu \in \Delta}$ of Ω such that $d(\pi_{\alpha(\mu)}, \varphi_\pi, N) = k$ for all $\mu \in \Delta$. Assume that (iii) holds. We apply Lemma 3.7 ((iii) $\Rightarrow$ (i)) to the net Ω_1, with $F_0 = \{\pi\}$ and $n_\pi = k + 1$. It follows that there exist (infinitely many) $\mu \in \Delta$ for which $d(\pi_{\alpha(\mu)}, \varphi_\pi, N) \geqslant k + 1$, a contradiction. ∎

The next result is a consequence of Theorem 4.2. It strengthens Theorem 3.2 of [3].

THEOREM 4.3. *Let A be a C^*-algebra and suppose that $\Omega = (\pi_\alpha)_{\alpha \in \Lambda}$ is a net in $\widehat{A}$.*

(i) *For each $a \in A^+$*

$$\liminf \mathrm{Tr}(\pi_\alpha(a)) \geqslant \sum_{\pi \in \widehat{A}} \mathrm{M_L}(\pi, \Omega)\, \mathrm{Tr}(\pi(a)) \tag{4.1}$$

(working in $[0, \infty]$ with the convention that $\infty \times 0 = 0$).

(ii) *Given $\pi_0 \in \widehat{A}$, there exists a subnet $\Omega_0 = (\pi_{\alpha(\mu)})_{\mu \in \Delta}$ such that for each $a \in A^+$*

$$\liminf \mathrm{Tr}(\pi_{\alpha(\mu)}(a)) \geqslant \mathrm{M_U}(\pi_0, \Omega)\mathrm{Tr}(\pi_0(a)) + \sum_{\pi \in \widehat{A} \setminus \{\pi_0\}} \mathrm{M_L}(\pi, \Omega)\, \mathrm{Tr}(\pi(a)). \tag{4.2}$$

REMARK 4.4. Recall that $\mathrm{M_L}(\pi, \Omega) = 0$ unless Ω converges to π, whereas $\mathrm{M_U}(\pi, \Omega) > 0$ whenever π is a cluster point of Ω.

Proof. (i) Suppose that (4.1) fails to hold for some $a \in A^+$. For this element a the left-hand side of (4.1) is finite (with value R, say) and there exists $\varepsilon > 0$ such that the right-hand side of (4.1) is greater then $R + 2\varepsilon$. There exists a subnet $\Omega_1 = (\pi_{\alpha(\mu)})_{\mu \in \Delta_1}$ of Ω such that

$$\mathrm{Tr}(\pi_{\alpha(\mu)}(a)) < R + \varepsilon \quad (\mu \in \Delta_1).$$

For each $\pi \in \widehat{A}$ with $\mathrm{M_L}(\pi, \Omega) > 0$, we have that $\pi_{\alpha(\mu)} \to \pi$ and that $\mathrm{M_L}(\pi, \Omega) \leqslant \mathrm{M_L}(\pi, \Omega_1)$. Hence, applying Theorem 4.2 to Ω_1, with $F = \widehat{A}$, there exist a subnet $\Omega_2 = (\pi_{\alpha(\nu)})_{\nu \in \Delta_2}$ of Ω, a Hilbert space H, and representations σ and $(\sigma_\nu)_{\nu \in \Delta_2}$ of A on H such that:

(a) $\mathrm{Ess}(\sigma_\nu) \simeq \pi_{\alpha(\nu)}$;

(b) $\mathrm{Ess}(\sigma) \simeq \bigoplus_{\pi \in \widehat{A}} \mathrm{M_L}(\pi, \Omega) \cdot \pi$ (where, for definiteness, if $\mathrm{M_L}(\pi, \Omega) = \infty$ we take it to be $\aleph_0$);

(c) $\sigma_\nu \to \sigma$ in the strong topology of $\mathrm{Rep}(A, H)$;

(d) $\mathrm{Tr}(\pi_{\alpha(\nu)}(a)) < R + \varepsilon$ for all $\nu \in \Delta_2$.

It follows from (b) that $\mathrm{Tr}(\sigma(a)) > R + 2\varepsilon$ and so there exists an orthonormal set $\{\xi_1, \xi_2, \ldots, \xi_n\}$ in H such that

$$\sum_{i=1}^{n} \langle \sigma(a)\xi_i, \xi_i \rangle > R + \varepsilon.$$

By (c), there exists $\nu_0 \in \Delta_2$ such that for $\nu \geqslant \nu_0$

$$\sum_{i=1}^{n} \langle \sigma_\nu(a)\xi_i, \xi_i \rangle > R + \varepsilon.$$

Hence $\mathrm{Tr}(\sigma_\nu(a)) > R + \varepsilon$ for all $\nu \geqslant \nu_0$. However, it follows from (a) and (d) that $\mathrm{Tr}(\sigma_\nu(a)) < R + \varepsilon$ for all $\nu \in \Delta_2$. This contradiction establishes (i).
(ii) Let $\pi_0 \in \widehat{A}$. By Proposition 2.3, there exists a subnet $\Omega_0 = (\pi_{\alpha(\mu)})_{\mu \in \Delta}$ of Ω such that $\mathrm{M_L}(\pi_0, \Omega_0) = \mathrm{M_U}(\pi_0, \Omega)$. Since $\mathrm{M_L}(\pi, \Omega_0) \geqslant \mathrm{M_L}(\pi, \Omega)$ for $\pi \in \widehat{A} \setminus \{\pi_0\}$, (ii) now follows by applying the result of (i) to the net Ω_0. ∎

It is natural to ask whether the inequality in (4.1) above can be reversed if 'lim inf' is replaced by 'lim sup'. Consideration of elementary examples shows that the desired result may fail in general. However, it is possible to obtain a result of this kind under the assumption of an auxiliary finiteness condition. Details of this will appear in a subsequent paper ([6]).

We show next how Theorems 4.1 and 4.2 specialize to give characterizations of $\mathrm{M_U}(\pi)$ and $\mathrm{M_L}(\pi)$ (Theorems 4.5 and 4.7).

THEOREM 4.5. *Let A be a C^*-algebra, let $\pi \in \widehat{A}$, and let $m \in \mathbf{P} \cup \{\infty\}$. The following are equivalent:*

(i) $\mathrm{M_U}(\pi) \geqslant m$.

(ii) *For every cardinal realization for m, and for every cardinal number r such that $\ell^2(r) \bigoplus (\ell^2(m) \otimes H_\pi)$ is infinite dimensional and large enough for A,* *there are a Hilbert space H, a representation σ of A on H, and a net $(\sigma_\alpha)_{\alpha\in\Lambda}$ in* $\mathrm{Irr}(A,H)$ *converging to σ in the strong topology, such that:*

(a) $\mathrm{Ess}(\sigma) \simeq m\cdot\pi$;

(b) $\dim\left(\sigma(A)H\right)^\perp = r$.

(iii) *For some cardinal realization for m there are a Hilbert space H, a representation σ of A on H, and a net $(\sigma_\alpha)_{\alpha\in\Lambda}$ in $\mathrm{Irr}(A,H)$ converging to σ in the strong topology such that σ contains a subrepresentation unitarily equivalent to $m\cdot\pi$.*

Proof. (i) $\Rightarrow$ (ii): If $\{\pi\}$ is open, let Ω be a net with constant value equal to π. Then Ω converges to π in $\widehat{A}$, and $\mathrm{M_U}(\pi) = \mathrm{M_U}(\pi,\Omega)$. If $\{\pi\}$ is not open, Proposition 2.2 provides a net Ω in $\widehat{A}\setminus\{\pi\}$ such that Ω converges to π and $\mathrm{M_U}(\pi) = \mathrm{M_U}(\pi,\Omega)$. Assuming (i), Theorem 4.1 implies that $P(\Omega, m\cdot\pi)$ holds for every cardinal realization of m. Thus (ii) follows.

(ii) $\Rightarrow$ (iii): immediate.

(iii) $\Rightarrow$ (i): Assume (iii). Let $\Omega = (\pi_\alpha)_{\alpha\in\Lambda}$ where $\pi_\alpha = \Phi(\sigma_\alpha)$. Then $P'(\Omega, m\cdot\pi)$ holds, with Ω itself playing the role of the subnet. Using Theorem 4.1 ((iii) $\Rightarrow$ (i)), we have

$$m \leqslant \mathrm{M_U}(\pi,\Omega) \leqslant \mathrm{M_U}(\pi). \quad \blacksquare$$

REMARK 4.6. It follows from Lemma 3.5 that, if (ii) and (iii) hold, then $\Phi(\sigma_\alpha) \to \pi$ in $\widehat{A}$. Hence, if these equivalent conditions are satisfied, then in (ii) the net (σ_α) can be chosen so that $\Phi(\sigma_\alpha)$ lies in $\widehat{A}\setminus\{\pi\}$ for all α if and only if $\{\pi\}$ is not open in $\widehat{A}$. With regard to this latter condition, we recall from [3], Proposition 4.11 that if $\{\pi\}$ is open in $\widehat{A}$ then $\mathrm{M_U}(\pi)$ equals 1 or ∞.

THEOREM 4.7. *Let A be a C^*-algebra, let $\pi\in\widehat{A}$, and suppose that $\{\pi\}$ is not open in $\widehat{A}$. Let $m\in\mathbf{P}\cup\{\infty\}$. The following are equivalent:*

(i) $\mathrm{M_L}(\pi) \geqslant m$.

(ii) *For every net Ω in $\widehat{A}\setminus\{\pi\}$ which converges to π, and for every cardinal realization for m, $P(\Omega, m\cdot\pi)$ holds.*

(iii) *For every net Ω in $\widehat{A}\setminus\{\pi\}$ which converges to π, there exists a cardinal realization for m such that $P'(\Omega, m\cdot\pi)$ holds.*

Proof. (i) $\Rightarrow$ (ii): Let Ω be a net in $\widehat{A}\setminus\{\pi\}$ converging to π. Assuming (i), $m \leqslant \mathrm{M_L}(\pi) \leqslant \mathrm{M_L}(\pi,\Omega)$, and it follows from Theorem 4.2 ((i) $\Rightarrow$ (ii)) that $P(\Omega, m\cdot\pi)$ holds for every cardinal realization for m.

(ii) $\Rightarrow$ (iii): immediate.

(iii) $\Rightarrow$ (i): Let Ω be a net in $\widehat{A}\setminus\{\pi\}$ converging to π such that $\mathrm{M_L}(\pi) = \mathrm{M_L}(\pi,\Omega)$ (from Proposition 2.2). Assuming (iii), we have that for every subnet

Ω_1 of Ω there exists a cardinal realization for m such that $P'(\Omega_1, m\cdot\pi)$ holds. By Theorem 4.2 ((iii) $\Rightarrow$ (i)), it follows that $m \leqslant \mathrm{M_L}(\pi,\Omega) = \mathrm{M_L}(\pi)$. ∎

Let H be an infinite dimensional Hilbert space which is large enough for A. Recall that $\mathrm{Rep}(A,H)$ is the set of all (possibly degenerate) representations of A on H with the strong topology, and that $\mathrm{Irr}(A,H)$ is the subspace consisting of those non-zero representations σ for which $\mathrm{Ess}(\sigma)$ is irreducible. Let $\Phi : \mathrm{Irr}(A,H) \to \widehat{A}$ be the canonical surjection. By using a related result of Fell ([11]), Gardner showed that Φ is continuous and open ([12]). The fact that Φ is continuous is generalized by Lemma 3.5. The following corollary, which is proved without recourse to the results of Fell and Gardner, generalizes the fact that Φ is open.

COROLLARY 4.8. *Let A be a C^*-algebra, let $\pi \in \widehat{A}$, and suppose that $\{\pi\}$ is not open in $\widehat{A}$. Let $m \in \mathbf{P}\cup\{\infty\}$. Let H be an infinite dimensional Hilbert space which is large enough for A. Let $\tau \in \mathrm{Rep}(A,H)$ satisfy $\mathrm{Ess}(\tau) \simeq m\cdot\pi$ for some cardinal realization of m. The following are equivalent:*

(i) $\mathrm{M_L}(\pi) \geqslant m$;

(ii) *For each strong neighbourhood U of τ, $\{\pi\}\cup\Phi(U\cap \mathrm{Irr}(A,H))$ is a neighbourhood of π in $\widehat{A}$.*

Proof. (i) $\Rightarrow$ (ii): Suppose that (ii) fails. Then there is a strong neighbourhood U of τ, and a net Ω in $\widehat{A}\setminus(\{\pi\}\cup\Phi(U\cap\mathrm{Irr}(A,H)))$ converging to π. Assuming (i), it follows from Theorem 4.7 that if we take $r = \dim(\tau(A)H)^\perp$ then there exist a subnet $(\pi_{\alpha(\mu)})_{\mu\in\Delta}$ of Ω, a Hilbert space K and representations σ and $(\sigma_\mu)_{\mu\in\Delta}$ of A on K such that:

(a) $\mathrm{Ess}(\sigma_\mu) \simeq \pi_{\alpha(\mu)}$;

(b) $\mathrm{Ess}(\sigma) \simeq m\cdot\pi \simeq \mathrm{Ess}(\tau)$;

(c) (σ_μ) converges to σ in $\mathrm{Rep}(A,K)$;

(d) $\dim(\sigma(A)K)^\perp = r$.

By (b) and (d), there exists a unitary operator V from K onto H such that

$$V\sigma(a)V^* = \tau(a) \quad (a\in A).$$

By (c), $V\sigma_\mu V^* \to \tau$ in $\mathrm{Rep}(A,H)$. From (a) we have that $\Phi(V\sigma_\mu V^*) = \pi_{\alpha(\mu)}$, which gives a contradiction.

(ii) $\Rightarrow$ (i): Assume (ii). By Theorem 4.7 it suffices to show that $P'(\Omega, m\cdot\pi)$ holds for every net Ω in $\widehat{A}\setminus\{\pi\}$ converging to π. Let $\Omega = (\pi_\alpha)_{\alpha\in\Lambda}$ be such a net. Let $\mathcal{U}$ be the neighbourhood base at τ in $\mathrm{Rep}(A,H)$. Set $\Delta = \Lambda\times\mathcal{U}$, directed by

$$(\beta,U)\leqslant(\beta',U') \iff \beta\leqslant\beta' \text{ and } U\supseteq U'.$$

For $\mu = (\beta, U) \in \Delta$ there exists $\alpha(\mu) \in \Lambda$ such that $\alpha(\mu) \geqslant \beta$ and $\pi_{\alpha(\mu)} \in \Phi(U \cap \mathrm{Irr}(A,H))$. Let $\sigma_\mu \in U$ satisfy $\mathrm{Ess}(\sigma_\mu) \simeq \pi_{\alpha(\mu)}$. Then $\Omega_1 = (\pi_{\alpha(\mu)})_{\mu\in\Delta}$ is a subnet of Ω. By construction, $\lim_\mu \sigma_\mu = \tau$ in $\mathrm{Rep}(A,H)$. This establishes $P'(\Omega, m\cdot\pi)$. ∎

We briefly indicate how the above result implies that Φ is an open map. Let $\sigma \in \mathrm{Irr}(A,H)$ and let U be any neighbourhood of σ in $\mathrm{Rep}(A,H)$. If $\{\Phi(\sigma)\}$ is not open in $\widehat{A}$, then $\mathrm{M_L}(\Phi(\sigma)) \geqslant 1$ and so Corollary 4.8 implies that $\Phi(U\cap\mathrm{Irr}(A,H))$ is a neighbourhood of $\Phi(\sigma)$. If $\{\Phi(\sigma)\}$ *is* open in $\widehat{A}$ then the containing set $\Phi(U \cap \mathrm{Irr}(A,H))$ is automatically a neighbourhood of $\Phi(\sigma)$.

The proof of the corollary indicates the importance of the cardinal r in the definition of property $P(\Omega,\rho)$. This acts as a substitute for Gardner's technique using 'too large Hilbert spaces' ([12]).

Finally, in Theorems 4.9 and 4.12 below, we adapt Theorems 4.5 and 4.7 to give exact formulae for $\mathrm{M_U}(\pi)$ and $\mathrm{M_L}(\pi)$ that do not involve the use of nets. Recall from Section 1 that, for $\sigma \in \overline{\mathrm{Irr}(A,H)}$, $m(\pi,\sigma) \in \mathbb{N}\cup\{\infty\}$ is the multiplicity of π in σ.

THEOREM 4.9. *Let A be a C^*-algebra and let H be infinite dimensional and large enough for A. Let $\pi \in \widehat{A}$.*

(i) $\mathrm{M_U}(\pi) = \infty$ *if and only if there exists $\sigma \in \overline{\mathrm{Irr}(A,H)}$ containing a subrepresentation unitarily equivalent to an infinite direct sum of copies of π.*

(ii) *If the equivalent conditions of (i) do not hold then*

$$\mathrm{M_U}(\pi) = \max\{m(\pi,\sigma) \mid \sigma \in \overline{\mathrm{Irr}(A,H)}\} < \infty.$$

Proof. Let m be a cardinal realization for $\mathrm{M_U}(\pi)$, where if $\mathrm{M_U}(\pi) = \infty$ we take $m = \aleph_0$. Then $\dim(\ell^2(m)\otimes H_\pi) \leqslant \dim(H)$. Let r be any cardinal number such that

$$\dim(\ell^2(r) \oplus (\ell^2(m)\otimes H_\pi)) = \dim(H).$$

By Theorem 4.5 ((i) $\Rightarrow$ (ii)), there exist a Hilbert space K, a representation σ of A on K and a net $(\sigma_\alpha)_{\alpha\in\Lambda}$ in $\mathrm{Irr}(A,K)$ converging to σ in the strong topology such that:

(a) $\mathrm{Ess}(\sigma) \simeq m\cdot\pi$;

(b) $\dim(\sigma(A)K)^\perp = r$.

In particular, $\dim(K) = \dim(H)$ and so there exists a unitary operator V from K onto H. Then $V\sigma_\alpha V^* \in \mathrm{Irr}(A,H)$, $V\sigma_\alpha V^* \to V\sigma V^*$ in the strong topology of $\mathrm{Rep}(A,H)$, and $\mathrm{Ess}(V\sigma V^*) \simeq m\cdot\pi$. This establishes the 'only if' part of (i) and shows that, in (ii), $\mathrm{M_U}(\pi)$ is contained in the set to be maximized.

To complete the proof, suppose that $\rho \in \overline{\mathrm{Irr}(A,H)}$, suppose that either $n \in \mathbf{P}$ or $n = \aleph_0$, and suppose that ρ contains a subrepresentation unitarily equivalent to $n \cdot \pi$. By Theorem 4.5 ((iii) $\Rightarrow$ (i)), we obtain that if $n = \aleph_0$ then $\mathrm{M_U}(\pi) = \infty$ and that if $n \in \mathbf{P}$ then $\mathrm{M_U}(\pi) \geqslant n$. ∎

To give an analogous characterization of lower multiplicity, we require the following lemma.

LEMMA 4.10. *Let A be a C^*-algebra and let H be infinite dimensional and large enough for A. Let $Y \subseteq \widehat{A}$, let $E = \overline{\Phi^{-1}(Y)}$, and let $\sigma \in E$. Let $\rho \in \mathrm{Rep}(A,H)$ satisfy:*

(i) $\mathrm{Ess}(\rho)$ *is unitarily equivalent to a subrepresentation of σ;*

(ii) $\dim(\rho(A)H)^\perp = \dim(H)$.

Then $\rho \in E$.

Proof. Let $\sigma = \sigma_0 \oplus \sigma_1$ with $\mathrm{Ess}(\rho) \simeq \sigma_0$, and let $H = H_0 \oplus H_1$ be the corresponding decomposition of H. Let $\widetilde{H} = H \otimes \ell^2(\mathbf{P})$, and let $\widetilde{\sigma} = \sigma \otimes e_{11}$, $\widetilde{\sigma}_0 = (\sigma_0 \oplus 0_{H_1}) \otimes e_{11}$ in $\mathrm{Rep}(A, \widetilde{H})$, where $\{e_{ij} \mid i,j \in \mathbf{P}\}$ is the usual system of matrix units in $L(\ell^2(\mathbf{P}))$. We will also let $\{\delta_j \mid j \in \mathbf{P}\}$ be the usual orthonormal basis in $\ell^2(\mathbf{P})$. Let $W : H \to \widetilde{H}$ be a unitary operator, and let $\rho_0 = W^*\widetilde{\sigma}_0 W$ in $\mathrm{Rep}(A,H)$. Note that $\mathrm{Ess}(\rho_0) \simeq \sigma_0$ and $\dim(\rho_0(A)H)^\perp = \dim(H)$, and so $\rho_0 \simeq \rho$. Since E is saturated with respect to unitary equivalence on H, it suffices to show that $\rho_0 \in E$.

By hypothesis there is a net (σ_α) in $\mathrm{Irr}(A,H)$ such that $\Phi(\sigma_\alpha) \in Y$ and $\sigma_\alpha \to \sigma$. Let $\widetilde{\sigma}_\alpha = \sigma_\alpha \otimes e_{11} \in \mathrm{Rep}(A, \widetilde{H})$. Let V be any unitary operator in $L(\widetilde{H})$. We have $W^*V^*\widetilde{\sigma}_\alpha VW \to W^*V^*\widetilde{\sigma}VW$ in $\mathrm{Rep}(A,H)$. Since $\Phi(W^*V^*\widetilde{\sigma}_\alpha VW) = \Phi(\sigma_\alpha) \in Y$, and since E is closed, we have that $W^*V^*\widetilde{\sigma}VW \in E$.

Now let P be the orthogonal projection of H onto H_1. Let

$$V_k = P \otimes (e_{1k} + e_{k1}) + 1 - P \otimes (e_{11} + e_{kk}).$$

Then V_k is a unitary operator in $L(\widetilde{H})$. We claim that $V_k^*\widetilde{\sigma}V_k \to \widetilde{\sigma}_0$. To see this, let $a \in A$. If $\xi \in H_0 \otimes \delta_1$ then

$$V_k^*\widetilde{\sigma}(a)V_k\xi = V_k^*\widetilde{\sigma}(a)\xi = \widetilde{\sigma}_0(a)\xi.$$

If $\xi \in (H_0 \otimes \delta_1)^\perp$, then since $(H_0 \otimes \delta_1)^\perp$ is invariant for each V_k we have

$$\begin{aligned}\|\widetilde{\sigma}(a)V_k\xi\| &\leqslant \|a\|\,\|(P \otimes e_{11})V_k\xi\| \\ &= \|a\|\,\|(P \otimes e_{1k})\xi\| \\ &\to 0 \quad \text{as } k \to \infty.\end{aligned}$$

Hence for all $a \in A$, $\xi \in \widetilde{H}$ we have $\lim_k V_k^* \widetilde{\sigma}(a) V_k \xi = \widetilde{\sigma}_0(a)\xi$. Thus $V_k^* \widetilde{\sigma} V_k \to \widetilde{\sigma}_0$, and so since E is closed,

$$\rho_0 = W^* \widetilde{\sigma}_0 W = \lim_k W^* V_k^* \widetilde{\sigma} V_k W \in E. \quad \blacksquare$$

DEFINITION 4.11. Let A be a C^*-algebra and let $\pi \in \widehat{A}$. Denote by $W(\pi)$ the collection $\{Y \subseteq \widehat{A} \setminus \{\pi\} \mid \pi \in \overline{Y}\}$.

THEOREM 4.12. *Let A be a C^*-algebra and let H be infinite dimensional and large enough for A. Let $\pi \in \widehat{A}$ and suppose that $\{\pi\}$ is not open in $\widehat{A}$. Then*

$$\mathrm{M_L}(\pi) = \inf_{Y \in W(\pi)} \sup\{m(\pi, \tau) \mid \tau \in \overline{\Phi^{-1}(Y)}\}.$$

Proof. ($\leqslant$): Let $Y \in W(\pi)$. Then there is a net $\Omega = (\pi_\alpha)_{\alpha \in \Lambda}$ in Y converging to π. By Theorem 4.7 ((i) $\Rightarrow$ (ii)), $P(\Omega, \mathrm{M_L}(\pi) \cdot \pi)$ holds (where in the case that $\mathrm{M_L}(\pi) = \infty$, we realize it as $\aleph_0$). Thus there is a subnet $(\pi_{\alpha(\mu)})_{\mu \in \Delta}$ of Ω, and representations σ, $(\sigma_\mu)_{\mu \in \Delta}$ of A on H, such that

$$\mathrm{Ess}(\sigma_\mu) \simeq \pi_{\alpha(\mu)}$$
$$\mathrm{Ess}(\sigma) \simeq \mathrm{M_L}(\pi) \cdot \pi$$
$$\sigma_\mu \to \sigma.$$

It follows that $\sigma \in \overline{\Phi^{-1}(Y)}$ and $m(\pi, \sigma) = \mathrm{M_L}(\pi)$. Therefore

$$\mathrm{M_L}(\pi) \leqslant \sup\{m(\pi, \tau) \mid \tau \in \overline{\Phi^{-1}(Y)}\}.$$

Taking the infimum on Y we obtain the desired inequality.

($\geqslant$): Let $k \in \mathbf{P}$ with

$$k \leqslant \inf_{Y \in W(\pi)} \sup\{m(\pi, \tau) \mid \tau \in \overline{\Phi^{-1}(Y)}\}.$$

We then have that for each $Y \in W(\pi)$,

$$k \leqslant \sup\{m(\pi, \tau) \mid \tau \in \overline{\Phi^{-1}(Y)}\}. \tag{4.3}$$

Fix $\rho \in \mathrm{Rep}(A, H)$ with $\mathrm{Ess}(\rho) \simeq k \cdot \pi$ and $\dim(\rho(A)H)^\perp = \dim(H)$. Let $\Omega = (\pi_\alpha)_{\alpha \in \Lambda}$ be a net in $\widehat{A} \setminus \{\pi\}$ converging to π. For each $\beta \in \Lambda$, let $Y_\beta = \{\pi_\alpha \mid \alpha \geqslant \beta\}$. Then $Y_\beta \in W(\pi)$, so by (4.3) there exists $\tau_\beta \in \overline{\Phi^{-1}(Y_\beta)}$ with $m(\pi, \tau_\beta) \geqslant k$. By Lemma 4.10 we have

$$\rho \in \overline{\Phi^{-1}(Y_\beta)} \quad (\beta \in \Lambda). \tag{4.4}$$

Let $\mathcal{U}$ be the neighbourhood base at ρ in $\mathrm{Rep}(A, H)$, and let $\Delta = \Lambda \times \mathcal{U}$ with the product direction. For $\mu = (\beta, G) \in \Delta$, it follows from (4.4) that there exists $\alpha(\mu) \geqslant \beta$ and $\sigma_\mu \in G$ with $\sigma_\mu \in \mathrm{Irr}(A, H)$ and $\Phi(\sigma_\mu) = \pi_{\alpha(\mu)}$. Thus $P'(\Omega, k \cdot \pi)$ holds. From Theorem 4.7 ((iii) $\Rightarrow$ (i)), it follows that $k \leqslant \mathrm{M_L}(\pi)$, and the desired inequality is established. $\blacksquare$

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