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THE PROPERTIES OF THE CAUCHY TRANSFORM ON A BOUNDED DOMAIN

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ABSTRACT. In this paper we find the norm and the asymptotic behavior of singular values of the Cauchy transform on a bounded domain.

KEYWORDS: *Cauchy transform, Laplace operator, singular values.*

AMS SUBJECT CLASSIFICATION: 47B10.

1. INTRODUCTION

Let $D \subset \mathbb{C}$ be a bounded domain. Denote by $L^2(D)$ the space of complex valued functions in D for which the norm

$$\|f\| = \Big(\int_D |f(\xi)^2 \,\mathrm{d}A(\xi)\Big)^{\frac{1}{2}}$$

is finite. Here the $\mathrm{d}A(\xi) = \mathrm{d}x\mathrm{d}y$, $\xi = x + \mathrm{i}y$. The Cauchy integral operator $C : L^2(D) \to L^2(D)$ is defined by

$$Cf(z) = -\frac{1}{\pi}\int_D \frac{f(\xi)}{\xi - z}\,\mathrm{d}A(\xi).$$

It is well known that C is a bounded operator on $L^2(D)$. If D is the unit disc, J.M. Anderson and A. Hinkkanen proved in [1] that $\|C\| = 2/\alpha$ where α is the smallest positive zero of the Bessel function

$$J_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}.$$

In [2], Anderson, Khavison and Lomosov determined all the eigenvalues and eigenvectors, of the operators C^*C and L, where L is defined on $L^2(D)$ (D is the unit disc

$$Lf(z) = \frac{1}{2\pi}\int_D \ln\frac{1}{|z-\xi|} f(\xi)\,\mathrm{d}A(\xi).$$

They also determined the eigenvalues and eigenvectors of the operator

$$Nf(x) = \frac{1}{(n-2)\omega_{n-1}}\int_D |x-y|^{2-n} f(y)\,\mathrm{d}y$$

(ω_{n-1} denotes the surface area of the unit sphere S^{n-1} in R^n) acting on $L^2(D)$, where D is the ball in R^n. In the case of an arbitrary domain D, estimates from below of the norms of C, L, N are also given [2] but there are no precise estimates from above. Our Theorem 2.1 gives the exact value of the norm of C in an arbitrary simply connected domain in **C**.

In [3], Arazy and Khavinson gave estimates from above and below of the singular values of the operator C in an arbitrary bounded domain in C. They also founded estimates for the singular values of the operators CP, PCP and LP, when P is the Bergman projector, and showed that these are *two times nicer* than the corresponding estimates for C or L. Our Theorem 2.6 gives an expression for the exact asymptotic behaviour of the singular values of the operator C in the terms of geometric properties of the domain D.

The exact asymptotic behaviour of the singular values of PC and LP and their dependence on the length of ∂D will be presented in a forthcoming paper.

Let T be a compact operator on Hilbert space $\mathcal{H}$. The singular values of the operator T are the eigenvalues of the operator $(T^*T)^{\frac{1}{2}}$. The eigenvalues of the operator $(T^*T)^{\frac{1}{2}}$, arranged in the decreasing order and repeated according to their multiplicity, form a sequence $s_1(T), s_2(T), \ldots$ tending to zero. Denote by C_p the Schatten–von Neumann class of operators. Let $\mathcal{N}_t(T)$ be the singular value distribution function

$$\mathcal{N}_t(T) = \sum_{s_n(T)\geqslant t} 1, \quad t > 0.$$

Denote by $\int_D K(x,y)\,\mathrm{d}y$ the integral operator on $L^2(D)$ with the kernel $K(\,\cdot\,,\cdot)$. By $a_n \sim b_n$, $n\to\infty$ and $f(x)\sim g(x)$, $x\to 0$ we denote the fact that

$$\lim_{n\to\infty}\frac{a_n}{b_n} = 1 \quad\text{and}\quad \lim_{x\to 0}\frac{f(x)}{g(x)} = 1.$$

2. RESULTS

THEOREM 2.1. *If $D \subset \mathbb{C}$ is a bounded domain with piecewise C^1 boundary, then*

$$\|C\| = \frac{2}{\sqrt{\lambda_1}}$$

where λ_1 is the smallest eigenvalue of the following boundary problem

$$\begin{cases} -\Delta u = \lambda u \\ u|\partial D = 0. \end{cases} \tag{2.1}$$

In what follows we need some lemmas.

LEMMA 2.2. *Let $f \in L^2(D)$ and*

$$\widehat{f}(x) = \frac{1}{2\pi}\int_D e^{-iux_1 - ivx_2} f(u,v)\,\mathrm{d}u\mathrm{d}v.$$

Then for $0 < \alpha \leqslant 1/2$

$$\int_{\mathbb{R}^2} \frac{|\widehat{f}(x)|^2}{|x|^{2\alpha}}\,\mathrm{d}x \leqslant \lambda_1^{-\alpha} \int_D |f(x)|^2\,\mathrm{d}x$$

where λ_1 is the smallest eigenvalue of the boundary problem (2.1). (Here $x = (x_1, x_2)$, $|x| = \sqrt{x_1^2 + x_2^2}$.)

Proof. Let $\varphi \in C_0^\infty(D)$ (infinitely many differentiable function with the compact support lying in D). Then $-\Delta\varphi = F^{-1}|x|^2F\varphi$. Here F is the Fourier transform, i.e.

$$F\varphi(x) = \widehat{\varphi}(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-itx}\varphi(t)\,\mathrm{d}t.$$

Therefore we have

$$(-\Delta\varphi, \varphi)_{L^2(D)} = (F^{-1}|x|^2F\varphi, \varphi)_{L^2(D)}. \tag{2.2}$$

Let $\{u_n\}_{n=1}^\infty$ be the orthonormal base of $L^2(D)$ consisting of eigenfunctions of the Laplace operator $-\Delta$ with the boundary condition $u|\partial D = 0$ corresponding to the eigenvalues $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ respectively. Since

$$(-\Delta\varphi, \varphi) = \sum_{n=1}^{\infty} \lambda_n |(\varphi, u_n)|^2 \geqslant \lambda_1 \|\varphi\|^2,$$

then from (2.2) it follows

$$\lambda_1(\varphi,\varphi) \leqslant (F^{-1}|x|^2F\varphi,\varphi) \quad \forall\,\varphi \in C_0^\infty(D).$$

Let A_0 and B be linear operators on $L^2(D)$ defined on the domains

$$\mathcal{D}(A_0) = C_0^\infty(D)$$

and

$$\mathcal{D}(B) = \Big\{\varphi \in L^2(D) : \int\limits_{\mathbb{R}^2} |x|^4|\widehat{\varphi}|^2\,\mathrm{d}x < \infty\Big\}$$

by

$$A_0\varphi = F^{-1}|x|^2F\varphi \quad (\varphi \in \mathcal{D}(A_0))$$
$$B\varphi = F^{-1}|x|^2F\varphi \quad (\varphi \in \mathcal{D}(B)).$$

The operator A_0 is symetric and hence closable. The operator B is selfadjoint. It can be easily proved that $\overline{A}_0 = B$, where $\overline{A}_0$ is closure of A_0. Let $\varphi \in \mathcal{D}(B)$. Then there exists a sequence $\varphi_n \in C_0^\infty(D)$ such that

$$\|\varphi - \varphi_n\|_{L^2(D)} \to 0 \quad \text{and} \quad A_0\varphi_n \to \overline{A}_0\varphi = B\varphi.$$

From (2.2) it follows

$$\lambda_1(\varphi_n,\varphi_n) \leqslant (F^{-1}|x|^2F\varphi_n,\varphi_n).$$

Putting $n \to \infty$ we obtain

$$\lambda_1(\varphi,\varphi) \leqslant (B\varphi,\varphi) \quad \forall\,\varphi \in \mathcal{D}(B)$$

and therefore

$$B + \lambda \geqslant \lambda_1 + \lambda \quad (\text{on } \mathcal{D}(B)),\ (\lambda > 0);$$

hence

$$(B+\lambda)^{-1} \leqslant (\lambda_1+\lambda)^{-1}.$$

Since

$$B^{-\alpha} = \frac{\sin\alpha\pi}{\pi}\int\limits_0^\infty \lambda^{-\alpha}(\lambda+B)^{-1}\,\mathrm{d}\lambda \quad \text{for } 0<\alpha<1 \quad ([6]),$$

we get

$$\begin{aligned}(B^{-\alpha}f,f) &= \frac{\sin\alpha\pi}{\pi}\int\limits_0^\infty \lambda^{-\alpha}((\lambda+B)^{-1}f,f)\,\mathrm{d}\lambda \\ &\leqslant \frac{\sin\alpha\pi}{\pi}\int\limits_0^\infty \lambda^{-\alpha}(\lambda_1+\lambda)^{-1}\,\mathrm{d}\lambda\cdot\|f\|^2_{L^2(D)} = \lambda_1^{-\alpha}\|f\|^2_{L^2(D)}.\end{aligned} \tag{2.3}$$

Substituting $B^{-\alpha} = F^{-1}|x|^{-2\alpha}F$ in (2.3) we obtain

$$(F^{-1}|x|^{-2\alpha}Ff, f)_{L^2(D)} \leqslant \lambda_1^{-\alpha}\|f\|^2_{L^2(D)},$$

i.e.,

$$\int_{\mathbf{R}^2} \frac{|\widehat{f}(x)|^2}{|x|^{2\alpha}}\,\mathrm{d}x \leqslant \lambda_1^{-\alpha}\int_D |f(x)|^2\,\mathrm{d}x. \quad \blacksquare$$

LEMMA 2.3. *Let* $\Theta_n(\xi) = \frac{1}{\sqrt{\lambda_n}}\left(\frac{\partial u_n}{\partial x} - \mathrm{i}\frac{\partial u_n}{\partial y}\right)$, $\xi = x + \mathrm{i}y$. *Then the following equalities hold:*

(i) $\int_D \Theta_n(\xi)\overline{\Theta_m(\xi)}\,\mathrm{d}A(\xi) = \delta_{nm}$;

(ii) $C^*\Theta_n = -\frac{2}{\sqrt{\lambda_n}}u_n$, *($C^*$ is the adjoint operator of C)*;

(iii) $Cu_n = \frac{2}{\sqrt{\lambda_n}}(-\Theta_n + k_n)$ *where* $k_n(z) = \frac{1}{2\pi\mathrm{i}}\int_{\partial D}\frac{\Theta_n(\xi)}{\xi - z}\,\mathrm{d}\xi$.

Proof. The relation (i) is obtained directly applying Green formula (having in mind that $-\Delta u_n = \lambda_n u_n$, $u_n|\partial D = 0$ and $\int_{\partial D} u_n(\xi)\overline{u_m(\xi)}\,\mathrm{d}A(\xi) = \delta_{nm}$).

(ii) By Cauchy-Green formula

$$f(z) = -\frac{1}{2\pi\mathrm{i}}\int_{\partial D}\frac{f(\xi)}{\overline{\xi} - \overline{z}}\,\mathrm{d}\overline{\xi} - C^*\left(\frac{\partial f}{\partial \xi}\right)$$

(which holds for $f \in C(\overline{D}) \cap C^1(D)$) we get

$$-C^*\left(\frac{\partial u_n}{\partial \xi}\right) = u_n.$$

Since $\frac{\sqrt{\lambda_n}}{2}\Theta_n = \frac{\partial u_n}{\partial \xi}$, we obtain

$$C^*\Theta_n = -\frac{2}{\sqrt{\lambda_n}}u_n.$$

(iii) By Cauchy-Green formula

$$f(z) = \frac{1}{2\pi\mathrm{i}}\int_{\partial D}\frac{f(\xi)}{\xi - z}\,\mathrm{d}\xi + C\left(\frac{\partial f}{\partial \overline{\xi}}\right)$$

we get

$$\begin{aligned} Cu_n &= C\left(-\frac{4}{\lambda_n}\frac{\partial}{\partial \overline{z}}\left(\frac{\partial u_n}{\partial z}\right)\right) = -\frac{4}{\lambda_n}C\left(\frac{\partial}{\partial \overline{z}}\left(\frac{\partial u_n}{\partial z}\right)\right) \\ &= -\frac{4}{\lambda_n}\left[\frac{\partial u_n}{\partial z} - \frac{1}{2\pi\mathrm{i}}\int_{\partial D}\frac{\frac{\partial u_n}{\partial \xi}}{\xi - z}\,\mathrm{d}\xi\right] = \frac{2}{\sqrt{\lambda_n}}(-\Theta_n + k_n). \quad \blacksquare \end{aligned}$$

Proof of Theorem 2.1. Let $\varphi, \psi \in C_0^\infty(D)$. By direct calculation we get

$$(C\varphi, \psi)_{L^2(D)} = \frac{2}{i}\int_{\mathbf{C}} \frac{\widehat{\varphi}(z)\overline{\widehat{\psi}(z)}}{z}\, \mathrm{d}A(z)$$

because $\widehat{\left(\frac{1}{z}\right)} = \frac{i}{z}$ in the sense of distributions theory and

$$|(C\varphi, \psi)| \leqslant 2\int_{\mathbf{C}} \frac{|\widehat{\varphi}(z)|\,|\widehat{\psi}(z)|}{|z|}\, \mathrm{d}A \leqslant 2\sqrt{\int_{\mathbf{C}} \frac{|\widehat{\varphi}(z)|^2}{|z|}\, \mathrm{d}A}\ \sqrt{\int_{\mathbf{C}} \frac{|\widehat{\psi}(z)|^2}{|z|}\, \mathrm{d}A}$$

$$\leqslant \frac{2}{\sqrt{\lambda_1}}\|\varphi\|_{L^2(D)} \cdot \|\varphi\|_{L^2(D)}$$

(acording to Lemma 2.2, case $\alpha = 1/2$). Then for $f, g \in L^2(D)$ the inequality $|(Cf, g)| \leqslant \frac{2}{\sqrt{\lambda_1}}\|f\|\,\|g\|$ holds and so $\|C\| \leqslant \frac{2}{\sqrt{\lambda_1}}$. By Lemma 2.3 it follows $C^*\Theta_1 = -\frac{2}{\sqrt{\lambda_1}}u_1$. Hence

$$\frac{2}{\sqrt{\lambda_1}} = \left\|-\frac{2}{\sqrt{\lambda_1}}u_1\right\| = \|C^*\Theta_1\| \leqslant \|C^*\| = \|C\|.$$

So $\|C\| = \frac{2}{\sqrt{\lambda_1}}$. (The inequality $\|C\| \geqslant \frac{2}{\sqrt{\lambda_1}}$ is also proved in [2] (Proposition 5.1, p. 402) by different argument.) ∎

REMARK 2.4. If $D = \{z : |z| < 1\}$ then $\lambda_1 = \alpha^2$, where α is the smallest positive zero of Bessel function J_0. According to Theorem 2.1 we get $\|C\| = 2/\alpha$. This result it obtained in [1].

REMARK 2.5. If D is bounded domain in $\mathbf{C}$ and D lies in the disc of radius R, than

$$\|C\| \leqslant \frac{2R}{\alpha}. \tag{2.4}$$

By Faber-Krahn inequality ([7]) we get $\lambda_1 \geqslant \frac{\pi\alpha^2}{|D|}$ where $|D|$ is area of domain D and by Theorem 2.1 we obtain

$$\|C\| \leqslant \frac{2}{\alpha}\sqrt{\frac{|D|}{\pi}}$$

which is a better estimate than (2.4).

Now consider a generatization of the Cauchy operator.

Let $m \in C(\overline{D})$ be a complex continuous function on $\overline{D}$ and let μ be the measure defined by

$$\mathrm{d}\mu(\xi) = m(\xi)\, dA(\xi).$$

Consider the operator $A : L^2(D) \to L^2(D)$ defined by

$$Af(z) = -\frac{1}{\pi}\int_D \frac{f(\xi)}{\xi - z}\, \mathrm{d}\mu(\xi).$$

It is well known that C is a compact operator, hence the operator A is also compact.

THEOREM 2.6. *If D is a bounded Jordan measurable domain in $\mathbb{C}$, then*

$$s_n(A) \sim \Big(\int_D |m(\xi)|^2 \, \mathrm{d}A(\xi)\Big)^{\frac{1}{2}} (\pi n)^{-\frac{1}{2}}.$$

CONSEQUENCE 2.7. $A \in C_p \Leftrightarrow p > 2$. (This corollary is contained in [3].)

REMARK 2.8. If $m \equiv 1$ then $A = C$ and from Theorem 2.6 it follows that $s_n \sim \sqrt{\frac{|D|}{\pi n}}$. Since $\lambda_n \sim \frac{4\pi n}{|D|}$, by Weyl theorem ([6]) we have $s_n(C) \sim \frac{2}{\sqrt{\lambda_n}}$.

Observe that from Theorem 2.1 it follows $s_1(C) = \frac{2}{\sqrt{\lambda_1}}$. For the other singular values we usually have $s_n \neq \frac{2}{\sqrt{\lambda_n}}$, but still the asymptotic relation

$$s_n(C) = \frac{2}{\sqrt{\lambda_n}}(1 + \mathrm{o}(1))$$

holds. We mention that when D is unit disc we have $s_n = \frac{2}{\sqrt{\lambda_n}}$ (see [2], Theorem 2.2), although the eigenfunctions and their multiplicities are slightly different.

Before the proof of Theorem 2.6 we need a number of lemmas.

LEMMA 2.9. *If D is bounded domain with piecewise C^1 boundary then*

$$C = \sum_{n \geqslant 1} \frac{2}{\sqrt{\lambda_n}} (\,\cdot\,, u_n) \cdot (-\Theta_n + k_n)$$

where $k_n(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\Theta_n(\xi)}{\xi - z} \, \mathrm{d}\xi$.

Proof. From Lemma 2.3 it follows

$$C u_n = \frac{2}{\sqrt{\lambda_n}}(-\Theta_n + k_n).$$

Since $\{u_n\}_{n=1}^{\infty}$ is an orthonormal basis of $L^2(D)$ then we have

$$f = \sum_{n=1}^{\infty} (f, u_n) \cdot u_n, \quad (f \in L^2(D)))$$

and therefore

$$Cf = \sum_{n=1}^{\infty} (f, u_n) C u_n = \sum_{n=1}^{\infty} \frac{2}{\sqrt{\lambda_n}} (f, u_n)(-\Theta_n + k_n). \quad \blacksquare$$

Let $C_1 = -\sum_{n=1}^{\infty} \frac{2}{\sqrt{\lambda_n}} (\,\cdot\,, u_n)\Theta_n$.

Now we prove that if $D = [0, \pi]^2$ then $C - C_1$ is a Hilbert-Schmidt operator. The eigenfunctions and the eigenvalues of the boundary problem

$$-\Delta u = \lambda u$$
$$u|\partial D = 0 \quad D = [0, \pi] \times [0, \pi]$$

are $u_{mn} = \frac{2}{\pi} \sin nx \sin my$ and $\mu_{mn} = m^2 + n^2$. In this case the operator $C - C_1$ has the kernel $R(\xi, z)$, where

$$\begin{aligned} R(\xi, z) = &-\frac{1}{\pi}\frac{1}{\xi - z} \\ &+ \frac{8}{\pi^2}\sum_{m,n=1}^{\infty} \frac{1}{m^2+n^2} \sin nu \sin mv (n \cos nx \sin my - im \sin nx \cos my) \end{aligned}$$

where $\xi = u + iv$ and $z = x + iy$. The series

$$\sum_{m,n=1}^{\infty} \frac{1}{m^2+n^2} \sin nu \sin mv (n \cos nx \sin my - im \sin nx \cos my)$$

is convergent in distributional sense.

LEMMA 2.10. $\iint\limits_{DD} |R(\xi, z)|^2 \, dA(\xi) dA(z) < \infty$.

Proof. Since

$$\sum_{m=1}^{\infty} \frac{\cos m\alpha}{m^2 + n^2} = -\frac{1}{2n^2} + \frac{\pi}{2n}\left(e^{-|\alpha| n} + \frac{2\cosh \alpha n}{e^{2\pi n} - 1}\right) \quad (-2\pi < \alpha < 2\pi),$$

applying simple transformations, we obtain

$$\begin{aligned} R(\xi, z) = -\frac{1}{\pi}\frac{1}{\xi - z} + \frac{8}{\pi^2}\left(\frac{\pi}{8}\sum_{n=1}^{\infty} \sin n(u - x) e^{-n|v-y|} - i\frac{\pi}{8}\sum_{n=1}^{\infty} \sin n(v - y) e^{-n|x-u|}\right) \\ + \text{ some function from } L^2(D \times D). \end{aligned}$$

Since

$$\sum_{n=1}^{\infty} e^{-an} \sin nb = \frac{\sin b}{2\cosh a - 2\cosh b} \quad a > 0, \ b \in \mathbf{R}$$

from the previous equation it follows

$$\begin{aligned} R(\xi, z) = &-\frac{1}{\pi}\frac{1}{\xi - z} + \frac{1}{\pi}\frac{\sin(u - x)}{2\cosh(v - y) - 2\cos(u - x)} \\ &- \frac{i}{\pi}\frac{\sin(v - y)}{2\cosh(x - u) - 2\cos(v - y)} \\ &+ \text{some function from } L^2(D \times D). \end{aligned} \tag{2.5}$$

Since the function

$$(a,b) \longmapsto \frac{\sin b}{2\cosh a - 2\cos b} - \frac{b}{a^2+b^2}$$

is bounded in a neighborhood of the point (0,0), from (2.5) it follows

$$R(\xi,z) = -\frac{1}{\pi}\frac{1}{\xi - z} + \frac{1}{\pi}\frac{u-x}{(u-x)^2+(v-y)^2} - \frac{i}{\pi}\frac{v-y}{(u-x)^2+(v-y)^2}$$
$$+ \text{ some function from } L^2(D\times D).$$

So,

$$R(\xi,z) = -\frac{1}{\pi}\frac{1}{\xi - z} + \frac{1}{\pi}\frac{\overline{\xi}-\overline{z}}{|\xi - z|^2} + \text{some function from } L^2(D\times D)$$

i.e., $R \in L^2(D\times D)$. ∎

LEMMA 2.11. *If* $D = [0,\pi^2]$, *then for the operator* $C : L^2(D) \to L^2(D)$, *defined by*

$$Cf(z) = -\frac{1}{\pi}\int_D \frac{f(\xi)}{\xi - z}\,\mathrm{d}A(\xi),$$

the relation $s_n(C) \sim \sqrt{\frac{\pi}{n}}$ *holds.*

Proof. According to Lemma 2.10, $C - C_1$ is a Hilbert-Schmidt operator and

(2.6) $$\lim_{n\to\infty} n^{\frac{1}{2}} s_n(C - C_1) = 0.$$

On the other hand, we have $s_n(C_1) = 2/\sqrt{\lambda_n}$, where λ_n are the eigenvalues of the following boundary problem

$$-\Delta u = \lambda u$$
$$u|\partial D = 0 \quad D = [0,\pi]\times[0,\pi].$$

The Weyl Theorem gives $\lambda_n \sim \frac{4n}{\pi}$. So from Ky Fan Theorem ([5]) and (2.6) it follows

$$s_n(C) \sim \sqrt{\frac{\pi}{n}}. \quad ∎$$

Observe that, by substitution, from Lemma 2.11 we get

(2.7) $$s_n\left(-\frac{1}{\pi}\int_D \cdot\,\frac{\mathrm{d}A(\xi)}{\xi - z}\right) \sim \sqrt{\frac{|D|}{\pi n}}$$

where D is an arbitrary square with the sides parallel to the coordinate axes and $|D|$ denotes its area.

The following two lemmas are direct consequences of Lemma 1 and 2 from [4].

LEMMA 2.12. *If T' and T'' are compact operators and*

$$T = T' + T'', \quad \lim_{t\to 0+} t^{\gamma}\mathcal{N}_t(T') = C(T') \quad \text{and} \quad s_n(T'') = o(n^{-\frac{1}{\gamma}}),\ \gamma > 0$$

then there exists the limit $\lim_{t\to 0+} t^{\gamma}\mathcal{N}_t(T)$ and it is equal to $C(T')$.

LEMMA 2.13. *Let T be compact operator and suppose that for every $\varepsilon > 0$ there exists a decomposition $T = T'_\varepsilon + T''_\varepsilon$ where $T'_\varepsilon, T''_\varepsilon$ are compact operators such that:*

(i) *There exists $\lim_{t\to 0+} t^{\gamma}\mathcal{N}_t(T'_\varepsilon) = C(T'_\varepsilon)$, $C(T'_\varepsilon)$ being a bounded function in the neighborhood of $\varepsilon = 0$;*

(ii) $\overline{\lim_{n\to\infty}} s_n(T''_\varepsilon) n^{\frac{1}{\gamma}} \leqslant \varepsilon$.

Then there exists $\lim_{\varepsilon\to 0+} C(T'_\varepsilon) = C(T)$ and

$$\lim_{t\to 0+} t^{\gamma}\mathcal{N}_t(T) = C(T).$$

Proof of Theorem 2.6. Consider first the case when $D = [0,\pi]^2$. Divide the square D in N squares D_i and denote by ξ_i the center of D_i. Let $\varepsilon > 0$. Since $m \in C(\overline{D})$ then for N large enough the inequality $|m(\xi) - m(\xi_i)| < \varepsilon$ holds for every $\xi \in D_i$. Then

$$\left|\sum_{j=1}^{N}(m(\xi) - m(\xi_j))\chi_{D_j}(\xi)\right| < \varepsilon \tag{2.8}$$

for every $\xi \in D$. ($\chi_S(\cdot)$ is characteristic function of the set S.) The operator

$$A = -\frac{1}{\pi}\int_D \cdot \frac{m(\xi)\mathrm{d}A(\xi)}{\xi - z}$$

can be represented in the form

$$A = B_N + H_N + E_N$$

where B_N, H_N, E_N are linear operators on $L^2(D)$ defined respectively by

$$B_N f = -\frac{1}{\pi}\int_D \frac{1}{\xi - z}\left[\sum_{j=1}^{N}(m(\xi) - m(\xi_j))\chi_{D_j}(\xi)\right] f(\xi)\,\mathrm{d}A(\xi)$$

$$H_N f = \sum_{j=1}^{N} m(\xi_j)\chi_{D_j}(z)\int_{D_j} -\frac{1}{\pi}\frac{1}{\xi - z} f(\xi)\,\mathrm{d}A(\xi)$$

and

$$E_N f = \sum_{\substack{i \neq j \\ i,j=1}} m(\xi_j)\chi_{D_i}(z) \int_{D_j} -\frac{1}{\pi}\frac{1}{\xi - z} f(\xi)\, \mathrm{d}A(\xi).$$

Since

$$\int_{D_i}\int_{D_j} \frac{\mathrm{d}A(\xi)\mathrm{d}A(\xi)}{|\xi - z|^2} < \infty$$

for $i \neq j$, we conclude that E_N is a Hilbert-Schmidt operator and hence

(2.9) $$s_n(E_N) = \mathrm{o}(n^{-\frac{1}{2}}).$$

From (2.7) and (2.8) it follows

$$s_n(B_N) \leqslant C \cdot \frac{\varepsilon}{n^{\frac{1}{2}}}$$

where the constant C does not depend on n and ε. From (2.9) and the property of singular values of the sum of two operators we obtain

(2.10) $$\varlimsup_{n\to\infty} n^{\frac{1}{2}} s_n(B_N + E_N) \leqslant C' \cdot \varepsilon$$

where C' does not depend on ε and N.

Define the operators $C_j^N : L^2(D_j) \to L^2(D_j)$, $j = 1, 2, \cdots, N$ by

$$C_j^N f(z) = -\frac{1}{\pi}\int_{D_j} m(\xi_j)\frac{f(\xi)}{\xi - z}\, \mathrm{d}A(\xi).$$

The operator H_N is a direct sum of the operators C_j^N, $(j = 1, 2, \ldots, N)$ and hence

$$\mathcal{N}_t(H_N) = \sum_{j=1}^{N} \mathcal{N}_t(C_j^N).$$

From (2.7) (consequence of Lemma 2.11), it follows

$$s_n(C_j^N) \sim |m(\xi_j)|\sqrt{\frac{|D_j|}{\pi n}} \quad n \to \infty$$

and therefore

$$\mathcal{N}_t(C_j^N) \sim |m(\xi_j)|^2 \frac{|D_j|}{\pi t^2} \quad t \to 0^+.$$

So

(2.11) $$\lim_{t\to 0+} t^2 \mathcal{N}_t(H_N) = \frac{1}{\pi}\sum_{j=1}^{N} |m(\xi_j)|^2 |D_j|.$$

From (2.10) and (2.11), by Lemma 2.13, we obtain

$$\lim_{t\to 0+} t^2\mathcal{N}_t(A) = \lim_{N\to\infty} \frac{1}{\pi}\sum_{j=1}^{N} |m(\xi_j)|^2|D_j| = \frac{1}{\pi}\int_D |m(\xi)|^2 \,\mathrm{d}A(\xi).$$

So

$$\mathcal{N}_t(A) \sim \frac{1}{\pi t^2}\int_D |m(\xi)|^2 \,\mathrm{d}A(\xi) \quad t\to 0^+. \tag{2.12}$$

Putting $t = s_n(A)$ we obtain $ns_n^2(A) \sim \frac{1}{\pi}\int_D |m(\xi)|^2 \,\mathrm{d}A(\xi)$, i.e.,

$$s_n(A) \sim (\pi n)^{-\frac{1}{2}}\Big(\int_D |m(\xi)|^2 \,\mathrm{d}A(\xi)\Big)^{\frac{1}{2}}$$

and Theorem 2.6 is proved in the case when D is a square.

LEMMA 2.14. *Let $D = \bigcup_{i=1}^{r} K_i$ where K_i are the squares with the property $K_i^\circ \cap K_j^\circ = \emptyset$ $(i \neq j)$, (K_i° is the interior of the square K_i), and m is a continuous complex function on $\overline{D}$. Then for the operator $A_1 : L^2(D) \to L^2(D)$ defined by*

$$A_1 f(z) = -\frac{1}{\pi}\int_D f(\xi)\frac{m(\xi)}{\xi - z}\,\mathrm{d}A(\xi)$$

the following asymptotic formula holds

$$\mathcal{N}_t(A_1) \sim \frac{1}{\pi t^2}\int_D |m(\xi)|^2 \,\mathrm{d}A(\xi) \quad t\to 0^+.$$

Proof. Let $P_j : L^2(D) \to L^2(D)$, $P_j f(z) = \chi_{K_j}(z) f(z)$. Then

$$A_1 = \sum_{i=1}^{r} P_i A_1 P_i + \sum_{\substack{i\neq j\\ i,j=1}} P_i A_1 P_j.$$

The operator $P_i A_1 P_j$ are the Hilbert-Schmidt ones; hence

$$\lim_{n\to\infty} n^{\frac{1}{2}} s_n\left(\sum_{i\neq j}^{r} P_i A_1 P_j\right) = 0.$$

By Lemma 2.12 we obtain

$$\lim_{t\to 0+} t^2\mathcal{N}_t(A_1) = \lim_{t\to 0+} t^2\mathcal{N}_t\Big(\sum_{i=1}^{r} P_i A_1 P_i\Big) = \sum_{i=1}^{r}\lim_{t\to 0+} t^2\mathcal{N}_t(P_i A_1 P_i).$$

Since by (2.12)

$$\lim_{t\to 0} t^2 \mathcal{N}_t(P_i A_1 P_i) = \frac{1}{\pi}\int\limits_{K_i} |m|^2 \, dA$$

we have

$$\lim_{t\to 0} t^2 \mathcal{N}_t(A_1) = \frac{1}{\pi}\int\limits_{D} |m|^2 \, dA. \quad \blacksquare$$

LEMMA 2.15. *If $\Omega_1, \Omega_2 \subset \mathbf{R}^2$ are bounded measurable sets, $\Omega_1 \subset \Omega_2$ and $B_i : L^2(\Omega_i) \to L^2(\Omega_i)$, $i = 1,2$, are the linear operators defined by*

$$B_i f(z) = -\frac{1}{\pi}\int\limits_{\Omega_i} \frac{m(\xi) f(\xi)}{\xi - z} \, dA(\xi)$$

then

$$\mathcal{N}_t(B_1) \leqslant \mathcal{N}(B_2).$$

Proof. We have $B_1 = P_1 B_2 P_2$ where $P_2 : L^2(\Omega_1) \to L^2(\Omega_2)$ and $P_1 : L^2(\Omega_2) \to L^2(\Omega_1)$ are defined by $P_1 f = f|\Omega_1$ and $P_2 f(z) = f(z)$ for $z \in \Omega_1$ and $P_2 f(z) = 0$ for $z \in \Omega_2 \setminus \Omega_1$. From $B_1 = P_1 B_2 P_2$ we have $s_n(B_1) \leqslant s_n(B_2)$. The statement of lemma follows. $\blacksquare$

Proof of Theorem 2.6 *in the general case.* Let D be a bounded Jordan measurable domain in $\mathbb{C}$. Let $\underline{D_N} \subset D \subset \overline{D_N}$ where $\underline{D_N}$ and $\overline{D_N}$ are finite unions of the squares of equal side length such that

$$|\underline{D_N}| \to |D|$$
$$|\overline{D_N}| \to |D| \quad N \to \infty$$

where $|W|$ denote the area of W. Let $\tilde{m}$ be a continuous extension of the function m in some neighborhood of the set $\overline{D}$. Let $\underline{A_N}$ and $\overline{A_N}$ be the linear operators on $L^2(\underline{D_N})$ and $L^2(\overline{D_N})$ respectively defined by

$$\underline{A_N} f(z) = -\frac{1}{\pi}\int\limits_{\underline{D_N}} \frac{m(\xi) f(\xi)}{\xi - z} \, dA(\xi)$$

$$\overline{A_N} f(z) = -\frac{1}{\pi}\int\limits_{\overline{D_N}} \frac{\tilde{m}(\xi) f(\xi)}{\xi - z} \, dA(\xi).$$

According to Lemma 2.15 we have

$$\mathcal{N}_t(\underline{A_N}) \leqslant \mathcal{N}_t(A) \leqslant \mathcal{N}_t(\overline{A_N}),$$

i.e.,

$$t^2\mathcal{N}_t(\underline{A_N}) \leqslant t^2\mathcal{N}_t(A) \leqslant t^2\mathcal{N}_t(\overline{A_N}).$$

So

$$\underline{\lim}_{t\to 0} t^2\mathcal{N}_t(\underline{A_N}) \leqslant \underline{\lim}_{t\to 0} t^2\mathcal{N}_t(A) \leqslant \overline{\lim}_{t\to 0} t^2\mathcal{N}_t(A) \leqslant \overline{\lim}_{t\to 0} t^2\mathcal{N}_t(\overline{A_N}). \tag{2.13}$$

Since by Lemma 2.14

$$\underline{\lim}_{t\to 0} t^2\mathcal{N}_t(\underline{A_N}) = \lim_{t\to 0} t^2\mathcal{N}_t(\underline{A_N}) = \frac{1}{\pi}\int\limits_{\underline{D_N}} |m(\xi)|^2\,\mathrm{d}A(\xi)$$

$$\overline{\lim}_{t\to 0} t^2\mathcal{N}_t(\overline{A_N}) = \lim_{t\to 0} t^2\mathcal{N}_t(A) = \frac{1}{\pi}\int\limits_{\overline{D_N}} |\widetilde{m}(\xi)|^2\,\mathrm{d}A(\xi),$$

then from (2.13) it follows

$$\frac{1}{\pi}\int\limits_{\underline{D_N}} |m(\xi)|^2\,\mathrm{d}A(\xi) \leqslant \underline{\lim}_{t\to 0} t^2\mathcal{N}_t(A) \leqslant \overline{\lim}_{t\to 0} t^2\mathcal{N}_t(A) \leqslant \frac{1}{\pi}\int\limits_{\overline{D_N}} |\widetilde{m}(\xi)|^2\,\mathrm{d}A(\xi).$$

From the last relation for $N \to \infty$ we obtain

$$\lim_{t\to 0} t^2\mathcal{N}_t(A) = \frac{1}{\pi}\int\limits_{D} |m(\xi)|^2\,\mathrm{d}A(\xi). \tag{2.14}$$

Putting $t = s_n(A)$ in (2.14) we get $ns_n^2(A) \sim \frac{1}{\pi}\int\limits_D |m(\xi)|^2\,\mathrm{d}A(\xi)$, i.e.,

$$s_n(A) \sim (\pi n)^{-\frac{1}{2}}\Big(\int\limits_D |m(\xi)|^2\,\mathrm{d}A(\xi)\Big)^{\frac{1}{2}}.$$

Theorem 2.6 is proved. ∎

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