J. OPERATOR THEORY
36(1996), 295–316

INFINITE SERIES OF QUANTUM SPECTRAL STOCHASTIC INTEGRALS

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Communicated by William B. Arveson

ABSTRACT. We obtain sufficient conditions for the convergence of infinite series of quantum spectral stochastic integrals. The resulting operator-valued processes are used to drive quantum stochastic integro-differential equations. Unitary solutions of these equations implement quantum stochastic flows and give rise to a new representation for the generators of a class of completely positive semigroups. As an application, we are able to construct a class of flows on algebras of operators which are driven by multidimensional Lévy processes.

KEYWORDS: *Quantum stochastic calculus, quantum spectral stochastic integral, Lévy process, Lévy flow.*

AMS SUBJECT CLASSIFICATION: Primary 81S25; Secondary 46L50, 46L55.

1. INTRODUCTION

The quantum stochastic calculus of R.L. Hudson and K.R. Parthasarathy ([13], [14]) has been developed into a highly effective tool for building non-commutative analogues of a number of important probabilistic notions such as Markov processes and stochastic flows and has given new insights into the nature of irreversible processes in quantum theory (see the monographs [21] and [19] for a panoramic view). The theory takes place in a Hilbert space $\mathfrak{H} = \mathfrak{H}_0 \otimes \Gamma(L^2(\mathbf{R}^+, \mathcal{K}))$ where $\mathfrak{H}_0$, the *initial* Hilbert space allows for the representation of operator algebras which then interact with *noise* in the symmetric Fock space $\Gamma(L^2(\mathbf{R}^+, \mathcal{K}))$. The basic noise martingales are built out of certain annihilation, creation and number-type operators therein. In the standard theory, the space $\mathcal{K}$ is, at most taken to be an infinite direct sum leading to a countable number of independent noises. In an earlier paper ([2]), the first author began the process of extending the Hudson-Parthasarathy theory to a continuum of noises where $\mathcal{K}$ is a direct integral with respect to a finite measure. From the point of view of stochastic integration this means that we introduce an additional integration over a *space variable* — which is described by a convenient projection-valued-measure, as well as the usual *time variable*. The resulting objects are called *quantum spectral stochastic integrals*. A related construction can be found in [7].

In the current paper, we take $\mathcal{K}$ to be an infinite direct sum of such direct integrals and so we consider infinite series of quantum spectral stochastic integrals. A further extension allows us to consider such integrals defined (spatially) over $\mathbf{R}^n$ rather than $\mathbf{R}$ as in [2]. We obtain three interesting applications for these integrals:

(a) *Semigroups.* Quantum stochastic integro-differential equations driven by our series of integrals give dilations of a class of quantum dynamical semigroups whose generators can be given a finer structure than the canonical one of [18];

(b) *Quantum physic.* The coupling between *system* and *noise* is more detailed than in the usual theory;

(c) *Probability.* The Hudson–Parthasarathy theory contains, as a special case, stochastic integration with respect to a countable number of independent Brownian motions and Poisson processes. The theory of [2] allowed us to extend this to Lévy measures with finite intensity measure. The theory contained herein enables us to include the most general Lévy process taking values in $\mathbf{R}^n$ and also some Lévy processes with infinitely many degrees of freedom. Indeed we close the paper by constructing a class of algebraic Lévy flows whose generators are a subclass of those discussed in (a) and which yield an algebraic generalisation of the Hunt-Courrège formula ([16], [10]) for the generators of Markov semigroups associated with semigroups of measures on $\mathbf{R}^n$ (this construction should not be confused with that of [6] which is from a different point of view and is far more general).

The organisation of this paper is as follows. In Section 2, we review the construction of quantum spectral stochastic integrals in [2] and indicate how to extend these to $\mathbf{R}^n$. Convergence of series is described in Section 3 and in Section 4 we prove existence and unitarity for a class of integro-differential equations with bounded operator coefficients and obtain the semigroups discussed above. Section 5 is an interlude where, using ideas about current group representations due to H. Araki, K.R. Parthasarathy and many others we explain how Lévy processes can be naturally represented in Fock space. Finally in Section 6 we combine the results of the previous two sections to obtain a non-commutative generalisation of a class of Lévy flows of diffeomorphisms of manifolds constructed in [1] and extend the operator theoretic construction to include a class of flows driven by Lévy processes with infinitely many degrees of freedom.

NOTATION. Einstein summation convention will be used throughout. If x is a column vector then x^{T} is its transpose. If V_j $(j = 1, 2)$ are vector spaces then $V_1 \underline{\otimes} V_2$ is their (algebraic) tensor product. $\mathcal{U}(\mathfrak{H})$ denotes the group of all unitary operators in a Hilbert space $\mathfrak{H}$. If T is a densely-defined closeable operator in $\mathfrak{H}$ then $\overline{T}$ is its closure. $\mathcal{B}(S)$ is the Borel σ-algebra of a topological space S.

2. QUANTUM SPECTRAL STOCHASTIC INTEGRALS

Let $\Gamma(\mathcal{J})$ denote the symmetric Fock space over the complex separable Hilbert space $\mathcal{J}$. If $\mathcal{M}$ is a dense linear subspace of $\mathcal{J}$, we note that the linear span $\mathcal{E}(\mathcal{M})$ of the exponential vectors $\{e(f), f \in \mathcal{J}\}$ is dense in $\Gamma(\mathcal{J})$.

In the sequel, we will always take $\mathcal{J}$ to be $L^2(\mathbf{R}^+, \mathcal{K})$ (which we identify with $L^2(\mathbf{R}^+) \otimes \mathcal{K}$) where $\mathcal{K}$ is infinite dimensional and $\mathcal{M}$ to be the algebraic tensor product of the locally bounded functions on $\mathbf{R}^+$ with a dense linear subspace $\mathcal{C}$ of $\mathcal{K}$ ($\mathcal{C}$ will be specified below). The inner product in $\mathcal{K}$ will usually be denoted as $(\cdot, \cdot)$.

Fix $x, y \in \mathcal{J}$ and let $H \in B(\mathcal{J})$ be self-adjoint. We denote by $A_x^\dagger(t)$ the *creation operator* $a^\dagger(\chi_{[0,t)} \otimes x)$, by $A_y(t)$ the *annihilation operator* $a(\chi_{[0,t)} \otimes y)$ and by $\Lambda_H(t)$ the *conservation operator* $\lambda(\chi_{[0,t)} \otimes H)$ which are all densely defined linear operators in $\Gamma(\mathcal{J})$ with $\mathcal{E}(\mathcal{M})$ contained in each of their domains.

Now let $\mathfrak{D}_0$ be a dense linear manifold in a complex separable Hilbert space $\mathfrak{H}_0$ and write $\mathfrak{H} = \mathfrak{H}_0 \otimes \Gamma(\mathcal{J})$. We assume that for $j = 1, 2, 3, 4$, $E_j = (E_j(t), t \in \mathbf{R}^+)$ are *regular processes* in $\mathfrak{H}$ i.e.

(i) $\mathfrak{D}_0 \underline{\otimes} \mathcal{E}(\mathcal{M}) \subseteq \mathrm{Dom}(E_j(t))$ for each $t \in \mathbf{R}^+$;

(ii) each E_j is adapted and locally square-integrable in the sense of [21].

We will assume familiarity with the construction and elementary properties of *quantum stochastic integrals* of the form

$$M(t) = \int_0^t (E_1(s)\,\mathrm{d}A_x^\dagger(s) + E_2(s)\,\mathrm{d}\Lambda_H(s) + E_3(s)\,\mathrm{d}A_y(s) + E_4(s)\,\mathrm{d}s)$$

and we note that the process $M = (M(t),\ t \in \mathbf{R}^+)$ is itself regular in $\mathfrak{H}$ (see [13] and [21] for further details).

Let S be a Borel subset of $\mathbf{R}^+$ which we equip with the Euclidean norm $\|\cdot\|_e$ and let $\mathcal{B}(S)$ be the σ-algebra of Borel subsets of S. P will denote a projection valued measure on $\mathcal{B}(\mathbf{R}^n)$ taking values in the lattice of projections in K.

Now let $E = \{E(t,\lambda),\ t \in \mathbf{R}^+,\ \lambda \in S\}$ be a family of densely defined linear operators in $\mathfrak{H}$ which satisfies the following conditions:

(i) each process $E(\lambda) = (E(t,\lambda),\ t \in \mathbf{R}^+)$ is regular in $\mathfrak{H}$;

(ii) the map $\lambda \to \sup_{0\leqslant s\leqslant t} \|E(s,\lambda)(u\otimes e(f))\|$ from S to $\mathbf{R}^+$ is bounded for all $u \in \mathfrak{D}_0$ and $f \in \mathcal{M}$;

(iii) for all $t \in \mathbf{R}^+$, $u \in \mathfrak{D}_0$ and $f \in \mathcal{M}$ we have that given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\lambda, \mu \in S$ with $\|\lambda - \mu\|_e < \delta$ then $\sup_{0\leqslant s\leqslant t} \|(E(s,\lambda) - E(s,\mu))(u \otimes e(f))\| < \varepsilon$.

Such families will be called *admissible*.

Let $\{E_j,\ j = 1,2,3\}$ be admissible families and assume that $G \in \mathcal{B}(S)$ is such that

$$P(G)x \neq 0 \quad \text{and} \quad P(G)y \neq 0.$$

By imitating the argument of [2], where S was taken to be $\mathbf{R}$, we can construct *quantum spectral stochastic integrals* to be regular processes $M^G = (M^G(t),\ t \in \mathbf{R}^+)$ having the symbolic form

$$\begin{aligned} M^G(t) = \int_{G\times[0,t)} &(E_1(s,\lambda)A^\dagger(\mathrm{d}s, P(\mathrm{d}\lambda)x) + E_2(s,\lambda)\Lambda(\mathrm{d}s, P(\mathrm{d}\lambda)) \\ &+ E_3(s,\lambda)A(\mathrm{d}s, P(\mathrm{d}\lambda)y). \end{aligned} \tag{2.1}$$

The procedure is as follows:

By a *partition* $\mathcal{P}$ of $G \in \mathcal{B}(S)$ we will mean a family of subsets $\{G_i,\ i \in I\}$ wherein each $G_i \in \mathcal{B}(S)$, $\bigcup_{i\in I} G_i = G$ and all but a possible finite number of the G_i's are mutually disjoint. Such a partition is *finite* if the set I has finite cardinality and in this case we will always assume that all the G_i's are mutually disjoint. First assume that G has finite diameter.

Now let $\mathcal{P}$ be a finite partition and define the operators $M^G_{\mathcal{P}}(t)$ on $\mathfrak{D}_0 \otimes \mathcal{E}(\mathcal{M})$ by

$$M^G_{\mathcal{P}}(t) = \sum_{j=1}^m \int_0^t (E_1(s,\mu_j)\,\mathrm{d}A^\dagger_{x_j}(s) + E_2(s,\mu_j)\,\mathrm{d}\Lambda_{H_j}(s) + E_3(s,\mu_j)\,\mathrm{d}A_{y_j}(s)) \tag{2.2}$$

where each $\mu_j \in G_j$, $H_j = P(G_j)$ and $x_j = H_j x$, $y_j = H_j y$.

Let $(\mathcal{P}_n, n \in \mathbb{N})$ be a sequence of finite partitions of G with each $\mathcal{P}_n = \{G_j^{(n)}, 1 \leqslant j \leqslant m_n\}$ such that $\lim_{n\to\infty} |\mathcal{P}_n| = 0$ where $|\mathcal{P}_n| = \max\{\operatorname{diam}(G_j^{(n)});\ 1 \leqslant j \leqslant m\}$, then we define

$$M^G(t)(u \otimes e(f)) = \lim_{n\to\infty} M^G_{\mathcal{P}_n}(t)(u \otimes e(f))$$

for each $u \in \mathfrak{D}_0$, $f \in \mathcal{M}$.

In the case where $\operatorname{diam}(G) = \infty$, we consider a sequence $(G_n, n \in \mathbb{N})$ where each $G_n \in \mathcal{B}(S)$ has finite diameter, $G_n \subset G_{n+1}$ and $\bigcup_{n=1}^{\infty} G_n = G$ and define

$$M^G(t)(u \otimes e(f)) = \lim_{n\to\infty} M^{G_n}(t)(u \otimes e(f))$$

for each $u \in \mathfrak{D}_0$, $f \in \mathcal{M}$. $M^G = (M^G(t),\ t \in \mathbb{R}^+)$ is a regular process for all $G \in \mathcal{B}(S)$.

The details of the above are a straightforward generalisation of the arguments of [2]. For an alternative point of view see [8], [9]. We refer to these two papers for a discussion of further properties of the integrals (2.1). For the next section, we will need the following result for a restricted class of these integrals.

THEOREM 2.1. *For $j = 1, 2$, let $(E_k^j,\ k = 1, 2)$ be admissible families such that for all $t \in \mathbb{R}^+$*

$$M^j(t) = \int_{G\times[0,t)} (E_1^j(s,\lambda)A^\dagger(\mathrm{d}s, P(\mathrm{d}\lambda)x) + E_2^j(s,\lambda)A(\mathrm{d}s, P(\mathrm{d}\lambda)y)$$

then for $u, v \in \mathfrak{D}_0$, $f, g \in \mathcal{E}(\mathcal{M})$, we have

$$\begin{aligned}
&\big\langle M^1(t)(u \otimes e(f)), M^2(t)(v \otimes e(g))\big\rangle \\
&\quad = \int_{G\times[0,t)} \Big\{\big\langle M^1(s)(u \otimes e(f)), [E_1^2(s,\lambda)(f(s), P(\mathrm{d}\lambda)x) \\
&\qquad + E_2^2(s,\lambda)(P(\mathrm{d}\lambda)y, g(s))](v \otimes e(g))\big\rangle \\
&\qquad + \big\langle [E_1^1(s,\lambda)(g(s), P(\mathrm{d}\lambda)x) \\
&\qquad + E_2^1(s,\lambda)(P(\mathrm{d}\lambda)y, f(s))](u \otimes e(f), M^2(s)(v \otimes e(g))\big\rangle \\
&\qquad + \big\langle E_1^1(s,\lambda)(u \otimes e(f)), E_1^2(s,\lambda)(v \otimes e(g))\big\rangle (x, P(\mathrm{d}\lambda)y)\Big\}\,\mathrm{d}s.
\end{aligned} \tag{2.3}$$

Proof. See [2], Theorem 3.4. ∎

3. CONVERGENCE OF INFINITE SERIES OF QUANTUM SPECTRAL STOCHASTIC INTEGRALS

We fix a partition $\{G_n,\ n \in \mathbf{N}\}$ of S and choose a sequence of vectors $(v_n,\ n \in \mathbf{N})$ in $\mathcal{K}$ such that each

$$P(G_n)v_n = v_n. \tag{3.1}$$

We extend $(v_n,\ n \in \mathbf{N})$ to a total set $\mathcal{T}$ in $\mathcal{K}$ and from now on we take $\mathcal{C}$ to be the linear span of $\mathcal{T}$. Hence for any $f \in \mathcal{M}$ and for all $t \in \mathbf{R}^+$, $f(t)$ is a linear combination of at most n_f members of $\mathcal{T}$ with $n_f < \infty$ (cf. [20]).

We define $M_f = \max\{\|v_r\|^2,\ 1 \leqslant r \leqslant n_f\}$. We are interested in sequences $(E_n,\ n \in \mathbf{N})$ of admissible families satisfying the integrability condition

$$\sum_{n=1}^{\infty} \int_{G_n \times [0,t)} \|E_n(s,\lambda)(u \otimes e(f))\|^2 \, \|P(\mathrm{d}\lambda)v_n\| \, \mathrm{d}s < \infty. \tag{3.2}$$

Such sequences will be called *Lévy-integrable* (for reasons that will become clearer in Section 6).

Now let E and F be Lévy-integrable sequences. We wish to investigate the convergence of the sequence of quantum spectral stochastic integrals $(M_n,\ n \in \mathbf{N})$ where each

$$M_n(t) = \sum_{r=1}^{n} \int_{G_r \times [0,t)} [E_r(s,\lambda)A^{\dagger}(\mathrm{d}s, P(\mathrm{d}\lambda)v_r) + F_r(s,\lambda)A(\mathrm{d}s, P(\mathrm{d}\lambda)v_r)]. \tag{3.3}$$

LEMMA 3.1. *For each $f \in \mathcal{M}$, there exists a monotonic increasing function B_f on $\mathbf{R}^+$ such that for all $u \in \mathfrak{D}_0$ and for $n \geqslant n_f$ we have*

$$\|M_n(t)(u \otimes e(f))\|^2 \leqslant B_f(t) \sum_{r=1}^{n} \int_{G_r \times [0,t)} \varphi_r(s,\lambda,u,f,E,F) \|P(\mathrm{d}\lambda)v_r\|^2 \, \mathrm{d}s \tag{3.4}$$

where $\varphi_r(s,\lambda,u,f,E,F) = \max\{\|E_r(s,\lambda)(u \otimes e(f))\|^2, \|F_r(s,\lambda)(u \otimes e(f))\|^2\}$.

Proof. By Theorem 2.1 and several applications of the Schwarz inequality (see also Lemma 1 of [8]), we have

$$\begin{aligned}
&\|M_n(t)(u\otimes e(f))\|^2\\
&\leqslant \int_0^t \Big\{ 2\,\mathrm{Re}\Big(\sum_{r=1}^{n_f}\int_{G_r}\langle M_n(t)(u\otimes e(f)), [(P(\mathrm{d}\lambda)f(s),v_r)E_r(s,\lambda)\\
&\qquad + (v_r, P(\mathrm{d}\lambda)f(s))F_r(s,\lambda)](u\otimes e(f))\rangle\Big)\\
&\qquad + \sum_{r=1}^{n}\int_{G_r}\|E_r(s,\lambda)(u\otimes e(f))\|^2\|P(\mathrm{d}\lambda)v_r\|^2\Big\}\,\mathrm{d}s\\
&\leqslant \int_0^t \Big\{\sum_{r=1}^{n_f}\|M_n(t)(u\otimes e(f))\|^2\\
&\qquad + \sum_{r=1}^{n_f}\Big\|\int_{G_r}[(P(\mathrm{d}\lambda)f(s),v_r)E_r(s,\lambda) + (v_r,P(\mathrm{d}\lambda)f(s))F_r(s,\lambda)](u\otimes e(f))\Big\|\\
&\qquad + \sum_{r=1}^{n}\int_{G_r}\|E_r(s,\lambda)(u\otimes e(f))\|^2\|P(\mathrm{d}\lambda)v_r\|^2\Big\}\mathrm{d}s\\
&\leqslant \int_0^t \Big\{ n_f\|M_n(t)(u\otimes e(f))\|^2 +\\
&\qquad + 2\sum_{r=1}^{n_f}\|v_r\|^2\int_{G_r}\|f(s)\|^2\varphi_r(s,\lambda,u,f,E,F)\|P(\mathrm{d}\lambda)v_r\|^2\\
&\qquad + \sum_{r=1}^{n}\int_{G_r}\|E_r(s,\lambda)(u\otimes e(f))\|^2\|P(\mathrm{d}\lambda)v_r\|^2\Big\}\mathrm{d}s.
\end{aligned}$$

The result now follows by a standard use of integrating factors and we have

$$B_f(t) = \mathrm{e}^{t\,n_f}\Big(1 + 2M_f \sup_{0\leqslant s\leqslant t}\|f(s)\|^2\Big). \quad ∎$$

THEOREM 3.2. *For each* $u\in\mathfrak{D}_0$, $f\in\mathfrak{M}$, $T\in\mathbb{R}^+$ *the sequence* $(M_n(t)(u\otimes e(f))_{n\in\mathbb{N}}$ *converges uniformly for* $t\in[0,T]$.

Proof. Replacing $M_n(t)$ in (3.4) by $(M_n(t)-M_m(t))$ we obtain the estimate

$$\|(M_n(t)-M_m(t))(u\otimes e(f))\|^2 \leqslant B_f(T)\sum_{r=m+1}^{n}\int_{G_r\times[0,T)}\varphi_r(s,\lambda,u,f,E,F)\|P(\mathrm{d}\lambda)v_r\|^2$$

for each $t\in[0,T]$ and since E and F are Lévy-integrable sequences it follows that $(M_n(t)(u\otimes e(f))_{n\in\mathbb{N}}$ is uniformly Cauchy and hence uniformly convergent on $[0,T]$. ∎

We define $M(t)(u \otimes e(f)) = \lim_{n\to\infty} M_n(t)(u \otimes e(f))$. Clearly each $M(t)$ is a linear operator in $\mathfrak{H}$ with $\mathfrak{D}_0 \underline{\otimes} \mathcal{E}(\mathcal{M}) \subseteq \mathrm{Dom}(M(t))$ for each $t \in \mathbf{R}^+$. Furthermore $(M(t),\ t \in \mathbf{R}^+)$ is a regular process in $\mathfrak{H}$ and from Lemma 3.1, we deduce the estimate

$$\text{(3.5)} \quad \|M(t)(u \otimes e(f))\|^2 \leqslant B_f(t) \sum_{r=1}^{\infty} \int_{G_r \times [0,t)} \varphi_r(s, \lambda, u, f, E, F) \|P(\mathrm{d}\lambda) v_r\|^2 \,\mathrm{d}s.$$

On $\mathfrak{D}_0 \underline{\otimes} \mathcal{E}(\mathcal{M})$, we have the representation as a strong limit

$$\text{(3.6)} \quad M(t) = \sum_{r=1}^{\infty} \int_{G_r \times [0,t)} [E_r(s, \lambda) A^\dagger(\mathrm{d}s, P(\mathrm{d}\lambda) u_r) + F_r(s, \lambda) A(\mathrm{d}s, P(\mathrm{d}\lambda) u_r)].$$

We leave as an exercise for the reader the derivation of appropriate extensions of Theorem 3.4 (i) and (ii) of [2] for matrix elements of the $M(t)$'s and of their weak products.

In the sequel we will be concerned with regular processes in $\mathfrak{H}$ of the form $(N(t),\ t \in \mathbf{R}^+)$ where

$$\text{(3.7)} \qquad N(t) = M(t) + \int_{S \times [0,t)} G(s, \lambda) \Lambda(\mathrm{d}s, P(\mathrm{d}\lambda)) + \int_{[0,t)} H(s) \,\mathrm{d}s$$

where G is an admissible family and H a regular process.

On combining (3.5) with the results of [2] or [8], we obtain the estimate

$$\text{(3.8)} \quad \|N(t)(u \otimes e(f))\|^2 \leqslant C_f(t) \Bigg\{ \sum_{r=1}^{\infty} \int_{G_r \times [0,t)} \varphi_r(s, \lambda, u, f, E, F) \|P(\mathrm{d}\lambda) v_r\|^2 \,\mathrm{d}s$$
$$+ \int_{S \times [0,t)} \|G(s, \lambda)(u \otimes e(f))\|^2 \|P(\mathrm{d}\lambda) f(s)\|^2 \,\mathrm{d}s + \int_{[0,t)} \|H(s)(u \otimes e(f))\|^2 \,\mathrm{d}s \Bigg\}$$

where each C_f is a monotonic increasing function on $\mathbf{R}^+$.

4. EXISTENCE AND UNITARITY FOR SOLUTIONS OF QUANTUM STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS

We define a σ-finite measure ν on $\mathcal{B}(\mathbf{R}^n)$ by

$$\nu(G) = \sum_{n=1}^{\infty} \|P(G)v_n\|^2 \tag{4.1}$$

and we will in this section take S to be the support of the measure ν.

For $j = 1, 2$, let $\{L_j^{(n)}(\lambda), \lambda \in S, n \in \mathbf{N}\}$ be families of linear operators in $\mathfrak{H}_0$ satisfying the conditions:

(i) Each $L_j^{(n)}(\lambda) \in \mathcal{B}(\mathfrak{H}_0)$.

(ii) The map $\lambda \to L_j^{(n)}(\lambda)$ is strongly continuous from S to $\mathcal{B}(\mathfrak{H}_0)$ for each $n \in \mathbf{N}$.

(iii) $\sup_{\lambda \in S} \|L_j^{(n)}(\lambda)\| < \infty$ for each $n \in \mathbf{N}$.

(iv) There exist $D_j < \infty$ ($j = 1, 2$) such that for each $u \in \mathfrak{H}_0$,

$$\sum_{n=1}^{\infty} \int_{G_r} \|L_j^{(n)}(\lambda)u\|^2 \|P(\mathrm{d}\lambda)v_n\|^2 < D_j \|u\|^2.$$

We will also require a family $\{L_3(\lambda),\ \lambda \in S\}$ of linear operators in S satisfying (i), (ii) and (iii) above only and we write

$$D_3 = \sup_{\lambda \in S} \|L_3(\lambda)\|^2.$$

Finally we will have need of $L_4 \in \mathcal{B}(\mathfrak{H}_0)$ and we will write $D_4 = \|E_4\|^2$.

Our aim in this section is to give meaning to quantum stochastic integro-differential equations of the form

(4.2)

$$X(t) = I + \sum_{n=1}^{\infty} \int_{G_n \times [0,t)} X(s)\{L_1^{(n)}(\lambda)A^\dagger(\mathrm{d}s, P(\mathrm{d}\lambda)v_n) + L_2^{(n)}(\lambda)A(\mathrm{d}s, P(\mathrm{d}\lambda)v_n)\}$$
$$+ \int_{S \times [0,t)} X(s)L_3(\lambda)\Lambda(P(\mathrm{d}\lambda)) + \int_0^t X(s)L_4 \mathrm{d}s.$$

We will use the standard approach of Picard iteration (cf. [13], [2]).

LEMMA 4.1. *There exists a sequence of regular processes* $(X_m,\ m \in \mathbf{N} \cup \{0\})$ *such that* $X_0(t) = 0$ *for all* $t \in \mathbf{R}^+$ *and for* $m \geqslant 1$, (i) *and* (ii) *below hold:*

(i)

$$X_m(t) = I + \sum_{n=1}^{\infty} \int_{G_n \times [0,t)} X_{m-1}(s)\{L_1^{(n)}(\lambda)A^\dagger(\mathrm{d}s, P(\mathrm{d}\lambda)v_n) + L_2^{(n)}(\lambda)A(\mathrm{d}s, P(\mathrm{d}\lambda)v_n)\} + \int_{S \times [0,t)} X_{m-1}(s)L_3(\lambda)\Lambda(P(\mathrm{d}\lambda)) + \int_0^t X_{m-1}(s)L_4 \mathrm{d}s.$$

(ii) *For each* $u \in \mathfrak{H}_0$, $f \in \mathcal{M}$, $T \in \mathbf{R}^+$ *and for all* $0 \leqslant t \leqslant T$

$$\|(X_m(t) - X_{m-1}(t))(u \otimes e(f))\|^2 \leqslant \gamma_f(T)^{m-1} \frac{t^{m-1}}{(m-1)!}\|u\|^2\|e(f)\|^2$$

where $\gamma(t) = C_f(t)(D_1 + D_2 + D_3 \sup_{0 \leqslant s \leqslant t} \|f(s)\|^2 + D_4)$.

Proof. We establish (i) and (ii) by induction. Since $U_1(t) = I$ for all $t \in \mathbf{R}^+$ it is immediate that the case $m = 1$ holds. Now assume that both propositions are valid for all $m \leqslant p$. In order for (i) to make sense for $m = p + 1$, we must show that for $j = 1, 2$, $E_j = (E_j^{(n)},\ n \in \mathbf{N})$ are Lévy-integrable sequences where each $E_j^{(n)}(t, \lambda) = X_p(t)L_j^{(n)}(\lambda)$ for $t \in \mathbf{R}^+$, $\lambda \in S$, that E_3 is an admissible family where each $E_3(t, \lambda) = X_p(t)L_3(\lambda)$ for $t \in R^+$, $\lambda \in S$ and that E_4 is a regular process where each $E_4(t) = X_p(t)L_4$. To demonstrate these, note first that by (ii), we have for $u \in \mathfrak{D}_0$, $f \in \mathcal{M}$,

$$\|(X_p(t)(u \otimes e(f))\| \leqslant \sum_{k=1}^{p} \|(X_k(t) - X_{k-1}(t))(u \otimes e(f))\| \leqslant \left(\sum_{k=1}^{p} \gamma_f(t)^{k-1} \frac{t^{k-1}}{(k-1)!}\right)^{\frac{1}{2}} \|u\|\,\|e(f)\|.$$

Hence we deduce that $\alpha_p(f, t) < \infty$ where

$$\alpha_p(f, t) = \sup\{\|(X_p(t)(u \otimes e(f))\|;\ \|u\| = 1\}.$$

So to establish the integrability condition (3.2) for E_j $(j = 1, 2)$, we find that

$$\sum_{n=1}^{\infty} \int_{G_n \times [0,t)} \|X_p(t)L_j^{(n)}(\lambda)(u \otimes e(f))\|^2 \|P(\mathrm{d}\lambda)v_n\|^2 \mathrm{d}s \leqslant \alpha_p(f,t)^2 \sum_{n=1}^{\infty} \int_{G_n \times [0,t)} \|L_j^{(n)}(\lambda)u\|^2 \|P(\mathrm{d}\lambda)v_n\| \,\mathrm{d}s \leqslant D_j \alpha_p(f,t)^2 \|u\|^2 < \infty \quad \text{by condition (iii) above.}$$

The other conditions are all proved similarly and so (i) is established.

The inductive argument for (ii) follows from (3.8) by a similar argument to that of [13]. ∎

THEOREM 4.2. *There exists a regular process* $(X(t),\ t \in \mathbf{R}^+)$ *which is the unique solution of Equation* 4.2.

Proof. As in [13], one uses the estimate (ii) to show that for each $u \in \mathfrak{D}_0$, $f \in \mathcal{M}, T \in \mathbf{R}^+$, the sequence $(X_m(t)(u \otimes e(f)),\ m \in \mathbf{N})$ is uniformly Cauchy and hence uniformly convergent on $[0, T)$. We may then define

$$X(t)(u \otimes e(f)) = \lim_{m \to \infty} X_m(t)(u \otimes e(f)). \quad ∎$$

We are interested in establishing necessary and sufficient conditions for the solution to (4.2) to be a *unitary process*, i.e. each $X(t)$ is a unitary operator in $\mathfrak{H}$. Prior to this we need the following result.

LEMMA 4.3. *Let* $\{L^{(n)}(\lambda),\ \lambda \in S,\ n \in \mathbf{N}\}$ *and* $\{L^{(n)}(\lambda)^*,\ \lambda \in S,\ n \in \mathbf{N}\}$ *both satisfy conditions* (i) *to* (iv) *at the beginning of this section, then the strong sum of* $B(\mathfrak{H}_0)$*-valued integrals*

$$\sum_{n=1}^{\infty} \int_{G_n} L^{(n)}(\lambda)^* L^n(\lambda) \|P(\mathrm{d}\lambda) v_n\|^2$$

exists and defines a positive self-adjoint operator $T \in B(\mathfrak{H}_0)$.

Proof. This follows by a similar argument to that of [21], Lemma 27.4, p. 225. ∎

THEOREM 4.4. *The solution to* (4.2) *is a unitary process if and only if there exists* $\{W(\lambda),\ \lambda \in S\}$ *wherein each* $W(\lambda)$ *is a unitary operator in* $\mathfrak{H}_0$ *and there exists* $H = H^* \in B(\mathfrak{H}_0)$ *such that, writing each* $L_1^{(n)}(\lambda) = L^{(n)}(\lambda)$, *we have for each* $n \in \mathbf{N}$, $\lambda \in S$

$$L_2^{(n)}(\lambda) = -L^{(n)}(\lambda)^* W(\lambda)$$
$$L_3^{(n)}(\lambda) = W(\lambda) - I$$
$$L_4^{(n)} = \mathrm{i}H - \frac{1}{2} \sum_{n=1}^{\infty} \int_{G_n} L^{(n)}(\lambda)^* L^{(n)}(\lambda) \|P(\mathrm{d}\lambda) v_n\|^2.$$

Proof. This is along similar lines to that of [13] — see also [2]. Note that the existence of L_4 is guaranteed by Lemma 4.3. ∎

Now let $\mathcal{A}$ be a $*$- subalgebra of $B(\mathfrak{H}_0)$ and consider the following families of maps from $\mathcal{A}$ into $B(\mathfrak{H}_0)$; $\{\sigma(\lambda),\ \lambda \in S\}$ where for each $x \in \mathcal{A}$,

$$\sigma(\lambda)(x) = W(\lambda)xW(\lambda)^* - x \tag{4.3}$$

$\{\alpha_n(\lambda),\ \lambda \in S,\ n \in \mathbf{N}\}$ where each

$$\alpha_n(\lambda)(x) = L^{(n)}(\lambda)x - W(\lambda)xW(\lambda)^* L^{(n)}(\lambda) \tag{4.4}$$

and $\tau : \mathcal{A} \to B(\mathfrak{H}_0)$ where

$$\begin{aligned} \tau(x) = \mathrm{i}[H, x] - \frac{1}{2}\sum_{n=1}^{\infty} \int_{G_n} & (L^{(n)}(\lambda)^* L^{(n)}(\lambda)x \\ & - L^{(n)}(\lambda)^* W(\lambda)xW(\lambda)^* L^{(n)}(\lambda) - xL^{(n)}(\lambda)^* L^{(n)}(\lambda))\|P(\mathrm{d}\lambda)v_n\|^2. \end{aligned} \tag{4.5}$$

The convergence in (4.5) is again in the strong topology and the existence of the middle term follows by a similar argument to that of Lemma 27.7 in [21], p. 228.

We assume that each $\sigma(\lambda)$, $\alpha_n(\lambda)$ and τ have range lying inside $\mathcal{A}$. Now let $U = (U(t),\ t \in \mathbf{R}^+)$ be a unitary process and define a family of $*$-homomorphisms $J = (j_t,\ t \in \mathbf{R}^+)$ from $\mathcal{A}$ into $B(\mathfrak{H})$ by

$$j_t(x) = U(t)(x \otimes I)U(t)^* \tag{4.6}$$

for each $x \in \mathcal{A}$, $t \in \mathbf{R}^+$. A standard argument (see e.g. [13], [2]) yields the following representation for J in terms of quantum spectral stochastic integrals

$$\begin{aligned} j_t(x) = \sum_{n=1}^{\infty} \int_{G_n \times [0,t)} & \{j_s(\alpha_n(\lambda)(x))A^\dagger(\mathrm{d}s, P(\mathrm{d}\lambda)v_n) \\ & + j_s(\widetilde{\alpha}_n(\lambda)(x))A(\mathrm{d}s, P(\mathrm{d}\lambda)v_n)\} \\ & + \int_{S\times[0,t)} j_s(\sigma(\lambda)(x))\Lambda(P(\mathrm{d}\lambda)) + \int_0^t j_s(\tau(x))\,\mathrm{d}s \end{aligned} \tag{4.7}$$

where each $x \in \mathcal{A}$, $t \in \mathbf{R}^+$ and each

$$\widetilde{\alpha}_n(\lambda)(x) = \alpha_n(\lambda)(x^*)^*.$$

From (4.6) we see that J is a generalisation of the *quantum stochastic flows* considered in [15] and [11].

NOTE. In the case $\mathcal{A} = B(\mathfrak{H}_0)$, the prescription

$$\langle u, T_t(x)v\rangle = \langle (u \otimes e(0),\ j_t(x)(v \otimes e(0))\rangle$$

for $t \in \mathbf{R}^+$, $x \in \mathcal{A}$ yields a norm continuous quantum dynamical semigroup $(T_t,\ t \in \mathbf{R}^+)$ on $B(\mathfrak{H}_0)$ with infinitesimal generator τ however (4.5) yields a different representation than that described in [18] which seems to be a consequence of the finer coupling to noise (cf. [14]).

(4.5) can clearly be generalised further to the case where each $\|P(\cdot)v_n\|^2$ is replaced by a σ-finite measure ν_n. The relationship between such generators and quantum stochastic flows will be investigated elsewhere.

5. LÉVY PROCESSES IN FOCK SPACE (CF. [20], [21])

Let $\mathcal{G}$ be a Borel group with identity e and $V : \mathcal{G} \to \mathcal{U}(\mathcal{K})$ a unitary Borel representation. Let $\eta : \mathcal{G} \to \mathcal{K}$ be a one-cocycle so that η is Borel-measurable and

$$V(g)\eta(h) = \eta(gh) - \eta(g) \tag{5.1}$$

for all $g, h \in \mathcal{G}$. Suppose there exists a Borel map $\beta : \mathcal{G} \to \mathbf{R}$ such that

$$\beta(gh) - \beta(g) - \beta(h) = \mathrm{Im}(\langle \eta(g^{-1}), \eta(h)\rangle) \tag{5.2}$$

for all $g, h \in \mathcal{G}$.

Now let W be the Weyl representation of the Euclidean group of $\mathcal{K}$ in $\Gamma(\mathcal{K})$, so that for each $U \in \mathcal{U}(\mathcal{K})$, $v \in \mathcal{K}$, $W(U, v) \in \mathcal{U}(\Gamma(\mathcal{K}))$ (see [13] or [3] for further details). We obtain a Borel representation of $\mathcal{G}$ in $\Gamma(\mathcal{K})$ (called a *type S representation* in [12]) by

$$\mathfrak{U}(g) = \mathrm{e}^{\mathrm{i}\beta(g)} W(V(g), \eta(g)). \tag{5.3}$$

Now replace $\mathcal{G}$ by the current group $C(\mathbf{R}^+, \mathcal{G})$ of Borel maps with compact support from $\mathbf{R}^+$ to $\mathcal{G}$ where the group operations are defined pointwise. We wish to obtain a type S representation of $C(\mathbf{R}^+, \mathcal{G})$ in $\Gamma(\mathcal{J})$ where $\mathcal{J} = L^2(\mathbf{R}^+, \mathcal{K})$. In order to do this we replace the triple (V, η, β) by $(\widetilde{V}, \widetilde{\eta}, \widetilde{\beta})$ where (see [22]), for $\psi \in C(\mathbf{R}^+, \mathcal{G})$, $t \in \mathbf{R}^+$,

$$\widetilde{V}(\psi)(t) = V(\psi(t))$$
$$\widetilde{\eta}(\psi)(t) = \eta(\psi(t))$$
$$\widetilde{\beta}(\psi) = \int_0^t \beta(\psi(t))\,\mathrm{d}t.$$

We denote by $\widetilde{\mathfrak{U}}$ the type S representation corresponding to $(\widetilde{V}, \widetilde{\eta}, \widetilde{\beta})$ via (5.3).

From now on we will take $\mathcal{G} = \mathbf{R}^+$. $\{G_n,\ n \in \widetilde{\mathbf{N}}\}$ will be our usual partition of S but we make two additional assumptions:

(a) $$G_1 = G_2 = \cdots = G_m = \{0\} \quad \text{and} \quad \dim \mathcal{V} = m < \infty$$

where $\mathcal{V} = \mathrm{Ran}P(\{0\})$. As a consequence of (a) we see that for all $E \in \mathcal{B}(\mathbf{R}^n)$ for ν as given by (4.4) we have

$$\nu(E) = m\delta_0(E) + \nu(E \setminus \{0\})$$

where δ_0 is Dirac measure at the origin. In the sequel we will find it convenient to identify $\mathcal{V}$ with $\mathbf{R}^m$. We will also, without loss of generality, take $\{v_1, v_2, \ldots, v_m\}$ to be the natural basis in $\mathbf{R}^m$.

Our second assumption is that ν is a *Lévy measure* i.e.

(b) $$\int_{\mathbf{R}^n \setminus \{0\}} (x^2 \wedge 1)\nu(\mathrm{d}x) < \infty.$$

For $1 \leqslant j \leqslant n$, let $Y_1, \ldots, Y_n$ be the mutually commuting self-adjoint operators in $\mathcal{K}$ defined by

$$Y_j = \int_{\mathbf{R}^n} y_j P(\mathrm{d}y).$$

In the sequel, we will need to consider $Y = (Y_1, Y_2, \ldots, Y_n)$ as a column vector and we will also find a use for the self-adjoint operator

$$|Y|^2 = |Y_1|^2 + |Y_2|^2 + \cdots + |Y_n|^2 = \int_{\mathbf{R}^n} (y_1^2 + y_2^2 + \cdots + y_n^2)P(\mathrm{d}y).$$

We are now ready to define the basic triple (V, η, β): $(V(x),\ x \in \mathbf{R}^n)$ is the strongly continuous n-parameter group given by

(5.4) $$V(x) = \exp(\mathrm{i}x^{\mathrm{T}}Y)$$

where $x = (x^1, x^2, \ldots, x^n)$. $\eta : \mathbf{R}^n \to \mathcal{K}$ is the one-cocycle given by

(5.5) $$\eta(x) = \sigma x + \sum_{n=m+1}^{\infty} (V(x) - I)v_n$$

where σ is an $(m \times n)$ real valued matrix.

Finally define a continuous map $\beta : \mathbf{R}^n \to \mathbf{R}$ by

$$\beta(x) = b^{\mathrm{T}}x + \sum_{n=m+1}^{\infty} \left\langle v_n, \left(\sin(x^{\mathrm{T}}Y) - \frac{x^{\mathrm{T}}Y}{1+|Y|^2}\right) v_n \right\rangle \tag{5.6}$$

where $b \in \mathbf{R}^n$ (see [21], p. 159, Exercise 21.12).

Now as above we construct the triple $(\widetilde{V}, \widetilde{\eta}, \widetilde{\beta})$ and consider the representation $\widetilde{\mathfrak{U}}$ of $C(\mathbf{R}^+, \mathbf{R}^+)$ in $\Gamma(\mathcal{J})$ given by (5.3). Choose $\psi \in C(\mathbf{R}^+, \mathbf{R}^n)$ to be

$$\psi = x\chi_{[0,t)} \quad \text{for some } x \in \mathbf{R}^n, \ t \in \mathbf{R}^+$$

then if we define $\mathcal{W}_t(x) = \widetilde{\mathcal{U}}(\psi)$ for fixed $t \in \mathbf{R}^+$, we see that $(\mathcal{W}_t(x), \ x \in \mathbf{R}^n)$ is a strongly continuous n-parameter unitary group on $\Gamma(\mathcal{J})$ so that we have

$$\mathcal{W}_t(x) = \exp(\mathrm{i}x^{\mathrm{T}}X(t)) \tag{5.7}$$

where each $X(t)$ is a column vector comprising self-adjoint operators in $\Gamma(\mathcal{J})$. Arguing as in [21], p. 159, we obtain the Lévy-Khintchine formula

$$\langle e(0), \mathcal{W}_t(x)e(0)\rangle = \mathrm{e}^{t\zeta(x)}$$

where

$$\zeta(x) = \mathrm{i}b^{\mathrm{T}}x - \frac{1}{2}ax + \int_{\mathbf{R}^n\setminus\{0\}} \left(\mathrm{e}^{\mathrm{i}y^{\mathrm{T}}x} - 1 - \frac{\mathrm{i}y^{\mathrm{T}}x}{1+\|y\|^2}\right) \nu(\mathrm{d}y) \tag{5.8}$$

where a is the $n \times n$ matrix $\sigma^{\mathrm{T}}\sigma$.

From (5.7) and (5.8) we conclude that $(X(t), \ t \in \mathbf{R}^+)$ is a representation in $\Gamma(\mathcal{J})$ of an n-dimensional Lévy-process i.e. a stochastic process taking values in $\mathbf{R}^n$ with stationary and independent increments which is continuous in probability and satisfies $X(0) = 0$ almost surely.

We have the following decomposition for $X(t) = (X_1(t), \ldots, X_n(t))$, for $t \in \mathbf{R}^+$ and $1 \leqslant j \leqslant n$

$$\begin{aligned} X_j(t) = {} & b_j tI + \mathrm{i}\sigma_j^k(a(\chi_{[0,t)} \otimes v_k) - a^\dagger(\chi_{[0,t)} \otimes v_k)) \\ & + \sum_{n=m+1}^{\infty} \Big\{ a^\dagger(\chi_{[0,t)} \otimes Y_j v_n) + \lambda(\chi_{[0,t)} \otimes Y_j P(G_n)) \\ & + a(\chi_{[0,t)} \otimes Y_j v_n) + t\Big\langle v_n, \frac{Y_j|Y|^2}{1+|Y|^2} v_n \Big\rangle \Big\}. \end{aligned} \tag{5.9}$$

To verify (5.9), in the case where ν has bounded support we have $\mathcal{E}(\mathcal{J}) \subset \mathrm{Dom}(X_j(t))$ for all $t \in \mathbf{R}^+$ and we can explicitly compute

$$\langle e(f), X_j(t)e(g)\rangle = \frac{1}{\mathrm{i}}\,\frac{\partial}{\partial x_j}\langle e(f), \mathcal{W}_t(x)e(g)\rangle\Big|_{x=0}.$$

In the general case there is no guarantee that $\mathrm{Dom}(X_j(t))$ contains a total set of exponential vectors and (5.9) must be verified by using the technique of [21], p. 155–159.

If we compare (5.9) with the Lévy-Itô decomposition for Lévy-processes (see e.g. [17], p. 65) then it is tempting to rewrite it as

$$\begin{aligned} X_j(t) &= b_j tI + \mathrm{i}\sigma_j^k(a(\chi_{[0,t)} \otimes v_k) - a^\dagger(\chi_{[0,t)} \otimes v_k)) \\ &\quad + \sum_{n=m+1}^{\infty} \int_{G_n \times [0,t)} \lambda_j \Big\{A^\dagger(\mathrm{d}s, P(\mathrm{d}\lambda)v_n) + \Lambda(\mathrm{d}s, P(\mathrm{d}\lambda)) \\ &\quad + A(\mathrm{d}s, P(\mathrm{d}\lambda)v_n) + \frac{|\lambda|^2}{1+|\lambda|^2}\Big\}\nu(\mathrm{d}\lambda)\,\mathrm{d}s. \end{aligned} \tag{5.10}$$

Such a representation was anticipated by H. Araki — see Equation 9.7 on p. 420 of [5].

As in [2], we may then write, for $t \in \mathbf{R}^+$, $1 \leqslant k \leqslant m$

$$B_k(t) = \mathrm{i}(a(\chi_{[0,t)} \otimes v_k) - a^\dagger(\chi_{[0,t)} \otimes v_k)) \tag{5.11}$$

and we regard $B(t) = (B_1(t), \ldots, B_m(t))$ as the *Brownian part* of the generator $X(t)$ and for $G \in \mathcal{B}(\mathbf{R}^n \setminus \{0\})$, we write

$$\begin{aligned} \widetilde{N}(t,G) = \sum_{n=m+1}^{\infty} \int_{(G\cap G_n)\times[0,t)} &\{A^\dagger(\mathrm{d}s, P(\mathrm{d}\lambda)v_n) \\ &+ \Lambda(\mathrm{d}s, P(\mathrm{d}\lambda)) + A(\mathrm{d}s, P(\mathrm{d}\lambda)v_n)\} \end{aligned} \tag{5.12}$$

and we regard $\widetilde{N}$ as the *Poisson part* of $X(t)$. More precisely it plays the role of a compensated Poisson random measure on $\mathbf{R}^+ \times (\mathbf{R}^n \setminus \{0\})$ where the intensity measure is ν.

We will find these ideas of great value in the next section.

6. LÉVY FLOWS ON $*$-SUBALGEBRAS OF $B(\mathfrak{H}_0)$

Let $(\Omega, \mathfrak{F}, P)$ be a probability space and M be a finite-dimensional, paracompact, connected C^∞-manifold. Let $Z_1, Z_2, \ldots, Z_n$ be complete, smooth vector fields on M which are such that the Lie algebra generated by $\{Z_1, Z_2, \ldots, Z_n\}$ is finite dimensional.

Consider the stochastic integral equation defined for all $f \in C^\infty(M)$, $p \in M$ by

$$\begin{aligned} j_t(f)(p) &= f(p) + \int_0^t \sigma_k^i j_s(Z^k(f(p)))\,\mathrm{d}B_i(s) \\ &+ \int_0^{t+} \int_{\mathbf{R}^n\setminus\{0\}} [(j_{s^-} \circ k(x))(f)(p) - j_{s^-}(f)(p)]\widehat{N}(\mathrm{d}x, \mathrm{d}s) \\ &+ \int_0^t j_s(\mathcal{L}(f)(p))\,\mathrm{d}s \end{aligned} \tag{6.1}$$

where $B = (B^1, B^2, \ldots, B^m)$ is a standard Brownian motion, $\widehat{N}$ is a compensated Poisson random measure on $\mathbf{R}^+ \times (\mathbf{R}^n \setminus \{0\})$ with intensity measure ν and for each $x \in \mathbf{R}^n$, $k(x)$ is the automorphism of $C^\infty(M)$ given by

$$k(x)f(p) = f(\exp(x^j Z_j(p))$$

and the generator $\mathcal{L}$ is given by

$$\begin{aligned} \mathcal{L}(f)(p) &= b^j Z_j(f)(p) + \frac{1}{2}a^{ij} Z_i Z_j f(p) \\ &+ \int_{\mathbf{R}^n\setminus\{0\}} \left(k(x)(f)(p) - f(p) - \frac{x^j Z_j(f)(p)}{1+|x|^2}\right)\nu(\mathrm{d}x) \end{aligned} \tag{6.2}$$

where b, σ and a are as in the previous section.

In [1] it was shown that there exists a unique family of homomorphisms $\{j_t, t \in \mathbf{R}^+\}$ of $C^\infty(M)$ into $L^\infty(M \times \Omega,\ \mu \times P)$ satisfying (6.2) where μ is a fixed *Lebesgue measure* on M (see [18], p. 158) such that

$$j_t(f) = f \circ \Phi_{0,t} \tag{6.3}$$

where $\Phi = \{\Phi_{s,t},\ 0 \leqslant s \leqslant t < \infty\}$ is a Lévy flow of diffeomorphisms of M. Furthermore in [4], a family of unitary operators $(U(t),\ t \in \mathbf{R}^+)$ was constructed on $L^2(\Omega, \mathfrak{F}, P; \mathfrak{H}_0)$ where $\mathfrak{H}_0$ is the intrinsic Hilbert space of (M, μ) such that

$$j_t(f) = U(t) f U(t)^* \tag{6.4}$$

and it was shown that $U(t)$ had the stochastic integral representation

$$U(t) = I + \int_0^t \sigma_k^i U(s)\widetilde{Z}^k \, \mathrm{d}B_i(s) + \int_0^t \int_{\mathbf{R}^n\setminus\{0\}} U(s)(S(x) - I)\widehat{N}(\mathrm{d}x, \mathrm{d}s) + \int_0^t U(s)\mathfrak{N} \, \mathrm{d}s \tag{6.5}$$

where the following are all densely-defined linear operators in $\mathfrak{H}_0$, $\widetilde{Z} = Z_k + \frac{1}{2}\mathrm{div}(Z_k)$ for $1 \leqslant k \leqslant m$ is essentially skew-adjoint, $S(x) = \exp(\overline{x^k \widetilde{Z}_k})$ for $x \in \mathbf{R}^n$ is unitary and

$$\mathfrak{N} = b^j \widetilde{Z}_j + \frac{1}{2} a^{ij} \widetilde{Z}_i \widetilde{Z}_j + \int_{\mathbf{R}^n\setminus\{0\}} \left(S(x) - I - \frac{x^j \widetilde{Z}_j}{1 + |x|^2} \right) \nu(\mathrm{d}x).$$

We wish to *quantise* this construction and in effect to *quantise* the Lévy flow Φ. Using the canonical isomorphism between $L^2(\Omega, \mathfrak{F}, P; \mathfrak{H}_0)$ and $\mathfrak{H}_0 \otimes L^2(\Omega, \mathfrak{F}, P)$, we now take $\mathfrak{H}_0$ to be arbitrary and inspired by the discussion of the previous section, we replace $L^2(\Omega, \mathfrak{F}, P)$ by $\Gamma(\mathcal{J})$. The role of $C^\infty(M)$ will be played by an arbitrary $*$-subalgebra $\mathcal{A}$ of $B(\mathfrak{H}_0)$. We will consider generalisations of Equations (6.1) and (6.5) wherein B and $\widehat{N}$ are replaced by their quantum analogues (5.11) and (5.12).

Our first task will then be to investigate the conditions under which there exists a unique unitary process $U = (U(t),\ t \in \mathbf{R}^+)$ in $\mathfrak{H}$ satisfying the equation

$$U(t) = I + \int_0^t \sigma_k^i U(s) T^k \, \mathrm{d}B_i(s) + \int_{(\mathbf{R}^n\setminus\{0\})\times[0,t)} U(s)(W(x) - I)\widetilde{N}(\mathrm{d}x, \mathrm{d}s) + \int_0^t U(s)\mathfrak{M} \, \mathrm{d}s \tag{6.6}$$

where $T_1, \ldots, T_n$ are arbitrary densely-defined essentially skew-adjoint operators in $\mathfrak{H}_0$ with common invariant core $\mathfrak{D}_0$ and are such that $x^j T_j$ is itself essentially skew-adjoint for each $x \in \mathbf{R}^n$, $W(x)$ is the unitary operator $\exp(\overline{x^j T_j})$ and

$$\mathcal{M} = b^j T_j + \frac{1}{2} a^{ij} T_i T_j + \int_{\mathbf{R}^n\setminus\{0\}} \left(W(x) - I - \frac{x^j T_j}{1 + |x|^2} \right) \nu(\mathrm{d}x).$$

We consider two cases in which the required process can be constructed.

(a) Let $T_1, T_2, \ldots, T_n$ all be bounded.

We will appeal to Theorem 4.4 and make the following choices therein with $\mathfrak{D}_0 = \mathfrak{H}_0$,

$$L^{(j)}(0) = \sigma_j^k T_k \quad \text{for } 1 \leqslant j \leqslant m$$
$$L^{(j)}(x) = W(x) - I \quad \text{for } j > m,\ x \neq 0$$
$$L_3(x) = W(x) - I \quad \text{for all } x \in \mathbf{R}^n$$

$$H = -\mathrm{i}b^j T_j + \sum_{n=m+1}^{\infty} \int_{G_n} \left(\sin(x^j T_j) + \mathrm{i}\frac{x^j T_j}{1+|x|^2} \right) \|P(\mathrm{d}x)v_n\|^2.$$

In order to apply Theorem 4.4, we need to show that condition (iv) at the beginning of Section 4 is satisfied. To see this note that each $W(x)$ can be embedded into a strongly-continuous one parameter group and so for each $x \in \mathbf{R}^n$ there exists a projection-valued measure Q_x on $\mathfrak{H}_0$ such that $W(x) = \int_{\mathbf{R}} \mathrm{e}^{\mathrm{i}z(x)} Q_x(\mathrm{d}z)$. Hence we find that for each $u \in \mathfrak{H}_0$

$$\sum_{n=m+1}^{\infty} \int_{G_n} \|L^{(n)}(x)u\|^2 \|P(\mathrm{d}x)v_n\|^2$$
$$= \int_{\mathbf{R}^n \setminus \{0\}} \|(W(x) - I)u\|^2 \nu(\mathrm{d}x)$$
$$\leqslant \int_{|x|\leqslant 1} \int_{\mathbf{R}} |\mathrm{e}^{\mathrm{i}z(x)} - 1|^2 \|Q_x(\mathrm{d}z)u\|^2 \nu(\mathrm{d}x) + 4\nu(|x| > 1)\|u\|^2.$$

Now using the inequality $|\mathrm{e}^{\mathrm{i}z(x)} - 1| \leqslant |z(x)|$, we find the above integral is majorised by

$$\int_{|x|\leqslant 1} \|x^j T_j u\|^2 \nu(\mathrm{d}x) \leqslant \max_{1\leqslant j\leqslant n} \|T_j u\|^2 \int_{|x|\leqslant 1} |x|^2 \nu(\mathrm{d}x)$$

and the result now follows from the fact that ν is a Lévy measure.

A similar argument shows that H is a well-defined bounded self-adjoint operator.

(b) Suppose $T_1, T_2, \ldots, T_n$ are all Borel functions of an unbounded self-adjoint operator

$$T = \int_{\mathbf{R}} z Q(\mathrm{d}z)$$

so that each $T_j = g_j(T)$ for $1 \leqslant j \leqslant n$ then we may follow the procedure of [3] and define

$$U(t) = \int_{\mathbb{R}} \mathcal{W}_t(g(z))Q(\mathrm{d}z)$$

where $\mathcal{W}_t$ is as given by (5.7) and $g(z) = (g_1(z), g_2(z), \ldots, g_n(z))$ for $z \in \mathbb{R}$.

In either of cases (a) and (b) we can define the flow J as in (4.6) on a unital $*$-subalgebra $\mathcal{A}$ of $B(\mathfrak{H}_0)$. We then obtain the representation

$$\begin{aligned} j_t(a) = a + \int_0^t \sigma_k^i j_s([T^k, a])\,\mathrm{d}B_i(s) \\ + \int_{(\mathbb{R}^n\setminus\{0\})\times[0,t)} j_s(W(x)aW(x)^* - a)\widetilde{N}(\mathrm{d}x, \mathrm{d}s) + \int_0^t j_s(\tau(a))\,\mathrm{d}s \end{aligned} \tag{6.7}$$

where the generator

$$\begin{aligned} \tau(a) = b^j[T_j, a] + \frac{1}{2}a^{ij}[T_i, [T_j, a]] \\ + \int_{\mathbb{R}^n\setminus\{0\}} \Big\{W(x)aW(x)^* - a - \frac{x^j}{1+|x|^2}[T_j, a]\Big\}\nu(\mathrm{d}x). \end{aligned} \tag{6.8}$$

Based on the analogy between (6.1) and (6.7) we call J a *Lévy flow on* $\mathcal{A}$. It is interesting to compare the form of the generator τ in (6.8) with that considered by Hunt in relation to convolution semigroups of measures on Lie groups [16] (see also [10]).

Finally, we can construct a class of Lévy flows on $\mathcal{A}$ with infinitely many degrees of freedom and bounded coefficients. For convenience we take $n = 1$.

We assume now that $\dim \mathrm{Ran}P(\{0\}) = \infty$ and we write

$$\{v_n,\ n \in \mathbb{N}\} = \{u_n,\ n \in \mathbb{N}\} \cup \{w_n,\ n \in \mathbb{N}\},$$

and take $\{u_n,\ n \in \mathbb{N}\}$ to be a maximal orthonormal set in $\mathrm{Ran}P(\{0\})$. We now take for each $n \in \mathbb{N}$, $W_n = \exp(xT_n)$ to be a strongly continuous one-parameter group wherein each T_n is a bounded skew-adjoint operator. In Theorem 4.4, we take

$$L^{(n)}(0) = T_n, \quad n \in \mathbb{N}$$
$$L^{(n)}(x) = W_n(x) - I, \quad n \in \mathbb{N},\ x \neq 0$$

then we can assert unitarity of solutions to the equation

$$
\begin{aligned}
U(t) = I + \sum_{n=1}^{\infty} \int_0^t U(s)T_n \, \mathrm{d}B_n(s) \\
+ \sum_{n=1}^{\infty} \int_{G_n \times [0,t)} U(s)(W_n - I)\widetilde{N}_n(\mathrm{d}x, \mathrm{d}s) + \int_0^t U(s)\mathcal{M} \, \mathrm{d}s
\end{aligned}
\tag{6.9}
$$

where each

$$B_n(t) = \mathrm{i}(a(\chi_{[0,t)} \otimes u_n) - a^{\dagger}(\chi_{[0,t)} \otimes u_n))$$

$$\widetilde{N}_n(\mathrm{d}s, \mathrm{d}x) = A^{\dagger}(\mathrm{d}s, P(\mathrm{d}\lambda)w_n) + \Lambda(\mathrm{d}s, P(\mathrm{d}\lambda)) + A(\mathrm{d}s, P(\mathrm{d}\lambda)w_n)$$

and

$$\mathcal{M} = \mathrm{i}H + \frac{1}{2}\sum_{n=1}^{\infty} T_n^2 + \sum_{n=1}^{\infty} \int_{G_n} (\cos(xT_n) - I)\|P(\mathrm{d}x)w_n\|^2 \tag{6.10}$$

where H is a self-adjoint operator in $\mathfrak{H}_0$.

The required flows may now be constructed by unitary conjugation in the usual way.

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Received October 23, 1995.