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ON THE ITERATES OF A PARACOMPLETE OPERATOR

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ABSTRACT. In this paper we obtain some density results concerning convenient topologies on the domains of the iterates of a (possibly unbounded and non-everywhere defined) linear operator acting in a Banach space, for which a suitable essential resolvent set is nonempty.

KEYWORDS: *Unbounded linear operators in Banach spaces, paracomplete operators, dense subsets of the domains of the iterates, upper and lower essential resolvent sets.*

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INTRODUCTION

This paper deals with the iterates of a linear operator acting in a Banach space.

The following result has been recently proved by C. Lennard ([10]), by means of the Mittag-Leffler theorem about inverse limits.

THEOREM 0.1. ([10], 1.3) *Let A be a closed densely defined linear operator, with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ in a Banach space X, having nonempty resolvent set. If $(\alpha_n)_{n\in\mathbf{N}}$ and $(\beta_n)_{n\in\mathbf{N}}$ are scalar sequences, such that $(\alpha_n A+\beta_n I_X)(\mathcal{D}(A))$ is dense in X for any $n\in\mathbf{N}$, then*

$$\bigcap_{n=1}^{\infty}(\alpha_0 A+\beta_0 I_X)\cdots(\alpha_{n-1}A+\beta_{n-1}I_X)(\mathcal{D}(A^n))$$

is dense in X. In particular, it follows that

(i) $\bigcap_{n\in\mathbf{N}}\mathcal{D}(A^n)$ *is dense in X;*

(ii) $\bigcap_{n \in \mathbf{N}} \mathcal{R}(A^n)$ *is dense in* X *if* $\mathcal{R}(A)$ *is dense in* X.

Here we extend Theorem 0.1 to a larger class of closed operators than the densely defined ones having nonempty resolvent set. Actually, we extend Theorem 0.1 to a set of linear operators with domain and range in X which is not contained in the set of all closed operators with domain and range in X, but is contained in the larger set of all paracomplete operators with domain and range in X (namely, the operators whose graph, under a convenient norm, is a Banach space continuously embedded in $X \times X$).

In Section 1 we collect some preliminaries, in the attempt of making our paper as self-contained as possible.

Section 2 contains the results of this paper. After proving some preliminary algebraic results, we introduce the upper and lower essential resolvent sets of a linear operator A with domain and range in X (see definitions between Remark 2.5 and Remark 2.6). The upper and lower essential resolvent sets of A contain the resolvent set of A. Indeed, they contain the set of all scalars λ for which $\lambda I_X - A$ is a closed Fredholm operator. It turns out that an operator A having nonempty lower essential resolvent set must be paracomplete, together with all polynomials in A, but need not be closed, contrary to the case of an operator having nonempty upper essential resolvent set (see Remark 2.6). In Theorem 2.10 and Corollary 2.11 we extend Theorem 0.1 to the operators having nonempty lower essential resolvent set; more generally, we obtain density results involving the topologies induced by the norms which make the domains of the iterates of such operators Banach spaces continuously embedded in X. In particular, Corollary 2.11 enables us to conclude that, if the lower essential resolvent set of a densely defined linear operator A with domain and range in X is nonempty, then the intersection of the domains of the iterates of A is dense in X; more generally, it is dense in the domain of every iterate of A, with respect to the corresponding topology. This generalizes a result obtained by K. Schmüdgen in [17] for densely defined closed symmetric operators acting in Hilbert spaces. In the remarks following Corollary 2.11, we also show that Theorem 2.10 and Corollary 2.11 cannot be extended to the operators having nonempty upper essential resolvent set.

Throughout this paper, when the scalar field is not specified we assume it may be either $\mathbf{R}$ or $\mathbf{C}$ and denote it by $\mathbf{K}$.

1. SECTION

We will denote the sets of nonnegative integers and of positive integers by $\mathbf{N}$ and $\mathbf{Z}_+$, respectively.

If V and W are vector spaces over $\mathbf{K}$, let $\Lambda(V,W)$ denote the set of all linear operators whose domain is a linear subspace of V and whose range is contained in W. In the special case $W=V$ we shall use the notation $\Lambda(V)$ instead of $\Lambda(V,V)$. I_V will denote the identity operator on V, namely the element of $\Lambda(V)$ which is defined on the whole of V and maps every $x\in V$ into x.

For any $A\in\Lambda(V,W)$ we will denote the domain of A by $\mathcal{D}(A)$, and the kernel and range of A by $\mathcal{N}(A)$ and $\mathcal{R}(A)$, respectively. Moreover, we define

$$\mathrm{nul}(A)=\dim(\mathcal{N}(A))$$

and

$$\mathrm{def}(A)=\dim(W/\mathcal{R}(A))$$

where $\dim(M)=\sup\{n\in\mathbf{N}: M \text{ contains } n \text{ linearly independent vectors}\}$ $(\in\mathbf{N}\cup\{\infty\})$ for any vector space M.

If $\mathcal{N}(A)=\{0\}$, then there exists a unique operator $A^{-1}\in\Lambda(W,V)$ such that $\mathcal{D}(A^{-1})=\mathcal{R}(A)$ and $A^{-1}Ax=x$ for any $x\in\mathcal{D}(A)$. We call A^{-1} the inverse of A. Notice that $\mathcal{N}(A^{-1})=\{0\}$, $\mathcal{R}(A^{-1})=\mathcal{D}(A)$ and $AA^{-1}y=y$ for any $y\in\mathcal{R}(A)$, namely $(A^{-1})^{-1}=A$.

If V_0, V_1, V_2 are vector spaces over $\mathbf{K}$, $A_1\in\Lambda(V_0,V_1)$ and $A_2\in\Lambda(V_1,V_2)$, the operator $A_2A_1\in\Lambda(V_0,V_2)$ is defined by

$$\mathcal{D}(A_2A_1)=\{x\in\mathcal{D}(A_1): A_1x\in\mathcal{D}(A_2)\}$$

and

$$(A_2A_1)x=A_2(A_1x)\quad\text{for any } x\in\mathcal{D}(A_2A_1).$$

It is not difficult to verify that, if V_3 is a vector space over $\mathbf{K}$ and $A_3\in\Lambda(V_2,V_3)$, then $A_3(A_2A_1)=(A_3A_2)A_1$ (so that parentheses can be omitted). The product $A_n\dots A_1$ of a finite number of linear operators is canonically defined by induction.

Now let V be a vector space over $\mathbf{K}$ and let $A\in\Lambda(V)$. Then the sequence $(A^n)_{n\in\mathbf{N}}$ of elements of $\Lambda(V)$ is defined in the following way: $A^0=I_V$ and $A^n=A_1\cdots A_n$ (where $A_k=A$ for any $k=1,\dots,n$) for any $n\in\mathbf{Z}_+$ (so that $A^1=A$). Thus $(\mathcal{D}(A^n))_{n\in\mathbf{N}}$ is a non-increasing sequence of linear subspaces of V. We also remark that, for any $j,k\in\mathbf{N}$, we have

$$\mathcal{D}(A^{k+j})=\{x\in\mathcal{D}(A^k): A^kx\in\mathcal{D}(A^j)\}$$

and

$$A^{k+j}x = A^j(A^k x) \quad \text{for any } x \in \mathcal{D}(A^{k+j}).$$

If $A_j \in \Lambda(V, W)$ and $\lambda_j \in \mathbf{K}$ for any $j = 1, \ldots, n$ (where V and W are vector spaces over $\mathbf{K}$ and $n \in \mathbf{Z}_+$), the operator $\sum_{j=1}^{n} \lambda_j A_j \in \Lambda(V, W)$ is defined by

$$\mathcal{D}\Big(\sum_{j=1}^{n} \lambda_j A_j\Big) = \bigcap_{j=1}^{n} \mathcal{D}(A_j)$$

and

$$\Big(\sum_{j=1}^{n} \lambda_j A_j\Big)x = \sum_{j=1}^{n} \lambda_j A_j x \quad \text{for any } x \in \bigcap_{j=1}^{n} \mathcal{D}(A_j).$$

Finally, let $n \in \mathbf{N}$ and let $a_0, \ldots, a_n \in \mathbf{K}$, with $a_n \neq 0$ if $n \in \mathbf{Z}_+$. If $\mathfrak{p}$ denotes the polynomial of degree n defined by

$$\mathfrak{p}(\lambda) = \sum_{k=0}^{n} a_k \lambda^k \quad \text{for any } \lambda \in \mathbf{K}$$

(notice that we assume the null polynomial to be of degree zero), then the operator $\mathfrak{p}(A) \in \Lambda(V)$ is defined by

$$\mathfrak{p}(A) = \sum_{k=0}^{n} a_k A^k.$$

Notice that $\mathcal{D}\big(\mathfrak{p}(A)\big) = \mathcal{D}(A^n)$. We also remark that, if $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are polynomials with coefficients in $\mathbf{K}$, of degrees n_1 and n_2, respectively, then

$$\mathfrak{p}_1(A)x + \mathfrak{p}_2(A)x = (\mathfrak{p}_1 + \mathfrak{p}_2)(A)x \quad \text{for any } x \in \mathcal{D}\big(A^{\max\{n_1,n_2\}}\big).$$

Furthermore, it is not difficult to verify that

$$\mathcal{D}(A^{n_1+n_2}) \subset \mathcal{D}\big(\mathfrak{p}_1(A)\mathfrak{p}_2(A)\big) \subset \mathcal{D}\big(\mathfrak{p}_1\mathfrak{p}_2(A)\big)$$

(where each of the inclusions above can be replaced by equality if in addition both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are nonzero), and

$$\mathfrak{p}_1(A)\mathfrak{p}_2(A)x = \mathfrak{p}_1\mathfrak{p}_2(A)x \quad \text{for any } x \in \mathcal{D}(A^{n_1+n_2}).$$

Namely,

$$\mathfrak{p}_1(A)\mathfrak{p}_2(A) = \mathfrak{p}_1\mathfrak{p}_2(A) = \mathfrak{p}_2(A)\mathfrak{p}_1(A)$$

when both polynomials $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are nonzero.

Now let Z be a Banach space. Following [9], Definition 2.1.1, we shall call a linear subspace W of Z, which can be endowed with a complete norm $\|\cdot\|_W$ such that the canonical embedding of $(W, \|\cdot\|_W)$ into Z is continuous, a *paracomplete* subspace of Z. Notice that every complete norm on W, under which the canonical embedding of W into Z is continuous, is equivalent to $\|\cdot\|_W$ in virtue of the closed graph theorem. Thus all such norms induce the same topology, which we will denote by τ_W, on W.

If X and Y are Banach spaces over $\mathbf{K}$, we denote by $L(X,Y)$ the Banach space of all linear bounded operators from X into Y (namely, the operators $A \in \Lambda(X,Y)$ such that $\mathcal{D}(A) = X$ and A is continuous). Furthermore, we set

$$C(X,Y) = \{A \in \Lambda(X,Y) : A \text{ is closed}\}.$$

We recall that $L(X,Y) \subset C(X,Y)$. In the special case $Y = X$, we will write $C(X)$ and $L(X)$ instead of $C(X,X)$ and $L(X,X)$, respectively.

We say that $A \in \Lambda(X,Y)$ has an inverse in $L(Y,X)$ if $\mathcal{N}(A) = \{0\}$ and $A^{-1} \in L(Y,X)$ (which implies that $\mathcal{R}(A) = Y$). Notice that $A^{-1} \in C(Y,X)$ for any $A \in C(X,Y)$ with $\mathcal{N}(A) = \{0\}$. Then, in virtue of the closed graph theorem, an operator $A \in C(X,Y)$ has an inverse in $L(Y,X)$ if and only if $\mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A) = Y$.

An operator $T \in \Lambda(X,Y)$ is called *paracomplete* (see [9], Definition 2.1.2) if its graph is a paracomplete subspace of $X \times Y$. We set

$$PC(X,Y) = \{T \in \Lambda(X,Y) : T \text{ is paracomplete}\}$$

(in the special case $Y = X$, we will write $PC(X)$ instead of $PC(X,X)$). Notice that $C(X,Y) \subset PC(X,Y)$. We also recall (see [9], remark preceding Proposition 2.1.3, and Proposition 2.1.4) that

FACT 1.1. *For any $A \in PC(X,Y)$, $\mathcal{D}(A)$ (respectively, $\mathcal{R}(A)$) is a paracomplete subspace of X (respectively, Y).*

The following result about closedness of range, which we will need in the sequel, is a consequence of Fact 1.1 and [9], Proposition 2.1.1.

THEOREM 1.2. *Let X and Y be Banach spaces, and let $A \in PC(X,Y)$. If $\operatorname{def}(A) < \infty$, then $\mathcal{R}(A)$ is closed.*

We will also need the following generalization of the closed graph theorem.

THEOREM 1.3. (see [9], Proposition 2.1.5) *Let X and Y be Banach spaces, and let $A \in PC(X,Y)$. If $\mathcal{D}(A) = X$, then $A \in L(X,Y)$.*

The following result shows that, unlike the closed operators, the class of paracomplete operators is closed under sum and product.

THEOREM 1.4. (see [9], Proposition 2.1.3) *Let X, Y, Z be Banach spaces, and let $A, B \in PC(X,Y)$, $D \in PC(Y,Z)$. Then $A + B \in PC(X,Y)$ and $DA \in PC(X,Z)$.*

We will also use the Mittag-Leffler theorem on inverse limits, which is recorded here as Theorem 1.5. We recall (see for instance [3], beginning of Section 2) that, if $(E_n)_{n\in\mathbf{N}}$ is a sequence of nonempty sets, and $(\theta_n)_{n\in\mathbf{N}}$ is a sequence of maps, with $\theta_n : E_{n+1} \to E_n$ for any $n \in \mathbf{N}$, then the projective limit $\varprojlim(E_n, \theta_n)_{n\in\mathbf{N}}$ is defined to be the set of all elements $(x_n)_{n\in\mathbf{N}}$ of the Cartesian product $\prod\limits_{n\in\mathbf{N}} E_n$ satisfying $x_n = \theta_n(x_{n+1})$ for any $n \in \mathbf{N}$. For any $k \in \mathbf{N}$, let π_k denote the k^{th} coordinate projection from $\prod\limits_{n\in\mathbf{N}} E_n$ onto E_k. Notice that

$$\pi_k\big(\varprojlim(E_n, \theta_n)_{n\in\mathbf{N}}\big) = \{x \in E_k : \text{ there exists } (x_n)_{n\geqslant k} \in \prod_{n\geqslant k} E_n \\ \text{satisfying } x_k = x \text{ and } x_n = \theta_n(x_{n+1}) \quad \text{for any } n \geqslant k\}.$$

THEOREM 1.5. (see for example [3], 2.2, or [1], II, 3.5, Theorem 1) *Let $(E_n)_{n\in\mathbf{N}}$ be a sequence of complete metric spaces, and let $\theta_n : E_{n+1} \to E_n$ be continuous for any $n \in \mathbf{N}$. If $\theta_n(E_{n+1})$ is dense in E_n for any $n \in \mathbf{N}$, then $\pi_k\big(\varprojlim(E_n, \theta_n)_{n\in\mathbf{N}}\big)$ is dense in E_k for any $k \in \mathbf{N}$.*

2. SECTION

We begin by proving a few purely algebraic results, which will be useful in the sequel.

LEMMA 2.1. *Let V be a vector space over $\mathbf{K}$, let $A \in \Lambda(V)$ and let $\mathfrak{p}$ be a nonzero polynomial with coefficients in $\mathbf{K}$. Then*

(i) $\mathcal{N}(\mathfrak{p}(A)) \subset \bigcap\limits_{m\in\mathbf{N}} \mathcal{D}(A^m)$;

(ii) $\mathcal{N}(\mathfrak{p}(A)) \subset \mathcal{D}(\mathfrak{q}(A))$ *and* $\mathfrak{q}(A)\Big(\mathcal{N}(\mathfrak{p}(A))\Big) \subset \mathcal{N}(\mathfrak{p}(A))$ *for any polynomial $\mathfrak{q}$ with coefficients in $\mathbf{K}$.*

Proof. We begin by proving (i). Let n denote the degree of $\mathfrak{p}$. Since the desired inclusion is straightforward if $n = 0$ (as $\mathcal{N}(\mathfrak{p}(A)) = \{0\}$, being $\mathfrak{p}$ nonzero), we assume $n \in \mathbf{Z}_+$.

Let $a_0, \ldots, a_n \in \mathbf{K}$ be such that $\mathfrak{p}(\lambda) = \sum_{k=0}^{n} a_k \lambda^k$ for any $\lambda \in \mathbf{K}$. Then $a_n \neq 0$. We prove by induction that $\mathcal{N}(\mathfrak{p}(A)) \subset \mathcal{D}(A^m)$ for any $m \geqslant n$. The desired inclusion holds for $m = n$, as $\mathcal{N}(\mathfrak{p}(A)) \subset \mathcal{D}(\mathfrak{p}(A)) = \mathcal{D}(A^n)$. Now suppose that $\mathcal{N}(\mathfrak{p}(A)) \subset \mathcal{D}(A^m)$ for some $m \geqslant n$. Then for any $x \in \mathcal{N}(\mathfrak{p}(A))$ we have

$$A^n x = -\frac{1}{a_n} \sum_{k=0}^{n-1} a_k A^k x \in \mathcal{D}(A^{m-n+1}),$$

and consequently $x \in \mathcal{D}(A^{m+1})$. We have thus established (i).

Now we prove (ii). Let $\mathfrak{q}$ be a polynomial with coefficients in $\mathbf{K}$. From (i) we get $\mathcal{N}(\mathfrak{p}(A)) \subset \mathcal{D}(\mathfrak{q}(A))$. Furthermore, since for any $x \in \mathcal{N}(\mathfrak{p}(A)) \subset \bigcap_{m \in \mathbf{N}} \mathcal{D}(A^m)$ we have

$$\mathfrak{p}(A)\mathfrak{q}(A)x = \mathfrak{q}(A)\mathfrak{p}(A)x = 0,$$

namely $\mathfrak{q}(A)x \in \mathcal{N}(\mathfrak{p}(A))$, we obtain $\mathfrak{q}(A)(\mathcal{N}(\mathfrak{p}(A))) \subset \mathcal{N}(\mathfrak{p}(A))$, which completes the proof. ∎

For any real vector space V, we denote the complexification of V (see [16], page 33) by $\widetilde{V}$. We recall that, if Y is a real Banach space, then $\widetilde{Y}$ is a complex Banach space under a convenient norm induced by the norm of Y (see [16], page 261). Now let V be a real vector space and let $T \in \Lambda(V)$. We recall that the complex extension of T is the operator $\widetilde{T} \in \Lambda(\widetilde{V})$ defined by

$$\mathcal{D}(\widetilde{T}) = \mathcal{D}(T) + \mathrm{i}\mathcal{D}(T)$$

and

$$\widetilde{T}(x + \mathrm{i}y) = Tx + \mathrm{i}Ty \quad \text{for any } x, y \in \mathcal{D}(T).$$

Notice that $(\widetilde{T})^n = \widetilde{T^n}$ for any $n \in \mathbf{N}$.

LEMMA 2.2. *Let V be a vector space over $\mathbf{K}$, let $A \in \Lambda(V)$ and let $\mathfrak{p}, \mathfrak{q}$ be polynomials with coefficients in $\mathbf{K}$. If $\mathfrak{p}$ and $\mathfrak{q}$ have no common roots in $\mathbf{C}$, then*

(i) $\mathcal{N}(\mathfrak{p}(A)) = \mathfrak{q}(A)\big(\mathcal{N}(\mathfrak{p}(A))\big)$;

(ii) $\mathcal{R}(\mathfrak{p}(A)) \cap \mathcal{R}(\mathfrak{q}(A)) = \mathcal{R}(\mathfrak{p}\mathfrak{q}(A))$.

Proof. We first prove (i). If $\mathfrak{p}$ is the null polynomial, then $\mathfrak{q}$ has no roots in $\mathbf{C}$, namely $\mathfrak{q}$ is a nonzero polynomial of degree zero, and consequently the desired equality is straightforward. Now suppose $\mathfrak{p}$ to be nonzero. Then Lemma 2.1

provides the inclusions $\mathcal{N}(\mathfrak{p}(A)) \subset \mathcal{D}(\mathfrak{q}(A))$ and $\mathfrak{q}(A)\big(\mathcal{N}(\mathfrak{p}(A))\big) \subset \mathcal{N}(\mathfrak{p}(A))$. Since $\mathfrak{p}$ and $\mathfrak{q}$ have no common roots in $\mathbf{C}$, and consequently have no nontrivial common factors, there exist polynomials $\mathfrak{r}$ and $\mathfrak{s}$ with coefficients in $\mathbf{K}$ such that

$$\mathfrak{r}(\lambda)\mathfrak{p}(\lambda) + \mathfrak{s}(\lambda)\mathfrak{q}(\lambda) = 1 \quad \text{for any } \lambda \in \mathbf{K}$$

(as in the algebra of polynomials with coefficients in $\mathbf{K}$ every ideal is principal). If n denotes the maximum between the sums of the degrees of $\mathfrak{r}$ and $\mathfrak{p}$ and of $\mathfrak{s}$ and $\mathfrak{q}$, then for any $x \in \mathcal{D}(A^n)$ we have

$$x = (\mathfrak{r}\mathfrak{p} + \mathfrak{s}\mathfrak{q})(A)x = \mathfrak{r}\mathfrak{p}(A)x + \mathfrak{s}\mathfrak{q}(A)x = \mathfrak{r}(A)\mathfrak{p}(A)x + \mathfrak{s}(A)\mathfrak{q}(A)x.$$

Since $\mathcal{N}(\mathfrak{p}(A)) \subset \bigcap_{m\in\mathbf{N}} \mathcal{D}(A^m) \subset \mathcal{D}(A^n)$ by Lemma 2.1, it follows that for any $x \in \mathcal{N}(\mathfrak{p}(A))$ we have

$$x = \mathfrak{s}(A)\mathfrak{q}(A)x = \mathfrak{q}(A)\mathfrak{s}(A)x.$$

Hence $\mathcal{N}(\mathfrak{p}(A)) \subset \mathfrak{q}(A)\big(\mathfrak{s}(A)(\mathcal{N}(\mathfrak{p}(A)))\big)$. Since $\mathfrak{s}(A)(\mathcal{N}(\mathfrak{p}(A))) \subset \mathcal{N}(\mathfrak{p}(A))$ in virtue of Lemma 2.1, it follows that $\mathcal{N}(\mathfrak{p}(A)) \subset \mathfrak{q}(A)(\mathcal{N}(\mathfrak{p}(A)))$, which establishes (i).

Now we prove (ii). It is easily seen that $\mathcal{R}(\mathfrak{p}\mathfrak{q}(A)) \subset \mathcal{R}(\mathfrak{p}(A)) \cap \mathcal{R}(\mathfrak{q}(A))$. Thus it remains to prove the opposite inclusion. It suffices to prove the desired inclusion for $\mathbf{K} = \mathbf{C}$; once the result is established in the complex case, it can be applied to the complex extension of A, and the real case follows by remarking that $\widetilde{\mathfrak{t}(A)} = \mathfrak{t}(\widetilde{A})$ for any polynomial $\mathfrak{t}$ with real coefficients. Thus we assume $\mathbf{K} = \mathbf{C}$.

We prove first that, for any $\alpha \in \mathbf{C}$ and for any $n \in \mathbf{N}$, we have

$$\mathcal{R}(A - \alpha I_V) \cap \mathcal{R}(\mathfrak{t}(A)) \subset \mathcal{R}((A - \alpha I_V)\mathfrak{t}(A)) \tag{2.1}$$

for any polynomial $\mathfrak{t}$ of degree n, with complex coefficients and such that α is not a root of $\mathfrak{t}$.

We proceed by induction. (2.1) clearly holds for $n = 0$. Now suppose (2.1) to be satisfied for some $n \in \mathbf{N}$, and let $\mathfrak{u}$ be a polynomial with complex coefficients, of degree $n+1$ and such that α is not a root of $\mathfrak{u}$. Let $\beta \in \mathbf{C}$ be a root of $\mathfrak{u}$. Then there exists a nonzero polynomial $\mathfrak{t}$ of degree n such that

$$\mathfrak{u}(\lambda) = (\lambda - \beta)\mathfrak{t}(\lambda) \quad \text{for any } \lambda \in \mathbf{C}$$

and consequently

$$\mathfrak{u}(A) = (A - \beta I_V)\mathfrak{t}(A) = \mathfrak{t}(A)(A - \beta I_V).$$

Notice that $\beta \neq \alpha$ and α is not a root of $\mathfrak{t}$. For any $x \in \mathcal{R}(A - \alpha I_V) \cap \mathcal{R}(\mathfrak{u}(A))$, there exist $y \in \mathcal{D}(A)$ and $z \in \mathcal{D}(A^{n+1})$ such that

$$x = (A - \alpha I_V)y = \mathfrak{t}(A)(A - \beta I_V)z.$$

Since $x \in \mathcal{R}(A - \alpha I_V) \cap \mathcal{R}(\mathfrak{t}(A))$ and $\mathfrak{t}$ has degree n, it follows that

$$x = (A - \alpha I_V)\mathfrak{t}(A)v = \mathfrak{t}(A)(A - \alpha I_V)v$$

for some $v \in \mathcal{D}(A^{n+1})$. Then $(A - \beta I_V)z - (A - \alpha I_V)v \in \mathcal{N}(\mathfrak{t}(A))$, which is contained in $\mathcal{R}(A - \alpha I_V)$ by (i), as α is not a root of $\mathfrak{t}$. Consequently, there exists $w \in \mathcal{D}(A)$ such that $(A - \beta I_V)z = (A - \alpha I_V)w$. Since $\beta \neq \alpha$, it follows that

$$z = \frac{1}{\alpha - \beta}(A - \alpha I_V)(w - z) \in \mathcal{R}(A - \alpha I_V).$$

Hence $x \in \mathcal{R}\big(\mathfrak{t}(A)(A-\beta I_V)(A-\alpha I_V)\big) = \mathcal{R}\big(\mathfrak{u}(A)(A-\alpha I_V)\big) = \mathcal{R}\big((A-\alpha I_V)\mathfrak{u}(A)\big)$ and (2.1) is established for $n + 1$.

Now, again by induction, we prove that for any $n \in \mathbf{N}$ we have

$$\mathcal{R}\big(\mathfrak{a}(A)\big) \cap \mathcal{R}\big(\mathfrak{b}(A)\big) \subset \mathcal{R}\big(\mathfrak{ab}(A)\big) \tag{2.2}$$

for any polynomial $\mathfrak{a}$ of degree n with coefficients in $\mathbf{C}$ and for any polynomial $\mathfrak{b}$ with coefficients in $\mathbf{C}$ having no roots in common with $\mathfrak{a}$.

It is easily seen that (2.2) holds for $n = 0$. Now suppose (2.2) to be satisfied for some $n \in \mathbf{N}$, and let $\mathfrak{a}$ and $\mathfrak{b}$ be polynomials with coefficients in $\mathbf{C}$ having no common roots, such that the degree of $\mathfrak{a}$ is $n + 1$. Then both $\mathfrak{a}$ and $\mathfrak{b}$ are nonzero and, if α is a root of $\mathfrak{a}$, there exists a nonzero polynomial $\mathfrak{c}$ with complex coefficients, of degree n, such that

$$\mathfrak{a}(\lambda) = (\lambda - \alpha)\mathfrak{c}(\lambda) \quad \text{for any } \lambda \in \mathbf{C}.$$

Then for any $x \in \mathcal{R}\big(\mathfrak{a}(A)\big) \cap \mathcal{R}\big(\mathfrak{b}(A)\big)$ there exist $y \in \mathcal{D}(A^{n+1})$ and $z \in \mathcal{D}\big(\mathfrak{b}(A)\big)$ such that

$$x = \mathfrak{c}(A)(A - \alpha I_V)y = \mathfrak{b}(A)z.$$

Since $\mathfrak{c}$ and $\mathfrak{b}$ have no common roots and $\mathfrak{c}$ has degree n, it follows that there exists $u \in \mathcal{D}\big(\mathfrak{cb}(A)\big)$ such that $x = \mathfrak{c}(A)\mathfrak{b}(A)u$. Then

$$(A - \alpha I_V)y - \mathfrak{b}(A)u \in \mathcal{N}\big(\mathfrak{c}(A)\big) \subset \mathcal{R}\big(\mathfrak{b}(A)\big)$$

by (i), namely $(A - \alpha I_V)y = \mathfrak{b}(A)v$ for some $v \in \mathcal{D}\big(\mathfrak{b}(A)\big)$. Now, since α is not a root of $\mathfrak{b}$, from (2.1) it follows that

$$(A - \alpha I_V)y = (A - \alpha I_V)\mathfrak{b}(A)w$$

for some $w \in \mathcal{D}\big((A - \alpha I_V)\mathfrak{b}(A)\big)$ and consequently

$$x = \mathfrak{c}(A)(A - \alpha I_V)\mathfrak{b}(A)w = \mathfrak{a}(A)\mathfrak{b}(A)w \in \mathcal{R}\big(\mathfrak{ab}(A)\big).$$

The proof is thus complete. ∎

LEMMA 2.3. *Let V be a vector space over* $\mathbf{K}$, *let* $A \in \Lambda(V)$, *let* $n \in \mathbf{N}$ *and let* $\mathfrak{p}$ *be a polynomial of degree* n, *with coefficients in* $\mathbf{K}$. *Then for any* $k \in \mathbf{N}$ *we have*

$$\mathcal{R}(\mathfrak{p}(A)) \cap \mathcal{D}(A^k) = \mathfrak{p}(A)(\mathcal{D}(A^{n+k})).$$

Proof. For any $k \in \mathbf{N}$, the inclusion $\mathfrak{p}(A)(\mathcal{D}(A^{n+k})) \subset \mathcal{R}(\mathfrak{p}(A)) \cap \mathcal{D}(A^k)$ is clear. We prove that $\mathcal{R}(\mathfrak{p}(A)) \cap \mathcal{D}(A^k) \subset \mathfrak{p}(A)(\mathcal{D}(A^{n+k}))$. Since this inclusion is straightforward if $\mathfrak{p}$ is the null polynomial, we assume $\mathfrak{p}$ to be nonzero. If $\mathfrak{q}$ is the polynomial defined by

$$\mathfrak{q}(\lambda) = \lambda^k \mathfrak{p}(\lambda) \quad \text{for any } \lambda \in \mathbf{K},$$

it follows that the degree of $\mathfrak{q}$ is $n+k$.

For any $y \in \mathcal{R}(\mathfrak{p}(A)) \cap \mathcal{D}(A^k)$, there exists $x \in \mathcal{D}(\mathfrak{p}(A)) = \mathcal{D}(A^n)$ such that $y = \mathfrak{p}(A)x$. Since $y \in \mathcal{D}(A^k)$, it follows that

$$x \in \mathcal{D}(A^k \mathfrak{p}(A)) = \mathcal{D}(\mathfrak{q}(A)) = \mathcal{D}(A^{n+k}).$$

Hence $y \in \mathfrak{p}(A)(\mathcal{D}(A^{n+k}))$. The proof is now complete. ∎

LEMMA 2.4. *Let V be a vector space and let* $A \in \Lambda(V)$. *If for some* $p \in \mathbf{N}$ $\dim(\mathcal{R}(A^p)/\mathcal{R}(A^{p+1})) < \infty$, *then* $\dim(\mathcal{R}(A^n)/\mathcal{R}(A^{n+k})) < \infty$ *for any* $n \geqslant p$ *and for any* $k \in \mathbf{N}$.

Proof. Let $n \geqslant p$. We prove our assertion by induction on k. Clearly, it holds for $k = 0$. Now suppose $\dim(\mathcal{R}(A^n)/\mathcal{R}(A^{n+k})) < \infty$ for some $k \in \mathbf{N}$. From [7], 3.2 and 2.2 it follows that the vector spaces $\mathcal{R}(A^p)/\mathcal{R}(A^{p+1})$ and $\mathcal{R}(A^{n+k})/\mathcal{R}(A^{n+k+1})$ are isomorphic to $(\mathcal{D}(A^p) + \mathcal{R}(A))/(\mathcal{N}(A^p) + \mathcal{R}(A))$ and $(\mathcal{D}(A^{n+k}) + \mathcal{R}(A))/(\mathcal{N}(A^{n+k}) + \mathcal{R}(A))$, respectively. Since $n + k \geqslant p$, and consequently

$$\mathcal{N}(A^p) \subset \mathcal{N}(A^{n+k}) \subset \mathcal{D}(A^{n+k}) \subset \mathcal{D}(A^p),$$

it follows that

$$\dim\Big((\mathcal{D}(A^{n+k}) + \mathcal{R}(A))/(\mathcal{N}(A^{n+k}) + \mathcal{R}(A))\Big) \leqslant \dim\Big((\mathcal{D}(A^p) + \mathcal{R}(A))/(\mathcal{N}(A^p) + \mathcal{R}(A))\Big).$$

Hence

$$\dim(\mathcal{R}(A^{n+k})/\mathcal{R}(A^{n+k+1})) \leqslant \dim(\mathcal{R}(A^p)/\mathcal{R}(A^{p+1})) < \infty.$$

Since the finite-dimensional vector space $\mathcal{R}(A^n)/\mathcal{R}(A^{n+k})$ is isomorphic to the quotient space

$$(\mathcal{R}(A^n)/\mathcal{R}(A^{n+k+1})) \Big/ (\mathcal{R}(A^{n+k})/\mathcal{R}(A^{n+k+1})),$$

it follows that $\mathcal{R}(A^n)/\mathcal{R}(A^{n+k+1})$ has finite dimension as well.

We have thus established the desired result. ∎

If X is a Banach space, let $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$ denote, respectively, the sets of all paracomplete Fredholm, upper semi-Fredholm and lower semi-Fredholm operators with domain and range in X. Namely,

$$\Phi_+(X) = \{A \in PC(X) : \mathcal{R}(A) \text{ is closed and } \mathrm{nul}(A) < \infty\},$$
$$\Phi_-(A) = \{A \in PC(X) : \mathrm{def}(A) < \infty\}$$

and

$$\Phi(X) = \Phi_+(X) \cap \Phi_-(X).$$

Notice that, for any $A \in \Phi_-(X)$, $\mathcal{R}(A)$ is closed in virtue of Theorem 1.2.

REMARK 2.5. We remark that $\Phi_+(X) \subset C(X)$ (and consequently $\Phi(X) \subset C(X)$). Indeed, for any $A \in \Phi_+(X)$, both $\mathcal{N}(A)$ and $\mathcal{R}(A)$ are closed subspaces of X, and consequently $A \in C(X)$ by [9], Proposition 2.2.3.

On the contrary, an element of $\Phi_-(X)$ need not be closed. For instance, if E is a Banach space which is isomorphic to its square (for example, an infinite-dimensional Hilbert space), $U \in L(E, E \times E)$ is bijective and W is a non-closed paracomplete subspace of E (notice that, in virtue of [6], Theorem 1, for every infinite-dimensional Banach space Y there exists $T \in L(Y)$ such that $\mathcal{R}(T)$ is not closed in Y; hence from Fact 1.1 it follows that every infinite-dimensional Banach space has a non-closed paracomplete subspace), then the linear operator

$$A : E \times W \ni (x, y) \longmapsto Ux \in E \times E$$

is paracomplete (as it is continuous and its domain is a paracomplete subspace of $E \times E$) and surjective. Hence $A \in \Phi_-(E \times E)$. Nevertheless, A is not closed, as $\mathcal{N}(A) = \{0\} \times W$, which is not a closed subspace of $E \times E$.

We recall that in [9] a class of paracomplete operators with domain and range in a Hilbert space H, called the quasi-Fredholm operators, is introduced (see [9], Definition 3.1.2). The quasi-Fredholm operators contain the closed semi-Fredholm operators with domain and range in H, namely all elements of $C(H)$ having either finite-dimensional kernel and closed range, or finite-codimensional range (see [9], Example 4 on page 197), and consequently, by Remark 2.5, contain $\Phi_+(H)$. On the contrary, the quasi-Fredholm operators do not contain $\Phi_-(H)$ when H has infinite dimension: indeed, a surjective operator $T \in PC(H)$ turns to be quasi-Fredholm if and only if $T \in C(H)$ (see [9], Remark (3.1.1)). Then the example contained in Remark 2.5, by choosing $E = H$, shows that $\Phi_-(H)$ is not contained in the quasi-Fredholm operators, as $U^{-1}AU \in \Phi_-(H)$ by Theorem 1.4, but is not closed, being A non-closed.

Let Y be a complex Banach space and let $A \in \Lambda(Y)$. Following [4], page 288, we define the resolvent set $\rho(A)$ of A in the following way:

$$\rho(A) = \{\lambda \in \mathbf{C} : \lambda I_Y - A \text{ has an inverse in } L(Y)\}.$$

We recall that $\rho(A) \neq \emptyset$ implies $A \in C(Y)$ (see [4], XIV, 1.2). We denote the essential resolvent set of A by $\rho_\Phi(A)$. Namely,

$$\rho_\Phi(A) = \{\lambda \in \mathbf{C} : \lambda I_Y - A \in \Phi(Y)\}.$$

We remark that $\rho(A) \subset \rho_\Phi(A)$ by [4], XIV, 1.2. Now we define the upper essential resolvent set $\rho_{\Phi_+}(A)$ of A and the lower essential resolvent set $\rho_{\Phi_-}(A)$ of A in the following way:

$$\rho_{\Phi_+}(A) = \{\lambda \in \mathbf{C} : \ \lambda I_Y - A \in \Phi_+(Y)\}$$

and

$$\rho_{\Phi_-}(A) = \{\lambda \in \mathbf{C} : \ \lambda I_Y - A \in \Phi_-(Y)\}.$$

Notice that $\rho_\Phi(A) = \rho_{\Phi_+}(A) \cap \rho_{\Phi_-}(A)$. Furthermore, since $\lambda I_Y + \mu T \in C(Y)$ for any $T \in C(Y)$ and for any $(\lambda, \mu) \in \mathbf{C} \times (\mathbf{C} \setminus \{0\})$, from Remark 2.5 it follows that $\rho_{\Phi_+}(A) \neq \emptyset$ implies $A \in C(Y)$. Notice also that $\rho_{\Phi_-}(A) \neq \emptyset$ implies $A \in PC(Y)$ in virtue of Theorem 1.4.

Now let Z be a real Banach space and let $T \in \Lambda(Z)$. We define the resolvent set, the essential resolvent set and the upper and lower essential resolvent sets of T to be the resolvent set, the essential resolvent set and the upper and lower essential resolvent sets, respectively, of the complex extension of T, and will denote them again by $\rho(T), \rho_\Phi(T), \rho_{\Phi_+}(T)$ and $\rho_{\Phi_-}(T)$, respectively.

It is not difficult to verify that $T \in C(Z)$ (respectively, $PC(Z)$) if and only if $\widetilde{T} \in C(\widetilde{Z})$ (respectively, $PC(\widetilde{Z})$). Hence $\rho_{\Phi_+}(T) \neq \emptyset$ implies $T \in C(Z)$ and $\rho_{\Phi_-}(T) \neq \emptyset$ implies $T \in PC(Z)$.

REMARK 2.6. Let X be a Banach space over $\mathbf{K}$, and let $A \in \Lambda(X)$. If $\rho_{\Phi_+}(A) \neq \emptyset$, then from Remark 2.5 and [5], IV.2.12 for complex Banach spaces, plus equivalence between closedness of $\mathfrak{p}(A)$ and of its complex extension $\widetilde{\mathfrak{p}}(A) = \mathfrak{p}(\widetilde{A})$ in the case of a real Banach space X, it follows that $\mathfrak{p}(A) \in C(X)$ for any polynomial $\mathfrak{p}$ with coefficients in $\mathbf{K}$.

If $\rho_{\Phi_-}(A) \neq \emptyset$, then the comments following the definitions of the upper and lower essential resolvent sets in the complex and in the real case, together with Theorem 1.4, give $\mathfrak{p}(A) \in PC(X)$ for any polynomial $\mathfrak{p}$ with coefficients in $\mathbf{K}$. The example provided in Remark 2.5 shows that A need not be closed.

Now let V be a vector space over $\mathbf{K}$, and let $A \in \Lambda(V)$. If $(\lambda_n)_{n\in\mathbf{N}}$ and $(\mu_n)_{n\in\mathbf{N}}$ are sequences of elements of $\mathbf{K}$, for any $k \in \mathbf{N}$ set

$$\mathcal{P}_k\big(A;(\lambda_n)_{n\in\mathbf{N}},(\mu_n)_{n\in\mathbf{N}}\big) = \pi_k\left(\varprojlim(\mathcal{D}(A^n),\theta_n)_{n\in\mathbf{N}}\right),$$

where $\theta_n : \mathcal{D}(A^{n+1}) \to \mathcal{D}(A^n)$ maps every $x \in \mathcal{D}(A^{n+1})$ into $(\lambda_n A + \mu_n I_V)x$ for any $n \in \mathbf{N}$. Then for any $k \in \mathbf{N}$ we have $\mathcal{P}_k\big(A;(\lambda_n)_{n\in\mathbf{N}},(\mu_n)_{n\in\mathbf{N}}\big) = \{x \in V :$ there exists a sequence $(x_n)_{n\geqslant k}$ of elements of V, with $x_n \in \mathcal{D}(A^n)$ for any $n \geqslant k$, satisfying $x_k = x$ and $x_n = (\lambda_n A + \mu_n I_V)x_{n+1}$ for any $n \geqslant k\}$. Notice also that $\mathcal{P}_k\big(A;(\lambda_n)_{n\in\mathbf{N}},(\mu_n)_{n\in\mathbf{N}}\big)$ is a linear subspace of V for any $k \in \mathbf{N}$. Besides, we have

$$\mathcal{P}_k\big(A;(\lambda_n)_{n\in\mathbf{N}},(\mu_n)_{n\in\mathbf{N}}\big) = (\lambda_k A + \mu_k I_V)\big(\mathcal{P}_{k+1}(A;(\lambda_n)_{n\in\mathbf{N}},(\mu_n)_{n\in\mathbf{N}})\big)$$

for any $k \in \mathbf{N}$.

We remark that $\mathcal{P}_0\big(A;(1)_{n\in\mathbf{N}},(0)_{n\in\mathbf{N}}\big)$ is the largest of all subspaces M of V satisfying $A\big(\mathcal{D}(A) \cap M\big) = M$: indeed, we have

$$\begin{aligned}\mathcal{P}_0\big(A;(1)_{n\in\mathbf{N}},(0)_{n\in\mathbf{N}}\big) &= A\Big(\mathcal{P}_1\big(A;(1)_{n\in\mathbf{N}},(0)_{n\in\mathbf{N}}\big)\Big)\\ &= A\Big(\mathcal{D}(A) \cap \mathcal{P}_0\big(A;(1)_{n\in\mathbf{N}},(0)_{n\in\mathbf{N}}\big)\Big),\end{aligned}$$

and conversely, for any subspace M of V satisfying $A\big(\mathcal{D}(A) \cap M\big) = M$ and for any $x \in M$, there exists a sequence $(x_n)_{n\in\mathbf{N}}$, with $x_n \in \mathcal{D}(A^n) \cap M$ for any $n \in \mathbf{N}$, satisfying $x_0 = x$ and $x_n = Ax_{n+1}$ for any $n \in \mathbf{N}$. Following for example [11], we call $\mathcal{P}_0\big(A;(1)_{n\in\mathbf{N}},(0)_{n\in\mathbf{N}}\big)$ the *core* of A, and denote it by $\mathcal{CO}(A)$ (see also [15], where the core of a map from a set into itself is defined). We recall that $\mathcal{CO}(A) \subset \bigcap_{n\in\mathbf{N}} \mathcal{R}(A^n)$, and the inclusion may be strict. The following is an example.

EXAMPLE 2.7. Let us consider the linear bounded operator

$$S : \ell_2 \ni (x_n)_{n\in\mathbf{N}} \mapsto \left(\sum_{n=1}^{\infty} \frac{1}{n}\, x_{\frac{n(n+1)}{2}}\right) e_0 + \sum_{n=1}^{\infty} \varepsilon_{n+1} x_{n+1} e_n \in \ell_2,$$

where $(e_n)_{n\in\mathbf{N}}$ denotes the canonical basis of ℓ_2 and the sequence $(\varepsilon_n)_{n\geqslant 2}$ is defined by

$$\varepsilon_n = \begin{cases} 0 & \text{if } n = \frac{k(k+1)}{2} \text{ for some } k \geqslant 2;\\ 1 & \text{if } n \neq \frac{k(k+1)}{2} \text{ for any } k \geqslant 2;\end{cases}$$

then for any $k \geqslant 2$ we have

$$\frac{1}{k-1} e_0 = S^k e_{\frac{k(k+1)}{2}-1}$$

and

$$\mathcal{R}(S^k) \subset \left\{ (x_n)_{n\in\mathbf{N}} \in \ell_2 : \ x_n = 0 \text{ for any } n \in \mathbf{Z}_+ \text{ satisfying } n < \frac{k(k+1)}{2} \right\}.$$

Hence $\bigcap_{n\in\mathbf{N}} \mathcal{R}(S^n)$ coincides with the linear span of e_0, whereas, since $Se_0 = 0$, we have $\mathcal{CO}(S) = \{0\}$.

We remark that, if V is a vector space over $\mathbf{K}$ and $A \in \Lambda(V)$, then for any two sequences $(\lambda_n)_{n\in\mathbf{N}}$ and $(\mu_n)_{n\in\mathbf{N}}$ of elements of $\mathbf{K}$ we have

$$\mathcal{P}_k(A; (\lambda_n)_{n\in\mathbf{N}}, (\mu_n)_{n\in\mathbf{N}}) \subset \bigcap_{n=k+1}^{\infty} (\lambda_k A + \mu_k I_V) \cdots (\lambda_{n-1} A + \mu_{n-1} I_V)(\mathcal{D}(A^n)).$$

The inclusion above may be strict, in virtue of Example 2.7.

LEMMA 2.8. *Let $(V_n)_{n\in\mathbf{N}}$ be a sequence of vector spaces over $\mathbf{K}$, let $\theta_n : V_{n+1} \to V_n$ be a linear map for any $n \in \mathbf{N}$, and let $(\lambda_n)_{n\in\mathbf{N}}$ be a sequence of nonzero scalars. Then $\pi_k \left(\varprojlim(V_n, \lambda_n\theta_n)_{n\in\mathbf{N}}\right) = \pi_k \left(\varprojlim(V_n, \theta_n)_{n\in\mathbf{N}}\right)$ for any $k \in \mathbf{N}$.*

Proof. Since $\lambda_n \neq 0$ for any $n \in \mathbf{N}$, and consequently $\theta_n = \frac{1}{\lambda_n}(\lambda_n\theta_n)$ for any $n \in \mathbf{N}$, it suffices to prove that $\pi_k \left(\varprojlim(V_n, \lambda_n\theta_n)_{n\in\mathbf{N}}\right) \subset \pi_k \left(\varprojlim(V_n, \theta_n)_{n\in\mathbf{N}}\right)$ for any $k \in \mathbf{N}$. Let $k \in \mathbf{N}$, and let $x \in \pi_k \left(\varprojlim(V_n, \lambda_n\theta_n)_{n\in\mathbf{N}}\right)$. Then there exists $(x_n)_{n\geqslant k} \in \prod_{n\geqslant k} V_n$ satisfying $x_k = x$ and $x_n = \lambda_n\theta_n x_{n+1}$ for any $n \geqslant k$. If $(y_n)_{n\geqslant k} \in \prod_{n\geqslant k} V_n$ is defined by $y_n = \left(\prod_{j=k}^{n-1} \lambda_j\right) x_n$ for any $n \geqslant k$ (where the product $\prod_{j=k}^{n-1} \lambda_j$ is understood to be equal to 1 for $n = k$), it follows that $y_k = x$ and

$$\theta_n y_{n+1} = \theta_n \left(\prod_{j=k}^{n} \lambda_j\right) x_{n+1} = \left(\prod_{j=k}^{n-1} \lambda_j\right) \lambda_n\theta_n x_{n+1} = \left(\prod_{j=k}^{n-1} \lambda_j\right) x_n = y_n$$

for any $n \geqslant k$. Hence $x \in \pi_k \left(\varprojlim(V_n, \theta_n)_{n\in\mathbf{N}}\right)$, which gives the desired inclusion. ∎

PROPOSITION 2.9. *Let V be a vector space over* $\mathbf{K}$, *let* $A \in \Lambda(V)$ *and let* $(\lambda_n)_{n\in\mathbf{N}}, (\mu_n)_{n\in\mathbf{N}}$ *be scalar sequences.*

(i) *If* $\lambda_n = 0$ *and* $\mu_n \neq 0$ *for any* $n \in \mathbf{N}$, *then*

$$\mathcal{P}_k\big(A; (\lambda_n)_{n\in\mathbf{N}}, (\mu_n)_{n\in\mathbf{N}}\big) = \bigcap_{n\in\mathbf{N}} \mathcal{D}(A^n)$$

for any $k \in \mathbf{N}$.

(ii) *If* $\lambda_n \neq 0$ *and* $\mu_n = 0$ *for any* $n \in \mathbf{N}$, *then*

$$\mathcal{P}_k\big(A; (\lambda_n)_{n\in\mathbf{N}}, (\mu_n)_{n\in\mathbf{N}}\big) = \mathcal{CO}(A) \cap \mathcal{D}(A^k)$$

for any $k \in \mathbf{N}$.

Proof. If $\lambda_n = 0$ and $\mu_n \neq 0$ for any $n \in \mathbf{N}$, from Lemma 2.8 it follows that

$$\mathcal{P}_k\big(A; (\lambda_n)_{n\in\mathbf{N}}, (\mu_n)_{n\in\mathbf{N}}\big) = \mathcal{P}_k\big(A; (0)_{n\in\mathbf{N}}, (1)_{n\in\mathbf{N}}\big) = \bigcap_{n\in\mathbf{N}} \mathcal{D}(A^n)$$

for any $k \in \mathbf{N}$, which establishes (i).

If $\lambda_n \neq 0$ and $\mu_n = 0$ for any $n \in \mathbf{N}$, from Lemma 2.8 it follows that

$$\begin{aligned}\mathcal{P}_k\big(A; (\lambda_n)_{n\in\mathbf{N}}, (\mu_n)_{n\in\mathbf{N}}\big) &= \mathcal{P}_k\big(A; (1)_{n\in\mathbf{N}}, (0)_{n\in\mathbf{N}}\big) \\ &= \mathcal{P}_0\big(A; (1)_{n\in\mathbf{N}}, (0)_{n\in\mathbf{N}}\big) \cap \mathcal{D}(A^k) = \mathcal{CO}(A) \cap \mathcal{D}(A^k)\end{aligned}$$

for any $k \in \mathbf{N}$, which establishes (ii). ∎

Theorem 2.10 below extends Theorem 0.1 to the paracomplete operators having nonempty lower essential resolvent set. We are going to use Lemma 2.1, Lemma 2.2, Lemma 2.3 and Lemma 2.4 in order to adapt the proof of Theorem 0.1 given in [10], 1.3 to the more general context.

THEOREM 2.10. *Let X be a Banach space over* $\mathbf{K}$, *and let* $A \in \Lambda(X)$ *be such that* $\rho_{\Phi_-}(A) \neq \emptyset$ *(which gives* $\mathfrak{p}(A) \in PC(X)$ *for any polynomial* $\mathfrak{p}$ *with coefficients in* $\mathbf{K}$ *by Remark 2.6; consequently, by Fact 1.1,* $\mathcal{D}(A^n)$ *is a paracomplete subspace of X for any* $n \in \mathbf{N}$*). If* $(\alpha_n)_{n\in\mathbf{N}}$ *and* $(\beta_n)_{n\in\mathbf{N}}$ *are scalar sequences, such that* $(\alpha_n A + \beta_n I_X)(\mathcal{D}(A))$ *is dense in X for any* $n \in \mathbf{N}$, *then* $\mathcal{P}_k\big(A; (\alpha_n)_{n\in\mathbf{N}}, (\beta_n)_{n\in\mathbf{N}}\big)$ *is* $\tau_{\mathcal{D}(A^k)}$*-dense in* $\mathcal{D}(A^k)$ *for any* $k \in \mathbf{N}$. *In particular, it follows that* $\mathcal{P}_0\big(A; (\alpha_n)_{n\in\mathbf{N}}, (\beta_n)_{n\in\mathbf{N}}\big)$ *is dense in X, and consequently*

$$\bigcap_{n\in\mathbf{Z}_+} (\alpha_0 A + \beta_0 I_X)\cdots(\alpha_{n-1}A + \beta_{n-1} I_X)\big(\mathcal{D}(A^n)\big)$$

is dense in X.

Proof. Since the result holds trivially if $X = \{0\}$, we may assume $X \neq \{0\}$. Furthermore, it suffices to prove Theorem 2.10 for a complex Banach space, as, since it is not difficult to verify that for any $k \in \mathbf{N}$ we have

$$\mathcal{P}_k(\widetilde{A}; (\alpha_n)_{n\in\mathbf{N}}, (\beta_n)_{n\in\mathbf{N}}) = \mathcal{P}_k(A; (\alpha_n)_{n\in\mathbf{N}}, (\beta_n)_{n\in\mathbf{N}}) + i\mathcal{P}_k(A; (\alpha_n)_{n\in\mathbf{N}}, (\beta_n)_{n\in\mathbf{N}})$$

if X is real, the real case can be derived from the complex one by going to the complex extension of A. Thus we suppose X to be a complex nonzero Banach space.

Now let $\|\cdot\|_0$ denote the norm of X. Furthermore, for any $n \in \mathbf{Z}_+$, let $\|\cdot\|_n$ denote a complete norm on $\mathcal{D}(A^n)$, such that the canonical injection Γ_n from $(\mathcal{D}(A^n), \|\cdot\|_n)$ into X is continuous. Clearly, $\|\cdot\|_n$ induces the topology $\tau_{\mathcal{D}(A^n)}$ on $\mathcal{D}(A^n)$ for any $n \in \mathbf{N}$.

For any $n \in \mathbf{N}$, we define the linear operators $A_n, J_n : (\mathcal{D}(A^{n+1}), \|\cdot\|_{n+1}) \to (\mathcal{D}(A^n), \|\cdot\|_n)$, in the following way:

$$A_n x = Ax \text{ and } J_n x = x$$

for any $x \in \mathcal{D}(A^{n+1})$. Fix $n \in \mathbf{N}$. Since each of $(\mathcal{D}(A^{n+1}), \|\cdot\|_{n+1})$ and $(\mathcal{D}(A^n), \|\cdot\|_n)$ is continuously embedded in X, it follows from the closed graph theorem that J_n is continuous. Furthermore, we have $A_n = \Gamma_n^{-1} A \Gamma_{n+1}$. Since $A \in PC(X)$, and in addition $\Gamma_{n+1} \in L\Big((\mathcal{D}(A^{n+1}), \|\cdot\|_{n+1}), X\Big)$ and $\Gamma_n \in L((\mathcal{D}(A^n), \|\cdot\|_n), X)$, which gives $\Gamma_n^{-1} \in C\Big(X, (\mathcal{D}(A^n), \|\cdot\|_n)\Big)$, then

$$A_n \in PC\Big((\mathcal{D}(A^{n+1}), \|\cdot\|_{n+1}), (\mathcal{D}(A^n), \|\cdot\|_n)\Big)$$

by Theorem 1.4. Since $\mathcal{D}(A_n) = \mathcal{D}(A^{n+1})$, from Theorem 1.3 it follows that A_n is continuous. Hence $\alpha A_n + \beta J_n \in L\Big((\mathcal{D}(A^{n+1}), \|\cdot\|_{n+1}), (\mathcal{D}(A^n), \|\cdot\|_n)\Big)$ for any $n \in \mathbf{N}$ and for any $\alpha, \beta \in \mathbf{C}$.

Now let $\alpha, \beta \in \mathbf{C}$ be such that $(\alpha A + \beta I_X)(\mathcal{D}(A))$ is dense in X. Then $(\alpha, \beta) \neq (0,0)$ as $X \neq \{0\}$. We prove that $\mathcal{R}(\alpha A_n + \beta J_n)$ is dense in $(\mathcal{D}(A^n), \|\cdot\|_n)$ for any $n \in \mathbf{N}$.

Since $\rho_{\Phi_-}(A) \neq \emptyset$, it follows that $\lambda I_X - A \in \Phi_-(X)$ for some $\lambda \in \mathbf{C}$.

Let $n \in \mathbf{N}$. We begin by proving that

$$\mathcal{N}((\lambda I_X - A)^n) \subset \mathcal{R}(\alpha A_n + \beta J_n)$$

and

$$\mathcal{R}((\lambda I_X - A)^n) \cap (\alpha A + \beta I_X)(\mathcal{D}(A)) = (\lambda I_X - A)^n (\mathcal{R}(\alpha A_n + \beta J_n)).$$

Indeed, if $\alpha = 0$, then $\beta \neq 0$ and consequently

$$\mathcal{R}(\alpha A_n + \beta J_n) = \mathcal{R}(J_n) = \mathcal{D}(A^{n+1}),$$

which contains $\mathcal{N}((\lambda I_X - A)^n)$ in virtue of Lemma 2.1. Furthermore, by Lemma 2.3 we have

$$\begin{aligned}(\lambda I_X - A)^n(\mathcal{R}(\alpha A_n + \beta J_n)) &= (\lambda I_X - A)^n(\mathcal{D}(A^{n+1})) = \mathcal{R}((\lambda I_X - A)^n) \cap \mathcal{D}(A) \\ &= \mathcal{R}((\lambda I_X - A)^n) \cap (\alpha A + \beta I_X)(\mathcal{D}(A)).\end{aligned}$$

If $\alpha \neq 0$ and $\beta = -\alpha\lambda$, then $\mathcal{R}(\lambda I_X - A) = (\alpha A + \beta I_X)(\mathcal{D}(A))$, which is dense in X. Since $\lambda I_X - A \in \Phi_-(X)$, and consequently $\mathcal{R}(\lambda I_X - A)$ is closed, it follows that $\mathcal{R}(\lambda I_X - A) = (\alpha A + \beta I_X)(\mathcal{D}(A)) = X$. Then by Lemma 2.3 we obtain

$$\mathcal{R}(\alpha A_n + \beta J_n) = (\lambda I_X - A)(\mathcal{D}(A^{n+1})) = \mathcal{R}(\lambda I_X - A) \cap \mathcal{D}(A^n) = \mathcal{D}(A^n),$$

which gives the inclusion $\mathcal{N}((\lambda I_X - A)^n) \subset \mathcal{R}(\alpha A_n + \beta J_n)$ as well as the equality $\mathcal{R}((\lambda I_X - A)^n) \cap (\alpha A + \beta I_X)(\mathcal{D}(A)) = (\lambda I_X - A)^n(\mathcal{R}(\alpha A_n + \beta J_n))$.

Finally, if $\alpha \neq 0$ and $\beta \neq -\alpha\lambda$, then the polynomials $\mathfrak{p}$ and $\mathfrak{q}$ defined by $\mathfrak{p}(\mu) = (\lambda - \mu)^n$ and $\mathfrak{q}(\mu) = \alpha\mu + \beta$ for any $\mu \in \mathbf{C}$ have no common roots in $\mathbf{C}$. Hence, in virtue of Lemma 2.2 and Lemma 2.1, we have

$$\begin{aligned}\mathcal{N}((\lambda I_X - A)^n) &= (\alpha A + \beta I_X)\Big(\mathcal{N}((\lambda I_X - A)^n)\Big) \subset (\alpha A + \beta I_X)(\mathcal{D}(A^{n+1})) \\ &= \mathcal{R}(\alpha A_n + \beta J_n)\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}((\lambda I_X - A)^n) \cap (\alpha A + \beta I_X)(\mathcal{D}(A)) &= \mathcal{R}((\lambda I_X - A)^n(\alpha A + \beta I_X)) \\ &= (\lambda I_X - A)^n\Big((\alpha A + \beta I_X)(\mathcal{D}(A^{n+1}))\Big) \\ &= (\lambda I_X - A)^n(\mathcal{R}(\alpha A_n + \beta J_n)).\end{aligned}$$

The proof of the assertion above is thus complete.

Now we prove that $\mathcal{R}(\alpha A_n + \beta J_n)$ is dense in $(\mathcal{D}(A^n), \|\cdot\|_n)$.

From Theorem 1.4 it follows that $(\lambda I_X - A)^n \Gamma_n \in PC\Big((\mathcal{D}(A^n), \|\cdot\|_n), X\Big)$. Since $\mathcal{R}(\Gamma_n) = \mathcal{D}(A^n) = \mathcal{D}((\lambda I_X - A)^n)$, it follows that $\mathcal{D}((\lambda I_X - A)^n \Gamma_n) = \mathcal{D}(A^n)$, and consequently $(\lambda I_X - A)^n \Gamma_n \in L\Big((\mathcal{D}(A^n), \|\cdot\|_n), X\Big)$ by Theorem 1.3. Furthermore, we have $\mathcal{R}((\lambda I_X - A)^n \Gamma_n) = \mathcal{R}((\lambda I_X - A)^n)$.

Since $\operatorname{def}(\lambda I_X - A) < \infty$, from Lemma 2.4 it follows that $\operatorname{def}((\lambda I_X - A)^n) < \infty$ as well. Hence $\mathcal{R}((\lambda I_X - A)^n \Gamma_n) = \mathcal{R}((\lambda I_X - A)^n)$ is a closed subspace of X

by Theorem 1.2. Now from [8], Lemma 322 it follows that there exists $\delta_n > 0$ such that

$$\|(\lambda I_X - A)^n x\|_0 \geqslant \delta_n \inf\{\|x - z\|_n : z \in \mathcal{N}((\lambda I_X - A)^n)\}$$

for any $x \in \mathcal{D}(A^n)$. Then for any $x \in \mathcal{D}(A^n)$ and for any $y \in \mathcal{R}(\alpha A_n + \beta J_n)$ there exists $z_{x\,y} \in \mathcal{N}((\lambda I_X - A)^n)$ satisfying

$$\|x - y - z_{x\,y}\|_n \leqslant \frac{2}{\delta_n}\|(\lambda I_X - A)^n x - (\lambda I_X - A)^n y\|_0.$$

Since $(\alpha A + \beta I_X)(\mathcal{D}(A))$ is dense in X and $\mathcal{R}((\lambda I_X - A)^n)$ is a finite-codimensional closed subspace of X, then we are enabled to apply [5], IV.2.8 and conclude that $\mathcal{R}((\lambda I_X - A)^n) \cap (\alpha A + \beta I_X)(\mathcal{D}(A))$ is dense in $\mathcal{R}((\lambda I_X - A)^n)$. Hence, by what we have proved above, $(\lambda I_X - A)^n(\mathcal{R}(\alpha A_n + \beta J_n))$ is dense in $\mathcal{R}((\lambda I_X - A)^n)$. Consequently, for any $x \in \mathcal{D}(A^n)$ and for any $\varepsilon > 0$ there exists $y_\varepsilon \in \mathcal{R}(\alpha A_n + \beta J_n)$ such that

$$\|(\lambda I_X - A)^n x - (\lambda I_X - A)^n y_\varepsilon\|_0 < \frac{\varepsilon \delta_n}{2},$$

which gives

$$\|x - y_\varepsilon - z_{x\,y_\varepsilon}\|_n < \varepsilon.$$

Since $z_{x\,y_\varepsilon} \in \mathcal{N}((\lambda I_X - A)^n) \subset \mathcal{R}(\alpha A_n + \beta J_n)$, we conclude that $\mathcal{R}(\alpha A_n + \beta J_n)$ is dense in $(\mathcal{D}(A^n), \|\cdot\|_n)$.

Hence $\mathcal{R}(\alpha_n A_n + \beta_n J_n)$ is dense in $(\mathcal{D}(A^n), \|\cdot\|_n)$ for any $n \in \mathbf{N}$.

Now, as in the proof of [10], 1.3, we are enabled to apply Theorem 1.5. We conclude that $\mathcal{P}_k(A; (\alpha_n)_{n\in\mathbf{N}}, (\beta_n)_{n\in\mathbf{N}})$ is $\tau_{\mathcal{D}(A^k)}$-dense in $\mathcal{D}(A^k)$ for any $k \in \mathbf{N}$. The remaining part of the statement of the theorem is a consequence of the inclusion

$$\mathcal{P}_0(A; (\alpha_n)_{n\in\mathbf{N}}, (\beta_n)_{n\in\mathbf{N}}) \subset \bigcap_{n\in\mathbf{Z}_+} (\alpha_0 A + \beta_0 I_X) \cdots (\alpha_{n-1} A + \beta_{n-1} I_X)(\mathcal{D}(A^n)). \quad \blacksquare$$

The following result is a consequence of Theorem 2.10 and of Proposition 2.9.

COROLLARY 2.11. *Let X be a Banach space, and let $A \in \Lambda(X)$ be such that $\rho_{\Phi_-}(A) \neq \emptyset$. Then:*

(i) $\bigcap_{n\in\mathbf{N}} \mathcal{D}(A^n)$ is $\tau_{\mathcal{D}(A^k)}$-dense in $\mathcal{D}(A^k)$ for any $k \in \mathbf{N}$ (in particular, $\bigcap_{n\in\mathbf{N}} \mathcal{D}(A^n)$ is dense in X) if $\mathcal{D}(A)$ is dense in X;

(ii) $\mathcal{CO}(A) \cap \mathcal{D}(A^k)$ is $\tau_{\mathcal{D}(A^k)}$-dense in $\mathcal{D}(A^k)$ for any $k \in \mathbf{N}$, and consequently $\left(\bigcap_{n\in\mathbf{N}} \mathcal{R}(A^n)\right) \cap \mathcal{D}(A^k)$ is $\tau_{\mathcal{D}(A^k)}$-dense in $\mathcal{D}(A^k)$ for any $k \in \mathbf{N}$ (in particular,

$\mathcal{CO}(A)$ is dense in X, and consequently $\bigcap_{n\in\mathbb{N}} \mathcal{R}(A^n)$ is dense in X), if $\mathcal{R}(A)$ is dense in X.

We recall that (ii) is proved in [3], 3.1 in the special case $A \in L(X)$.

We also recall that K. Schmüdgen ([17], Theorem 1.9, (b); see also [17], Corollary 1.4) proved that, if H is a complex infinite-dimensional Hilbert space and T is a linear closed densely defined symmetric operator with domain and range in H, such that at least one of the orthogonal complements of the spaces $\mathcal{R}(T+\mathrm{i}I_H)$ and $\mathcal{R}(T-\mathrm{i}I_H)$ has finite dimension, then $\bigcap_{n\in\mathbb{N}} \mathcal{D}(T^n)$ is $\tau_{\mathcal{D}(T^k)}$-dense in $\mathcal{D}(T^k)$ for any $k \in \mathbb{N}$. We remark that this result can be derived from Corollary 2.11. Indeed, since T is symmetric, it follows that

$$\mathcal{N}(T+\mathrm{i}I_H) = \mathcal{N}(T-\mathrm{i}I_H) = \{0\},$$

and in addition both $\mathcal{R}(T+\mathrm{i}I_H)$ and $\mathcal{R}(T-\mathrm{i}I_H)$ are closed subspaces of H, which gives $\rho_{\Phi_+}(T) \neq \emptyset$. Furthermore, if at least one of the orthogonal complements of $\mathcal{R}(T+\mathrm{i}I_H)$ and $\mathcal{R}(T-\mathrm{i}I_H)$ has finite dimension, it follows that $\rho_{\Phi}(T) \neq \emptyset$, which implies that $\rho_{\Phi_-}(T) \neq \emptyset$. Since $\mathcal{D}(T)$ is dense in H, we are enabled to apply Corollary 2.11 and conclude that $\bigcap_{n\in\mathbb{N}} \mathcal{D}(T^n)$ is $\tau_{\mathcal{D}(T^k)}$-dense in $\mathcal{D}(T^k)$ for any $k \in \mathbb{N}$.

We remark that the condition "$\rho_{\Phi_-}(A) \neq \emptyset$" cannot be replaced by "$\rho_{\Phi_+}(A) \neq \emptyset$" in Theorem 2.10 and Corollary 2.11. Indeed, M.A. Naimark ([12] and [13]; see also [2], XII, 9.21) provided an example of a densely defined closed symmetric operator T, with domain and range in an infinite-dimensional Hilbert space H, such that $\mathcal{D}(T^2) = \{0\}$ (see [17], Theorem 5.2 for a more general related result). From the remarks above it follows that $\rho_{\Phi_+}(T) \neq \emptyset$. Nevertheless, $\bigcap_{n\in\mathbb{N}} \mathcal{D}(T^n) = \{0\}$, which is not dense in H.

Finally, we remark that the examples T_1 and T_3 of closed densely defined operators with domain and range in $L_p([0,1])$ $(1 \leqslant p < \infty)$, having empty resolvent set and dense intersection of the domains of the iterates, provided on page 625 of [10], satisfy $\rho_{\Phi}(T_1) = \rho_{\Phi}(T_3) = \mathbb{C}$, and consequently fulfil the hypotheses of Corollary 2.11. The linear densely defined operator T, with domain and range in ℓ_2, considered in the Introduction of [10], which also has empty resolvent set and dense intersection of the domains of the iterates, is not paracomplete in virtue of Fact 1.1, as its domain, namely the space of all finitely nonzero sequences in ℓ_2, cannot be endowed with a complete norm by the Baire theorem. Hence $\rho_{\Phi_-}(T) = \emptyset$ by Remark 2.6. In order to obtain an example of a paracomplete operator having empty lower essential resolvent set and dense intersection of the domains of the iterates, it suffices to consider an unbounded paracomplete densely defined operator A, with domain and range in a Banach space X, satisfying $\mathcal{R}(A) \subset \mathcal{D}(A)$ (see [14], 3.1 for an example of a unbounded closed densely defined operator T with domain and range in ℓ_2, satisfying $\mathcal{R}(T) \subset \mathcal{D}(T)$). Indeed, since A is paracomplete and unbounded, it follows from Theorem 1.3 that $\mathcal{D}(A)$ is not closed in X. By applying Theorem 1.2 to the canonical injection of $\mathcal{D}(A)$ into X, we conclude that $\mathcal{D}(A)$ has infinite codimension in X. Since, if X is complex (respectively, real), we have $\mathcal{R}(\lambda I_X - A) \subset \mathcal{D}(A)$ (respectively, $\mathcal{R}(\lambda I_{\widetilde{X}} - \widetilde{A}) \subset \mathcal{D}(\widetilde{A})$) for any $\lambda \in \mathbf{C}$, it follows that $\rho_{\Phi_-}(A) = \emptyset$. Nevertheless, we have $\bigcap_{n \in \mathbf{N}} \mathcal{D}(A^n) = \mathcal{D}(A)$, which is dense in X as A is densely defined.

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