J. OPERATOR THEORY
36(1996), 379–385

BOOK REVIEW:

Schur Parameters, Factorization and Dilation Problems, **T. Constantinescu**, in Operator Theory: Advances and Applications, vol. 82, Birkhäuser Verlag, Basel–Boston–Berlin, 1996, 253+ix pages, ISBN 3–7643–X and 0–8176–5285–X

KEYWORDS: *Schur parameters, positive block operator matrices, contractive operators, displacement structure, Kolmogorov decomposition, factorizations, interpolation problems, moment problems, continuation, completion, transmission-line, signal processing.*
AMS SUBJECT CLASSIFICATION: Primary 47A20, 47A57, 47A68; Secondary 30E05, 93B28.

At the beginning of this century, the complex function theory was one of the major domain of investigation in mathematics. At that time, C. Carathéodory ([14], [15]) and L. Fejer ([16]) were considering some problems of interpolation for analytic functions in the unit disc in terms of Taylor coefficients. Motivated by this problem, I. Schur ([40]) produced what is now called *the Schur algorithm*, which was initially viewed as a continued fraction algorithm: if F is a function in the Hardy space H^∞ (consisting on functions which are bounded and analytic in the complex open disc) of norm $\leqslant 1$, then the *Schur parameters* associated to F is the sequence $\{\gamma_n\}_{n\leqslant 0}$ of complex numbers, $|\gamma_n| \geqslant 1$ for all $n \in \mathbf{N}$, and defined recursively by

$$F_0 = F, \quad \gamma_0 = F_0(0), \quad F_{n+1}(z) = \frac{F_n(z) - \gamma_n}{z(1 - \overline{\gamma}_n F_n(z))}, \quad \gamma_n = F_n(0).$$

If for some $k \in \mathbf{N}$ we have $|\gamma_k| = 1$ then $\gamma_n = 0$ for all $n > k$.

Some years later G. Szegö (e.g. see [41]) developed the theory of orthogonal polynomials on the unit circle and obtained formulae relating these polynomials with numbers which are called now *Szegö parameters* and are similar to the Schur parameters. In the same time G. Pick ([36]) and R. Nevanlinna ([35]) studied another interpolation problem, known since then as the *Nevanlinna-Pick problem*, and in [35] an algorithm similar with the Schur algorithm was obtained.

Due to some dilation approach initiated by M.A. Naĭmark ([33]) and M.G. Kreĭn([28], [29]), the celebrated works of H. Weyl, J. von Neumann and O. Friedrichs on selfadjoint extensions of symmetric operators were related with interpolation problems. Soon after that, Z. Nehari ([34]) solved an H^∞ approximation problem for functions in L^∞ which turned out to be general enough to contain both the Carathéodory and the Nevanlinna-Pick problem. Classical moment problems as the trigonometric moment problem, the Hamburger and Stieltjes moment problems, etc. were also related to the these topics.

After this rather long period of deep but still theoretical approaches, the technological era starting modestly in the fifties and fully exploding in the sixties put all these into a different perspective. Motivated by N. Wiener ([43]) filtering problem, N. Levinson ([32]) developed an efficient algorithm for solving normal equations which turned out to be strongly related with the Szegö formulae. About the same time, the Schur algorithm was rediscovered in seismic oil prospecting. It is actually geophysics which provides the most intuitive description of the Schur algorithm: the Schur parameters are exactly the reflection coefficients of a layered media, while the association of a sequence of Schur parameters is like a peeling-off process revealing the hidden internal structure of the studied phenomenon, e.g. [23].

Further on, a transmission-line model was associated to the Schur algorithm in problems on circuit synthesis and linear estimation. J.P. Burg ([13]) created a technique in spectral analysis of stationary time series which clarified the connection between the Szegö theory and the maximum entropy method. These problems made a successful career in electrical engineering where the Nevanlinna-Pick interpolation problem became a main tool in robust control, network analysis, and signal processing. A method which encodes a generalization of the Schur algorithm, and hence, can be successfully used in interpolation, was developed over years by T. Kailath and his school, [26], [31], and [39].

In parallel to the scalar case, multi-input multi-output problems in system theory motivated matrix valued, or even more general, operator valued, theory. The dilation theoretical achievements in the works of M.S. Livshits and M.S. Brodskii ([12]), C. Foiaş and B. Sz–Nagy ([42]), and L. de Branges and J.L. Rovnyak ([11]) concerning modelling of operators on Hilbert spaces created the condition of the development starting with the work of D. Sarason ([38]) who represented the Carathéodory and Nevanlinna-Pick interpolation problems as a contractive commutant lifting problem. C. Foiaş and B. Sz–Nagy almost instantly put this into an abstract commutant lifting problem which since then didn't cease to be a source of more and more applications. The scattering theory of P. Lax and R.S. Phillips ([30]) was soon recognized as another parallel way of these investigations.

In the operator valued formulation, a sequence of Schur parameters is a sequence of operators $\{\Gamma_n\}_{n\in\mathbb{N}}$ such that $\Gamma_0 \in \mathcal{L}(\mathcal{H}_0, \mathcal{K})$ is a contraction (here $\mathcal{H}_0$ and $\mathcal{K}$ are Hilbert spaces) and, for $n \geqslant 1$, $\Gamma_n \in \mathcal{L}(\mathcal{H}_n, \mathcal{D}_{\Gamma_{n-1}^*})$ is contraction. Here, for a *contractive operator* Γ between Hilbert spaces we denote by Γ^* its *adjoint operator*, $D_\Gamma = (I - \Gamma^*\Gamma)^{1/2}$ is its *defect operator* and $\mathcal{D}_\Gamma = \overline{\mathcal{R}(D_\Gamma)}$ is its *defect* space. The Schur parameters were first called *choice sequence* ([17]) and a tremendous amount of work was done to clarify the operator-valued Schur analysis in connection with contractive intertwining dilations and Ando dilations ([5]). The parametrization of contractive intertwining dilations culminated with the work of Gr. Arsene, Z. Ceauşescu and C. Foiaş, [8].

About the same time, another parallel highway was opened by the investigations of V.M. Adamyan, D.Z. Arov and M.G. Kreĭn([1], [2], and [3]), on the singular values of Hankel operators and Nehari type problems. These works were later related to the commutant lifting approach, e.g. see C. Foiaş and A.J. Frazho ([24]) and M. Rosenblum and J.L. Rovnayk ([37]). Another method, called the *band method* was initiated by H. Dym and I. Gohberg ([22]). Using the Bergman-Aronszajn ([6]) reproducing kernel method, filtered by the de Branges complementation theory, a Schur analysis was developed by D. Alpay and H. Dym ([4]).

Tiberiu Constantinescu is a mathematician who was educated and found his maturity in the operator theory group at Institute of Mathematics (for about ten years, for political reasons, this institute was disguised as INCREST) in Bucharest, Romania. The leader and the engine of this group was until 1977 C. Foiaş whose influence was so big that, with the exception of some outstanding personalities, he pushed the investigations of many of his colleagues in the directions of his mathematical interest. As an outcome, this connected the group in a very short time to the major mathematical centers in the world and made Bucharest one of the most powerful center in operator theory. Foiaş's mathematical activity was always many-sided and his interest in high quality applied mathematics oriented problems directed some mathematicians in his group towards system theory, time series analysis, geophysics, etc. The N. Wiener award on 1995 is a recognition of the mathematical merits of this outstanding mathematician.

Fate made that Constantinescu started his mathematical career immediately after Foiaş fled, first to France and then to, the major brain attractor, the USA. The reader might think that this was an impediment. On the contrary, the opposite was true, as almost nothing grows under the shadow of an oak tree. C. Foiaş left behind him important open problems but the approaches that the mathematicians around him were using proved to be not sufficiently adequate. Constantinescu found the area ready to feed his remarkably original ideas and personality. He started by performing the Schur analysis ([18]) through a general and abstract pattern applied to the operator valued variant of the A.N. Kolmogorov decomposition ([27]). He further used the Schur analysis to incorporate the M.A. Naĭmark and M.G. Kreĭn unitary extensions of isometries, see [19] and [20], and applied all these to positive completion problems, cf. [7], in the time-varying case, too. A general and simple pattern for all these is presented in [21]. The completions of partial matrices, e.g. [25], is another domain where the Schur analysis turned out to show its value. The connection with the graph theory proved to be benefic and under Constantinescu's influence some young mathematicians in Bucharest, e.g. M. Bakonyi, started to work successfully on these problems. Constantinescu then worked for a while in the group at the Electrical Engineer Department of Stanford University where he brought his expertise in Schur analysis, e.g. see [39], and benefited from the overwhelming influence of T. Kailath.

This book is the second (the first one is more an introduction and it was written in collaboration with M. Bakonyi ([9])) from a programme which reveals the architect vocation of the author. A through presentation of the contents of the volume under review would run as follows:

The first two chapters of the book describe the structure of positive definite kernels in terms of Schur parameters, Kolmogorov decompositions, Cholesky factorizations, triangular contractions, realizations for unitary systems, as well as the models for families of contractions of Sz.-Nagy and Foiaş, and of de Branges and Rovnyak.

Following the pattern of unitary extensions of isometric operators Chapter 3 deals with interpolation problems, moment problems and commutant lifting theorem. The concept of *structured matrix* is explained in Chapter 4 by the introduction of the concept of matrix with displacement structure. Generalized Schur algorithms are explained and many important applications as to adaptive filtering, interpolation problems are carefully treated. In the next chapter the previous results are unified through the existence of the spectral factor for positive definite kernels. Then applications to the study of non-stationary processes, prediction theory and Szegö limit theorems are presented.

In the last chapters, the structured matrices are replaced with arbitrary patterns, e.g. chordal graphs, and applications to completion problems and determinantal formulae are obtained.

One of the remarkable features of this book is the constant effort of the author of unifying different problems under general, geometric and simple formulations. Apparently the ideas behind all these are very simple, but a more thorough examination reveals a complicated and intricate structure that needs a lot of ability to manipulate. This book represents an operator theorist point of view on the many faces of the Schur analysis and its applications. The Schur analysis can be compared, to a certain extent, with the Fourier analysis. It is a procedure to reach the bricks of the matter and if you are clever enough and have the necessary ability, you can play with it and explain a lot of the, otherwise weird, behaviour of the phenomenon under investigation.

But, as in almost all the other science's domains, there is a barrier just at the beginning. The readers are required to make the effort of overpassing the complexity of the formal manipulation which might be discouraging for some of them. However, if this is accomplished then the beauty and the efficiency of this theory is a deserved award. The author made efforts to keep the messy calculations as clear as possible and it is mostly appreciated that the geometric aspects, which are very important for understanding what's all about, are emphasized.

I predict a successful career for this book, as well for the others to continue the author's programme. It comes to fill a need of all the insiders of this theory and I'm sure that other mathematicians, electrical engineers and physicists will soon recognize its value and utility.

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