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BOOK REVIEW:

Boundary Value Problems, Schrödinger Operators, Deformation Quantization, **Michael Demuth, Elmar Schrohe, Bert-Wolfgang Schulze** (Editors), Akademie Verlag, Berlin 1995, 353 pag., ISBN 3-05-501699-8.

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AMS SUBJECT CLASSIFICATION: 35S15, 35A27, 35J11.

The study of *Partial differential equations* proved to be a central matter for a very large number of domains from pure mathematics, theoretical physics and very different fields of applications. In the last period this subject had very important developments originating mainly in the theory of pseudodifferential operators and microlocal analysis to which a large number of books and papers has been devoted. In this context, the *Max Planck Research Group for Partial Differential Equations and Complex Analysis* at the University of Potsdam proposes a series of books under the title: "*Advances in Partial Differential Equations*", containing research expository papers as well as some short communications dealing with theoretical developments and applications in this field. The articles are intended to be largely self-contained and written by specialists and addressing a wide range of readers from graduate students to active researchers.

The Volume 8 in the *"Mathematical Topics"* series of Akademie Verlag, contains the second issue of the "*Advances in Partial Differential Equations*" series entitled "*Boundary Value Problems, Schrödinger Operators, Deformation Quantization*" published in 1995 and edited by Michael Demuth, Elmar Schrohe and Bert-Wolfgang Schulze (editor in chief). This volume gathers five contributions; the first two are continuations of two previous papers published in the first volume of the same series (volume 5 in the *"Mathematical Topics"* series of Akademie Verlag) are written by Bert-Wolfgang Schulze and Elmar Schrohe and deal with *boundary value problems on manifolds with boundary and conical singularities*; the following two contributions are due to Boris Fedosov and are treating the subject of *deformation quantization*, while the third paper by M. Birman presents some new results of the author concerning the estimation of the *number of eigenvalues appearing in a spectral gap of a periodic hamiltonian due to a regular perturbation decaying at infinity*.

In the analysis of boundary value problems on manifolds with boundaries and conical singularities, when one studies composition of operators and parametrices some new classes of symbols are needed and some new algebras of associated operators are put into evidence. The first contribution in the volume, by Bert-Wolfgang Schulze is entitled "*The Variable Discrete Asymptotics in Pseudo-Differential Boundary Value Problems II*" and is the second part of an expository paper presenting the classes of symbols and the symbolic calculus appearing when treating boundary value problems in the class with variable discrete asymptotics. The paper deals with manifolds with boundary and with boundary value problems violating the transmission property, the asymptotics near the boundary being the substitute for the smoothness of solutions up to the boundary from the case with transmission property. The techniques can be generalized for the case of manifolds with edges. The second contribution, by Elmar Schrohe and Bert-Wolfgang Schulze is entitled "*Boundary Value Problems in Boutet de Monvel's Algebra for Manifolds with Conical Singularities II*" and is also the second part of an expository paper presenting the algebras of symbols and their associated operators of interest in the study of pseudo-differential operators on manifolds with boundary and conical singularities. The first chapter introduces the main definitions needed in the sequel; the notions of conical singularity, Green and Mellin operators, spaces with asymptotics and the Boutet de Monvel's algebra are briefly reviewed. The second chapter analysis the Mellin symbols, their operator kernels, their composition law and their adjoints and introduces the so called Mellin quantization providing a correspondence between two different classes of symbols. The third chapter is devoted to the development of the symbolic calculus for the cone algebra with asymptotics based on meromorphic Mellin symbols.

Deformation quantization, a structure first introduced in the 1978 paper by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer, consists in defining the algebra of quantum observables as an algebra of formal power series with respect to a formal parameter h. More precisely, given a symplectic manifold M, one considers the linear space $Z = C^\infty(M)[[h]]$ of formal power series in h with coefficients smooth functions on M and defines on it an associative product (the star-product), satisfying a locality condition and a correspondence principle, meaning that the leading term of the star-product is just the usual product of the leading terms of the factors and the commutator for the star-product has a leading term linear in h and given by the usual Poisson bracket. The advantage of this construction is that it provides a good candidate for generalizing the usual Weyl Calculus from the case of a symplectic space to any symplectic manifold. The problem that one has to solve after rigorously defining the algebra of observables is to provide a Hilbert space representation for it, in the spirit of quantum theory and this task proves to be a rather difficult one. One defines a kind of symbolic calculus by associating the so called *Formal Weyl Algebras Bundle* to the algebra Z previously defined and generalizes the notion of topological index for this case. Then one can show that this index is in fact a formal Laurent series in the parameter h. In "*The Index Theorem for Deformation Quantization*", the third paper in the volume, Boris Fedosov makes a rather self-contained presentation of the main notions and techniques involved in the above construction. The first chapter introduces the notions of K-functor and K-functor with compact supports and their usual representations, the characteristic classes and the Thom Isomorphism Theorems and that of Fredholm family. The second chapter brings in the main notions from the technique of deformation quantization. Starting from an arbitrary symplectic manifold one defines its Formal Weyl Algebras Bundle (FWAB) by considering formal power series in the formal parameter h with coefficients in the tangent bundle of the symplectic manifold and with a special composition law inspired from the classical Weyl calculus. On this bundle one considers connections induced by symplectic connections on the symplectic manifold, defines the notion of an Abelian connection and shows that such an Abelian connection always exists. The subalgebra of flat sections of the FWAB is defined as the algebra of quantum observables ant it is shown to be in a bijective correspondence with the linear space Z; moreover the star-product on Z is defined to be the product induced by the composition law of the FWAB. One can then prove existence and uniqueness of the solution of the Heisenberg Equation, a theorem stating that any quantum algebra is locally trivial and the existence and uniqueness of a trace on any quantum algebra. The third chapter defines the index of an elliptic element of the algebra of quantum observables and proves an index formula for this. One also presents a similar construction to the Thom isomorphism. The fourth contribution to the volume is "*A Trace Density in Deformation Quantization*" by Boris Fedosov. In this paper the author proves that the trace of any flat section of the Weyl algebras bundle may be represented as an integral with respect to a trace density and a procedure is given allowing in principle to calculate the coefficients of this trace density.

The last contribution to the volume is the paper "*The Discrete spectrum in Gaps of the Perturbed Periodic Schrödinger Operator. I. Regular Perturbations*" by M.Sh. Birman. One considers in $L^2(\mathbf{R}^d)$ for $d \geqslant 2$ a second order elliptic operator of the form:

$$A = -\text{div}(g(x)\text{grad}) + p(x)$$

with p and g periodic and a perturbation of this operator with a non-negative potential V of class $L^{d/2}(\mathbf{R}^d)$ (for $d \geqslant 3$, and a more complicated condition for $d = 2$), coupled to the operator A with a real coupling constant α. One defines the families of operators $A_{\pm}(\alpha) = A \mp \alpha V$ with the sum defined in the sense of quadratic forms. One considers a spectral gap of A and proves asymptotic estimations for the number of eigenvalues of $A_{\pm}(\alpha)$ in the chosen gap for $\alpha \to \infty$.

The contributions included in this second volume of the "*Advances in Partial Differential Equations*" series present some self-contained and detailed analysis of some very important advances in the fields of boundary value problems, Schrödinger operators and Weyl quantization and are addressing a very large auditory among mathematicians and theoretical physicists.

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