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INDUCED REPRESENTATIONS OF TWISTED C^*-DYNAMICAL SYSTEMS

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ABSTRACT. Let (A, G, α, u) be a twisted C^*-dynamical system in the sense of Busby and Smith. Then for any closed subgroup H of G, $A \times_{\alpha,u} H$ is Morita equivalent to $C_0(G/H, A) \times_{\tilde{\alpha},\tilde{u}} G$, where $(\tilde{\alpha}, \tilde{u})$ is the diagonal twisted action. We show that the space of compactly supported bounded Borel functions $B_c(G, A)$ can be given a natural pre-imprimitivity bimodule structure which implements the equivalence, and use this to induce representations from $A \times_{\alpha,u} H$ to $A \times_{\alpha,u} G$. We prove an imprimitivity theorem for this inducing process, and show how the inducing processes of Busby and Smith and Mackey are special cases of ours.

KEYWORDS: *C^*-algebra, dynamical system, Morita equivalence.*

AMS SUBJECT CLASSIFICATION: 46L55.

INTRODUCTION

Consider an action α of a locally compact group G on a C^*-algebra A and a closed subgroup H of G. Green ([7]) proved that the space $C_c(G, A)$ of continuous, compactly supported functions from G into A carries a natural $C_0(G/H, A) \times_{\tilde{\alpha}} G$ – $A \times_\alpha H$ pre-imprimitivity bimodule structure, where $\tilde{\alpha}$ is the diagonal action of G on $C_0(G/H, A)$. Then he showed that $A \times_\alpha G$ maps into the multiplier algebra of $C_0(G/H, A) \times_{\tilde{\alpha}} G$, so that Rieffel's Morita equivalence framework could be used to induce representations of $A \times_\alpha H$ up to $A \times_\alpha G$. These results were essential to Green's generalization of the Mackey apparatus, and have proved to be among the most significant developments in the theory of C^*-dynamical systems. Placed in the Morita equivalence context, the theory of induced representations has turned out to be both powerful and elegant.

In the present paper, we prove analogous results for the twisted C^*-dynamical systems introduced by Busby and Smith in [1]. Roughly speaking, a Busby-Smith twisted action of a group G on a C^*-algebra A is a map α of G into $\mathrm{Aut}(A)$ which is allowed to be only Borel, together with an A-valued 2-cocycle u for G which measures the extent to which α is not a homomorphism. These systems have recently been studied from a representation-theoretic point of view by Packer and Raeburn ([12], [13]), who showed among other things that their theory contains that of the separable Green-twisted systems. Special cases of these systems appear in the work of Zeller-Meier ([15], for discrete G) and Mackey ([11], for $A = \mathbb{C}$), among many others.

Our approach is modeled on that of Green, although technical complications arise because of the Borel nature of the twisted actions. In [9] we used the stabilization trick of Packer and Raeburn ([12]) to show that if (A, G, α, u) is a Busby-Smith twisted C^*-dynamical system and H is a closed subgroup of G, there exists a diagonal twisted action $(\tilde{\alpha}, \tilde{u})$ of G on $C_0(G/H, A)$ such that $C_0(G/H, A) \times_{\tilde{\alpha},\tilde{u}} G$ is Morita equivalent to $A \times_{\alpha,u} H$; in order for this to be useful, we need an explicit, manageable bimodule which implements the equivalence. In Section 2 we show that the space $B_c(G, A)$ of bounded, compactly supported Borel functions from G into A can be given a natural right $A \times_{\alpha,u} H$-rigged space structure, and that then $C_0(G/H, A) \times_{\tilde{\alpha},\tilde{u}} G$ is isomorphic to $\mathcal{K}_{A\times_{\alpha,u}H}(B_c(G, A))$. While $B_c(G, A)$ has no natural left $C_0(G/H, A) \times_{\tilde{\alpha},\tilde{u}} G$-rigged space structure, we show in Section 3 that it does carry a left action of $A \times_{\alpha,u} G$, and that this is sufficient for inducing representations of $A \times_{\alpha,u} H$ to $A \times_{\alpha,u} G$. Taking advantage of the Morita equivalence framework, we prove an imprimitivity theorem, describing exactly which representations of $A \times_{\alpha,u} G$ are induced from representations of $A \times_{\alpha,u} H$. Our results also include a theorem on induction in stages: if K is a closed subgroup of H, then inducing from $A \times_{\alpha,u} K$ to $A \times_{\alpha,u} H$ followed by inducing from $A \times_{\alpha,u} K$ to $A \times_{\alpha,u} G$ is the same as inducing from $A \times_{\alpha,u} K$ to $A \times_{\alpha,u} G$ in one step. Finally, in Section 4 we show that the original induced representations of Busby and Smith ([1]) and the induced multiplier representations of Mackey ([11]) coincide with certain special cases of ours.

1. PRELIMINARIES

We begin by establishing our notation, conventions, and basic definitions. Throughout this paper, A and B will be separable C^*-algebras. The multiplier algebra of A is $\mathcal{M}(A)$, with unitary group $\mathcal{UM}(A)$. If $\mathcal{H}$ is a Hilbert space, we use the standard notations $\mathcal{KH}$, $\mathcal{B}(\mathcal{H})$, $\mathcal{UB}(\mathcal{H})$ for the compact operators, the bounded operators, and the unitary operators, respectively, on $\mathcal{H}$. All groups will be second countable, locally compact groups, and all integrals over these groups will be with respect to a left-invariant Haar measure. We denote the modular function on a group G by Δ_G. If H is a closed subgroup of G, we define for convenience $\gamma_H^G(t) = (\Delta_G(t)/\Delta_H(t))^{1/2}$, and we adopt the convention of [12] of writing $\dot{t}$ for an element tH of G/H. For a Borel measurable function f taking values in a C^*-algebra, we use the Bochner integral to make sense of the expression $\int_G f(s)\,ds$; for strictly Borel functions with values in multiplier algebras, the integral can be interpreted strictly. See [12], Section 1 for a brief discussion and further references.

TWISTED DYNAMICAL SYSTEMS. A *(Busby-Smith) twisted dynamical system* is a quadruple (A, G, α, u) consisting of a C^*-algebra A and a group G, together with a strongly Borel map $\alpha : G \to \mathrm{Aut}(A)$ and a strictly Borel map $u : G \times G \to \mathcal{UM}(A)$, such that:

(i) $\alpha_e = \mathrm{id}_A$ and $u(e,t) = u(t,e) = \mathbf{1}_A$ for $t \in G$;

(ii) $\alpha_s \circ \alpha_t(a) = u(s,t)\,\alpha_{st}(a)\,u(s,t)^*$ for $a \in A$ and $s,t \in G$;

(iii) $\alpha_r(u(s,t))\,u(r,st) = u(r,s)\,u(rs,t)$ for $r,s,t \in G$.

The pair (α, u) is called a *(Busby-Smith) twisted action* of G on A. (Cf. [1], Definition 2.1; [12], Definition 2.1.)

A *covariant homomorphism* of (A, G, α, u) into a multiplier algebra $\mathcal{M}(C)$ is a pair (π, U) consisting of a non-degenerate homomorphism $\pi : A \to \mathcal{M}(C)$ and a strictly Borel map $U : G \to \mathcal{UM}(C)$ such that:

(i) $\pi(\alpha_s(a)) = U(s)\,\pi(a)\,U(s)^*$ for $a \in A$ and $s \in G$;

(ii) $U(s)U(t) = \pi(u(s,t))\,U(st)$ for $s,t \in G$.

(Sometimes we write U_s for $U(s)$.) If $C = \mathcal{K}(\mathcal{H})$ for a (separable) Hilbert space $\mathcal{H}$, then we say that (π, U) is a *covariant representation* of (A, G, α, u) on $\mathcal{H}$. (Cf. [13], Definition 1.1; [12], Definition 2.3.)

For each twisted system (A, G, α, u) there is a *crossed product* C^*-algebra $A \times_{\alpha,u} G$ and a canonical covariant homomorphism (i_A, i_G) of (A, G, α, u) into $\mathcal{M}(A \times_{\alpha,u} G)$ such that:

(i) for any covariant representation (π, U) of (A, G, α, u) on a Hilbert space $\mathcal{H}$, there is a non-degenerate representation $\pi \times U$ of $A \times_{\alpha,u} G$ on $\mathcal{H}$, called the *integrated form* of (π, U), such that $(\pi \times U) \circ i_A = \pi$ and $(\pi \times U) \circ i_G = U$;

(ii) the set $\{i_A \times i_G(z) \mid z \in L^1(G, A)\}$ is dense in $A \times_{\alpha,u} G$, where $i_A \times i_G(z)$ denotes the strictly defined Bochner integral $\int_G i_A(z(s))\, i_G(s)\, ds$.

The set of compactly supported bounded Borel functions from G into A forms a $*$-algebra $B_c(G, A; \alpha, u)$ when equipped with the twisted convolution and involution given by:

$$f * g(t) = \int_G f(s)\, \alpha_s(g(s^{-1}t))\, u(s, s^{-1}t)\, ds,$$
$$f^*(t) = u(t, t^{-1})\, \alpha_t(f(t^{-1})^*)\, \Delta_G(t^{-1}).$$

The map $f \mapsto i_A \times i_G(f)$ is a $*$-homomorphism, and thus identifies $B_c(G, A; \alpha, u)$ with a dense $*$-subalgebra of $A \times_{\alpha,u} G$. (Cf. [12], Definition 2.4, Proposition 2.7.)

MORITA EQUIVALENCE OF TWISTED ACTIONS. Suppose (A, G, α, u) and (B, G, β, v) are twisted dynamical systems, and X is an A–B imprimitivity bimodule ([14], Definition 6.10]). We say the twisted actions (α, u) and (β, v) are *Morita equivalent* if there exists a strongly Borel map $\gamma : G \to \mathrm{Aut}(X)$ such that:

(i) $\alpha_s({}_A\langle x, y\rangle) = {}_A\langle \gamma_s(x), \gamma_s(y)\rangle$ for $x, y \in X$ and $s \in G$;

(ii) $\beta_s(\langle x, y\rangle_B) = \langle \gamma_s(x), \gamma_s(y)\rangle_B$ for $x, y \in X$ and $s \in G$;

(iii) $\gamma_s \circ \gamma_t(x) = u(s, t) \cdot \gamma_{st}(x) \cdot v(s, t)^*$ for $x \in X$ and $s, t \in G$.

(Here $\mathrm{Aut}(X)$ is the set of bicontinuous linear bijections φ of X which satisfy the ternary homomorphism identity $\varphi(x \cdot \langle y, z\rangle_B) = \varphi(x) \cdot \langle \varphi(y), \varphi(z)\rangle_B$.) We write $(A, G, \alpha, u) \sim_{X,\gamma} (B, G, \beta, v)$, and call (X, γ) a *system of imprimitivity* implementing the equivalence. (Cf. [2], Definition 2.1; [8], Definition 2.1.1; compare [5], [4], [3].) Morita equivalent twisted systems have (strongly) Morita equivalent crossed products ([2], Theorem 2.3). In fact, $B_c(G, X)$ is a $B_c(G, A; \alpha, u)$–$B_c(G, B; \beta, v)$ pre-imprimitivity bimodule when equipped with actions and inner products defined by

$$f \cdot x(s) = \int_G f(t) \cdot \gamma_t(x(t^{-1}s)) \cdot v(t, t^{-1}s)\, dt$$

$$x \cdot g(s) = \int_G x(t) \cdot \beta_t(g(t^{-1}s))\, v(t, t^{-1}s)\, dt$$

$$_{B_c(G,A;\alpha,u)}\langle x, y\rangle(s) = \int_G {}_A\langle x(t), \gamma_s(y(s^{-1}t)) \cdot v(s, s^{-1}t)\rangle\, \Delta_G(s^{-1}t)\, dt$$

$$\langle x, y\rangle_{B_c(G,B;\beta,v)}(s) = \int_G v(t^{-1}, t)^* \beta_{t^{-1}}(\langle x(t), y(ts)\rangle_B)\, v(t^{-1}, ts)\, dt.$$

2. THE IMPRIMITIVITY BIMODULE

Let (A, G, α, u) be a Busby-Smith twisted dynamical system, and let H be a closed subgroup of G. In [9], we showed that there is a twisted action $(\tilde{\alpha}, \tilde{u})$ of G on $C_0(G/H, A)$ given by

$$\tilde{\alpha}_s(f)(\dot{t}) = \alpha_s(f(s^{-1}\dot{t})) \quad \text{and} \quad (\tilde{u}(s,t)f)(\dot{r}) = u(s,t)\, f(\dot{r}),$$

and that then we have $C_0(G/H, A) \times_{\tilde{\alpha},\tilde{u}} G \sim A \times_{\alpha,u} H$. Our method there was to produce a system of imprimitivity $(A \otimes \mathcal{H}, \delta)$ implementing a Morita equivalence between (A, G, α, u) and an ordinary system $(A \otimes \mathcal{K}, G, \beta)$. (The specific formula for δ, which came from [12], Equation 3.1, is irrelevant here; we only need to know that δ interacts with (α, u) and $(\beta, 1)$ as in conditions (i)–(iii) immediately above. For example, we have $\delta_s \circ \delta_t(a \otimes \xi) = \delta_{st}(a \otimes \xi) \cdot u(s,t)^*$.) This gave rise to two equivalences:

$$C_0(G/H, A) \times_{\tilde{\alpha},\tilde{u}} G \sim C_0(G/H, A \otimes \mathcal{K}) \times_{\tilde{\beta}} G$$

and

$$(A \otimes \mathcal{K}) \times_\beta H \sim A \times_{\alpha,u} H,$$

which are linked by Green's result ([7], Proposition 3) for ordinary systems:

$$C_0(G/H, A \otimes \mathcal{K}) \times_{\tilde{\beta}} G \sim (A \otimes \mathcal{K}) \times_\beta H.$$

The desired result was then obtained by transitivity; a corollary of this argument is that the tensor product Y of the three bimodules involved implements the equivalence. The goal of this section is to show that (a completion of) $X_0 = B_c(G, A)$ is also a $C_0(G/H, A) \times_{\tilde{\alpha},\tilde{u}} G - A \times_{\alpha,u} H$ imprimitivity bimodule. Because its elements are only Borel, X_0 carries no natural $B_c(G, C_c(G/H, A); \tilde{\alpha}, \tilde{u})$-valued inner product, making our work a bit harder than one might have expected.

Let $B = A \times_{\alpha,u} H$, with dense subalgebra $B_0 = B_c(H, A; \alpha, u)$. Then X_0 carries the natural right B_0-rigged space structure defined as follows:

$$x \cdot g\,(r) = \int_H x(rt)\, \alpha_{rt}(g(t^{-1}))\, u(rt, t^{-1})\, \gamma_H^G(t)\, dt$$

$$\langle x, y\rangle_{B_0}(t) = \int_G u(s^{-1}, s)^*\, \alpha_{s^{-1}}(x(s)^* y(st))\, u(s^{-1}, st)\, \gamma_H^G(t)\, ds.$$

Our strategy will be to exhibit a right B_0-rigged space isomorphism of a dense subspace Y_0 of Y onto a dense subspace of X_0. It follows that the completion X of X_0 is isomorphic to Y as a right B-rigged space, and therefore that $\mathcal{K}_B(X) \cong \mathcal{K}_B(Y) = C_0(G/H, A) \times_{\tilde{\alpha},\tilde{u}} G$, as desired.

We begin by reviewing the pre-imprimitivity bimodule structure for the three components of which will make up Y_0. We will be concerned here with functions into reversed modules. If Z is an imprimitivity bimodule, recall that the reversed module $\bar{Z} = \{\bar{z} \mid z \in Z\}$ is identical to Z as a set, but employs the reversed actions and inner products. Accordingly, functions from a set S into $\bar{Z}$ differ only in attitude from functions into Z. A function $F : S \to Z$ will be written $\bar{F}$ when we want to treat it as taking values in $\bar{Z}$.

Let $E_0 = B_c(G, C_c(G/H, A); \tilde{\alpha}, \tilde{u})$ and $D_0 = B_c(G, C_c(G/H, A \otimes \mathcal{K}); \tilde{\beta})$. Then $B_c(G, C_c(G/H, \overline{A \otimes \mathcal{H}}))$ is an E_0–D_0 pre-imprimitivity bimodule when equipped with actions and inner products as follows:

$$F \cdot \bar{x}(r, \dot{t}) = \overline{\int_G \delta_s(x(s^{-1}r, s^{-1}\dot{t}))\, F(s, \dot{t})^*\, ds}$$

$$\bar{x} \cdot G(r, \dot{t}) = \overline{\int_G \beta_s(G(s^{-1}r, s^{-1}\dot{t}))^*\, x(s, \dot{t})\, ds}$$

$${}_{E_0}\langle \bar{x}, \bar{y}\rangle(r, \dot{t}) = \int_G \langle x(s, \dot{t}), \delta_r(y(r^{-1}s, r^{-1}\dot{t}))\rangle_A\, \Delta_G(r^{-1}s)\, ds$$

$$\langle \bar{x}, \bar{y}\rangle_{D_0}(r, \dot{t}) = \int_G \beta_{s^{-1}}({}_{A\otimes\mathcal{K}}\langle x(s, s\dot{t}), y(sr, s\dot{t})\rangle)\, ds.$$

(Here and throughout the paper we will identify functions from G into some space of functions on G/H with functions on $G \times G/H$.) Similarly, if $C_0 = B_c(H, A \otimes$

$\mathcal{K};\beta)$, then $B_c(G, A\otimes\mathcal{K})$ can be made into a D_0–C_0 pre-imprimitivity bimodule:

$$F\cdot x(r) = \int_G F(s,\dot r)\,\beta_s(x(s^{-1}r))\,\mathrm{d}s$$

$$x\cdot g(r) = \int_H x(rt)\,\beta_{rt}(g(t^{-1}))\,\gamma_H^G(t)\,\mathrm{d}t$$

$$_{D_0}\langle x,y\rangle(s,\dot r) = \int_H x(rt)\,\beta_s(y(s^{-1}rt)^*)\,\Delta_G(s^{-1}rt)\,\mathrm{d}t$$

$$\langle x,y\rangle_{C_0}(t) = \int_G \beta_{s^{-1}}(x(s)^*y(st))\,\mathrm{d}s\,\gamma_H^G(t).$$

Finally, we have the C_0–B_0 pre-imprimitivity bimodule $B_c(H, A\otimes\mathcal{H})$:

$$f\cdot x(h) = \int_H f(t)\cdot\delta_t(x(t^{-1}h))\cdot u(t,t^{-1}h)\,\mathrm{d}t$$

$$x\cdot g(h) = \int_H x(t)\cdot\alpha_t(g(t^{-1}h))\,u(t,t^{-1}h)\,\mathrm{d}t$$

$$_{C_0}\langle x,y\rangle(h) = \int_H {}_{A\otimes K}\langle x(t), \delta_h(y(h^{-1}t))\cdot u(h,h^{-1}t)\rangle\,\Delta_H(h^{-1}t)\,\mathrm{d}t$$

$$\langle x,y\rangle_{B_0}(h) = \int_H u(t^{-1},t)^*\,\alpha_{t^{-1}}(\langle x(t),y(th)\rangle_A)\,u(t^{-1},th)\,\mathrm{d}t.$$

Now let Y_0 be the balanced algebraic tensor product

$$Y_0 = B_c(G, C_c(G/H, \overline{A\otimes\mathcal{H}}))\odot_{D_0} B_c(G, A\otimes\mathcal{K})\odot_{C_0} B_c(H, A\otimes\mathcal{H});$$

by transitivity, Y_0 is an E_0–B_0 pre-imprimitivity bimodule, and completes to give

$$C_0(G/H, A)\times_{\tilde\alpha,\tilde u} G\sim_Y A\times_{\alpha,u} H.$$

For $(\bar F, f, \tau)$ in $B_c(G, C_c(G/H, \overline{A\otimes\mathcal{H}}))\times B_c(G, A\otimes\mathcal{K})\times B_c(H, A\otimes\mathcal{H})$, define $\varphi_0(\bar F, f, \tau)$ in $B_c(G, A)$ by

$$\varphi_0(\bar F, f,\tau)(s)$$
$$= \int_H \Big\langle \int_G \beta_r\,(f(r^{-1}st^{-1})^*)\cdot F(r,\dot s)\,\mathrm{d}r, \delta_{st^{-1}}\,(\tau(t))\Big\rangle_A u(st^{-1},t)\,\Delta_G(t^{-1})\gamma_H^G(t)\,\mathrm{d}t.$$

Then φ_0 is trilinear, and routine calculations show that it is balanced appropriately in both spots. Thus there exists a map $\varphi: Y_0\to B_c(G,A)$ such that $\varphi(\bar F\otimes f\otimes\tau) =$

$\varphi_0(\bar{F}, f, \tau)$. In order to see that φ is a B_0-rigged space isomorphism onto its range, it suffices to check that it preserves the B_0-valued inner product and intertwines the right action. The latter follows from straightforward calculations; to see the former, we have the following computation. For $t \in H$:

$$
\begin{aligned}
&\langle \bar{F} \otimes f \otimes \tau, \bar{G} \otimes g \otimes \sigma \rangle_{B_0}(t) \\
&= \langle \tau, \langle f, \langle \bar{F}, \bar{G} \rangle_{D_0} \cdot g \rangle_{C_0} \cdot \sigma \rangle_{B_0}(t) \\
&= \int_G \int_G \int_G \int_H \int_H u(h^{-1}, h)^* \, \alpha_{h^{-1}}\Big(\Big\langle \tau(h) \, , \, \beta_{s^{-1}}\Big(f(s)^* \beta_{x^{-1}}\big({}_{A\otimes\mathcal{K}}\langle F(x, x\dot{s}k) \, , \\
&\qquad G(xy, x\dot{s}k) \rangle\big) \beta_y\big(g(y^{-1}sk)\big)\Big) \cdot \delta_k\big(\sigma(k^{-1}ht)\big) \cdot u(k, k^{-1}ht) \Big\rangle_A \Big) \\
&\qquad \cdot u(h^{-1}, ht) \, \gamma_H^G(k) \, dh \, dk \, ds \, dx \, dy \\
&= \int_G \int_G \int_G \int_H \int_H u(h^{-1}, h)^* \, \alpha_{h^{-1}}\Big(\Big\langle \tau(h) \, , \, {}_{A\otimes\mathcal{K}}\langle \beta_{s^{-1}}\big(f(s)^*\big) \\
&\qquad \cdot \delta_{s^{-1}x^{-1}}\big(F(x, x\dot{s}k)\big) \cdot u(s^{-1}, x^{-1})^* \, , \, \beta_{s^{-1}y}\big(g(y^{-1}sk)^*\big) \\
&\qquad \cdot \delta_{s^{-1}x^{-1}}\big(G(xy, x\dot{s}k)\big) \cdot u(s^{-1}, x^{-1})^* \rangle \cdot \delta_k\big(\sigma(k^{-1}ht)\big) \cdot u(k, k^{-1}ht) \Big\rangle_A \Big) \\
&\qquad \cdot u(h^{-1}, ht) \, \gamma_H^G(k) \, dh \, dk \, ds \, dx \, dy \\
&= \int_G \int_G \int_G \int_H \int_H u(h^{-1}, h)^* \, \alpha_{h^{-1}}\Big(\big\langle \beta_{s^{-1}}\big(f(s)^*\big) \cdot \delta_{s^{-1}x^{-1}}\big(F(x, x\dot{s}k)\big) \, , \, \tau(h) \big\rangle_A^* \\
&\qquad \big\langle \beta_{s^{-1}y}\big(g(y^{-1}sk)^*\big) \cdot \delta_{s^{-1}x^{-1}}\big(G(xy, x\dot{s}k)\big) \, , \, \delta_k\big(\sigma(k^{-1}ht)\big) \cdot u(k, k^{-1}ht) \big\rangle_A \Big) \\
&\qquad \cdot u(h^{-1}, ht) \, \gamma_H^G(k) \, dh \, dk \, ds \, dx \, dy \\
&= \int_G \int_G \int_H \int_H \int_G u(h^{-1}, h)^* \, \alpha_{h^{-1}}\big(u(s^{-1}x^{-1}, xs)^*\big) \, u(h^{-1}, s^{-1}x^{-1}) \\
&\qquad \cdot \alpha_{h^{-1}s^{-1}x^{-1}}\Big(\big\langle \beta_x\big(f(s)^*\big) \cdot F(x, x\dot{s}k) \, , \, \delta_{xs}\big(\tau(h)\big) \big\rangle_A^* \\
&\qquad \cdot \big\langle \beta_{xy}\big(g(y^{-1}sk)^*\big) \cdot G(xy, x\dot{s}k) \, , \, \delta_{xsk}\big(\sigma(k^{-1}ht)\big) \big\rangle_A \Big) u(h^{-1}, s^{-1}x^{-1})^* \\
&\qquad \cdot \alpha_{h^{-1}}\big(u(s^{-1}x^{-1}, xsk)\big) \, \alpha_{h^{-1}}\big(u(k, k^{-1}ht)\big) u(h^{-1}, ht) \\
&\qquad \cdot \gamma_H^G(k) \, ds \, dk \, dh \, dy \, dx
\end{aligned}
$$

which becomes, by the substitutions $s \mapsto x^{-1}sh^{-1}$, $y \mapsto x^{-1}y$, $k \mapsto htk^{-1}$

$$
\begin{aligned}
&\langle \bar{F} \otimes f \otimes \tau, \bar{G} \otimes g \otimes \sigma \rangle_{B_0}(t) \\
&= \int_G \int_G \int_H \int_H \int_G u(h^{-1}, h)^* \, \alpha_{h^{-1}}\big(u(hs^{-1}, sh^{-1})^*\big)\, u(h^{-1}, hs^{-1}) \\
&\qquad \cdot \alpha_{s^{-1}}\Big(\big\langle \beta_x\big(f(x^{-1}sh^{-1})^*\big) \cdot F(x, \dot{s}) \,,\, \delta_{sh^{-1}}(\tau(h))\big\rangle_A^* \big\langle \beta_y\big(g(y^{-1}stk^{-1})^*\big) \\
&\qquad \cdot G(y, \dot{s}t) \,,\, \delta_{stk^{-1}}(\sigma(k))\big\rangle_A\Big) u(h^{-1}, hs^{-1})^* \, \alpha_{h^{-1}}(u(hs^{-1}, stk^{-1}) \\
&\qquad \cdot u(htk^{-1}, k)) u(h^{-1}, ht)\, \gamma_H^G(htk^{-1}) \Delta_G(h^{-1}) \Delta_H(k^{-1}) \, \mathrm{d}s \, \mathrm{d}k \, \mathrm{d}h \, \mathrm{d}y \, \mathrm{d}x \\
&\overset{\dagger}{=} \int_G \int_G \int_G \int_H \int_H u(s^{-1}, s)\, \alpha_{s^{-1}}\big(u(sh^{-1}, h)^*\big)\, \alpha_{s^{-1}}\Big(\big\langle \beta_x\big(f(x^{-1}sh^{-1})^*\big) \\
&\qquad \cdot F(x, \dot{s}) \,,\, \delta_{sh^{-1}}(\tau(h))\big\rangle_A^* \big\langle \beta_y\big(g(y^{-1}stk^{-1})^*\big) \cdot G(y, \dot{s}t) \,,\, \delta_{stk^{-1}}(\sigma(k))\big\rangle_A\Big) \\
&\qquad \cdot \alpha_{s^{-1}}(u(stk^{-1}, k))\, u(s^{-1}, st)\, \Delta_G(h^{-1}) \gamma_H^G(h)\, \Delta_G(k^{-1}) \gamma_H^G(k) \\
&\qquad \cdot \gamma_H^G(t) \, \mathrm{d}h \, \mathrm{d}k \, \mathrm{d}s \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_G u(s, s^{-1})^* \, \alpha_{s^{-1}}\big(\varphi(\bar{F} \otimes f \otimes \tau)(s)^* \varphi(\bar{G} \otimes g \otimes \sigma)(st)\big)\, u(s^{-1}, st) \, \mathrm{d}s \, \gamma_H^G(t) \\
&= \big\langle \varphi(\bar{F} \otimes f \otimes \tau) \,,\, \varphi(\bar{G} \otimes g \otimes \sigma) \big\rangle_{B_0}(t),
\end{aligned}
$$

which proves the assertion. The equality at † follows from two cocycle calculations included here separately:

$$
\begin{aligned}
&u(h^{-1}, h)^* \, \alpha_{h^{-1}}(u(hs^{-1}, sh^{-1})^*)\, u(h^{-1}, hs^{-1}) \\
&\qquad = u(h^{-1}, h)^* \, u(s^{-1}, sh^{-1})^* \\
&\qquad = u(s^{-1}, s)^* \, u(h^{-1}, hs^{-1})^* \, \alpha_{h^{-1}}(u(hs^{-1}, s))\, u(s^{-1}, sh^{-1})^* \\
&\qquad = u(s^{-1}, s)^* \, u(h^{-1}, hs^{-1})^* \, u(s^{-1}, sh^{-1})^* \, \alpha_{s^{-1}}(\alpha_{sh^{-1}}(u(hs^{-1}, s))) \\
&\qquad = u(s^{-1}, s)^* \, \alpha_{s^{-1}}(u(sh^{-1}, hs^{-1})^* \alpha_{sh^{-1}}(u(hs^{-1}, s))) \\
&\qquad = u(s^{-1}, s)^* \, \alpha_{s^{-1}}(u(sh^{-1}, h)^*);
\end{aligned}
$$

$$
\begin{aligned}
&u(h^{-1}, hs^{-1})^* \, \alpha_{h^{-1}}(u(hs^{-1}, stk^{-1})\, u(htk^{-1}, k)) u(h^{-1}, ht) \\
&\qquad = u(h^{-1}, hs^{-1})^* \, \alpha_{h^{-1}}(\alpha_{hs^{-1}}(u(stk^{-1}, k))\, u(hs^{-1}, st))\, u(h^{-1}, ht) \\
&\qquad = \alpha_{s^{-1}}(u(stk^{-1}, k))\, u(h^{-1}, hs^{-1})^* \, \alpha_{h^{-1}}(u(hs^{-1}, st))\, u(h^{-1}, ht) \\
&\qquad = \alpha_{s^{-1}}(u(stk^{-1}, k))\, u(s^{-1}, st).
\end{aligned}
$$

We now want to show that $\varphi(Y_0)$ is dense in X_0. We will need the following lemma regarding the norm on X_0 induced by the B_0-valued inner product.

LEMMA 2.1. *Suppose $\{f_n\}$ is a sequence in $B_c(G, A)$ such that $\|f_n\|_1 \to 0$ and there exists a positive constant M and a compact set $K \subseteq G$ with $\|f_n\|_\infty < M$ and* $\operatorname{supp} f_n \subseteq K$ *for all n. Then $\|f_n\|_B \to 0$.*

Proof. Let $\{f_n\}$, M, and K be as above. Then simply compute:

$$\begin{aligned}
\|f_n\|_B^2 &= \|\langle f_n, f_n\rangle_{B_0}\|_B \leqslant \|\langle f_n, f_n\rangle_{B_0}\|_{L^1(H,A)} \\
&= \int_H \Big\| \int_G u(s^{-1}, s)^* \,\alpha_{s^{-1}}(f_n(s)^* f_n(sh))\, u(s^{-1}, sh)\,\mathrm{d}s \Big\| \,\mathrm{d}h \\
&\leqslant \int_H \int_G \|f_n(s)\|\, \|f_n(sh)\|\,\mathrm{d}s\,\mathrm{d}h \\
&= \int_H \chi_{K^{-1}K}(h) \int_G \|f_n(s)\|\,\|f_n(sh)\|\,\mathrm{d}s\,\mathrm{d}h \\
&\leqslant \mu_H(K^{-1}K \cap H)\, M\, \|f_n\|_1 \to 0.
\end{aligned}$$

Note that $\mu_H(K^{-1}K \cap H) < \infty$ because $K^{-1}K \cap H$ is compact in H. ∎

We will also need the following technical lemma. The proof is an adaptation of the proof of [10], Satz 1, which also appears in the appendix of [13].

LEMMA 2.2. *Let H be a closed subgroup of a locally compact group G, and fix $g \in L^1(G)$. For each $\varepsilon > 0$, there exists $f \in L^1(H)$ such that*

$$\|g - g *_H f\|_1 < \varepsilon,$$

where $g *_H f(s) = \int_H g(st^{-1})\, f(t)\,\mathrm{d}t$.

Proof. For $h \in H$ and $g \in L^1(G)$, define $R_h(g)(s) = g(sh^{-1})$. Then it is easy to check that R_h is an isometry of $L^1(G)$ which satisfies for each $f \in L^1(H)$

$$R_h(g *_H f) = g *_H R_h(f) \quad \text{and} \quad g *_H f = \int_H R_t(g) f(t)\,\mathrm{d}t.$$

Now $h \mapsto R_h(g)$ is Borel from H into $L^1(G)$, so it is continuous on some compact set $K \subseteq H$ of positive measure. A standard compactness argument gives an open set $U \subseteq H$ with positive measure and compact closure such that $\|R_h(g) - R_t(g)\|_1 < \varepsilon$ for $h, t \in U$. Let w be the characteristic function of U in H, normalized so that $\int_H w(t)\,\mathrm{d}t = 1$, and put $f = R_h^{-1}(w)$, where h is some fixed element of U. (So

f is in fact the normalized characteristic function of Uh^{-1}, a relatively compact open neighborhood of e in H.) Then

$$\begin{aligned}\|g - g *_H f\|_1 = \|R_h(g) - g *_H R_h(f)\|_1 &= \Big\|R_h(g) - \int_H R_t(g)\, w(t)\, dt\Big\|_1 \\ &= \Big\|\int_H (R_h(g) - R_t(g))\, w(t)\, dt\Big\|_1 \\ &\leqslant \int_H \|R_h(g) - R_t(g)\|_1\, w(t)\, dt < \varepsilon. \quad \blacksquare\end{aligned}$$

THEOREM 2.3. *Let (A, G, α, u) be a twisted dynamical system, and let H be a closed subgroup of G. Then the completion of the right $B_0 = B_c(H, A; \alpha, u)$-rigged space $X_0 = B_c(G, A)$ is a $C_0(G/H, A) \times_{\tilde{\alpha},\tilde{u}} G$ — $A \times_{\alpha,u} H$ imprimitivity bimodule.*

Proof. Thus far we have shown that there is a right B_0-rigged space isomorphism φ of Y_0 into X_0; as outlined in the beginning of this section, the theorem will follow if we can show that the range of φ is dense in X_0. For this, it suffices to approximate functions $g \otimes_\alpha a$ of the form $s \mapsto g(s)\alpha_s(a)$ for $g \in B_c(G)$ and $a \in A$; of course this approximation will be done in the sense of Lemma 2.1. So, fix g and a as above, and let $K = \operatorname{supp} g$. Fix relatively compact open neighborhoods U' of e in G and V' of e in H. For each $\varepsilon > 0$, we will construct an element $\bar{F} \otimes f \otimes \tau$ of Y_0 such that:

(i) $\|g \otimes_\alpha a - \varphi(\bar{F} \otimes f \otimes \tau)\|_1 < \varepsilon$;
(ii) $\|\varphi(\bar{F} \otimes f \otimes \tau)\|_\infty < \|a\|\,\|g\|_\infty$;
(iii) $\operatorname{supp} \varphi(\bar{F} \otimes f \otimes \tau) \subseteq K\overline{U'V'}$.

Given this, we can select a sequence $\{\varphi(\bar{F}_n \otimes f_n \otimes \tau_n)\}$ in X_0 such that $\{g \otimes_\alpha a - \varphi(\bar{F}_n \otimes f_n \otimes \tau_n)\}$ satisfies the hypotheses of Lemma 2.1, and therefore $\varphi(\bar{F}_n \otimes f_n \otimes \tau_n) \to g \otimes_\alpha a$ in X_0.

So, fix $\varepsilon > 0$. Let ξ be a unit vector in $\mathcal{H}$, and let $T \in \mathcal{K}(\mathcal{H})$ be the operator defined by $T(\zeta) = \langle \zeta, \xi \rangle_{\mathcal{H}} \xi$. For any $\eta > 0$ we can find elements b and c in a norm 1 approximate identity for A such that $\|a - c^* b^* a\| < \eta$. Thus $\|a - \langle x, y \rangle_A\| < \eta$, where $x = b \otimes T \cdot (c \otimes \xi)$ and $y = a \otimes \xi$; note that $\|x\| \leqslant 1$ and $\|y\| = \|a\|$.

Now the action δ of G on $A \otimes \mathcal{H}$ is strongly Borel, so the map $s \mapsto \delta_s(y)$ of G into $A \otimes \mathcal{H}$ is Borel. It follows from the generalised Lusin's Theorem ([6], II.15.15) that there is a continuous function $D : G \to A \otimes \mathcal{H}$ and a set $E \subseteq G$ with $\mu(E) < \eta$ and $D(s) = \delta_s(y)$ for $s \in G \setminus E$.

Next, Lemma 2.2 (taking G itself for the closed subgroup) provides a normalized characteristic function n_U of a relatively compact open neighborhood U of e in G such that $\|g - g *_G n_U\|_1 < \eta$. A compactness argument shows that U may be taken small enough to also guarantee that $\|D(s) - D(r)\| < \eta$ whenever $r \in K$ and $r^{-1}s \in U$. Of course we may take U smaller still to ensure that $U \subseteq U'$.

Similarly, use Lemma 2.2 (with H for the closed subgroup) to find a normalized characteristic function n_V of a relatively compact open neighborhood V of e in H such that $\|n_U - n_U *_H n_V\|_1 < \eta$. V may also be chosen so that $V \subseteq V'$.

Finally, an application of Urysohn's Lemma provides a function $N \in C_c(G/H)$ which is identically 1 on the image of $K\overline{U}$ in G/H.

We make the following definitions:

$$\begin{aligned} F(r, \dot{s}) &= \overline{g(r)}\, N(\dot{s})\, \delta_r(c \otimes \xi) \\ f(s) &= n_U(s)\, b^* \otimes T^* \\ \tau(t) &= n_V(t)\, \delta_t(y)\, \Delta_G(t) \gamma_H^G(t^{-1}). \end{aligned}$$

Then $F \in B_c(G, C_c(G/H, A \otimes \mathcal{H}))$, $f \in B_c(G, A \otimes \mathcal{K})$, and $\tau \in B_c(H, A \otimes \mathcal{H})$. Moreover, we can compute:

$$\begin{aligned} &\|g \otimes_\alpha a - \varphi(\bar{F} \otimes f \otimes \tau)\|_1 \\ &= \int_G \Big\| g(s)\alpha_s(a) - \varphi(\bar{F} \otimes f \otimes \tau)(s) \Big\|\, ds \\ &= \int_G \Big\| g(s)\,\alpha_s(a) - \int_G g(r)\, N(\dot{s}) \Big(\int_H n_U(r^{-1}st^{-1})\, n_V(t)\, dt \Big) \langle \delta_r(x), \delta_s(y)\rangle_A\, dr \Big\|\, ds \\ &\leqslant \int_G |g(s)| \Big\| \alpha_s(a) - \langle \delta_s(x), \delta_s(y)\rangle_A \Big\|\, ds \\ &\quad + \int_G \Big| g(s) - \int_G g(r) N(\dot{s})\, n_U(r^{-1}s)\, dr \Big|\, \|\langle \delta_s(x), \delta_s(y)\rangle_A\|\, ds \\ &\quad + \int_G \int_G |g(r)\, N(\dot{s})\, n_U(r^{-1}s)|\, \|\langle \delta_s(x) - \delta_r(x), \delta_s(y)\rangle_A\|\, dr\, ds \\ &\quad + \int_G \int_G |g(r)\, N(\dot{s})| \Big| n_U(r^{-1}s) - \int_H n_U(r^{-1}st^{-1})\, n_V(t)\, dt \Big|\, \|\langle \delta_r(x), \delta_s(y)\rangle_A\|\, dr\, ds \\ &\leqslant \|a - \langle x, y\rangle_A\| \int_G |g(s)|\, ds \\ &\quad + \|x\|\, \|y\| \int_G |g(s) - g *_G n_U(s)|\, ds \end{aligned}$$

$$
\begin{aligned}
&+ \|y\| \int_{G\setminus E} \int_{G\setminus E} \left|g(r)\, n_U(r^{-1}s)\right| \|D(s) - D(r)\| \,\mathrm{d}r\,\mathrm{d}s \\
&+ 2\|x\|\,\|y\| \int_{G\setminus E} \int_{E} \left|g(r)\, n_U(r^{-1}s)\right| \,\mathrm{d}r\,\mathrm{d}s \\
&+ 2\|x\|\,\|y\| \int_{E} \int_{G\setminus E} \left|g(r)\, n_U(r^{-1}s)\right| \,\mathrm{d}r\,\mathrm{d}s \\
&+ 2\|x\|\,\|y\| \int_{E} \int_{E} \left|g(r)\, n_U(r^{-1}s)\right| \,\mathrm{d}r\,\mathrm{d}s \\
&+ \|x\|\,\|y\| \int_G |g(r)| \,\mathrm{d}r \int_G |n_U(s) - n_U *_H n_V(s)| \,\mathrm{d}s \\
&\leqslant \eta\|g\|_1 + \eta\|x\|\,\|y\| + \eta\|y\|\,\|g\|_1 + 6\eta\|x\|\,\|y\|\,\|g\|_\infty + \eta\|x\|\,\|y\|\,\|g\|_1 \\
&= \eta(\|g\|_1 + \|a\| + 2\,\|a\|\,\|g\|_1 + 6\,\|a\|\,\|g\|_\infty).
\end{aligned}
$$

Thus by a judicious choice of η we may ensure that $\|g \otimes_\alpha a - \varphi(\bar{F} \otimes f \otimes \tau)\|_1 < \varepsilon$.

Also, we have

$$
\begin{aligned}
&\sup_{s\in G} \|\varphi(\bar{F} \otimes f \otimes \tau)(s)\| \\
&\qquad = \sup_{s\in G} \left\| \int_G g(r)\, N(\dot{s}) \left(\int_H n_U(r^{-1}st^{-1})\, n_V(t)\,\mathrm{d}t \right) \langle \delta_r(x), \delta_s(y)\rangle_A \,\mathrm{d}r \right\| \\
&\qquad \leqslant \|x\|\,\|y\|\,\|g\|_\infty \int_G \int_H \left| n_U(r^{-1}t^{-1})\, n_V(t) \right| \,\mathrm{d}t\,\mathrm{d}r \\
&\qquad = \|a\|\,\|g\|_\infty .
\end{aligned}
$$

Finally, one can check that supp $\varphi(\bar{F} \otimes f \otimes \tau)$ is contained in $K\overline{UV} \subseteq K\overline{U'V'}$. As described above, the theorem now follows from Lemma 2.1. ∎

3. THE IMPRIMITIVITY THEOREM

In this section we use Theorem 2.3 to define a process for inducing representations of $A \times_{\alpha,u} H$ to $A \times_{\alpha,u} G$ when H is a closed subgroup of G. We then prove an imprimitivity theorem, which describes which representations of $A \times_{\alpha,u} G$ are induced from $A \times_{\alpha,u} H$. The section concludes with a theorem on induction in stages. We will retain the notation of the preceding section; in particular, E denotes the *imprimitivity algebra* $C_0(G/H, A) \times_{\tilde{\alpha},\tilde{u}} G$ and E_0 denotes its dense subalgebra $B_c(G, C_c(G/H, A); \tilde{\alpha}, \tilde{u})$, identified with a subalgebra of $B_c(G \times G/H, A)$.

PROPOSITION 3.1. *There exists a non-degenerate $*$-homomorphism φ of $A \times_{\alpha,u} G$ into $\mathcal{M}(E)$.*

Proof. For each $f \in B_c(G, A; \alpha, u)$ and $F \in E_0$, put

$$(\varphi(f)F)(s,\dot{t}) = \int_G f(r)\,\alpha_r(F(r^{-1}s, r^{-1}\dot{t}))\,u(r, r^{-1}s)\,dr$$

and

$$(F\varphi(f))(s,\dot{t}) = \int_G F(r,\dot{t})\,\alpha_r(f(r^{-1}s))\,u(r, r^{-1}s)\,dr.$$

Then it is easy to see that $\varphi(f)$ determines a multiplier of E_0 (the cocycles do not disrupt the continuous second variable), and thus an element of $\mathcal{M}(E)$. Indeed, $\varphi(f)$ is just the element of $B_c(G, C_b(G/H, A); \alpha, u) \subseteq \mathcal{M}(E_0)$ given by $\varphi(f)(s,\dot{t}) = f(s)$; this makes it clear that the map $f \mapsto \varphi(f)$ is a $*$-homomorphism.

It only remains to show that φ is non-degenerate. Let $\{e_k\}$ be an approximate identity for A, and let $\{\tilde{e}_k\}$ be the sequence of functions defined by $\tilde{e}_k(\dot{t}) = e_k$, which converges strictly to 1 in $\mathcal{M}(C_0(G/H, A))$. Then an approximate identity for $B_c(G, A; \alpha, u)$ constructed from the $\{e_k\}$ as in the appendix of [13] is carried by φ to an approximate identity constructed from the $\{\tilde{e}_k\}$ for E_0. Thus φ is non-degenerate as claimed. ∎

Suppose we have a non-degenerate representation $\pi \times U$ of $A \times_{\alpha,u} H$ on a Hilbert space $\mathcal{H}$. Then we get the induced representation $(\pi \times U)^X$ of E on $\mathcal{H}^X$, which extends to $\mathcal{M}(E)$. Composing with φ, we thus obtain a non-degenerate representation $(\pi \times U)^G_H = (\pi \times U)^X \circ \varphi$ of $A \times_{\alpha,u} G$ on $\mathcal{H}^G_H = \mathcal{H}^X$; we say that $(\pi \times U)^G_H$ is *induced from* $A \times_{\alpha,u} H$. The map which sends each representation $\pi \times U$ of $A \times_{\alpha,u} H$ to the representation $(\pi \times U)^G_H$ of $A \times_{\alpha,u} G$ induces a map on unitary equivalence classes, which will be denoted by Ind^G_H.

One can deduce from the definition that $(\pi \times U)^G_H$ is the integrated form of the covariant representation (π^G_H, U^G_H) defined for $a \in A$, $s \in G$, and $x \otimes \xi \in \mathcal{H}^G_H$ by

$$\pi^G_H(a)(x \otimes \xi) = (a \cdot x) \otimes \xi \quad \text{and} \quad U^G_H(s)(x \otimes \xi) = {}^s x \otimes \xi,$$

where $(a \cdot x)(t) = ax(t)$ and ${}^s x(t) = \alpha_s(x(s^{-1}t))u(s, s^{-1}t)$. Thus $(\pi \times U)^G_H$ is given for f in $B_c(G, A; \alpha, u)$ by

$$(\pi \times U)^G_H(f)(x \otimes \xi) = (f * x) \otimes \xi,$$

where $*$ is the usual twisted convolution in $B_c(G, A; \alpha, u)$. These formulas will be useful in Section 4, and in the proof of Theorem 3.3. First, we have the following imprimitivity theorem for Ind^G_H:

THEOREM 3.2. *Let (A, G, α, u) be a twisted dynamical system, and let H be a closed subgroup of G. A representation $(\pi \times U)$ of $A \times_{\alpha,u} G$ on a Hilbert space $\mathcal{H}$ is induced from $A \times_{\alpha,u} H$ if and only if there is a representation ρ of $C_0(G/H, A) \times_{\tilde{\alpha},\tilde{u}} G$ on $\mathcal{H}$ such that*

$$\pi \times U(f)\,\rho(F) = \rho(f \cdot F)$$

for all $f \in A \times_{\alpha,u} G$ and $F \in C_0(G/H, A) \times_{\tilde{\alpha},\tilde{u}} G$. (By $f \cdot F$ is meant the product $\varphi(f)F$ in $\mathcal{M}(E)$, which will be an element of E.)

Proof. The theorem is immediate from Theorem 2.3, Proposition 3.1, and [14], Theorem 6.29. ∎

THEOREM 3.3. *Let (A, G, α, u) be a twisted dynamical system and suppose H and K are closed subgroups of G with $K \subset H$. Then for each covariant representation (π, U) of (A, K, α, u), $((\pi \times U)_K^H)_H^G$ is unitarily equivalent to $(\pi \times U)_K^G$.*

Proof. Fix a covariant representation (π, U) of (A, K, α, u). In order to show that $((\pi \times U)_K^H)_H^G$ is equivalent to $(\pi \times U)_K^G$, it suffices by [14], Theorem 5.9, to show that $B_c(G, A) \otimes_{A \times_{\alpha,u} H} B_c(H, A)$ is isomorphic to $B_c(G, A)$ as a pre-Hermitian $A \times_{\alpha,u} K$-rigged $A \times_{\alpha,u} G$-module. Let ψ be the map determined by the rule $\psi(x \otimes f) = x \cdot f$, treating f in $B_c(H, A)$ as an element of $A \times_{\alpha,u} H$. In other words, for x in $B_c(G, A)$, f in $B_c(H, A)$ and s in G, define

$$\psi(x \otimes f)(s) = \int_H x(st)\,\alpha_{st}(f(t^{-1}))\,u(st, t^{-1})\,\gamma_H^G(t)\,dt.$$

Then it is clear from the definition that ψ intertwines the left action of $A \times_{\alpha,u} G$, and it is straightforward to check that it also intertwines the right action of $A \times_{\alpha,u} K$. To see that ψ preserves the $A \times_{\alpha,u} K$-valued inner product, fix x and y in $B_c(G, A)$, f and g in $B_c(H, A)$, and k in K, and compute:

$$\begin{aligned}
&\langle x \otimes f, y \otimes g\rangle_{A \times_{\alpha,u} K}(k) \\
&\quad = \langle f, \langle x, y\rangle_{A \times_{\alpha,u} H} \cdot g\rangle_{A \times_{\alpha,u} K} \\
&\quad = \int_H u(h, h^{-1})^*\,\alpha_h\left(f(h^{-1})^*\,(\langle x, y\rangle_{A \times_{\alpha,u} H} \cdot g)(h^{-1}k)\right)\,u(h, h^{-1}k) \\
&\qquad\quad \cdot \gamma_K^H(k)\,\Delta_H(h^{-1})\,dh \\
&\quad = \int_H \int_H u(h, h^{-1})^*\,\alpha_h\left(f(h^{-1})^*\,\langle x, y\rangle_{A \times_{\alpha,u} H}(t)\,\alpha_t(g(t^{-1}h^{-1}k))\,u(t, t^{-1}h^{-1}k)\right) \\
&\qquad\quad \cdot u(h, h^{-1}k)\,\gamma_K^H(k)\,\gamma_H^G(t)\,\Delta_H(h^{-1})\,dt\,dh \\
&\quad = \int_H \int_H \int_G u(h, h^{-1})^*\,\alpha_h\big(f(h^{-1})^*\,u(s^{-1}, s)^*\,\alpha_{s^{-1}}(x(s)^* y(st))\,u(s^{-1}, st) \\
&\qquad\quad \cdot \alpha_t(g(t^{-1}h^{-1}k))\,u(t, t^{-1}h^{-1}k)\big)\,u(h, h^{-1}k)\,\gamma_K^H(k)\,\gamma_H^G(t)\,\Delta_H(h^{-1})\,ds\,dt\,dh
\end{aligned}$$

$$= \int_H \int_H \int_G u(h,h^{-1})^* \alpha_h(f(h^{-1})^*)\, \alpha_h(u(s^{-1},s)^*)\, u(h,s^{-1})\, \alpha_{hs^{-1}}(x(s)^* y(st))$$
$$\cdot u(h,s^{-1})^* \alpha_h(u(s^{-1},st))\, u(h,t)\, \alpha_{ht}(g(t^{-1}h^{-1}k))\, u(h,t)^*$$
$$\cdot \alpha_h(u(t,t^{-1}h^{-1}k))\, u(h,h^{-1}k)\, \gamma_K^H(k)\, \gamma_H^G(t)\, \Delta_H(h^{-1})\, \mathrm{d}s\, \mathrm{d}t\, \mathrm{d}h$$

which becomes, by the substitutions $s \mapsto sh$, $t \mapsto h^{-1}kt$

$$= \int_H \int_H \int_G u(h,h^{-1})^* \alpha_h(f(h^{-1})^*)\, \alpha_h(u(h^{-1}s^{-1},sh)^*)\, u(h,h^{-1}s^{-1})$$
$$\cdot \alpha_{s^{-1}}\big(x(sh)^* y(skt)\big)\, u(h,h^{-1}s^{-1})^* \alpha_h(u(h^{-1}s^{-1},skt))\, u(h,h^{-1}kt)$$
$$\cdot \alpha_{kt}(g(t^{-1}))\, u(h,h^{-1}kt)^* \alpha_h(u(h^{-1}kt,t^{-1}))\, u(h,h^{-1}k)$$
$$\cdot \gamma_K^H(k)\, \gamma_H^G(h^{-1}kt)\, \Delta_H(h^{-1})\, \Delta_G(h)\, \mathrm{d}s\, \mathrm{d}t\, \mathrm{d}h$$
$$= \int_H \int_H \int_G u(s^{-1},s)^* \alpha_{s^{-1}}(u(sh,h^{-1})^*)\, u(s^{-1},sh)\, \alpha_h(f(h^{-1})^*)\, u(s^{-1},sh)^*$$
$$\cdot \alpha_{s^{-1}}\big(x(sh)^* y(skt)\big)\, u(s^{-1},skt)\, \alpha_{kt}(g(t^{-1}))\, u(s^{-1},skt)^* \alpha_{s^{-1}}(u(skt,t^{-1}))$$
$$\cdot u(s^{-1},sk)\, \gamma_H^G(ht)\, \gamma_K^G(k)\, \mathrm{d}s\, \mathrm{d}t\, \mathrm{d}h$$
$$= \int_G u(s^{-1},s)^* \alpha_{s^{-1}}\Big(\big(\int_H x(sh)\, \alpha_{sh}(f(h^{-1}))\, u(sh,h^{-1})\, \gamma_H^G(h)\, \mathrm{d}h\big)^*$$
$$\cdot \int_H y(skt)\, \alpha_{skt}(g(t^{-1}))\, u(skt,t^{-1})\, \gamma_H^G(t)\, \mathrm{d}t\Big)\, u(s^{-1},sk)\, \gamma_K^G(k)\, \mathrm{d}s$$

$$= \int_G u(s^{-1},s)^* \alpha_{s^{-1}}\big(\psi(x\otimes f)(s)^* \psi(y\otimes g)(sk)\big)\, u(s^{-1},sk)\, \gamma_K^G(k)\, \mathrm{d}s$$
$$= \langle \psi(x\otimes f), \psi(y\otimes g)\rangle_{A\times_{\alpha,u}K}(k).$$

It only remains to check that the range of ψ is dense in $B_c(G,A)$. This follows from the fact that an approximate identity for $B_c(H,A;\alpha,u)$ is also an approximate identity for the right action of $B_c(H,A;\alpha,u)$ on $B_c(G,A)$. ∎

4. BUSBY AND SMITH'S AND MACKEY'S INDUCED REPRESENTATIONS

In [1], Busby and Smith discuss two related inducing processes. The more general process consists of inducing from a C^*-algebra A to a twisted system (A,G,α,u); a special case yields a process for inducing representations of a closed normal subgroup up to the larger group. In this section we show that Ind_H^G can be used to produce representations equivalent to Busby and Smith's in these two situations.

We also show that Ind_H^G produces representations equivalent to Mackey's induced multiplier representations of groups [11], Section 4.

Let us begin by recalling the method for inducing from a C^*-algebra to a twisted system. Suppose (A, G, α, u) is a twisted dynamical system and π is a representation of A on a Hilbert space $\mathcal{H}$. Define the maps $\tilde{\pi}$ and U^π of A and G, respectively, into $\mathcal{B}(L^2(G, \mathcal{H}))$ as follows:

$$(\tilde{\pi}(a) f)(s) = \pi(\alpha_s(a))\, f(s)$$
$$(U^\pi(t) f)(s) = \pi(u(s,t))\, f(st)\, \Delta_G(t)^{\frac{1}{2}}.$$

Then [1], Theorem 4.1 states that $(\tilde{\pi}, U^\pi)$ is a covariant representation of (A, G, α, u) on $L^2(G, \mathcal{H})$.

We also can induce representations from A to (A, G, α, u), using Ind_H^G with $H = \{e\}$ and the fact that in this case $A \times_{\alpha,u} H = A$. More explicitly, let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a representation of A, so that defining $U(e) = 1_{\mathcal{B}(\mathcal{H})}$, (π, U) is trivially a covariant representation of the restricted system (A, H, α, u). Then Ind_H^G provides the induced covariant representation (π^G, U^G) of (A, G, α, u) on the completion $\mathcal{H}^G$ of $B_c(G, A) \otimes_A \mathcal{H}$. The operation on elementary tensors is calculated as follows:

$$\pi^G(a)(f \otimes \xi) = a \cdot f \otimes \xi$$
$$U^G(s)(f \otimes \xi) = {}^s f \otimes \xi,$$

where ${}^s f(t) = \alpha_s(f(s^{-1}t))u(s, s^{-1}t)$.

PROPOSITION 4.1. *Let (A, G, α, u) be a twisted system. Then for each representation π of A on a Hilbert space $\mathcal{H}$, Busby and Smith's induced representation $(\tilde{\pi}, U^\pi)$ of (A, G, α, u) is unitarily equivalent to (π^G, U^G).*

Proof. Define $\varphi_0 : B_c(G, A) \times \mathcal{H} \to L^2(G, \mathcal{H})$ by

$$\varphi_0(f, \xi)(s) = \pi(\alpha_s(f(s^{-1}))u(s, s^{-1}))\xi\, \Delta_G(s)^{-\frac{1}{2}}.$$

Clearly φ_0 is bilinear, and maps into $L^2(G, \mathcal{H})$ since f is bounded with compact support. Thus there exists a map $\varphi_1 : B_c(G, A) \otimes \mathcal{H} \to L^2(G, \mathcal{H})$ such that $\varphi_1(f \odot \xi) = \varphi_0(f, \xi)$.

We will show that φ_1 extends to a unitary map between the completions of the spaces. First compute:

$$\begin{aligned}
\langle f\otimes\xi, g\otimes\eta\rangle &= \langle\pi(\langle g,f\rangle_A)\xi,\eta\rangle_{\mathcal{H}}\\
&= \Big\langle\pi\Big(\int_G u(s^{-1},s)^*\alpha_{s^{-1}}(g(s)^*f(s))u(s^{-1},s)\,ds\Big)\xi,\eta\Big\rangle_{\mathcal{H}}\\
&= \int_G \Big\langle\pi(\alpha_s(f(s^{-1}))u(s,s^{-1}))\xi, \pi(\alpha_s(g(s^{-1}))\,u(s,s^{-1}))\eta\Big\rangle_{\mathcal{H}}\Delta_G(s^{-1})\,ds\\
&= \int_G \langle\varphi_1(f\otimes\xi)(s),\varphi_1(g\otimes\eta)(s)\rangle_{\mathcal{H}}\,ds\\
&= \langle\varphi_1(f\otimes\xi),\varphi_1(g\otimes\eta)\rangle_{L^2(G,\mathcal{H})}.
\end{aligned}$$

It follows that φ_1 is isometric, and so extends to a unitary operator φ provided it has dense range in $L^2(G,\mathcal{H})$. To see this, note that $\psi(f)(s)=\alpha_s(f(s^{-1}))u(s,s^{-1})$ defines an automorphism ψ of $B_c(G,A)$, so that functions of the form $s\mapsto \varphi_1(\psi^{-1}(g)\otimes\xi)(s)=\pi(g(s))\xi$ (for $g\in B_c(G,A)$) are in the range of φ_1. But functions of this form have dense span in $L^2(G,\mathcal{H})$, because π is non-degenerate.

Thus far we have established the unitary equivalence of the Hilbert spaces in question. We need only to check that φ intertwines the induced representations. Compute, for a in A, f in $B_c(G,A)$, ξ in $\mathcal{H}$, and s,t in G:

$$\begin{aligned}
\varphi(\pi^G(a)f\otimes\xi)(s) &= \varphi(a\cdot f\otimes\xi)(s)\\
&= \pi(\alpha_s(af(s^{-1}))u(s,s^{-1}))\,\xi\,\Delta_G(s)^{-\frac12}\\
&= \pi(\alpha_s(a))\pi(\alpha_s(f(s^{-1}))u(s,s^{-1}))\,\xi\,\Delta_G(s)^{-\frac12}\\
&= \tilde\pi(a)\varphi(f\otimes\xi)(s).
\end{aligned}$$

Also:

$$\begin{aligned}
\varphi(U^G(t)(f\otimes\xi))(s) &= \varphi({}^tf\otimes\xi)(s)\\
&= \pi(\alpha_s({}^tf(s^{-1}))u(s,s^{-1}))\,\xi\,\Delta_G(s)^{-\frac12}\\
&= \pi(\alpha_s(\alpha_t(f(t^{-1}s^{-1}))u(t,t^{-1}s^{-1}))u(s,s^{-1}))\,\xi\,\Delta_G(s)^{-\frac12}\\
&= \pi(u(s,t)\alpha_{st}(f((st)^{-1}))u(s,t)^*\alpha_s(u(t^{-1},t^{-1}s^{-1}))u(s,s^{-1}))\xi\,\Delta_G(s)^{-\frac12}\\
&= \pi(u(s,t)\alpha_{st}(f((st)^{-1}))u(st,(st)^{-1}))\,\xi\,\Delta_G(st)^{-\frac12}\Delta_G(t)^{-\frac12}\\
&= \pi(u(s,t))\varphi(f\otimes\xi)(st)\,\Delta_G(t)^{-\frac12}\\
&= U^\pi(t)(\varphi(f\otimes\xi))(s).
\end{aligned}$$

This completes the proof of the proposition. ∎

In order to induce representations from a closed normal subgroup N of a group G, Busby and Smith essentially define a twisted action (β, v) of G/N on $C^*(N)$ such that $C^*(N) \times_{\beta,v} G/N \cong C^*(G)$, and then use the process described above to induce from $C^*(N)$. They show ([1], Theorem 4.2) that the representations induced in this way are equivalent to Mackey's. Now our process can be used to induce representations from $C^*(N)$ to $C^*(G)$ by applying ind_N^G to the trivially twisted system $(\mathbf{C}, N, \mathrm{id}, \mathbf{1})$. In this special case our inducing process is essentially Rieffel's, which was shown in [14], Theorem 5.12, also to produce representations equivalent to Mackey's. Thus we see in a roundabout way that ind_N^G reproduces the induced group representations of Busby and Smith, and Mackey.

More generally, Mackey described in [11], Section 4, how to induce multiplier representations from subgroups. A *multiplier* of G is a Borel function $\sigma : G \times G \to \mathbf{T}$ such that $\sigma(s,e) = \sigma(e,t) = 1$, and

$$\sigma(s,t)\sigma(r,st) = \sigma(r,s)\sigma(rs,t) \quad \forall\, r,s,t \in G.$$

A *σ-representation* of G on a Hilbert space $\mathcal{H}$ is then a Borel map $U : G \to \mathcal{UB}(\mathcal{H})$ such that $U_e = \mathrm{id}$ and

$$U_s U_t = \sigma(s,t) U_{st}.$$

(This definition is actually conjugate to Mackey's.) Thus the multiplier is just a twist for the trivial action of G on $\mathbf{C}$, and a σ-representation is just a covariant representation of the twisted system $(\mathbf{C}, G, \mathrm{id}, \sigma)$.

Now fix a multiplier σ of G, a closed subgroup H of G, and a σ-representation U of H on $\mathcal{H}$. In what follows we outline Mackey's construction for inducing σ-representations, based on the formulation in [6], XI.10.9, for ordinary group representations. Choose a quasi-invariant measure μ on G/H; by [6], III.14.5 and 7, we can take $\mu = \rho^\#$ (see [6], III.13.8), where ρ is an everywhere positive continuous rho-function on G. Thus by [6], III.13.2,

$$\rho(sh) = (\Delta_H(h)/\Delta_G(h))\,\rho(s) \quad \forall s \in G, h \in H,$$

and by [6], III.13.10, we may normalise the Haar measures on H and G so that

$$\int_G f(s)\rho(s)\,\mathrm{d}s = \int_{G/H}\int_H f(sh)\,\mathrm{d}h\,\mathrm{d}\mu(\dot{s}) \quad \forall f \in B_c(G).$$

Then the space $\mathcal{K}$ of Borel functions $f : G \to \mathcal{H}$ satisfying

$$f(sh) = \sigma(s,h)U_h^{-1}(f(s)) \quad \forall s \in G, h \in H$$

and such that $\int\limits_{G/H} \langle f(s), f(s)\rangle_{\mathcal{H}}\, d\mu(\dot{s}) < \infty$ is a Hilbert space (identifying functions which are equal almost everywhere), with inner product given by

$$\langle f, g\rangle_{\mathcal{K}} = \int\limits_{G/H} \langle f(s), g(s)\rangle_{\mathcal{H}}\, d\mu(\dot{s}).$$

The representation V of G induced from U is given on $\mathcal{K}$ by

$$V_s(f)(t) = \sigma(s, s^{-1}t) f(s^{-1}t) \left(\frac{\rho(s^{-1}t)}{\rho(t)}\right)^{\frac{1}{2}}.$$

The next proposition shows that Mackey's induced σ-representations are equivalent to the representations we get by applying ind_N^G to the twisted system $(\mathbb{C}, H, \mathrm{id}, \sigma)$. We are grateful to the referee for suggesting this possibility. Here we consider only multipliers which are *normalised* in the sense that $\sigma(s, s^{-1}) = 1$ for all $s \in G$; since an arbitrary multiplier is always similar to a normalised one, this causes no loss of generality.

PROPOSITION 4.2. *Let σ be a normalised multiplier of a group G, and let H be a closed subgroup of G. Then for each σ-representation U of H on a Hilbert space $\mathcal{H}$, Mackey's induced representation V of G is unitarily equivalent to U_H^G.*

Proof. Define $\psi_0 : B_c(G) \times \mathcal{H} \to \mathcal{K}$ by

$$\psi_0(f, \xi)(s) = \int\limits_H f(sh)\sigma(s^{-1}, sh) U_h(\xi) \rho(sh)^{-\frac{1}{2}}\, dh.$$

Then ψ_0 is clearly bilinear, and each $\psi_0(f, \xi)$ is a Borel mapping of G into $\mathcal{H}$; thus ψ_0 determines a map ψ_1 of $B_c(G) \odot \mathcal{H}$ into the space of all Borel functions from G into $\mathcal{H}$. To see that ψ_1 maps into $\mathcal{K}$, we need to check that each $\psi_1(f \otimes \xi)$ transforms properly, and we must verify the norm condition. For the former, compute:

$$\begin{aligned}
\psi_1(f \otimes \xi)(sk) &= \int\limits_H f(skh)\sigma(k^{-1}s^{-1}, skh) U_h(\xi) \rho(skh)^{-\frac{1}{2}}\, dh \\
&\overset{h \mapsto k^{-1}h}{=} \int\limits_H f(sh)\sigma(k^{-1}s^{-1}, sh) U_{k^{-1}h}(\xi) \rho(sh)^{-\frac{1}{2}}\, dh \\
&= \int\limits_H f(sh)\sigma(k^{-1}s^{-1}, sh)\overline{\sigma(k^{-1}, h)} U_{k^{-1}} U_h(\xi) \rho(sh)^{-\frac{1}{2}}\, dh \\
&= U_k^{-1}\left(\int\limits_H f(sh)\sigma(s, k)\sigma(s^{-1}, sh) U_h(\xi) \rho(sh)^{-\frac{1}{2}}\, dh\right) \\
&= \sigma(s, k) U_k^{-1} \left(\psi_1(f \otimes \xi)(s)\right).
\end{aligned}$$

For the latter, we have the following calculation, which also shows that ψ_1 is isometric, and so extends to an isometry ψ of $\mathcal{H}_H^G$ into $\mathcal{K}$.

$$\begin{aligned}
\langle f\otimes\xi, g\otimes\eta\rangle_{\mathcal{H}_H^G} &= \langle U\left(\langle g,f\rangle_{C^*(H)}\right)(\xi),\eta\rangle_{\mathcal{H}} \\
&= \int_H \langle\langle g,f\rangle_{C^*(H)}(k)U_k(\xi),\eta\rangle_{\mathcal{H}}\,\mathrm{d}k \\
&= \int_G\int_H \langle\overline{g(s)}f(sk)\sigma(s^{-1},sk)U_k(\xi),\eta\rangle_{\mathcal{H}}\gamma_H^G(k)\,\mathrm{d}k\,\mathrm{d}s \\
&= \int_{G/H}\int_H\int_H \langle f(shk)\sigma(h^{-1}s^{-1},shk)U_k(\xi), g(sh)\eta\rangle_{\mathcal{H}}\rho(sh)^{-1}\gamma_H^G(k)\,\mathrm{d}k\,\mathrm{d}h\,\mathrm{d}\mu(\dot{s}) \\
&\overset{k\mapsto h^{-1}k}{=} \int_{G/H}\int_H\int_H \langle f(sk)\sigma(h^{-1}s^{-1},sk)U_{h^{-1}k}(\xi), g(sh)\eta\rangle_{\mathcal{H}}\rho(sh)^{-1}\gamma_H^G(h^{-1}k) \\
&\qquad \mathrm{d}k\,\mathrm{d}h\,\mathrm{d}\mu(\dot{s}) \\
&= \int_{G/H}\int_H\int_H \langle f(sk)\sigma(h^{-1}s^{-1},sk)\overline{\sigma(h^{-1},k)}U_k(\xi), g(sh)U_h(\eta)\rangle_{\mathcal{H}}\rho(sh)^{-\frac12}\rho(sk)^{-\frac12} \\
&\qquad \mathrm{d}k\,\mathrm{d}h\,\mathrm{d}\mu(\dot{s}) \\
&= \int_{G/H}\Big\langle\int_H f(sk)\sigma(s^{-1},sk)U_k(\xi)\rho(sk)^{-\frac12}\,\mathrm{d}k \\
&\qquad \int_H g(sh)\sigma(s^{-1},sh)U_h(\eta)\rho(sh)^{-\frac12}\,\mathrm{d}h\Big\rangle_{\mathcal{H}}\,\mathrm{d}\mu(\dot{s}) \\
&= \int_{G/H}\langle\psi_1(f\otimes\xi)(s),\psi_1(g\otimes\eta)(s)\rangle_{\mathcal{H}}\,\mathrm{d}\mu(\dot{s}) \\
&= \langle\psi_1(f\otimes\xi),\psi_1(g\otimes\eta)\rangle_{\mathcal{K}}.
\end{aligned}$$

To see that ψ maps onto $\mathcal{K}$, and therefore implements an isometric isomorphism between $\mathcal{H}_H^G$ and $\mathcal{K}$, we show that $\psi(\mathcal{H}_H^G)^\perp = \{0\}$. Suppose we have $g \in \mathcal{K}$ such that $\langle\psi(f\otimes\xi), g\rangle_{\mathcal{K}} = 0$ for all $f \in B_c(G)$ and $\xi \in \mathcal{H}$. Then:

$$\begin{aligned}
0 &= \int_{G/H}\langle\psi(f\otimes\xi)(s), g(s)\rangle_{\mathcal{H}}\,\mathrm{d}\mu(\dot{s}) \\
&= \int_{G/H}\int_H \langle f(sh)\sigma(s^{-1},sh)U_h(\xi)\rho(sh)^{-\frac12}, g(s)\rangle_{\mathcal{H}}\,\mathrm{d}h\,d\mu(\dot{s}) \\
&= \int_{G/H}\int_H f(sh)\langle\xi,\overline{\sigma(s^{-1},sh)}U_h^{-1}(g(s))\rangle_{\mathcal{H}}\rho(sh)^{-\frac12}\,\mathrm{d}h\,\mathrm{d}\mu(\dot{s})
\end{aligned}$$

$$= \int_{G/H} \int_{H} f(sh)\langle \xi, \overline{\sigma(s^{-1}, sh)\sigma(s,h)} g(sh)\rangle_{\mathcal{H}} \rho(sh)^{-\frac{1}{2}}\, \mathrm{d}h\, \mathrm{d}\mu(\dot{s})$$

$$= \int_{G/H} \int_{H} f(sh)\langle \xi, g(sh)\rangle_{\mathcal{H}} \rho(sh)^{-\frac{1}{2}}\, \mathrm{d}h\, \mathrm{d}\mu(\dot{s})$$

$$= \int_{G} \rho(s)^{\frac{1}{2}} f(s)\langle \xi, g(s)\rangle_{\mathcal{H}}\, \mathrm{d}s.$$

Since ρ is everywhere positive and $f \in B_c(G)$ was arbitrary, this implies that for each $\xi \in \mathcal{H}$, the function $s \mapsto \langle \xi, g(s)\rangle_{\mathcal{H}} = 0$ almost everywhere in G; hence $g(s) = 0$ for almost every s, which is to say $g = 0$ in $\mathcal{K}$.

It only remains to show that ψ intertwines U_H^G and V. For this, simply compute:

$$\psi(U_N^G(f)(f \otimes \xi))(s) = \psi({}^r f \otimes \xi)(s)$$

$$= \int_H {}^r f(sh)\sigma(s^{-1}, sh) U_h(\xi)\rho(sh)^{-\frac{1}{2}}\, \mathrm{d}h$$

$$= \int_H f(r^{-1}sh)\sigma(r, r^{-1}sh)\sigma(s^{-1}, sh) U_h(\xi)\rho(sh)^{-\frac{1}{2}}\, \mathrm{d}h$$

$$= \int_H f(r^{-1}sh)\sigma(r, r^{-1}s)\sigma(s^{-1}r, r^{-1}sh) U_h(\xi)\rho(r^{-1}sh)^{-\frac{1}{2}} \left(\frac{\rho(r^{-1}s)}{\rho(s)}\right)^{\frac{1}{2}} \mathrm{d}h$$

$$= \sigma(r, r^{-1}s)\psi(f \otimes \xi)(r^{-1}s) \left(\frac{\rho(r^{-1}s)}{\rho(s)}\right)^{\frac{1}{2}}$$

$$= V_r(\psi(f \otimes \xi))(s). \quad \blacksquare$$

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