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THE TOPOLOGY OF IDEALS IN SOME TRIANGULAR AF ALGEBRAS

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ABSTRACT. A set of meet-irreducible ideals is described for a class of maximal triangular almost finite algebras. This set forms a topological space under the hull-kernel closure, and there is a one-to-one correspondence between closed sets in this space and ideals in the AF algebra. Some connections with nest representations and nest-primitive ideals are also described.

KEYWORDS: *Almost finite algebra, lattice of ideals, meet-irreducible, nests of subspaces.*

AMS SUBJECT CLASSIFICATION: 46K50, 47D25, 46M40.

INTRODUCTION

There are many instances in the study of operator algebras where one can describe the ideal structure of an algebra by identifying the set of ideals with a collection of closed sets in some topological space. The canonical example is in C^*-theory, where one may construct the primitive spectrum $\mathrm{Prim}(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ as the set of primitive ideals in $\mathcal{A}$, and introduce the Jacobson (or hull-kernel) topology to create a topological space whose closed sets are in one-to-one correspondence with the closed ideals in $\mathcal{A}$. More precisely, in this case every ideal is described as the intersection of all those primitive ideals which contain it. When the algebra $\mathcal{A}$ arises as the C^*-crossed product of a group acting on a space X, there is a close connection between the topology of $\mathrm{Prim}(\mathcal{A})$ and the structure of the orbits on X, and so the ideal structure can be understood in terms of these orbits. (See for instance [4] or [17].) In [7] and [8], this notion is extended to nonselfadjoint crossed products, where it is shown that so-called nest-primitive ideals corresponding to arcs (subintervals of orbits) in X could be used to build a topological space which again describes the total ideal structure of the nonselfadjoint algebra. Indeed, in this case it was the structures on X and their topology which provided the motivation for studying this class of ideals.

However, it seems worthwhile to ask directly the question: given an algebra of operators, what set of ideals can be used to build a topological space, and when is this space rich enough to describe all the ideals in the algebra? Having asked the question, we will show via Propositions 1.1 and 1.3 that the ideals of interest are precisely the meet-irreducible ideals; that is, those ideals which cannot be written as the intersection of two strictly larger ideals. For C^*-algebras, these are nothing more than the primitive ideals (see Proposition 2.1), so the more interesting examples occur for nonselfadjoint operator algebras.

One area where much work has been done to understand the ideal structure for nonselfadjoint algebras is in the study of direct limits of upper triangular matrix algebras, and more generally triangular almost finite algebras, where questions about ideal structure have been an important consideration in understanding the classification of the algebras. For instance, in [14] Power has identified both the lattice of all ideals as well as the meet-irreducible ideals for an infinite tensor product of upper triangular matrix algebras. As noted in [14], this lattice of ideals is distributive, thus by Propositions 1.1 and 1.3 of the present work, the set of meet-irreducible ideals is in fact a topological space. Moreover, the closed sets in this space are in one-to-one correspondence with closed ideals in the tensor product, for every ideal is the intersection of the meet-irreducible ideals which contain it, as can be readily verify from the constructions in [14]. Other authors (as in [3], [6] and [12]) have looked at the lattice structure of ideals in other AF algebras, including the dual notion of join-irreducibility (for instance, in [6]); one might expect from the C^*-case that such ideals are rather rare. Meet-irreducible ideals also appeared in [7] and [8], in the context of nest-primitive ideals in nonselfadjoint crossed products, but in a rather curious way: the ideals constructed were all nest-primitive, and happened to form a topological space; by Propositions 1.1 and 1.3, these are necessarily meet-irreducible.

The purpose of this paper is to describe explicitly how a topology arises on a set of ideals, and then apply this to a class of maximally triangular AF algebras. The paper is structured as follows. Section 1 identifies the conditions necessary on a set of ideals to become a topological space. Section 2 considers a number of motivating examples; for C^*-algebras, we see the meet-irreducible ideals are just the primitive ideals, while for two types of nonselfadjoint algebras (the compact nest algebra, and the disk algebra), the meet-irreducibles are a generalization of the primitive ideals, namely the nest-primitive ideals. Section 3 contains the core of the paper, where a large set of meet-irreducible ideals is identified in a strongly maximal triangular (regular canonical) AF algebra, which produces a topological space whose closed sets are in one-to-one correspondence with ideals in the AF algebra.

Throughout the paper, we will discuss the connection of these meet-irreducible ideals with the nest-primitive ideals, which were introduced in [7] to describe the topology of the nonselfadjoint crossed products. However, except for some motivating examples, it is not clear what general relationship exists between these two types of ideals.

Our notation is as follows. An almost finite (AF) C^*-algebra $\mathcal{B}$ is the norm-closed union of an increasing sequence of finite dimensional C^*-algebras $\mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \cdots$; for such a sequence it is always possible to choose a system of matrix units $\{e^k_{ij}\}_{ij}$ for each $\mathcal{B}_k$, where each e^k_{ij} is a sum of elements of $\{e^{k+1}_{ij}\}_{ij}$. The abelian C^*-algebra $\mathcal{C}$ generated by the diagonal elements e^k_{ii} is a masa in $\mathcal{B}$ and is called the regular canonical masa associated with the system of matrix units.

A triangular (regular canonical) AF algebra is a norm-closed subalgebra $\mathcal{A} \subseteq \mathcal{B}$ with $\mathcal{A} \cap \mathcal{A}^*$ equal to the canonical masa $\mathcal{C}$.

We say $\mathcal{A}$ is strongly maximal triangular if each finite algebra $\mathcal{A}_k = \mathcal{A} \cap \mathcal{B}_k$ is maximal triangular in $\mathcal{B}_k$; equivalently, we may assume the algebras $\mathcal{B}_k$ are chosen so that $\mathcal{A}_k \subseteq \mathcal{B}_k$ is a direct sum of upper triangular matrix algebras. (See [16] for a well-developed account of these algebras.)

We will make use of the fact that the set of closed, two-sided ideals in a Banach algebra form a multiplicative lattice under the inclusion ordering; the greatest lower bound (meet) of two ideals is their intersection, the least upper bound (join) is the closed linear span of the union, and the product is the closed linear span of pairwise element products. A maximal ideal is thus a proper ideal which is maximal in the inclusion order, while a primary ideal is a proper ideal which is contained in a unique maximal ideal. Throughout this paper, an *ideal* means a norm-closed, two-sided ideal.

A bounded representation π of a Banach algebra $\mathcal{A}$ on Hilbert space $\mathcal{H}$ is called a *nest representation* if the closed, $\pi(A)$-invariant subspaces in $\mathcal{H}$ are linearly ordered by inclusion; that is, they form a nest (see [7]). This is a generalization of the notion of an irreducible representation, where the invariant subspaces form the trivial nest $\{0, \mathcal{H}\}$. The kernel of an irreducible representation is a primitive ideal; the kernel of a nest representation will be called a *nest-primitive* (or n-primitive) ideal.

1. TOPOLOGICAL SPACES OF IDEALS

Given a set Ω of closed, two-sided ideals in a Banach algebras A, we can attempt to introduce a topology on Ω by defining formally the closure of a subset F in Ω as

$$\overline{F} = \text{hull}(\ker(F))$$

where $I = \ker(F)$ is the ideal obtained as the intersection of all the ideals in set F and hull(I) is the subset of ideals in Ω which contain I. To ensure that this operation gives a topology, it suffices to verify that the four Kuratowski axioms for a closure operation are satisfied. Namely, the closure of the empty set ϕ is empty,

(K1) $\overline{\phi} = \phi$;

and for any subsets F, G in Ω, we have

(K2) $\overline{F} \supseteq F$;

(K3) $\overline{\overline{F}} = \overline{F}$;

(K4) $\overline{F \cup G} = \overline{F} \cup \overline{G}$.

It is not hard to see that $\overline{\phi} = \phi$ if and only if Ω consists of proper ideals. The equality $\overline{\overline{F}} = \overline{F}$ and the containments $\overline{F} \supseteq F$ and $\overline{F \cup G} \supseteq \overline{F} \cup \overline{G}$ follow almost immediately from definition of hull and ker, so to obtain a topology, one only requires the reverse containment for (K4). We state a convenient equivalency in the following proposition.

PROPOSITION 1.1. *Let Ω be a set of proper ideals in algebra $\mathcal{A}$. Then the hull-kernel operation produces a topological closure operation if and only if for any ideal I in Ω and subsets F, G we have*

$$I \supseteq \ker(F) \cap \ker(G) \quad \textit{implies} \quad I \supseteq \ker(F) \textit{ or } I \supseteq \ker(G).$$

Proof. By the comments above, we obtain a topological closure if and only if we obtain the containment in (K4) of $\overline{F \cup G} \subseteq \overline{F} \cup \overline{G}$; that is

$$I \in \overline{F \cup G} \Rightarrow I \in \overline{F} \cup \overline{G}.$$

But by definition, $I \in \overline{F \cup G}$ is equivalent to $I \supseteq \ker(F) \cap \ker(G)$ while $I \in \overline{F} \cup \overline{G}$ is equivalent to $I \supseteq \ker(F)$ or $I \supseteq \ker(G)$. Thus (K4) holds if and only if

$$I \supseteq \ker(F) \cap \ker(G) \Rightarrow I \supseteq \ker(F) \text{ or } I \supseteq \ker(G). \quad \blacksquare$$

As our desire is to find a set Ω of ideals which is rich enough that every ideal in the algebra can be written as $\ker(F)$ for some subset F of Ω, it is not too restrictive to generalize the condition in the above proposition as follows.

DEFINITION 1.2. An ideal I in algebra A is called (K4) if, for any ideals J, K in the algebra, one has that

$$I \supseteq J \cap K \Rightarrow I \supseteq J \text{ or } I \supseteq K.$$

Note that the condition (K4) is similar to the condition that an ideal be prime (i.e. $I \supseteq J \cdot K \Rightarrow I \supseteq J$ or $I \supseteq K$) as well as the condition that it be meet-irreducible (i.e. $I = J \cap K \Rightarrow I = J$ or $I = K$). We use the terminology (K4) because of the connection with Kuratowski's fourth axiom; however, in light of the following proposition, the term "meet-prime" might be appropriate as well.

PROPOSITION 1.3. *For any closed, two-sided ideal I in Banach algebra A,*
(i) *if I is prime, then I is* (K4)*;*
(ii) *if I is* (K4)*, then I is meet-irreducible.*

Proof. If I is prime and $I \supseteq J \cap K$, then $I \supseteq J \cap K \supseteq J \cdot K$, so by primeness, either $I \supseteq J$ or $I \supseteq K$, hence I is (K4).

If I is (K4) and $I = J \cap K$, then $J, K \supseteq I \supseteq J \cap K$, and so either $J \supseteq I \supseteq J$ or $K \supseteq I \supseteq K$. That is, $I = J$ or $I = K$, so I is meet-irreducible. ∎

Note that the conditions prime and (K4) are quite distinct. For instance, in the algebra of upper triangular four by four matrices, the ideal

$$I = \begin{pmatrix} * & * & * & * \\ & 0 & 0 & * \\ & & 0 & * \\ & & & * \end{pmatrix}$$

is (K4) but not prime. On the other hand, meet-irreducible and (K4) are equivalent notions for distributive lattices of ideals, for if I is meet-irreducible, and $I \supseteq J \cap K$, then using $\vee$ for the joint operation, $I = (J \cap K) \vee I = (J \vee I) \cap (K \vee I)$, so by irreducibility, $I = J \vee I$ or $I = K \vee I$, which means I contains J or I contains K. That is, I is (K4).

However, if the lattice of ideals can be described by closed sets in a topological space, then the lattice is distributive (since the closed sets in a topological space form a distributive lattice, under the operations of union and intersection), so if we have any hope of finding a "good" space of ideals, we need to look at the meet-irreducibles.

We will make use of the following connections between an ideal and its quotient algebra. Note this is a standard result when applied to prime ideals in C^*-algebras.

PROPOSITION 1.4. *Let I be a closed, two-sided ideal in Banach algebra A.*
(i) *If I is prime in A, then 0 is prime in A/I;*
(ii) *if I is (K4) in A, then 0 is (K4) in A/I;*
(iii) *if I is meet-irreducible in A, then 0 is meet-irreducible in A/I.*

Proof. The three cases are demonstrated via the same technique. We will show (iii). Assume I is meet-irreducible in A. If 0 is not meet-irreducible in A/I, then there exist two non-zero ideals J_0 and K_0 in A/I with zero intersection. Thus, there are two ideals J and K in A properly containing I with $J/I = J_0$ and $K/I = K_0$. Notice $(J \cap K)/I = J_0 \cap K_0 = 0$ and so $J \cap K = I$, contradicting that I was meet-irreducible. ∎

The following section describes several examples where the connections between ideals types can be made quite clear, and are the motivation to consider more general triangular algebras.

2. C^*-ALGEBRAS AND COMPACT NEST ALGEBRAS

For C^*-algebras, there is an equivalence of five conditions.

THEOREM 2.1. *Let I be a closed, two-sided ideal in a separable C^*-algebra. Then the following are equivalent:*
(i) *I is nest-primitive;*
(ii) *I is primitive;*
(iii) *I is prime;*
(iv) *I is (K4);*
(v) *I is meet-irreducible.*
Moreover, every ideal in a separable C^*-algebra is equal to the intersection of meet-irreducible ideals.

Proof. The equivalence of (i) and (ii) is shown in the comments before Theorem I.4 of [7]; to summarize, since a nest representation is cyclic, one can use a result of U. Haagerup to conclude it is similar to a $*$-representation, and thus the nest of invariant subspaces is trivial. In particular, a nest representation is similar to an irreducible $*$-representation, and thus its kernel is a primitive ideal.

The implication (ii) $\Rightarrow$ (iii) is well-known for C^*-algebras; see for instance Proposition 3.13.10 of [9]. The converse is true for separable C^*-algebras, as shown in Proposition 4.3.6 of the same text.

The implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) follow from Proposition 1.3. To complete the proof, we will show that (v) implies (ii).

To this end, suppose the ideal I is meet-irreducible. The set $\mathrm{Prim}(A/I)$ of primitive ideals in the quotient C^*-algebra A/I forms a topological space in the hull-kernel topology, which is second countable since A/I is separable. Let $\{G_n\}$ be a countable basis for the topology, consisting of open, non-empty sets in $\mathrm{Prim}(A/I)$. As G_n is non-empty, the ideal $\ker(\mathrm{Prim}((A/I)\backslash G_n))$ is non-zero, while the intersection $\ker(G_n) \cap \ker(\mathrm{Prim}((A/I)\backslash G_n))$ is the zero ideal, which is meet-irrreducible by Proposition 1.4. Thus $\ker(G_n)$ is the zero ideal, so each set G_n is dense in the space $\mathrm{Prim}(A/I)$, which is a Baire space. By the Baire category theorem, $\cap G_n$ is non-empty and contains a point J which is dense in $\mathrm{Prim}(A/I)$. Hence J is an ideal contained in every primitive ideal, so in fact $J = 0$, consequently 0 is a primitive ideal. Letting π be an irreducible representation of A/I with 0 kernel, its pullback to A is an irreducible representation with kernel I. Hence I is primitive. ∎

REMARK 2.2. It is interesting to note that the proof of (v) implies (ii) above uses the same techniques as in the demonstration of the well-known result in C^*-theory that prime and primitive are equivalent for separable C^*-algebras. Moreover, by the correspondence between closed ideals in a C^*-algebra A and closed sets in the primitive ideal space, it is clear that the lattice structure is isomorphic to the lattice of closed sets in $\mathrm{Prim}(A)$. Meet-irreducible ideals correspond to minimal closed sets, namely single points. Join-irreducible ideals correspond to maximal closed sets, namely those that cannot be expressed as the intersection of two larger closed sets. In the Hausdorff case, these correspond to the set complements of isolated points. In particular, while there are lots of meet-irreducible ideals around, join-irreducible ideals are rather rare.

We now consider two nonselfadjoint examples, where the description of the meet-irreducible ideals can also be made precise. The interest in these compact nest algebras and the disk algebra rises from their connection with nonselfadjoint crossed products of dynamical systems. They are, in some sense, the elementary building blocks for the crossed products, as described in [7] and [8].

Recall a nest in a Hilbert space $\mathcal{H}$ is a family of closed subspaces which is linearly ordered by inclusion, contains 0 and $\mathcal{H}$, and is closed under arbitrary unions and intersections. An interval is the orthogonal difference $N_1 \ominus N_0$ of two elements of the nest. A nest algebra is the set of all bounded linear operators on $\mathcal{H}$ leaving invariant the elements of a fixed nest. We say a compact nest algebra is the set of all compact linear operators on $\mathcal{H}$ leaving invariant a fixed nest. Notice the identity representation of a compact nest algebra $\mathcal{A}$ on a Hilbert space is a nest representation, as is its compression to any interval $N_1 \ominus N_0$, for the invariant subspaces are just the nest itself, or its restriction to the interval. (See

Theorems I.4 and II.1 of [7]). The nest representations and nest-primitive ideals of an arbitrary compact nest algebra can thus be described completely, and we observe the close relation between nest-primitive ideals and meet-irreducibles.

PROPOSITION 2.3. *Let $\mathcal{A}$ be a compact nest algebra on Hilbert space $\mathcal{H}$, and I a closed, two-sided ideal in $\mathcal{A}$. Then the following are equivalent:*

(i) *I is nest-primitive;*

(ii) *I is (K4);*

(iii) *I is meet-irreducible;*

(iv) *I is kernel of the compression map of $\mathcal{A}$ to some interval of the nest.*

Moreover, every ideal in $\mathcal{A}$ is an intersection of the meet-irreducible ideals which contain it.

Proof. These results are a summary of work in [7]. The equivalence (iv) $\Leftrightarrow$ (i) and the implication (i) $\Rightarrow$ (ii) follow from Theorems II.1 and II.2 in [7], while (ii) $\Rightarrow$ (iii) is Corollary II.3, although this is also a general consequence of Proposition 1.3 of the present paper. The key construction in these results is the observation that every ideal in $\mathcal{A}$ is of the form

$$I_\varphi = \{a \in \mathcal{A} : aN \leqslant \varphi(N), \text{ for all } N \in \mathcal{N}\},$$

where $\varphi : \mathcal{N} \to \mathcal{N}$ is a left continuous order homomorphism of the nest $\mathcal{N}$ into itself, with $\varphi(N) \leqslant N$ for every subspace N in the nest. The nest-primitive ideals are determined by a pair of subspaces $N_0 \leqslant N_1$, with corresponding order homomorphism

$$\varphi(N) = \begin{cases} N, & N < N_0; \\ N_0, & N_0 \leqslant N \leqslant N_1; \\ N, & N > N_1. \end{cases}$$

That is, φ determines a corner, and I_φ is the kernel of the compression of the identity representation onto the interval $N_1 \ominus N_0$ in the nest.

To complete the proof of the proposition, we need to show that (iii) implies (iv). Assume that the ideal I is meet-irreducible. If $I = I_\varphi$ is not the kernel of a compression map, then φ is not of the form of a corner, as above, so we may construct left continuous order homomorphisms φ_1, φ_2 not equal to φ with

$$\varphi = \max\{\varphi_1, \varphi_2\}.$$

Thus $I_\varphi = I_{\varphi_1} \cap I_{\varphi_2}$, contradicting that I is not meet-irreducible. Hence I is indeed the kernel of a compression.

The fact that every ideal in $\mathcal{A}$ is the intersection of meet-irreducibles (equivalently, nest-primitives) is Corollary II.4 of [7]. ∎

We have immediately the following.

COROLLARY 2.4. *Let $\mathcal{A}$ be a compact nest algebra and Ω the set of meet-irreducible ideals in $\mathcal{A}$. Then the hull-kernel operation gives a topology on Ω, and there is a one-to-one correspondence between ideals in $\mathcal{A}$ and closed sets in Ω, given by the hull and kernel maps.*

The paper [7] also considers the disk algebra $A(\Delta)$, the Banach algebra of continuous function on the unit disk in the complex plane, which are analytic in the interior. This is an interesting example where the meet-irreducible ideals are not a rich enough set to describe all the ideals in the algebra. Indeed, it is well-known that every ideal in $A(\Delta)$ is principal and is generated by some inner function in $A(\Delta)$, with a singular part determined by a measure on the unit circle. Maximal ideals correspond to ideals of functions vanishing at a single point in the disk; thus primary ideals have a singleton for their zero set. However, not every ideal is the intersection of primary ideals; we miss the ones with non-discrete measures in the singular part of the generating function (see [5]). We may extend the relevant results in [7] as follows.

PROPOSITION 2.5. *Let $A(\Delta)$ be the disk algebra, and I a closed, two-sided ideal in $A(\Delta)$. Then the following are equivalent:*

(i) *I is nest-primitive;*

(ii) *I is* (K4)*;*

(iii) *I is meet-irreducible;*

(iv) *I is either primary, or zero.*

However, not every ideal is the intersection of meet-irreducible ideals.

REMARK 2.6. This proposition is essentially Theorems IV.1 and IV.2 of [7], although these only show (i) $\Leftrightarrow$ (iv) $\Rightarrow$ (ii), while the implication (ii) $\Rightarrow$ (iii) follows from Proposition 1.3. To see that (iii) $\Rightarrow$ (iv), the same argument as used in Theorem IV.1 of [7]. That is, if I is a non-zero meet-irreducible ideal, then I is generated by some function F in $A(\Delta)$. The zero set of F must be a singleton (and hence I is primary), for if not, we factor $F = fg$ as a product of two functions with strictly smaller zero sets. Then if J denotes the ideal generated by f and K the ideal generated by g, we see that $fg \in J \cap K$, so $I = J \cap K$, contradicting that I is meet-irreducible.

3. FINITE AND ALMOST FINITE TRIANGULAR ALGEBRAS

The algebra $\mathcal{T}_n$ of n by n upper triangular matrices is a special case of a compact nest algebra and thus is covered by Proposition 2.3 above. Note that the typical nest-primitive ideal is formed by a corner of zeroes, such as

$$I = \begin{pmatrix} * & * & * & * \\ & 0 & 0 & * \\ & & 0 & * \\ & & & * \end{pmatrix},$$

and thus the nest-primitive ideals are in one-to-one correspondence with matrix units in $\mathcal{T}_n$. To make this correspondence explicit, fix a system of matrix units $\{e_{ij}\}_{i,j=1}^n$ for $\mathcal{T}_n$, and for any unit e_{ij} from the system, let $I(e_{ij})$ denote the largest ideal in $\mathcal{T}_n$ not containing e_{ij}. Conversely, given a meet-primitive ideal I in $\mathcal{T}_n$, let $e(I)$ denote the matrix unit in the system of units which is not contained in I, and maximal of all such units with respect to the partial order

$$e_{ij'j} \leqslant_p e_{i'j'} \Leftrightarrow j \leqslant j' \text{ and } i \geqslant i'.$$

This order $\leqslant_p$ is closely related to the Peters-Poon-Wagner order on the diagonal (see [12]), where two projections p, q are ordered $p \preceq q$ if there is some element w in the normalizer of the diagonal with range projection $r(w) = p$ and domain projection $d(w) = q$. Thus we may write for two units e, f in $\mathcal{T}_n$ that

$$e \leqslant_p f \Leftrightarrow d(e) \preceq d(f) \text{ and } r(e) \succeq r(f).$$

In terms of the nest-primitive ideals $I(e), I(f)$ defined above, we may also write

$$e \leqslant_p f \Leftrightarrow I(e) \supseteq I(f).$$

This description extends in a straightforward manner to a finite direct sum of upper triangulars. Namely, if $\mathcal{A} = \mathcal{T}_{n_1} \oplus \cdots \oplus \mathcal{T}_{n_k}$ is a direct sum of upper triangular matrices of sizes $n_1, \ldots, n_k$, we fix a system of matrix units $\mathcal{E}$ and introduce the partial order

$$e \leqslant_p f \Leftrightarrow d(e) \preceq d(f) \text{ and } r(e) \succeq r(f)$$

for any two units e, f in $\mathcal{E}$. For such a unit e, let $I(e)$ denote the largest ideal in $\mathcal{A}$ not containing e. That is, $I(e)$ is the closed linear span, or join, of all ideals in $\mathcal{A}$ not containing e. Since every matrix unit e can be taken in the form $0 \oplus \cdots \oplus e^j \oplus \cdots \oplus 0$ in $\mathcal{T}_{n_1} \oplus \cdots \oplus \mathcal{T}_{n_k}$, where e^j is a matrix unit in $\mathcal{T}_{n_j}$, it is easy to see that

$$I(e) = \mathcal{T}_{n_1} \oplus \cdots \oplus I(e^j) \oplus \cdots \oplus \mathcal{T}_{n_k},$$

where $I(e^j)$ is the meet-irreducible ideal in $\mathcal{T}_{n_j}$ corresponding to the unit e^j.

PROPOSITION 3.1. *The ideal* $I(e) = \mathcal{T}_{n_1} \oplus \cdots \oplus I(e^j) \oplus \cdots \oplus \mathcal{T}_{n_k}$ *is meet-irreducible in* $\mathcal{A} = \mathcal{T}_{n_1} \oplus \cdots \oplus \mathcal{T}_{n_k}$, *and every meet-irreducible ideal is of this form.*

Proof. We will show $I(e)$ is (K4) and hence meet-irreducible by Proposition 1.3. Suppose $I(e) \supseteq J \cap K$, where J, K are ideals in the algebra $\mathcal{A}$. Since $e \notin I(e)$, we must have either $e \notin J$ or $e \notin K$. Since $I(e)$ is the join of all ideals not containing e, we have either $I(e) \supseteq J$ or $I(e) \supseteq K$. Thus $I(e)$ is (K4) and hence meet-irreducible.

Conversely, suppose I is a meet-irreducible ideal in $\mathcal{A}$. Since $\mathcal{A}$ is a direct sum, we may write $I = I_1 \oplus \cdots \oplus I_k$, where each I_j is an ideal in the corresponding $\mathcal{T}_{n_j}$. If two or more of the I_j are proper ideals, say $I_{j_1} \neq \mathcal{T}_{n_{j_1}}$ and $I_{j_2} \neq \mathcal{T}_{n_{j_2}}$, then we can write

$$\begin{aligned} I = &(I_1 \oplus \cdots \oplus \mathcal{T}_{n_{j_1}} \oplus \cdots \oplus I_{j_2} \oplus \cdots \oplus I_k) \\ &\cap (I_1 \oplus \cdots \oplus I_{j_1} \oplus \cdots \oplus \mathcal{T}_{n_{j_2}} \oplus \cdots \oplus I_k) \end{aligned}$$

contradicting that I is meet-irreducible. Thus all but at most one of the ideals I_j equal the full algebra $\mathcal{T}_{n_j}$, so we write

$$I = \mathcal{T}_{n_1} \oplus \cdots \oplus I_j \oplus \cdots \oplus \mathcal{T}_{n_k},$$

where I_j is a (possibly proper) ideal in $\mathcal{T}_{n_j}$.

If I_j is not meet-irreducible in $\mathcal{T}_{n_j}$, we can write $I_j = J_j \cap K_j$ for two strictly larger ideals J_j and K_j in $\mathcal{T}_{n_j}$, so

$$I = (\mathcal{T}_{n_1} \oplus \cdots \oplus J_j \oplus \cdots \oplus \mathcal{T}_{n_k}) \cap (\mathcal{T}_{n_1} \oplus \cdots \oplus K_j \oplus \cdots \oplus \mathcal{T}_{n_k}),$$

again contradicting that I is meet-irreducible.

Thus I_j is meet-irreducible in $\mathcal{T}_{n_j}$, so by Proposition 2.3, we have $I_j = I(e^j)$ for some matrix unit in $\mathcal{T}_{n_j}$, and thus

$$I = \mathcal{T}_{n_1} \oplus \cdots \oplus I(e^j) \oplus \cdots \oplus \mathcal{T}_{n_k} = I(\mathbf{0} \oplus \cdots \oplus e^j \oplus \cdots \oplus \mathbf{0}) = I(e),$$

where $e = \mathbf{0} \oplus \cdots \oplus e^j \oplus \cdots \oplus \mathbf{0}$ is a matrix unit for $\mathcal{A}$. ∎

Notice this proof also shows an ideal in $\mathcal{T}_{n_1} \oplus \cdots \oplus \mathcal{T}_{n_k}$ is (K4) if and only if it is meet-irreducible, and is thus of the form $I(e)$, for some matrix unit e. It is also easy to see that $I(e)$ is the kernel of a nest representation, namely the one obtained by compressing the natural component representation

$$\mathcal{T}_{n_1} \oplus \cdots \oplus \mathcal{T}_{n_k} \to \mathcal{T}_{n_j} \to \mathcal{B}(\mathbb{C}^{n_j})$$

to an interval in $\mathbb{C}^{n_j}$, corresponding to the matrix unit e^j in $e = \mathbf{0}\oplus\cdots\oplus e^j\oplus\cdots\oplus\mathbf{0}$. Thus $I(e)$ is also a nest-primitive ideal.

Conversely, if $I = I_1\oplus\cdots\oplus I_k$ is a nest-primitive ideal, then it is the kernel of some nest representation π. If two or more of the I_j are proper, say $I_{j_1} \neq \mathcal{T}_{n_{j_1}}$ and $I_{j_2} \neq \mathcal{T}_{n_{j_2}}$, then the ideals $J = \mathbf{0}\oplus\cdots\oplus\mathcal{T}_{n_{j_1}}\oplus\cdots\mathbf{0}$ and $K = \mathbf{0}\oplus\cdots\oplus\mathcal{T}_{n_{j_2}}\oplus\cdots\mathbf{0}$ are not in the kernel of π. Since π is a nest representation, the non-zero invariant subspaces $\overline{\pi(J)\mathcal{H}}$ and $\overline{\pi(K)\mathcal{H}}$ are ordered, so we may assume $0 \neq \overline{\pi(J)\mathcal{H}} \subseteq \overline{\pi(K)\mathcal{H}}$. Multiplying by $\pi(J)$, we see

$$0 \neq \overline{\pi(J^2)\mathcal{H}} \subseteq \overline{\pi(JK)\mathcal{H}} = 0,$$

a contradiction. Hence, at most only one of the I_j can be proper. Thus $I = \mathcal{T}_{n_1}\oplus\cdots\oplus I_j\oplus\cdots\oplus\mathcal{T}_{n_k}$, and applying Proposition 2.3 to the component $\mathcal{T}_{n_j}$, we conclude $I_j = I(e^j)$ for some matrix unit e^j in $\mathcal{T}_{n_j}$. Thus,

$$I = \mathcal{T}_{n_1}\oplus\cdots\oplus I(e^j)\oplus\cdots\oplus\mathcal{T}_{n_k} = I(\mathbf{0}\oplus\cdots\oplus e^j\oplus\cdots\oplus\mathbf{0}) = I(e).$$

That is, the nest-primitive ideals are all of the special form constructed above. We may summarize as follows.

PROPOSITION 3.2. *Let $\mathcal{A} = \mathcal{T}_{n_1}\oplus\cdots\oplus\mathcal{T}_{n_k}$ be a direct sum of upper triangular n_j by n_j matrix algebras, and I a two-sided ideal in $\mathcal{A}$. Then the following are equivalent:*

(i) *I is nest-primitive;*

(ii) *I is* (K4)*;*

(iii) *I is meet-irreducible;*

(iv) *There exists a matrix unit e in $\mathcal{A}$ with $I = I(e) = \overline{\text{span}}\{J$ an ideal: $e \notin J\}$.*

Moreover, for a given system of matrix units $\mathcal{E}$ in $\mathcal{A}$, there is a one-to-one correspondence between nest-primitive ideals in $\mathcal{A}$ and matrix units in $\mathcal{E}$, given by (iv).

From this, we may now build a finite topological space which describes the lattice of ideals for $\mathcal{A}$.

PROPOSITION 3.3. *Let $\mathcal{A} = \mathcal{T}_{n_1}\oplus\cdots\oplus\mathcal{T}_{n_k}$ be a direct sum of upper triangular matrices, and Ω the set of nest-primitive (equivalently, meet-irreducible) ideals in $\mathcal{A}$. Then the hull-kernel closure produces a topology on Ω, and there is a one-to-one correspondence between closed sets $F \subseteq \Omega$ and ideals $I \in \mathcal{A}$, given by*

$$I = \ker(F)$$

and

$$F = \text{hull}(I).$$

Proof. The fact that the meet-irreducible ideals are (K4) provides the topology, by Proposition 1.1. We only need to show that every ideal in $\mathcal{A}$ is the intersection of the meet-irreducible ideals which contain it. But, any ideal J is determined by the matrix units it contains, equivalently by the units it does not contain, so

$$\begin{aligned} J &= \cap\{I(e) : e \notin J\} = \cap\{I(e) : I(e) \supseteq J\} \\ &= \ker\{I \in \Omega : I \supseteq J\} = \ker(\text{hull}(J)). \quad \blacksquare \end{aligned}$$

It is interesting to see just what the topology on Ω is. Since Ω is in one-to-one correspondence with the matrix units in system $\mathcal{E}$, we can simply transport the topology to $\mathcal{E}$. The closure of a point $e \in \mathcal{E}$ is just the set

$$\overline{e} = \{f \in \mathcal{E} : f \leqslant_p e\},$$

that is, the set of units smaller than e in the partial order $\leqslant_p$. In particular, this topology is generally not T_1. For an arbitrary subset F of $\mathcal{E}$, we can verify

$$\overline{F} = \{f \in \mathcal{E} : f \leqslant_p e, \text{ some } e \in F\}.$$

This is a discrete topology, with a wedge condition on the closures.

We now consider a certain subset of the almost finite dimensional triangular algebras. Namely, let $\mathcal{A}$ be a norm-closed, strongly maximal triangular (regular canonical) subalgebra of an AF C^*-algebra $\mathcal{B}$. We will attempt to construct a topological space of meet-irreducible (and nest-primitive) ideals which describe the ideal structure of $\mathcal{A}$, based on the finite dimensional constructions above.

Following Power in [16], fix an increasing sequence $\mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \cdots$ of finite dimensional C^*-subalgebras of $\mathcal{B}$ with dense union, where $\mathcal{A}_k = \mathcal{B}_k \cap \mathcal{A}$ is maximal triangular in $\mathcal{B}_k$. The diagonal $\mathcal{C} = \mathcal{A} \cap \mathcal{A}^*$ is a regular canonical masa, and is the closure of the union of finite dimensional masas $\mathcal{C}_k = \mathcal{A}_k \cap \mathcal{A}_k^*$ in $\mathcal{B}_k$. We choose a system of matrix units $\{e_{ij}^k\}$ for each C^*-algebra $\mathcal{B}_k$ so that the matrix units in $\mathcal{B}_k$ are sums of matrix units in $\mathcal{B}_{k+1}$, the self-adjoint matrix units are precisely the ones in the diagonal $\mathcal{C}_k$, and each algebra $\mathcal{A}_k$ is a direct sum of upper triangular matrices in $\mathcal{B}_k$.

We introduce the notion of an infinite chain of matrix units

$$\mathrm{e}^{k_o} \longrightarrow \mathrm{e}^{k_o+1} \longrightarrow \mathrm{e}^{k_o+2} \longrightarrow \cdots,$$

where each e^k is a matrix unit in $\mathcal{B}_k$ from the system of matrix units $\{e_{ij}^k\}$, and the notation $\mathrm{e}^k \to \mathrm{e}^{k+1}$ indicates that e^{k+1} is a summand of the matrix unit e^k.

Since the system $\{e^k_{ij}\}$ was constructed so that each matrix unit in $\mathcal{B}_k$ is a sum of matrix units in $\mathcal{B}_{k+1}$, a finite chain

$$\mathrm{e}^{k_o} \longrightarrow \mathrm{e}^{k_o+1} \longrightarrow \mathrm{e}^{k_o+2} \longrightarrow \cdots \longrightarrow \mathrm{e}^{k_o+n}$$

may be extended to an infinite chain by chasing down summands in the matrix unit system. With a finite choice of summands at each stage k, there are typically uncountably many different ways of extending a chain.

If a matrix unit e^k is in the maximal triangular algebra $\mathcal{A}$, then so are all its summands. Consequently, if the first matrix unit in an infinite chain of matrix unit lies in $\mathcal{A}$, then so do all the units in the chain.

Let

$$\mathrm{e}^{k_o} \longrightarrow \mathrm{e}^{k_o+1} \longrightarrow \mathrm{e}^{k_o+2} \longrightarrow \cdots$$

be a chain of matrix units in $\mathcal{A}$, and for each $k \geqslant k_o$, let I_k be the largest ideal in $\mathcal{A}_k$ not containing e^k. By the last section, I_k is meet-irreducible and (K4) in $\mathcal{A}_k$. Observe that since I_{k+1} does not contain e^{k+1}, the intersection $I_{k+1} \cap \mathcal{A}_k$ cannot contain e^k; since I_k is the largest of such ideals, we have

$$I_k \supseteq I_{k+1} \cap \mathcal{A}_k. \tag{3.1}$$

In general, we don't have equality of these two ideals in $\mathcal{A}_k$ (an example is given at the end of this section). However, if $I_k = I_{k+1} \cap \mathcal{A}_k$ for all $k \geqslant k_o$, then by Theorem 2.5 of [12] the closure of the union of I_k defines an ideal $I = \overline{\cup I_k}$ in standard form.

PROPOSITION 3.4. *Let $\mathcal{A} = \overline{\cup \mathcal{A}_k}$ be a closed, strongly maximal AF algebra and $\{I_k\}$ a sequence of ideals constructed as above. If $I_k = I_{k+1} \cap \mathcal{A}_k$, for all $k \geqslant k_o$, then the ideal $I = \overline{\cup I_k}$ is both* (K4) *and meet-irreducible.*

Proof. To see that I is (K4), let J and K be closed ideals in $\mathcal{A}$ with I containing the intersection $J \cap K$. Thus

$$I_k = I \cap \mathcal{A}_k \supseteq (J \cap K) \cap \mathcal{A}_k = (J \cap \mathcal{A}_k) \cap (K \cap \mathcal{A}_k) = J_k \cap K_k.$$

Since I_k is (K4) when $k \geqslant k_o$, we conclude that for all large k, I_k contains either J_k or K_k; in particular, either $I_k \supseteq J_k$ for infinitely many k, or $I_k \supseteq K_k$ for infinitely many k. Without loss of generality, say $I_k \supseteq J_k$ infinitely often. Notice if $k' \leqslant k$, where k is one of the indices where containment holds, then $I_{k'} = I_k \cap \mathcal{A}_{k'} \supseteq J_k \cap \mathcal{A}_{k'} = J_{k'}$, so we conclude $I_k \supseteq J_k$ for all k, hence $I \supseteq J$. Thus I is (K4).

Now by Proposition 1.3, I is meet-irreducible as well. ∎

Note that since $I = \overline{\cup I_k}$ is in canonical form, we have that $I_k = I \cap \mathcal{A}_k$ for all $k \geqslant k_o$. In particular, since the matrix unit e^k is not an element of I_k, it follows that every matrix unit in the chain is not in the ideal I. This provides a very rich family of meet-irreducible ideals, provided we can assert the equality $I_k = I_{k+1} \cap \mathcal{A}_k$ for all k. It is easy to verify that for both the standard embedding and refinement embeddings of upper triangular matrix algebras, we always have equality. For if a matrix unit f is not in $I_{k+1} \cap \mathcal{T}_{n_k}$, then writing $f = \sum f_i$ as a sum of matrix units in $\mathcal{T}_{n_{k+1}}$, we find some f_i is not in I_{k+1}, thus $f_i \leqslant_p e_{k+1}$. By the block structure of the standard and refinement embeddings, we can conclude $f \leqslant_p e_k$ and thus f is not in I_k. Since ideals are sums of the matrix units they contain, we obtain the reverse containment of (3.1) above, namely $I_k \supseteq I_{k+1} \cap A_k$, and hence equality.

More generally, given an embedding $\mathcal{T}_{n_1} \oplus \cdots \oplus \mathcal{T}_{n_k} \to \mathcal{T}_{m_1} \oplus \cdots \oplus \mathcal{T}_{m_{k'}}$, if each component map $\mathcal{T}_{n_j} \to \mathcal{T}_{m_{j'}}$ is either a standard or refinement embedding, or the zero map, then the result $I_k = I_{k+1} \cap \mathcal{A}_k$ is similarly obtained. Thus in these cases, for any chain of matrix units, we can construct a corresponding meet-irreducible ideal. We note in the following proposition that this is a very rich family of meet-irreducibles.

PROPOSITION 3.5. *Let $\mathcal{A}$ be a strongly triangular AF algebra, where each non-zero component map of the embeddings $\mathcal{A}_k \to \mathcal{A}_{k+1}$ is either a standard or a refinement embedding. Then every closed ideal in $\mathcal{A}$ equals the intersection of the meet-irreducible ideals which contain it. Moreover, it suffices to intersect only meet-irreducible ideals of the form constructed in Proposition* 3.4.

Proof. Let J be a closed ideal in $\mathcal{A}$ and let M be the intersection of all meet-irreducible ideals of the constructed form above, which contain J. Clearly J is a subset of M.

To show the reverse containment, suppose e^{k_o} is a matrix unit from $\mathcal{A}_{k_o}$ which is not in J. Since e^{k_o} is a sum of matrix units in $\mathcal{A}_{k_o+1}$, there must be at least one summand e^{k_o+1} of e^{k_o} in $\mathcal{A}_{k_o}$ which is not in the ideal J. Thus we may construct inductively an infinite chain of matrix units

$$\mathrm{e}^{k_o} \to \mathrm{e}^{k_o+1} \to \mathrm{e}^{k_o+2} \to \cdots,$$

none of which lies in J. With I_k the largest ideal in $\mathcal{A}_k$ not containing e_k, we note that $I_k \supseteq J \cap \mathcal{A}_k$, for all $k \geqslant k_o$. Thus the meet-irreducible ideal $I = \overline{\cup I_k}$ is of the constructed from above, I contains J, and e^{k_o} is not in I. Thus e^{k_o} is not in the intersection of all such meet-irreducibles, so e^{k_o} is not in M.

Since ideals equal the span of the matrix units they contain, we conclude $J = M$. ∎

COROLLARY 3.6. *With $\mathcal{A}$ as in Proposition 3.5, let Ω be the set of all meet-irreducible ideals constructed from unit chains $e^k \to e^{k+1} \to \cdots$, then the hull-kernel operation defines a topology on Ω, and there is a one-to-one correspondence between closed sets in Ω and ideals in $\mathcal{A}$, via the hull and kernel maps.*

Notice from the comments following Proposition 1.3, we can conclude that the lattice of ideals in $\mathcal{A}$ is distributive, and so the condition (K4) is equivalent to meet-irreducible. This raises a number of questions, including what does the space Ω look like, and whether every meet-irreducible ideal in $\mathcal{A}$ corresponds to a point in Ω; that is, is it of the special form constructed above. We will leave these questions to future work.

However, there is an interesting connection with nest representations which we can describe. With X denoting the Gelfand space of the diagonal $\mathcal{C}$, we can build a representation of A on the Hilbert space $l^2(X)$ by lifting the natural representation of each finite algebra $\mathcal{A}_k$ on its diagonal in the obvious way. To be explicit, we follow Powers description in [16] of the Gelfand space and identify each point x in X with a sequence

$$q_x^1 \geqslant q_x^2 \geqslant q_x^3 \geqslant \cdots$$

of diagonal projections q_x^k in $\mathcal{C}_k$, where q_x^k is the unique matrix unit in $\mathcal{C}_k$ with $x(q_x^k) = 1$. (The order $\geqslant$ is the usual range containment order for commuting projections.) Of course, given only the tail of a sequence

$$q_x^k \geqslant q_x^{k+1} \geqslant q_x^{k+2} \geqslant \cdots$$

the order condition $q_x^{k-1} \geqslant q_x^k$ determines uniquely the beginning of the sequence

$$q_x^1 \geqslant q_x^2 \geqslant \cdots \geqslant q_x^{k-1} \geqslant q_x^k \geqslant \cdots,$$

so these sequences of diagonal projections corresponding to points in X are uniquely determined by their tails. To build the representation of $\mathcal{A}$ on $l^2(X)$, let δ_x denote the basis function in $l^2(X)$ supported at the point x in X, corresponding to some sequence of units $q_x^1 \geqslant q_x^2 \geqslant \cdots$, and for any matrix unit f in $\mathcal{A}_k$, let fx be the point in X corresponding to the tail sequence

$$fq_x^k f^* \geqslant fq_x^{k+1} f^* \geqslant fq_x^{k+2} f^* \geqslant \cdots,$$

when this sequence doesn't terminate in zero. That is, there is the possibility that the sequence collapses to $fq_x^{k+n} f^* = 0$ for n sufficiently large, in which case this sequence does not represent a point in X, so fx is left undefined.

Define the map $\pi : \mathcal{A} \to \mathcal{B}(l^2(X))$ on matrix units by its action on basis elements, as

$$\pi(f)\delta_x = \begin{cases} \delta_{fx}, & \text{if } fx \in X \text{ is defined,} \\ 0, & \text{if } fq_x^{k+n}f^* = 0 \text{ for some } n \geqslant 0. \end{cases}$$

This map π, when restricted to the units of $\mathcal{A}_k$, is just the natural representation of $\mathcal{A}_k$ on its diagonal, so it is easy to see that the linear extension of π to $\mathcal{A}_k$ creates a coherent representation of the finite subalgebras and thus extends to a faithful representation of $\mathcal{A}$.

To obtain a nest representation, fix a chain of matrix units

$$\mathrm{e}^{k_o} \to \mathrm{e}^{k_o+1} \to \mathrm{e}^{k_o+2} \to \cdots,$$

let $I_{k_o}, I_{k_o+1}, \ldots$ be the corresponding sequence of meet-irreducible ideals in algebras $\mathcal{A}_{k_o}, \mathcal{A}_{k_o+1}, \ldots$, and let $X' \subseteq X$ be the subset of points x corresponding to sequences of units

$$q_x^{k_o} \geqslant q_x^{k_o+1} \geqslant q_x^{k_o+2} \geqslant \cdots$$

such that

$$q_x^k \notin I_k, \quad \text{for all } k \geqslant k_o.$$

Let π' be the compression of the representation π to the subspace $l^2(X') \subseteq l^2(X)$.

PROPOSITION 3.7. *Under the assumptions of Proposition* 3.5, *the linear map* π' *is a nest representation of algebra* $\mathcal{A}$, *whose kernel is the meet-irreducible ideal* I *corresponding to the chain of matrix units*

$$\mathrm{e}^{k_o} \to \mathrm{e}^{k_o+1} \to \mathrm{e}^{k_o+2} \to \cdots.$$

Proof. As in the construction of π, the map π' restricted to any finite subalgebra $\mathcal{A}_k$ is just the compression of the natural representation of a triangular algebra $\mathcal{T}_{n_1} \oplus \cdots \oplus \mathcal{T}_{n_r}$ onto an interval of its diagonal, and is thus a (nest) representation of $\mathcal{A}_k$. Taking limits, we conclude π' is a bounded representation of the AF algebra $\mathcal{A}$. Note that for any matrix unit f in $\mathcal{A}_k$, we have $f \notin \ker(\pi')$ if and only if $f \leqslant_p \mathrm{e}^k$, which is equivalent to $f \notin I_k = I \cap \mathcal{A}_k$. Since the ideal I is determined by the matrix units it contains, we have $\ker(\pi') = I$.

To see that π' is a nest representation, observe first that the Peters-Poon-Wagner order on the diagonal matrix units introduces a total order on the X' via the relation

$$x \prec y \Leftrightarrow \begin{cases} q_x^k = q_y^k, & k = 1, 2, \ldots, n-1, \\ q_x^n \prec q_y^n, & \text{some } n. \end{cases}$$

Transitivity of this order relation follows from the observation that a refinement embedding component of $\mathcal{A}_k \to \mathcal{A}_{k+1}$ will preserve the Peters-Poon-Wagner order for those diagonal elements not in I_k, while a standard embedding component is a one-to-one map on the diagonal elements not in I_k. Thus we may observe that $q_x^n \preceq q_y^n$ implies $q_x^k \preceq q_y^k$ for all $k \geqslant n$, from which transitivity follows. This order is total, since the diagonal elements q_x^k, q_y^k in $\mathcal{A}_k$ which are not in I form an interval along a diagonal in $\mathcal{A}_k$, and are thus totally ordered by $\preceq$.

The π'-invariant subspaces of $l^2(X)$ include the subspaces $l^2[-\infty, x]$ of functions supported on the interval $[-\infty, x] = \{y \in X' : y \preceq x\}$, and more generally, they are the subspaces $l^2[-\infty, c]$, where c represents a Dedekind cut in X'. These form a nest via the order $\preceq$ on X', thus π' is a nest representation. ∎

Thus once again, we find a topological space of meet-irreducibles which coincidently are also nest-primitives, that completely describes the ideal structure of the ideal space.

It is worth noting that if one considers more general $*$-extendable embeddings of upper triangular matrix algebras, the above construction need not work. In particular, given a map $\varphi : \mathcal{A}_k \to \mathcal{A}_{k+1}$ and a meet-irreducible ideal $I_k \subseteq \mathcal{A}_k$, it is not always possible to choose a corresponding meet-irreducible ideal $I_{k+1} \subseteq \mathcal{A}_{k+1}$ with $I_k = I_{k+1} \cap \mathcal{A}_k$. For instance, consider the following embedding of T_4 into T_8,

$$\begin{pmatrix} a & b & c & d \\ & e & f & g \\ & & h & i \\ & & & j \end{pmatrix} \to \begin{pmatrix} a & b & & & c & d & & \\ & e & & & f & g & & \\ & & a & b & & & c & d \\ & & & e & & & f & g \\ & & & & h & i & & \\ & & & & & j & & \\ & & & & & & h & i \\ & & & & & & & j \end{pmatrix},$$

which is the refinement embedding, amplified by two. Let I_4 be the meet-irreducible ideal in T_4 with corner f; that is, the ideal with entries e, f, h set to zero. There are two possible choices for meet-irreducible ideals $I_8 \subseteq T_8$ with corner f. The first one (choosing the upper f in T_8) has $I_8 \cap T_4$ equal to the ideal with entries a, b, e, f, h set to zero, which is strictly smaller than I_4. The second choice for I_8 has $I_8 \cap T_4$ equal to the ideal with entries e, f, h, i, j set to zero, which is also smaller than I_4. By modifying this example with a twist in the last two pairs of rows/columns, we obtain an example where one choice of I_8 gives $I_4 = I_8 \cap T_4$ while the other give $I_4 \neq I_8 \cap T_4 = 0$. We conclude that the general case of maximal triangular AF algebras may be quite complicated.

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REFERENCES

1. G. BIRKHOFF *Lattice Theory*, Amer. Math. Soc., Providence, 1961.
2. K.R. DAVIDSON, *Nest algebras*, Pitman Res. Notes Math. Ser., vol. 191, Longman, Essex 1988.
3. A.P. DONSIG, T.D. HUDSON, The lattice of ideals of a triangular AF algebra, *J. Funct. Anal.* **138**(1996), 1–39.
4. E. GOOTMAN, J. ROSENBERG, The structure of crossed product C^*-algebras: A proof of the generalized Effros-Hahn conjecture, *Invent. Math.* **53**(1979), 283–298.
5. K. HOFFMAN, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cloffs, New Jersey 1962.
6. T.D. HUDSON, Ideals in triangular AF algebras, *Proc. London Math. Soc. (3)* **69** (1994), 345–376.
7. M.P. LAMOUREUX, Nest representations and dynamical systems, *J. Funct. Anal.* **114**(1993), 467–492.
8. M.P. LAMOUREUX, Ideals in some continuous nonselfadjoint crossed product algebras, *J. Funct. Anal.*, to appear.
9. G. PEDERSEN, *C^*-algebras and their Automorphism Groups*, Academic Press London, 1979.
10. J. PETERS, Semi-crossed products of C^*-algebras, *J. Funct. Anal.* **59**(1984), 498–534.
11. J. PETERS, The ideal structure of certain nonselfadjoint operator algebras, *Trans. Amer. Math. Soc.* **305**(1988), 333–352.
12. J.R. PETERS, Y.T. POON, B.H. WAGNER, Triangular AF algebras, *J. Operator Theory* **23**(1990), 81–114.
13. J.R. PETERS, Y.T. POON, B.H. WAGNER, Analytic TAF algebras, *Canad. J. Math* **45**(1993), 1009–1031.
14. S.C. POWER, Infinite tensor products of upper triangular matrix algebras, *Math. Scand.* **65**(1989), 291–307.
15. S.C. POWER, On ideals of nest subalgebras of C^*-algebras, *Proc. London Math. Soc. (3)* **50**(1985), 314–332.
16. S.C. POWER, *Limit algebras: an introduction of subsalgebras of C^*-algebras*, Pitman Res. Notes Math. Ser., vol. 278, Essex 1992.
17. D. WILLIAMS, The topology on the primitive ideal space of transformation group C^*-algebras and CCR transformation group C^*-algebras, *Trans. Amer. Math. Soc.* **266**(1981), 335–359.

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