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BOOK REVIEW:

Spectral Theory of Indefinite Krein-Feller Differential Operators, **Andreas Fleige**, Mathematical Research, vol. 98, Akademie Verlag, Berlin 1996, 133 pages, ISBN 3–05–501742-0.

KEYWORDS: *Sturm-Liouville operators, spectral theory, selfadjoint definitizable operators, spaces with indefinite inner products, Krein-Feller differential operator, Riesz bases.*.

AMS SUBJECT CLASSIFICATION: Primary 34L40; Secondary 34L10, 47B50, 47E05.

The spectral theory of selfadjoint operators on Hilbert spaces proved to be remarkably useful for investigation of differential operators from various points of view: determination of the different types of spectra, calculation of the resolvent function, expansions in eigenfunctions, etc. and thus providing powerful tools for solving the corresponding differential equations, mostly when plugged with boundary values problems. One of the most investigated class of differential operators is corresponding to the Sturm-Liouville operators and there is a huge amount of literature dealing with these problems.

Among the many extensions of the spectral theory of selfadjoint operators on Hilbert spaces, the spectral theory of selfadjoint definitizable operators on Kreĭn spaces initiated by H. Langer ([8]) plays a special role. The class of selfadjoint definitizable operators on Kreĭn spaces turns out to be the most tractable within the class of selfadjoint operators on Kreĭn spaces and for which a rich class of applications exists. The most powerful tool available for this class of operators is the *spectral function* which, roughly speaking, is a spectral measure with singularities, usually called *critical points.* In the neighbourhood of such a critical point the spectral function may be unbounded, in this case the critical point is called *singular.* In the opposite case, when the critical point is *regular*, the only encountered difficulties are related with the length of the Jordan chains which can be larger than one. A pertinent survey of the many applications of the spectral theory of selfadjoint definitizable operators was recently delivered by A. Dijksma and H. Langer ([3]).

A useful application of the spectral theory is illustrated by the mathematical model corresponding to the vibrations of a string $= [a, b]$ of a density $r \in L^1([a, b])$, extended without mass to the points $a - \tan\alpha$ and $b + \cot\beta$ and fastened at the endpoints. This corresponds to the Sturm-Liouville problem

$$-f''(x) = \lambda r(x) f(x), \quad x \in [a, b], \tag{1}$$

$$\cos\alpha\, f(a) - \sin\alpha\, f'(a) = 0, \tag{2}$$

$$\sin\beta\, f(b) + \cos\beta\, f'(b) = 0. \tag{3}$$

The classical spectral theoretical approach associates to the problem (1)–(3) a selfadjoint operator A on the Hilbert space $L^2_r([a, b])$, equipped with the inner product

$$[f, g] = \int_a^b f(x)\overline{g(x)} r(x)\, \mathrm{d}x, \quad f, g \in L^2_r([a, b]), \tag{4}$$

and enables us to prove that the eigenfunctions of the operator A form and orthonormal basis of $L^2_r([a, b])$.

Now consider the case when the weight r changes sign. In this case the inner product (4) is indefinite and $L^2_r([a, b])$ becomes a Kreĭn space. The problem (1)–(3) still yields a selfadjoint operator A (now in the Kreĭn space sense) and one of the problem to be faced is whether the eigenfunctions system of A still forms a Riesz basis of $L^2_r([a, b])$. An approach making heavy use of the spectral theory of selfadjoint definitizable operators of this problem was used by K. Daho and H. Langer ([2]). In general, this problem may have a negative answer. Criteria for the Riesz basis property of the eigenfunctions of A have been found by H.G. Kaper, M.K. Kwong, C.G. Lekkerkerker, and A. Zettl ([6]), R. Beals ([1]), and others.

The Sturm-Liouville problem (1)–(3) can be generalized (cf. M.G. Kreĭn ([7]) and W. Feller ([4])) in the following way: consider the mass distribution function of the string m defined by

$$m(x) = \int_a^x r(t)\, \mathrm{d}t, \quad x \in [a, b], \tag{5}$$

and then, letting formally $\mathrm{d}m(x) = r(x)\,\mathrm{d}x$, the differential equation (1) can be written as

$$-\frac{\mathrm{d}^2 f}{\mathrm{d}m\,\mathrm{d}x} = \lambda f, \tag{6}$$

where the differential expression has to be interpreted as the Radon-Nikodym derivative of the complex Lebesgue-Stieltjes measure f', with respect to the measure induced by m. In general the measure m is not positive and also it is not absolutely continuous.

The aim of the monograph under review is to study this generalized problem from the point of view of the Riesz basis property of the eigenfunctions system. Apparently it is a refined version of the author Dissertation ([5]). The approach is based on the spectral theory of selfadjoint definitizable operators on Kreĭn spaces and the problem is reformulated, in most of the interesting cases, as referring to the regularity of the critical point infinity of the corresponding operator.

This book surely has remarkable qualities: it is a rigorous treatment of the topics, it is accessible beginning with graduate student level and brings new and interesting results.

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