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A DESCRIPTION OF COMMUTATIVE SYMMETRIC OPERATOR ALGEBRAS IN A PONTRYAGIN SPACE Π_1

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ABSTRACT. We construct a system of model commutative symmetric operator algebras (c.s.o.a.) in a Pontryagin space Π_1 such that both the weak operator and the uniform operator closures of any c.s.o.a. in Π_1 can be described in terms of the models found. We then use that representation to obtain the theorem of bicommutant for a c.s.o.a. in Π_1.

KEYWORDS: *Commutative, algebra, Pontryagin space, unitary equivalence, singular, weak topology, closure, bicommutant.*

AMS SUBJECT CLASSIFICATION: Primary 47D25; Secondary 47B50.

INTRODUCTION

Commutative symmetric operator algebras (c.s.o.a.) in Pontryagin space Π_1 were initiated by M.A. Naimark ([9], [10]) and later studied by A.I. Loginov ([5], [6]). In [10] all the c.s.o.a. acting in Π_1 space were divided into three non-intersecting classes I, II, III. For every class, a system of models was constructed so that every algebra of each class was equivalent to a certain model of this system. For the algebras of class III those constructions suggested that the algebras were unital and uniformly separable. A.I. Loginov found out the conditions of uniform closedness (completeness) of model algebras of class III. Modifying models of [10] and introducing the notions of a fundamental regular (singular) algebra, he showed that a complete regular (singular) algebra was necessarily a fundamental one, i.e.

its components appeared to be independent. An analogous investigation for arbitrary general (i.e., those having a neutral invariant subspace) complete symmetric operator algebras in space Π_1 was made by V.S. Shul'man ([12], [13]).

The present work deals with the description of the algebraic structure and the weak and also the uniform closures of c.s.o.a. in space Π_1. We start with considering the models for algebras of the classes I, II ([10]), write them down in a convenient form and give a description of the closures of these algebras. The suggested models for class III essentially differ from those of [6] and [10]. Firstly, they are constructed without supposing their uniform separability; our class of regular (singular) algebras is larger than that of [10]. Secondly, the construction of our models is made in such a way that the difference between the models of regular and singular algebras appears only in the absence or, respectively, in the presence of a certain anti-Hilbert space Q. Note that, contrary to [13], we do not suppose that the considered algebras are complete. The constructed models allow us to describe not only the uniform but also the weak closures of algebras of class III, that would be impossible in terms of the models of [6] and [10].

A key problem for symmetric operator algebras is the problem of relation between the algebra and its bicommutant. The theorem of bicommutant was proved for some classes of operator algebras in Pontryagin spaces ([1], [3], [4], [13]). The last part of the present paper is devoted to the theorem of bicommutant, which turns out to be always true for c.s.a.o. in Π_1 as we show using the constructed models and results of [13].

1. PRELIMINARIES

A Hilbert space $(\mathcal{H}, [\cdot,\cdot])$ is called a *Krein Space* if another Hermitian form $(\cdot,\cdot) = [J\cdot,\cdot]$ is given with $J = P_+ - P_-$, P_+ an orthogonal projection, and $P_- = I - P_+$. Generally, the form $(\cdot,\cdot)$ is not positive definite and is called *indefinite*. As a consequence, positive definite, non-negative, neutral, non-positive, and negative definite subspaces of $\mathcal{H}$ are naturally defined. Besides, the subspace $L \subset \mathcal{H}$ is called *non-degenerate* (*degenerate*) if $L \cap L^{\perp} = \{0\}$ (respectively, if $L \cap L^{\perp} \neq \{0\}$).

If we denote $\mathcal{H}_+ = P_+\mathcal{H}$, $\mathcal{H}_- = P_-\mathcal{H}$, then it is obvious that the direct J-orthogonal decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ holds where $\mathcal{H}_+$ is a positive definite subspace and $\mathcal{H}_-$ is negative definite.

Let $B(\mathcal{H})$ be the algebra of all linear operators acting in $\mathcal{H}$ which are continuous with respect to the Hilbert topology. For any operator $A \in B(\mathcal{H})$ there exists a unique operator $A^* \in B(\mathcal{H})$ such that $(A\xi, \eta) = (\xi, A^*\eta)$ for any $\xi, \eta \in \mathcal{H}$. An algebra $R \subset B(\mathcal{H})$ is said to be *symmetric* if $A \in R$ implies $A^* \in R$.

The algebras $R^{(1)}$ and $R^{(2)}$ in the Krein spaces $(\mathcal{H}_1, (\cdot,\cdot)_1)$ and $(\mathcal{H}_2, (\cdot,\cdot)_2)$, respectively, are called *equivalent*, if $R^{(2)} = UR^{(1)}U^{-1}$ for some J-unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$.

The topology in $B(\mathcal{H})$ generated by the system of semi-norms $p_{\xi,\eta}(A) = |(A\xi, \eta)|$, $\xi, \eta \in \mathcal{H}$, is called the *weak topology* and is denoted w. Obviously, this topology coincides with the weak topology in $B(\mathcal{H})$ generated by the Hilbert form.

If dim $P_+\mathcal{H} = k < +\infty$, then $\mathcal{H}$ is called a *Pontryagin space* and is denoted by Π_k.

In the sequel we intend to use some well-known facts avoiding any reference.

The dimension of any non-negative subspace in Π_k does not exceed k ([2]).

The subspace $L \subset \Pi_k$ is non-degenerate if and only if $\Pi_k = L \oplus L^\perp$ ([3]).

We will also need the following fact: any c.s.o.a. in Π_k has an invariant non-negative subspace of dimension k ([7], [8]).

Below R is a c.s.o.a. in $\Pi_1, \xi_0 \in \Pi_1$ is a non-negative, invariant under R, vector, $\lambda(A)$ is the *eigen-functional* (e.f.) of R corresponding to ξ_0 which means $A\xi_0 = \lambda(A)\xi_0$ for any operator $A \in R$.

2. ALGEBRAS OF CLASSES I AND II

An algebra R is said to be an *algebra of class* I if there exists a positive, invariant for R, vector ξ_0. Let $\lambda(A)$ be the e.f. corresponding to ξ_0. Let $\mathfrak{M} = \{\xi \in \Pi_1 : A\xi = \lambda(A)\xi$ for any $A \in R\}$, $\mathfrak{H} = \mathfrak{M}^\perp$ and $A_1 = A|\mathfrak{H}$ for $A \in R$. If $A_1 = 0$ (for $A \in R$) implies $\lambda(A) = 0$, then R is called an *algebra of class* $\mathrm{I_a}$; in the opposite case it is called an *algebra of class* $\mathrm{I_b}$. R is said to be an *algebra of class* II if any non-negative vector, invariant under R is neutral, but among them there exists a ξ_0 such that the corresponding e.f. $\lambda(A)$ is non-Hermitian. In this case R has a neutral invariant vector η_0 which is skewly linked with ξ_0 (that is $(\xi_0, \eta_0) \neq 0$) ([10]). We may suppose that $(\xi_0, \eta_0) = 1$ and define $\Pi_1' = \langle\xi_0\rangle + \langle\eta_0\rangle$ (Below the space Π_1' is a space of Π_1 type with a biorthogonal basis $\{\xi_0, \eta_0\}$.), where $\langle\xi_0\rangle$ is used for the linear span of the vector ξ_0. Let $\mathfrak{H} = (\Pi_1')^\perp$ and $A_1 = A|\mathfrak{H}$ for $A \in R$. *Classes* $\mathrm{II_a}$ and $\mathrm{II_b}$ are defined analogously with the definition of the classes $\mathrm{I_a}$ and $\mathrm{I_b}$.

If $\mathfrak{M}$ is either a one-dimensional positive space or a space of Π_1 type, then R_1 is a c.s.o.a. in the anti-Hilbert space $\mathfrak{H}$. (The space $(\mathfrak{H}, (\cdot,\cdot))$ is called *anti-Hilbert* if $(\mathfrak{H}, -(\cdot,\cdot))$ is a Hilbert space.) Consider the $*$-algebra

$$G_{\mathrm{I}} = \mathbb{C} \times R_1$$

with the component-wise algebraic operations and involution. Define the mapping $\mathrm{Op_I} : G_{\mathrm{I}} \to B(\Pi_1)$, where $\Pi_1 = \mathfrak{M} \oplus \mathfrak{H}$, and, for $\omega = \{\lambda, A_1\} \in G_{\mathrm{I}}$, the operator $A = \mathrm{Op_I}(\omega)$ acts in Π_1 by the formulas

$$A\eta = \lambda\eta, \quad Ah = A_1 h, \quad \eta \in \mathfrak{M}, \quad h \in \mathfrak{H}.$$

It is clear that $\mathrm{Op_I}$ is an injective $*$-homomorphism from G_{I} into $B(\Pi_1)$. Furthermore, let $\mathcal{E}_{\mathrm{I}}$ be a $*$-subalgebra in G_{I} such that:

(i) $\{A_1 : \omega \in \mathcal{E}_{\mathrm{I}}\} = R_1$;

(ii) for any $A_1 \in R_1$ there exists a unique $\lambda = \lambda(A_1)$ such that $\{\lambda, A_1\} \in \mathcal{E}_{\mathrm{I}}$, and $\lambda(A_1)$ is not an e.f. of R_1.

If Π_1' is a space of Π_1 type with a biorthogonal basis $\{\xi_0, \eta_0\}$, then we can consider the $*$-algebra

$$G_{\mathrm{II}} = \mathbb{C} \times \mathbb{C} \times R_1$$

also with the component-wise algebraic operations, and the involution given by $\{\lambda, \mu, A_1\}^* = \{\overline{\mu}, \overline{\lambda}, A_1^*\}$. Define the mapping $\mathrm{Op_{II}} : G_{\mathrm{II}} \to B(\Pi_1)$ so that $\Pi_1 = \Pi_1' \oplus \mathfrak{H}$, and, for $\omega = \{\lambda, \mu, A_1\} \in G_{\mathrm{II}}$, the operator $A = \mathrm{Op_{II}}(\omega)$ acts in Π_1 by the formulas

$$A\xi_0 = \lambda\xi_0, \quad A\eta_0 = \mu\eta_0, \quad Ah = A_1 h, \quad h \in \mathfrak{H}.$$

It is clear that $\mathrm{Op_{II}}$ is an injective $*$-homomorphism from G_{II} into $B(\Pi_1)$. Let $\mathcal{E}_{\mathrm{II}}$ be a $*$-subalgebra in G_{II} such that:

(i) $\{A_1 : \omega \in \mathcal{E}_{\mathrm{II}}\} = R_1$;

(ii) there exists a non-Hermitian function $\lambda : R_1 \to \mathbb{C}$ such that, for any $\{\lambda, \mu, A_1\} \in \mathcal{E}_{\mathrm{II}}$, we have $\lambda = \lambda(A_1)$, $\mu = \overline{\lambda(A_1^*)}$.

The following theorem gives a model representation for the c.s.o.a. of classes I and II (cf. [10]).

THEOREM 2.1. *The following conditions are equivalent:*

(i) *R is a c.s.o.a. of class* $\mathrm{I_a}, (\mathrm{I_b}, \mathrm{II_a}, \mathrm{II_b})$;

(ii) $R = \mathrm{Op_{II}}(\mathcal{E}_{\mathrm{I}})$

(Below the equality $R = R'$ means the equivalence of the algebras.) (respectively, $\mathrm{Op_I}(G_{\mathrm{I}})$, $\mathrm{Op_{II}}(\mathcal{E}_{\mathrm{II}})$, $\mathrm{Op_{II}}(G_{\mathrm{II}})$) *for some* $\mathcal{E}_{\mathrm{I}}respectivelyG_{\mathrm{I}}, \mathcal{E}_{\mathrm{II}}, G_{\mathrm{II}}$.

We shall call $\mathcal{E}_{\mathrm{I}}, G_{\mathrm{I}}, \mathcal{E}_{\mathrm{II}}, G_{\mathrm{II}}$ *definable manifolds* of the corresponding algebras.

Let us study now the closures of the described models. Let $\mathcal{E}_{\mathrm{I}} \subset G_{\mathrm{I}}$ and $\mathcal{E}_{\mathrm{II}} \subset G_{\mathrm{II}}$ be respectively the definable manifolds of c.s.o.a. of classes I and II. Denote

$$\widetilde{G}_{\mathrm{I}} = \mathbb{C} \times \overline{R}_1^{\mathrm{w}} \quad \text{and} \quad \widetilde{G}_{\mathrm{II}} = \mathbb{C} \times \mathbb{C} \times \overline{R}_1^{\mathrm{w}}.$$

Then the following inclusions hold:

$$\mathrm{Op_I}(\mathcal{E}_\mathrm{I}) \subset \mathrm{Op_I}(G_\mathrm{I}) \subset \mathrm{Op_I}(\widetilde{G}_\mathrm{I}) \tag{2.1}$$

$$\mathrm{Op_{II}}(\mathcal{E}_\mathrm{II}) \subset \mathrm{Op_{II}}(G_\mathrm{II}) \subset \mathrm{Op_{II}}(\widetilde{G}_\mathrm{II}). \tag{2.2}$$

It is easy to verify the following.

LEMMA 2.2. $\mathrm{Op_I}(\widetilde{G}_\mathrm{I})$ *and* $\mathrm{Op_{II}}(\widetilde{G}_\mathrm{II})$ *are* w*-closed algebras.*

From the definition of operations in G_I and G_II we can see that, for algebras of classes $\mathrm{I_a}$ and $\mathrm{II_a}$, the arising function $\lambda : R_1 \to \mathbb{C}$ is a character.

LEMMA 2.3. *Let* $B(\mathfrak{H}) \supset \mathcal{M}$ *be a* $*$*-algebra and let* λ *be a non-zero character on* $\mathcal{M}$*. Then* λ *is* w*-continuous if and only if* λ *is an eigenfunctional of* $\mathcal{M}$.

Proof. Suppose that λ is a w-continuous character on $\mathcal{M}$. Let $\widetilde{\lambda}$ be the continuous extension of λ to the W^*-algebra $\mathcal{A} = \overline{\mathcal{M}}^{\mathrm{w}}$. Its kernel $M = \operatorname{Ker}\widetilde{\lambda}$ is a w-closed two-sided ideal in $\mathcal{A}$, therefore there exists a central projection E in $\mathcal{A}$ such that $M = E\mathcal{A}$. It is easy to see that $G = 1_\mathcal{A} - E$ is an atom in $\mathcal{A}$, i.e. there exists no projection $G_1 \in \mathcal{A}$ such that $0 \neq G_1 < G$. Therefore $G\mathcal{A} = \{\mu G : \mu \in \mathbb{C}\}$ and $\widetilde{\lambda}(G) = 1$. Consequently, for any $GA = \mu G \in G\mathcal{A}$, we have $\mu = \widetilde{\lambda}(\mu G) = \widetilde{\lambda}(GA) = \widetilde{\lambda}(A)$, i.e. $G\mathcal{A} = \{\widetilde{\lambda}(A)G : A \in \mathcal{A}\}$. Consider now $0 \neq g \in G\mathfrak{H}$. Then, for any $A \in \mathcal{M}$, we get $Ag = AGg = GAg = \widetilde{\lambda}(A)Gg = \lambda(A)g$. Thus, λ is an e.f. of the algebra $\mathcal{M}$. The converse assertion of the lemma is obvious. ∎

The next lemma follows immediately from the definition of the $*$-algebra $\mathcal{E}_\mathrm{I}$ and Lemma 2.3.

LEMMA 2.4. *If the algebra* $R = \mathrm{Op_I}(\mathcal{E}_\mathrm{I})$ *is such that the corresponding character* λ *is not identically zero, then* λ *is* w*-discontinuous on* R_1.

THEOREM 2.5. (i) *If the algebra* $R = \mathrm{Op_I}(\mathcal{E}_\mathrm{I})$ *is such that the corresponding character* λ *is not identically zero (respectively* $\equiv 0$*) on* R_1*, then* $\overline{R}^{\mathrm{w}} = \mathrm{Op_I}(\widetilde{G}_\mathrm{I})$ *(respectively,* $\overline{R}^{\mathrm{w}} = 0 \oplus \overline{R_1}^{\mathrm{w}}$ *on* $\mathfrak{M} \oplus \mathfrak{H}$*);*
(ii) $\overline{\mathrm{Op_{II}}(\mathcal{E}_\mathrm{II})}^{\mathrm{w}} = \mathrm{Op_{II}}(\widetilde{G}_\mathrm{II})$.

Proof. First we prove (ii). Note that the corresponding character λ can not be w-continuous on R_1. Indeed, otherwise λ extends to a w-continuous (consequently, u-continuous) character on the C^*-algebra $\overline{R_1}^{\mathrm{w}}$ which must be Hermitian by the Gelfand-Naimark theorem which would contradict the non-Hermitianity of λ on R_1. Thus, λ is a non-Hermitian (consequently, non-zero) w-discontinuous character on R_1. Consider another character μ on R_1 defined by $\mu(A_1) = \overline{\lambda(A_1^*)}$ for $A_1 \in R_1$. Then, obviously, μ is also w-discotinuous and different from the character λ on R_1. Therefore the subspaces $H_\lambda = \operatorname{Ker}\lambda$ and $H_\mu = \operatorname{Ker}\mu$ do not coincide and are w-dense in R_1. Let us show that $\lambda_1 = \lambda|H_\mu$ and $\lambda_2 = \lambda|H_\lambda$ are w-discontinuous. Indeed, let, for example, λ_1 be w-continuous on H_μ, and let the net $\{D_\alpha\}_{\alpha\in\Lambda} \subset R_1$ w-converge to $D \in R_1$. Since λ and μ are different on R_1, $\lambda_1(B) \neq 0$ for some $B \in H_\mu$. Then $D_\alpha B \xrightarrow{\text{w}} DB$ and, from the fact that H_μ is an ideal in R_1, we get $\lambda_1(D_\alpha B) \to \lambda_1(DB)$, or $\lambda(D_\alpha)\lambda_1(B) \to \lambda(D)\lambda_1(B)$, hence $\lambda(D_\alpha) \to \lambda(D)$. Further, λ_1 and λ_2 are w-discontinuous characters on H_μ and H_λ, respectively, $R_1 \subset \overline{H_\mu}^{\text{w}}$ and $R_1 \subset \overline{H_\lambda}^{\text{w}}$ and, for any $\lambda_0, \mu_0 \in \mathbb{C}$, $A_1 \in R_1$, we find nets $\{(B_\alpha)_1\}_{\alpha\in\Lambda} \subset \lambda^{-1}(\lambda_0) \cap H_\mu$ and $\{(C_\alpha)_1\}_{\alpha\in\Lambda} \subset \mu^{-1}(\mu_0) \cap H_\lambda$, such that $(B_a)_1 \xrightarrow{\text{w}} A_1, (C_a)_1 \xrightarrow{\text{w}} 0$. Then, for the net $\{(A_\alpha)_1 = (B_\alpha)_1 + (C_\alpha)_1\}$, the properties $\{(A_\alpha)_1\} \subset R_1$, $(A_\alpha)_1 \xrightarrow{\text{w}} A_1$ and $\lambda((A_\alpha)_1) = \lambda_0$, $\mu((A_\alpha)_1) = \mu_0$ hold for any $\alpha \in \Lambda$. It is easy to see that the net

$$\{\mathrm{Op}_{\mathrm{II}}(\{\overline{\lambda((A_\alpha)_1)}, \lambda((A_\alpha)_1^*), (A_\alpha)_1\})\}_{\alpha\in\Lambda} \subset R$$

w-converges to the operator $\mathrm{Op}_{\mathrm{II}}(\{\lambda, \mu, A_1\}) \in \mathrm{Op}_{\mathrm{II}}(\widetilde{G}_{\mathrm{II}})$. Thus, taking into account Lemmas 2.2 and 2.3, we get $\overline{R}^{\text{w}} = \mathrm{Op}_{\mathrm{II}}(\widetilde{G}_{\mathrm{II}})$.

(i) is proved similarly by using Lemma 2.4. ∎

Now taking into account (2.1) and (2.2), from Lemma 2.2 and Theorem 2.5 we obtain the following:

COROLLARY 2.6. $\overline{\mathrm{Op}_{\mathrm{I}}(\mathcal{E}_{\mathrm{I}})}^{\text{w}} = \mathrm{Op}_{\mathrm{I}}(\widetilde{G}_{\mathrm{I}})$, $\overline{\mathrm{Op}_{\mathrm{II}}(\mathcal{E}_{\mathrm{II}})}^{\text{w}} = \mathrm{Op}_{\mathrm{II}}(\widetilde{G}_{\mathrm{II}})$.

COROLLARY 2.7. (i) *If R is either $\mathrm{Op}_{\mathrm{I}}(G_{\mathrm{I}})$ or $\mathrm{Op}_{\mathrm{II}}(G_{\mathrm{II}})$, then $R = \overline{R}^{\text{w}}$ if and only if $R_1 = \overline{R_1}^{\text{w}}$;*

(ii) *for $R = \mathrm{Op}_{\mathrm{II}}(\mathcal{E}_{\mathrm{II}})$, w-closedness is equiavalent to $R = \mathbf{0} \oplus R_1$ on $\mathfrak{M} \oplus \mathfrak{H}$ together with $R_1 = \overline{R_1}^{\text{w}}$;*

(iii) *there are no w-closed algebras of the class* $\mathrm{II_a}$.

If we denote $\widehat{G}_{\mathrm{I}} = \mathbb{C}\times\overline{R_1}^{\text{u}}$ and $\widehat{G}_{\mathrm{I}} = \mathbb{C} \times \mathbb{C}\times\overline{R_1}^{\text{u}}$ then, obviously, the following propositions concerning the closure of c.s.o.a. of classes I and II in the uniform topology are true (cf. [10]):

THEOREM 2.8. *Let $R = \mathrm{Op}_{\mathrm{I}}(\mathcal{E}_{\mathrm{I}})$. Then:*

(i) *if the function λ is u-continuous on R_1, then $\overline{R}^{\text{u}} = \mathrm{Op}_{\mathrm{I}}(\widetilde{\mathcal{E}}_{\mathrm{I}})$, where $\{A_1 : w \in \widetilde{\mathcal{E}}_{\mathrm{I}}\} = \overline{R_1}^{\text{u}}$ and $\widetilde{\lambda}$ is the continuous extension of λ to $\overline{R_1}^{\text{u}}$;*

(ii) *if the function λ is u-discontinuos on R_1, then $\overline{R}^{\text{u}} = \mathrm{Op}_{\mathrm{I}}(\widehat{G}_{\mathrm{I}})$; if $R = \mathrm{Op}_{\mathrm{II}}(\mathcal{E}_{\mathrm{II}})$, then $\overline{R}^{\text{u}} = \mathrm{Op}_{\mathrm{II}}(\widehat{G}_{\mathrm{II}})$.*

COROLLARY 2.9. $\overline{\mathrm{Op}_{\mathrm{I}}(G_{\mathrm{I}})}^{\mathrm{u}} = \mathrm{Op}_{\mathrm{I}}(\widehat{G}_{\mathrm{I}})$, $\overline{\mathrm{Op}_{\mathrm{II}}(G_{\mathrm{II}})}^{\mathrm{u}} = \mathrm{Op}_{\mathrm{II}}(\widehat{G}_{\mathrm{II}})$.

COROLLARY 2.10. (i) *If R is one of the algebras* $\mathrm{Op}_{\mathrm{I}}(G_{\mathrm{I}})$, $\mathrm{Op}_{\mathrm{II}}(G_{\mathrm{II}})$, *or* $\mathrm{Op}_{\mathrm{I}}(\mathcal{E}_{\mathrm{I}})$ *then* $R = \overline{R}^{\mathrm{u}}$ *if and only if* $R_1 = \overline{R_1}^{\mathrm{u}}$;

(ii) *there are no* u-*closed algebras of class* $\mathrm{II_a}$.

3. ALGEBRAS OF THE CLASS III

An algebra R is said to be of *class* III if it is not in any of the classes I and II. It is easy to see that for an algebra of class III there exists a unique invariant neutral subspace $\langle\xi_0\rangle$ ([10]). Let $\lambda(A)$ be the e.f. (Hermitian) on R corresponding to ξ_0. If η_0 is some neutral vector from Π_1 such that $(\xi_0, \eta_0) = 1$, $\Pi_1' = \langle\xi_0\rangle + \langle\eta_0\rangle$, $\mathfrak{H} = (\Pi_1')^{\perp}$, $M = \operatorname{Ker}\lambda$, and π is the projection from $\Pi_1 = \Pi_1' \oplus \mathfrak{H}$ on $\mathfrak{H}$, then the c.s.o.a. $M_1 = \pi M \pi$ does not depend on the choice of η_0 up to equivalence, and any operator $A' \in M$ acts on $\Pi_1 = \Pi_1' \oplus \mathfrak{H}$ by the formulas

$$A'\xi_0 = 0, A'\eta_0 = \gamma_{A'}\xi_0 + h_{A'}, A'h = (h, h_{A'^*})\xi_0 + A_1 h, \tag{3.1}$$

where $h \in \mathfrak{H}$, $\gamma_{A'} \in \mathbb{C}$, $h_{A'} = \pi A'\eta_0$ and $A_1 = \pi A' \pi \in M_1$ ([10]). It follows immediately from (3.1) that

$$A_1 h_{B'} = B_1 h_{A'}, \tag{3.2}$$

$$(h_{A'}, h_{B'^*}) = (h_{B'}, h_{A'^*}) \tag{3.3}$$

for any $A', B' \in M$. If E is the main unit of M_1, $\mathbf{1}_{\mathfrak{H}}$ is the identity operator in $\mathfrak{H}$, $G = \mathbf{1}_{\mathfrak{H}} - E$ and $P = E\mathfrak{H}$, $Q = G\mathfrak{H}$, then $\mathfrak{H} = P \oplus Q$, and the system (3.2), (3.3) may be rewritten in the form

$$A_1 p_{B'} = B_1 p_{A'}, \tag{3.4}$$

$$(p_{A'}, p_{B'^*}) + (q_{A'}, q_{B'^*}) = (p_{B'}, p_{A'^*}) + (q_{B'}, q_{A'^*}) \tag{3.5}$$

for any $A', B' \in M$, where $p_{A'}, p_{B'} \in P$ and $q_{A'}, q_{B'} \in Q$ are such that $p_{A'} + q_{A'} = h_{A'}$ and $p_{B'} + q_{B'} = h_{B'}$.

Below we assume that the space Π_1 is separable. Then $\overline{M_1}^{\mathrm{w}}$ is a commutative W^*-algebra which acts in the separable anti-Hilbert space P. Making use of a well-known theorem ([11]) we decompose P into a direct integral of separable anti-Hilbert spaces $\{P(t),\ t \in T\}$ over some compact T with the second axiom of countability and a regular Borel measure σ:

$$P = \int_T^{\oplus} P(t)\,\mathrm{d}\sigma(t)$$

such that the algebra $\overline{M_1}^{\mathrm{w}}$ is equivalent to the algebra of the multiplicators $\{L_{\widetilde{A}(t)} : \widetilde{A}(t) \in L_\infty(T,\sigma)\}$ acting on this direct integral.

It is easy to see that if we substitute the algebra M_1 by some equivalent algebra, then the resulting c.s.a.o. will be equivalent to R. Therefore we do not distinguish between the algebras $\overline{M_1}^{\mathrm{w}}$ and $\{L_{\widetilde{A}(t)} : \widetilde{A}(t) \in L_\infty(T,\sigma)\}$, and instead of $\overline{M_1}^{\mathrm{w}}$ we write simply $L_\infty(T,\sigma)$.

Denote by $\widetilde{A}(t)$ the element of $L_\infty(T,\sigma)$ corresponding to the operator $A_1 \in \overline{M_1}^{\mathrm{w}}$ and write $A_1 = \widetilde{A}(t)$. Let $A(t)$ be a class representative of $\widetilde{A}(t) \in L_\infty(T,\sigma)$. We denote by the symbols u, w etc. the topologies in $L_\infty(T,\sigma)$ corresponding to the appropriate operator topologies in $\overline{M_1}^{\mathrm{w}}$.

LEMMA 3.1. *The condition* (3.4) *is satisfied if and only if there exists a measurable function $\zeta(t)$ defined σ-a.e. on T with values in $P(t)$ such that $p_{A'}(t) = A(t)\zeta(t)$ σ-a.e. on T for any $A' \in M$.*

Proof. For the beginning let us suppose that $A_1 p_{B'} = B_1 p_{A'}$, or, equivalently, $A(t)p_{B'}(t) = B(t)p_{A'}(t)$ σ-a.e. for any $A', B' \in M$. By the lemma of separability ([11]) we can fix the sequence $\{A_n'\}_1^\infty \subset M$ such that $\overline{\{A_n'\}}^{\mathrm{w}} = \overline{M}^{\mathrm{w}}$.

Since $\overline{\{(A_n')_1\}}^{\mathrm{w}} = \overline{\{\pi A_n' \pi\}}^{\mathrm{w}} = \pi\overline{\{A_n'\}}^{\mathrm{w}}\pi = \pi\overline{M}^{\mathrm{w}}\pi = \overline{\pi M \pi}^{\mathrm{w}} = \overline{M_1}^{\mathrm{w}}$ and $\overline{M_1}^{\mathrm{w}}$ is equivalent to $L_\infty(T,\sigma)$, it is easy to see that for the sets $E_n = \{t \in T : A_n(t) \neq 0\}$, $n = 1, 2, \ldots$, the equality $\sigma\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \sigma(T)$ holds. Let N be a subset of T with $\sigma(N) = 0$ and

$$A_k(t)p_{A_m'}(t) = A_m(t)p_{A_k'}(t) \tag{3.6}$$

for any k and m, on $T \setminus N$. Then it follows that (3.6) hold for $K = \bigcup_{n=1}^{\infty} (E_n \setminus N)$ and $\sigma(K) = \sigma(T)$.

Define the function $\zeta(t)$ on K by

$$\zeta(t) = \frac{1}{A_n(t)} p_{A_n'}(t)$$

if $t \in E_n \setminus N$. Then $\zeta(t)$ is well-defined because if $t \in (E_n \cap E_{n}^{'}) \setminus N$, then (3.6) implies $\frac{1}{A_n(t)} p_{A_n'}(t) = \frac{1}{A_{n'}(t)} p_{A_{n'}'}(t)$. Thus, the formula

$$p_{A_n'}(t) = A_n(t)\zeta(t) \tag{3.7}$$

holds on $E_n \setminus N$. Show now that it is true everywhere on K. Indeed, if $t \in K \setminus (E_n \setminus N)$, then there exists a number $n' \neq n$ such that $t \in E_{n'} \setminus N$. Then $A_{n'}(t) \neq 0$ and, in addition, $A_n(t) = 0$, so (3.6) with $k = n, m = n'$ implies $p_{A_n'}(t) = 0$ and again $p_{A_n'}(t) = A_n(t)\zeta(t)$. Therefore the formulas (3.7) hold for any t in K, i.e. σ-a.e. on T.

Let now $A' \in M$ and take a net $\{A_a'\} \subset \{A_n'\}$ with $A_a' \xrightarrow{w} A'$. From (3.1), considering the decomposition $\mathfrak{H} = P \oplus Q$, we obtain

$$(A_a)_1 \xrightarrow{w} A_1 \quad \text{and} \quad p_{A_a'} \to p_{A'} \text{ weakly in } P. \tag{3.8}$$

Since $p_{A_\alpha'}(t) = A_\alpha(t)\zeta(t)$ for every α σ-a.e. on T we get $(p_{A_\alpha'}(t), \overline{A_m(t)}p(t)) = (A_m(t)A_\alpha(t)\zeta(t), p(t)) = (A_\alpha(t)A_m(t)\zeta(t), p(t))$ for arbitrary $p(t) \in P$ and m.

Passing to the limit with respect to α, by (3.8), we get $(p_{A'}(t), \overline{A_m(t)}p(t)) = (A(t)A_m(t)\zeta(t), p(t))$ for any $p(t) \in P$, hence $A_m(t)p_{A'}(t) = A(t)A_m(t)\zeta(t)$ σ-a.e. on T for any m. Since $\sigma(T) = \sigma\Big(\bigcup_{m=1}^{\infty} E_m\Big) = \sigma\Big(\bigcup_{m=1}^{\infty} \{t \in T : A_m(t) \neq 0\}\Big)$ we have

$$p_{A'}(t) = A(t)\zeta(t) \tag{3.9}$$

σ-a.e. on T for any $A' \in M$.

Finally, note that, by construction, the function $\zeta(t)$ is measurable and defined σ-a.e. on T.

The converse assertion of the lemma is obvious. ∎

It is easy to verify that $p_{A'}(t) = A(t)\zeta(t)$, $A' \in M$, σ-a.e. on T implies $(p_{A'}, p_{B'^*}) = (p_{B'}, p_{A'^*})$, therefore

$$(q_{A'}, q_{B'^*}) = (q_{B'}, q_{A'^*}), \tag{3.10}$$

for any A', $B' \in M$.

Finally, denote $\widetilde{L} = \{q_{A'} : A' \in M\}$. As in [10] we can state the following proposition:

LEMMA 3.2. *Condition* (3.10) *is satisfied if and only if for any* $A' \in M$, $q_{A'^*} = Vq_{A'}$ *where* V *is some anti-isometric operator from* $\widetilde{L}$ *onto* $\widetilde{L}$ *such that* $V^2 = 1$.

DEFINITION 3.3. If the main unit E of the algebra M does not coincide with $\mathbf{1}_{\mathfrak{H}}$, i.e. $G \neq 0$, then the algebra R is called *singular*. In the opposite case ($G = 0$), it will be called *regular*. It is clear that two equivalent c.s.o.a. of class III are simultaneously singular or simultaneously regular.

From now on we assume that R is a singular algebra with the unit $\mathbf{1}$. Let $A \in R$. Then $A' = A - \lambda(A)\mathbf{1} \in M$, and from (3.1) together with Lemmas 3.1 and 3.2 and the decomposition $\mathfrak{H} = P \oplus Q$, we get

$$\begin{aligned}
&(A - \lambda(A)\mathbf{1})\xi_0 = 0,\\
&(A - \lambda(A)\mathbf{1})\eta_0 = \gamma_{A'}\xi_0 + A(t)\zeta(t) + q_{A'},\\
&(A - \lambda(A)\mathbf{1})p = (p, p_{A'^*})\xi_0 + A_1 p,\\
&(A - \lambda(A)\mathbf{1})q = (q, Vq_{A'})\xi_0 + A_1 q = (q, Vq_{A'})\xi_0,
\end{aligned}$$

or, denoting $\lambda(A) = \lambda$, $\gamma_{A'} = \gamma$, $q_{A'} = l$,

$$\begin{aligned}
&A\xi_0 = \lambda\xi_0, A\eta_0 = \gamma\xi_0 + \lambda\eta_0 + A(t)\zeta(t) + l,\\
&A(t) = \int_T A(t)(p(t), \zeta(t))\, d\sigma \cdot \xi_0 + (A(t) + \lambda)p(t),\\
&Aq = (q, Vl)\xi_0 + \lambda q,
\end{aligned} \tag{3.11}$$

where $p \in P$, $q \in Q$.

Let us denote $d\mu(t) = (\zeta(t), \zeta(t))d\sigma(t)$, $M(T) = L_\infty(T, \sigma) \cap L_2(T, \mu)$, $L = \overline{\widetilde{L}}$, $G_S = \mathbb{C} \times \mathbb{C} \times L \times M(T)$ and $\mathcal{E}_S = \{\omega = \{\lambda, \mu, l, A(t)\} : A \in R\}$, where $\gamma \in \mathbb{C}$, $l \in L$ and $A(t) \in M(T)$ (it follows from the definition of $M(T)$ that $M_1 \subset M(T)$) are the components which correspond to $A \in R$ in (3.11). In general, the measure μ is infinite on T. We call the set $\mathcal{E}_S$ *a definable manifold* of the algebra R.

It is easy to verify that if in G_S one considers the component-wise addition and scalar multiplication as well as the following multiplication and involution

$$\begin{aligned}
&\{\lambda_1, \gamma_1, l_1, A_1(t)\} \cdot \{\lambda_2, \gamma_2, l_2, A_2(t)\}\\
&\quad = \Big\{\lambda_1\lambda_2, \lambda_1\gamma_2 + \lambda_2\gamma_1 + \int_T A_1(t)A_2(t)\, d\mu + (l_2, Vl_1),\\
&\qquad \lambda_1 l_2 + \lambda_2 l_1, \lambda_1 A_2(t) + \lambda_2 A_1(t) + A_1(t)A_2(t)\Big\},\\
&\{\lambda, \gamma, l, A(t)\}^* = \{\overline{\lambda}, \overline{\gamma}, Vl, \overline{A(t)}\},
\end{aligned} \tag{3.12}$$

then the mapping $\mathrm{Op}_S : G_S \to B(\Pi_1)$ with $\Pi_1 = \Pi_1' \oplus P \oplus Q$ and the operator $A = \mathrm{Op}_S(\omega)$, $\omega = \{\lambda, \gamma, l, A(t)\} \in G_S$, acting in Π_1 by (3.11) is an injective $*$-homomorphism from the algebra $\mathcal{E}_S$ into $B(\Pi_1)$, and $R = \mathrm{Op}_S(\mathcal{E}_S)$. Consequently, $\mathcal{E}_S$ is a $*$-subalgebra in G_S with the unit $\{1, 0, 0, 0\}$. Note also that

$$
(3.13) \qquad \begin{aligned} &\text{(i) } \{\gamma : \omega \in \mathcal{E}_S\} = \mathbb{C} \text{ (see [10]);} \\ &\text{(ii) } \{l : \omega \in \mathcal{E}_S\} = L; \\ &\text{(iii) if } M_1 = \{A(t) : \omega \in \mathcal{E}_S\}, \text{ then } \overline{M_1}^{\mathrm{w}} = L_\infty(T, \sigma). \end{aligned}
$$

Let now $R = \mathrm{Op}_S(\mathcal{E}_S)$ for some $*$-subalgebra $\mathcal{E}_S \subset G_S$ such that the conditions (3.13) hold. Let us show that then $\overline{R}^{\mathrm{w}} = \mathrm{Op}_S(G_S)$ for the corresponding manifold G_S. First, the following can be easily verified:

LEMMA 3.4. *The net* $\{\mathrm{Op}_S(\{\lambda_\nu, \gamma_\nu, l_\nu, A_\nu(t)\})\}_{\nu \in \Gamma} \subset \mathrm{Op}_S(G_S)$ *converges weakly to the operator* $\mathrm{Op}_S(\{\lambda, \gamma, l, A(t)\}) \in \mathrm{Op}_S(G_S)$ *if and only if the following conditions hold:*

(i) $\lambda_\nu \to \lambda$, $\gamma_\nu \to \gamma$;
(ii) $l_\nu \to l$ *weakly in* Q;
(iii) $A_\nu(t) \xrightarrow{w} A(t)$;
(iv) $\int_T A_\nu(t)(p(t), \zeta(t))\, d\sigma \to \int_T A(t)(p(t), \zeta(t))\, d\sigma$ *for any* $p(t) \in P$.

LEMMA 3.5. $\overline{\mathrm{Op}_S(G_S)}^{\mathrm{w}} = \mathrm{Op}_S(G_S)$.

Proof. Let $\mathrm{Op}_S(G_S) \ni A_\nu = \mathrm{Op}_S(\{\lambda_\nu, \gamma_\nu, l_\nu, A_\nu(t)\}) \xrightarrow{\mathrm{w}} A \in B(\Pi_1)$ and let $A\xi_0 = \alpha\xi_0 + \beta\eta_0 + p(t) + q$, where $p(t) \in P$, $q \in Q$. Let us show that $A \in \mathrm{Op}_S(G_S)$. Without loss of generallity assume that $A = A^*$ (since the involution is continuous in the weak topology and $\overline{\mathrm{Op}_S(G_S)}^{\mathrm{w}}$ is a $*$-algebra). Using (3.11) we get $\beta = (A\xi_0, \xi_0) = \lim_\nu (A_\nu \xi_0, \xi_0) = 0$, $(p(t), p'(t)) = (A\xi_0, p'(t)) = \lim_\nu (A_\nu \xi_0, p'(t)) = 0$ for any $p(t) \in P$, hence $p(t) = 0$. Analogously, we get $q = 0$. Therefore $A\langle \xi_0 \rangle \subset \langle \xi_0 \rangle$, and the corresponding eigenvalue is

$$
(3.14) \qquad \lambda = \alpha = (A\xi_0, \eta_0) = \lim_\nu (A_\nu \xi_0, \eta_0) = \lim_\nu \lambda_\nu .
$$

Since $A = A^*$ it follows from $A\langle \xi_0 \rangle \subset \langle \xi_0 \rangle$ that $A\langle \xi_0 \rangle^\perp \subset \langle \xi_0 \rangle^\perp$. But, since $\langle \xi_0 \rangle^\perp = \langle \xi_0 \rangle \oplus \mathfrak{H}$ with $\mathfrak{H} = P \oplus Q$, for any $h \in \mathfrak{H}$, there exists a number τ such that $Ah = \tau\xi_0 + A_1 h$, where $A_1 = \pi A \pi$, π is the projection of Π_1 onto $\mathfrak{H}$. Therefore, for $p = p(t) \in P$, by (3.11) and (3.14), we get

$$
\begin{aligned} ((A_1 - \lambda)p, p) = ((A - \lambda)p, p) &= \lim_\nu (A_\nu p, p) - \lim_\nu \lambda_\nu (p, p) \\ &= \lim_\nu ((A_\nu - \lambda_\nu)p, p) = \lim_\nu (A_\nu(t)p(t), p(t)). \end{aligned}
$$

Now $\{A_\nu(t)\}_{\nu\in\Gamma} \subset L_\infty(T,\sigma)$ implies $A_1 - \lambda \in L_\infty(T,\sigma)$.

If we denote $A_1 - \lambda$ by $A(t)$, then

$$A_\nu(t) \xrightarrow{\text{w}} A(t). \tag{3.15}$$

Let now $A\eta_0 = \gamma\xi_0 + \delta\eta_0 + p_A(t) + l$, where $p_A(t) \in P$, $l \in Q$. Then $\delta = (A\eta_0, \xi_0) = \lim_\nu (A_\nu\eta_0, \xi_0) = \lim_\nu \lambda_\nu = \lambda$. Analogously,

$$\gamma = \lim_\nu \gamma_\nu, \tag{3.16}$$

and for any $p(t) \in P$ we have

$$(A_\nu(t)\zeta(t), p(t)) \to (p_A(t), p(t)). \tag{3.17}$$

Moreover, for any $q \in Q$ we have $(l,q) = (A\eta_0, q) = \lim_\nu (A_\nu\eta_0, q) = \lim_\nu (l_\nu, q)$, so $l \in L$ and

$$l_\nu \to l \text{ weakly in } Q. \tag{3.18}$$

Prove that $p_A(t) = A(t)\zeta(t)$ σ-a.e. on T. From this, in particular, it would follow that $A(t) \in M(T)$. Let $B(t) \in M(T)$, that is $B(t)\zeta(t) \in P$. Then, by (3.15), $(A_\nu(t)B(t)\zeta(t), p(t)) \to (A(t)B(t)\zeta(t), p(t))$ for any $p(t) \in P$. Now, by (3.17) we have $(A_\nu(t)B(t)\zeta(t), p(t)) = (A_\nu(t)\zeta(t), \overline{B(t)}p(t)) \to (p_A(t), \overline{B(t)}p(t)) = (B(t)p_A(t), p(t))$.

Thus, $(A(t)B(t)\zeta(t), p(t)) = (B(t)p_A(t), p(t))$ for any vector-function $p(t) \in P$, hence $A(t)B(t)\zeta(t) = B(t)p_A(t)$ σ-a.e. for any $B(t) \in M(T)$. Since $\overline{M(T)}^{\text{w}} = L_\infty(T,\sigma)$, as in the proof of Lemma 3.1 we obtain $p_A(t) = A(t)\zeta(t)$ σ-a.e. on T.

Now we consider the operator $B = \mathrm{Op}_S(\{\lambda, \gamma, l, A(t)\}) \in \mathrm{Op}_S(G_S)$. Using (3.14)–(3.18) and Lemma 3.4 we get $A_\nu \xrightarrow{\text{w}} B$. Hence $A = B \in \mathrm{Op}_S(G_S)$ and, consequently, $\overline{\mathrm{Op}_S(G_S)}^{\text{w}} = \mathrm{Op}_S(G_S)$. ∎

If $\mathcal{N}$ is an algebra, then by $\mathcal{N}^n$ we shall denote the linear envelope of all products of n elements of $\mathcal{N}$. We need the following known fact.

LEMMA 3.6. *Let $\mathcal{N}$ be a subalgebra of a W^*-algebra $\mathcal{M}$, such that $\overline{\mathcal{N}}^{\text{w}} = \mathcal{M}$. Then $\overline{\mathcal{N}^n}^{\text{w}} = \mathcal{M}$ for any n.*

Proof. It is easy to see that $\overline{\mathcal{N}^n}^{\text{w}}$ is a two-sided, w-closed and symmetric ideal in $\mathcal{M}$. Hence $\overline{\mathcal{N}^n}^{\text{w}} = e\mathcal{M}$ for some central projection $e \in \mathcal{M}$. Let $g = 1 - e$ and $a = a^* \in \mathcal{N}$. From $a^n \in \mathcal{N}^n$ it follows that $ga^n = 0$ or $(ga)^n = 0$. Hence $0 = (ga)^n(ga)^n = ((ga)^*ga)^n$, thus $0 = (ga)^*ga = (ga)^2$ and $ga = 0$. Consequently, $ga = 0$ for any $a = a^* \in \mathcal{N}$. From the symmetry of $\mathcal{N}$ it follows that $ga = 0$ for any $a \in \mathcal{M}$. Further, since $g \in \mathcal{M}$ and $\mathcal{M} = \overline{\mathcal{N}}^{\text{w}}$, we have $g = 0$. Thus, $\overline{\mathcal{N}^n}^{\text{w}} = \mathcal{M}$. ∎

Since, by (3.13), for $M_1 = \{A(t) : \omega \in \mathcal{E}_S\}$, we get $\overline{M_1}^{\mathrm{w}} = L_\infty(T,\sigma)$, the next result follows.

COROLLARY 3.7. $\overline{M_1}^{\mathrm{w}} = L_\infty(T,\sigma)$ *for any* n.

REMARK 3.8. Below we assume that:
(i) $\zeta(t) \neq 0$ σ-a.e. on T;
(ii) $\int_T \mathrm{d}\mu = -\infty$.

After the proof of the main result of the section it will be shown that we can impose these restrictions without loss of generality.

LEMMA 3.9. *Let $\mathcal{N}$ be a* w*-dense subalgebra in $M(T)$. Then*

$$\{B(t)\zeta(t) : B(t) \in M(T)\} \subset \overline{\{A(t)\zeta(t) : A(t) \in \mathcal{N}\}}^{\|\cdot\|}.$$

Proof. Let $B(t) \in M(T)$. Consider a net $\{A_\alpha(t)\}_{\alpha\in\Gamma} \subset \mathcal{N}$ such that $A_\alpha(t) \xrightarrow{\mathrm{w}} B(t)$. Now take some net $\{C_\beta\}_{\beta\in\Lambda} \subset \mathcal{N}$ w-converging to $\mathbf{1} \in L_\infty(T,\sigma)$ (such a net exists since $L_\infty(T,\sigma) = \overline{M_1^n}^{\mathrm{w}} \subset \overline{M(T)}^{\mathrm{w}} = \overline{\mathcal{N}}^{\mathrm{w}}$). Then for any $\beta \in \Lambda$ and $p(t) \in P$ we obtain $(A_\alpha(t)C_\beta(t)\zeta(t), p(t)) \to (B(t)C_\beta(t)\zeta(t), p(t)) = (C_\beta(t)B(t)\zeta(t), p(t))$.

On the other hand $(C_\beta(t)B(t)\zeta(t), p(t)) \to (B(t)\zeta(t), p(t))$, so the point $B(t)\zeta(t)$ belongs to the weak closure of the set $\{A(t)\zeta(t) : A(t) \in \mathcal{N}\}$. Note that this set is convex and hence its weak closure coincides with the closure in the norm of the space P. ∎

Denote now M_n ($n \geqslant 2$) the set of all such $A(t) \in M(T)$ there exists a sequence $\{A_k(t)\}_1^\infty \subset M_1^n$ satisfying:

(i) $|A_k(t)| \leqslant C(T)$ σ-a.e. on T for each k and some function $C(t) \subset L_\infty(T,\sigma)$ with $\int_T C(t)\,\mathrm{d}\mu > -\infty$;
(ii) $A_k(t) \xrightarrow{\mathrm{s}} A(t)$ and
(iii) $\|A_k(t)\zeta(t) - A(t)\zeta(t)\|_P \to 0$.

Then, obviously,

$$M_1^n \subset M_n \subset M(T) \subset L_\infty(T,\sigma), \quad n \geqslant 2. \tag{3.19}$$

LEMMA 3.10. *M_n is a $*$-ideal in $L_\infty(T,\sigma)$.*

Proof. Let $A(t) \in M_n$, $B(t) \in L_\infty(T,\sigma)$, $\|B(t)\|_\infty \leqslant 1$. Since, by Corollary 3.7, $\overline{M_1^n}^{\mathrm{w}} = L_\infty(T,\sigma)$, using the Kaplansky density theorem, we find a sequence $\{B_k(t)\}_1^\infty \subset M_1^n$ such that $\|B_k(t)\|_\infty \leqslant 1$ for any k and $B_k(t) \xrightarrow{\mathrm{s}} B(t)$. Further, we fix a sequence $\{A_k(t)\}_1^\infty \subset M_1^n$ with a majorant $C(t)$ integrable with respect to the measure μ such that $A_k(t) \xrightarrow{s} A(t)$ and $\|A_k(t)\zeta(t) - A(t)\zeta(t)\|_P \to 0$.

Then $\{A_k(t)B_k(t)\}_1^\infty \subset M_1^n$, $A_k(t)B_k(t) \xrightarrow{s} A(t)B(t)$ and $|A_k(t)B_k(t)| \leqslant |A_k(t)|\ \|B_k(t)\|_\infty \leqslant C(t)$ σ-a.e. on T for any k.

Now,

$$\begin{aligned}
&\|A_k(t)B_k(t)\zeta(t) - A(t)B(t)\zeta(t)\|_P \\
&\quad\leqslant \|A_k(t)B_k(t)\zeta(t) - A(t)B_k(t)\zeta(t)\|_P + \|A(t)B_k(t)\zeta(t) - A(t)B(t)\zeta(t)\|_P \\
&\quad\leqslant \|B_k(t)\|_\infty \|A_k(t)\zeta(t) - A(t)\zeta(t)\|_P + [-(|B_k(t) - B(t)|^2 A(t)\zeta(t)), A(t)\zeta(t)]^{\frac{1}{2}}
\end{aligned}$$

converges to 0 because the both summands converge to zero: the first by assumption, and the second one because $B_k(t) \xrightarrow{s} B(t)$ implies $|B_k(t) - B(t)|^2 \xrightarrow{w} 0$. Thus, $A(t)B(t) \in M_n$ for $B(t) \in L_\infty(T, \sigma)$ with $\|B_k(t)\|_\infty \leqslant 1$ which means that M_n is an ideal in $L_\infty(T, \sigma)$. The symmetry is immediately verified. ∎

Let us denote $\mathcal{E}_S^0 = \{\omega \in \mathcal{E}_S : \lambda = 0\}$ and $M = \mathrm{Op}_S(\mathcal{E}_S^0)$.

Since by assumption the unit of G_S, which is equivalent to $\{1, 0, 0, 0\}$, belongs to $\mathcal{E}_S$ and $\{A(t) : \omega \in \mathcal{E}_S\} = M_1$ we have $\{A(t) : \omega \in \mathcal{E}_S^0\} = M_1$. Considering $M^3 = \mathrm{Op}_S((\mathcal{E}_S^0)^3)$, by the formulas (3.12) we get

$$M^3 = \mathrm{Op}_S\left(\left\{\omega = \left\{0, \int_T A(t)\,\mathrm{d}\mu, 0, A(t)\right\} : A(t) \in M_1^3\right\}\right). \tag{3.20}$$

In addition, we have

$$M^3 \subset M \subset R = \mathrm{Op}_S(\mathcal{E}_S) = \mathrm{Op}_S(G_S) = \overline{\mathrm{Op}_S(G_S)}^{\mathrm{w}}. \tag{3.21}$$

Consider the $*$-algebras

$$M^{(n)} = \mathrm{Op}_S\omega = \left(\left\{\omega = \left\{0, \int_T A(t)\,\mathrm{d}\mu, 0, A(t)\right\} : A(t) \in M_n\right\}\right), \quad n \geqslant 2.$$

Note that $\int_T |A_k(t) - A(t)|^2\,\mathrm{d}\mu \to 0$ (see the definition of M_n) implies that the sequence $\{A_k(t)\}$ converges to $A(t)$ in the measure μ. Then existence of a μ-integrable majorant from $L_\infty(T, \sigma)$ for $\{A_k(t)\}$ implies, by Lebesgue Theorem, that $A(t)$ is μ-integrable on T and $\int_T |A_k(t) - A(t)|\,\mathrm{d}\mu \to 0$. In particular, $\int_T A_k(t)\,\mathrm{d}\mu \to \int_T A(t)\,\mathrm{d}\mu$.

Since $M_1^3 \subset M_3$ (see (3.19)) the inclusion $M^3 \subset M^{(3)}$ holds. Making use of Lemma 3.4 one immediately verifies that

$$M^{(3)} \subset \overline{M_3}^{\mathrm{w}}. \tag{3.22}$$

LEMMA 3.11. *If* $R = \mathrm{Op}_S(\mathcal{E}_S)$, *then* $\overline{R}^{\mathrm{w}} = \mathrm{Op}_S(G_S)$.

Proof. Following [6], we shall prove this in several steps.

Step I. Let us show that $\overline{M^{(3)}}^{\mathrm{w}} \ni \mathrm{Op}(\{0,1,0,0\})$. Since, by Lemma 3.10, $\overline{M_3}^{\mathrm{w}}$ is a $*$-ideal in $L_\infty(T,\sigma)$, there exists a maximal family of pairwise orthogonal projections $\{\mathcal{X}_{E_n}(t)\}_1^\infty \subset M_3$. Let $\mathcal{X}_E = \sup \mathcal{X}_E(t)$. If $\mathcal{X}_E \neq 1 = \mathcal{X}_{T_n}$ then $\mathcal{X}_G \neq 0$, where $G = T \setminus E$, that is $\sigma(G) > 0$. Suppose that there exists $A(t) \in M_3$ such that $A(t)\mathcal{X}_G(t) \neq 0$. Then $|A(t)|^2\mathcal{X}_G(t) \neq 0$, so there exist a measurable set $G' \subset G$ and a number $\varepsilon > 0$ with $\sigma(G') > 0$ and $|A(t)|^2\mathcal{X}_{G'}(t) \geqslant \varepsilon\mathcal{X}_{G'}(t)$. Consider the function

$$f(t) = \begin{cases} \frac{1}{|A(t)|^2} & t \in G' \\ 0 & t \in T \setminus G'. \end{cases}$$

Then $f(t) \in L_\infty(T,\sigma)$ and since M_3 is an ideal in $L_\infty(T,\sigma)$, and by symmetry of M_3 it follows that $|A(t)|^2 \in M_3$, we have $\mathcal{X}_{G'}(t) = f(t)|A(t)|^2 \in M_3$. Because we have $\sigma(G' \cap E) = 0$ this contradicts the maximality of the family $\{\mathcal{X}_{E_n}(t)\}$. Consequently, $A(t)\mathcal{X}_G(t) = 0$ σ-a.e. for any $A(t) \in M_3$ and therefore, by $\overline{M_3}^{\mathrm{w}} = L_\infty(T,\sigma)$ (see (3.19) and Corollary 3.7), $\mathcal{X}_G(t) = 0$, i.e. $\sigma\Big(\bigcup\limits_{n=1}^{\infty} E_n\Big) = \sigma(T)$ or $\mu\Big(\bigcup\limits_{n=1}^{\infty} E_n\Big) = \sigma(T)$ by the definition of the measure μ.

According to the assumption (Remark 3.8), the measure μ is infinite on T, therefore $-\infty = \mu(T) = \mu\Big(\bigcup\limits_{n=1}^{\infty} E_n\Big) = \sum\limits_{n=1}^{\infty} \mu(E_n)$ and we can construct the following sequence of pairwise orthogonal projections in M_3 :

$$\mathcal{X}_{G_1}(t) = \sum_{n=1}^{\infty} \mathcal{X}_{E_n}(t), \quad \mu(G_1) < -2,$$

$$\mathcal{X}_{G_2}(t) = \sum_{n=n_1+1}^{n_1} \mathcal{X}_{E_n}(t), \quad \mu(G_2) < -2^2,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\mathcal{X}_{G_k}(t) = \sum_{n=n_{k-1}+1}^{n_k} \mathcal{X}_{E_n}(t), \quad \mu(G_k) < -2^k, \dots .$$

Since $\{\mathcal{X}_{E_n}(t)\}_1^\infty \subset M(T)$, we have $\mu(G_k) > -\infty$ for any k. Further, for any

n, we put $f_n(t) = \sum\limits_{k=1}^{\infty} \frac{1}{\mu(G_k)} \mathcal{X}_{G_k}(t)$. Then $\{f_n(t)\}_1^\infty \subset M_3$, $\|f_n(t)\|_\infty \leqslant 1$ and

$$\int_T f_n(t)\,\mathrm{d}\mu = \int_T \sum_{k=1}^{\infty} \frac{1}{\mu(G_k)} \mathcal{X}_{G_k}(t)\mathrm{d}\mu$$
$$= \sum_{k=1}^{\infty} \int_{G_k} \frac{1}{\mu(G_k)}\,\mathrm{d}\mu = n, \quad n = 1, 2, \dots .$$

Besides,

$$\int_T f_n^2(t)\,\mathrm{d}\mu = \int_T \sum_{k=1}^{\infty} \frac{1}{\mu(G_k)^2} \mathcal{X}_{G_k}(t)\mathrm{d}\mu = \sum_{k=1}^{\infty} \frac{1}{\mu(G_k)} > -\sum_{k=1}^{\infty} \frac{1}{2^k} = -1.$$

Now we consider the sequence

$$\left\{ A_n = \mathrm{Op}_S\left(\left\{ 0, \int_T \frac{f_n(t)}{\int_T f_n(t)\,\mathrm{d}\mu}\,\mathrm{d}\mu, 0, \frac{f_n(t)}{\int_T f_n(t)\mathrm{d}\mu} \right\} \right) \right\}_1^\infty \subset M^{(3)}.$$

According to the Lemma 3.4, $A_n \xrightarrow{\mathrm{w}} \mathrm{Op}_S(\{0,1,0,0\})$, that is

$$\mathrm{Op}_S(\{0,1,0,0\}) \in \overline{M^{(3)}}^{\mathrm{w}}.$$

Step II. We show now that $\mathrm{Op}_S(\{0,0,0,A(t)\}) \in \overline{M^{(3)}}^{\mathrm{w}}$ for any $A(t) \in M(T)$. Let $A(t) \in M(t)$. Applying (3.19) and Corollary 3.7, by Lemma 3.9 we find a sequence $\{A_n(t)\}_1^\infty \subset M_3$ such that $A_n(t)\zeta(t) \to A(t)\zeta(t)$.

Denote $G_n = \{t \in T : |A(t) - A_n(t)| \leqslant 1\}$, $F_n = T \setminus G_n$; $n = 1, 2, \dots$. Then

$$0 \geqslant \int_{F_n} \mathrm{d}\mu \geqslant \int_{F_n} |A(t) - A_n(t)|^2\,\mathrm{d}\mu \geqslant \int_T |A(t) - A_n(t)|^2\,\mathrm{d}\mu$$
$$= -\|A(t)\zeta(t) - A_n(t)\zeta(t)\|_P^2 \to 0,$$

and so

$$\int_{F_n} \mathrm{d}\mu \to 0. \tag{3.23}$$

If we exclude from our consideration the trivial case $\|A_n(t)\|_\infty = 0$, then we can assume $\|A_n(t)\|_\infty \neq 0$ for $n = 1, 2, \dots$ and denote

$$B_n(t) = A_n(t)\mathcal{X}_{G_n} + \frac{A_n(t)}{\|A_n(t)\|_\infty} \mathcal{X}_{F_n}(t); \quad n = 1, 2, \dots .$$

By Lemma 3.10, $\{B_n(t)\}_\infty \subset M_3$. Further, we have

$$|B_n(t)| = |A_n(t)|\mathcal{X}_{G_n}(t) + \mathcal{X}_{F_n}(t) \leqslant (1 + |A(t)|)\mathcal{X}_{G_n}(t) + \mathcal{X}_{F_n}(t) \leqslant 1 + \|A_n(t)\|_\infty,$$

i.e.

$$\|B_n(t)\|_\infty \leqslant 1 + \|A_n(t)\|_\infty \tag{3.24}$$

for any n. Also,

$$\begin{aligned}
&\|A(t)\zeta(t) - B_n(t)\zeta(t)\|_P \\
&\leqslant \|(A(t) - A_n(t))\mathcal{X}_{G_n}(t)\zeta(t)\|_P + \left\| \left(A(t) - \frac{A_n(t)}{\|A_n(t)\|_\infty} \right) \mathcal{X}_{F_n}(t)\zeta(t) \right\|_P \\
&\leqslant \|A(t)\zeta(t) - A_n(t)\zeta(t)\|_P + \|A(t)\mathcal{X}_{F_n}(t)\zeta(t)\|_P + \left\| \frac{A_n(t)}{\|A_n(t)\|_\infty} \mathcal{X}_{F_n}(t)\zeta(t) \right\|_P .
\end{aligned}$$

Here all the summands converge to zero: the first one by the choice of $\{A_n(t)\}$;

$$\|A(t)\mathcal{X}_{F_n}(t)\zeta(t)\|_P^2 = -\int_{F_n} A(t)|^2 \mathrm{d}\mu \leqslant \|A(t)\|_\infty^2 \int_{F_n} \mathrm{d}\mu$$

converges to 0 by (3.23), and

$$\left\| \frac{A_n(t)}{\|A_n(t)\|_\infty} \mathcal{X}_{F_n}(t)\zeta(t) \right\|_P^2 \leqslant -\int_{F_n} \mathrm{d}\mu$$

converges to 0 also by (3.23).

Therefore $\|A(t)\zeta(t) - B_n(t)\zeta(t)\|_P \to 0$. Using the separability of the space P and the boundedness of the sequence $\{B_n(t)\}$ (see (3.24)) we can choose a subsequence $\{B_{n_k}(t)\} \subset \{B_n(t)\}$ such that $B_{n_k}(t) \xrightarrow{\text{w}} B(t) \in L_\infty(T, \sigma)$. In this situation we can assume that $B_n(t) \xrightarrow{\text{w}} B(t) \in L_\infty(T, \sigma)$. Moreover, we have the convergence $B(t)\zeta(t) \to A(t)\zeta(t)$ in the norm of the space P. Hence, for any function $C(t) \in M(t)$, we get $(B_n(t)C(t)\zeta(t), p(t)) \to (B(t)C(t)\zeta(t), p(t))$ and also $(B_n(t)C(t)\zeta(t), p(t)) = (C(t)B_n(t)\zeta(t), p(t)) = (B_n(t)\zeta(t), \overline{C(t)}p(t)) \to (A(t)\zeta(t), \overline{C(t)}p(t))$ for each vector-function $p(t) \in P$. Therefore $B(t)C(t)\zeta(t) = C(t)A(t)\zeta(t)$ for any $C(t) \in M(T)$. Since by the assumption (see Remark 3.8) $\zeta(t) \neq 0$ σ-a.e. on T it follows that $B(t)C(t) = C(t)A(t)$ σ-a.e. on T for every $C(t) \in M(T)$, hence, noting that $\overline{M(T)}^{\text{w}} = L_\infty(T, \sigma)$, we get $A(t) = B(t)$ σ-a.e.

Thus, we proved that for each $A(t) \in M(T)$ there exists a sequence $\{B_n(t)\} \subset M_3$ such that $B_n(t) \xrightarrow{\text{w}} A(t)$ and $\|B_n(t)\zeta(t) - A(t)\zeta(t)\|_P \to 0$.

Further, we introduce the sequence

$$\Big\{B_n = \mathrm{Op}\Big(\Big\{0, \int\limits_T B_n(t)\,\mathrm{d}\mu, 0, B_n(t)\Big\}\Big)\Big\}_1^\infty \subset M^{(3)}.$$

It follows from Step I that $\{A_n = B_n - B_n^{'} = \mathrm{Op}_S(\{0,0,0,B_n(t)\})\}_1^\infty \subset \overline{M^{(3)}}^{\mathrm{w}}$ and, by Lemma 3.4, $A_n \xrightarrow{\mathrm{w}} \mathrm{Op}_S(\{0,0,0,A(t)\})$. Therefore

$$\mathrm{Op}_S(\{0,0,0,A(t)\}) \subset \overline{M^{(3)}}^{\mathrm{w}}$$

for any $A(t) \in M(T)$.

Step III. By (3.22) from Steps I and II it follows that for every $\gamma \in \mathbb{C}$ and $A(t) \in M(T)$ the operators $\mathrm{Op}_S(\{0,\gamma,0,0\})$ and $\mathrm{Op}_S(\{0,0,0,A(t)\})$ belong to $\overline{M^{(3)}}^{\mathrm{w}}$ and hence to $\overline{R}^{\mathrm{w}}$ (see (3.21)). Let us show now that for any $l \in L$ the operator $\mathrm{Op}_S(\{0,0,l,0\})$ belongs to $\overline{R}^{\mathrm{w}}$. It follows from (3.13) that for all $l \in L$ there exists a sequence $\{l_n\}_1^\infty \subset \{l : \omega \in \mathcal{E}_S\}$ such that $l_n \to l$ weakly in Q. Given some $\{\lambda_n\}_1^\infty \subset \mathbb{C}$, $\{\gamma_n\}_1^\infty \subset \mathbb{C}$, $\{A_n(t)\}_1^\infty \subset M(T)$ we have $\{\{\lambda_n,\gamma_n,l_n,A_n(t)\}\}_1^\infty \subset \mathcal{E}_S$, therefore $\{\mathrm{Op}_S(\{\lambda_n,\gamma_n,l_n,A_n(t)\})\}_1^\infty \subset R \subset \overline{R}^{\mathrm{w}}$. Since by the assumption the unit $\mathrm{Op}_S(\{1,0,0,0\}) \in R \subset \overline{R}^{\mathrm{w}}$, $\{\mathrm{Op}_S(\{\lambda_n,0,0,0\})\}_1^\infty \subset \overline{R}^{\mathrm{w}}$, and having also $\{\mathrm{Op}_S(\{0,\gamma_n,0,0\})\}_1^\infty \subset \overline{R}^{\mathrm{w}}$ and $\{\mathrm{Op}_S(\{0,0,0,A_n(t)\})\}_1^\infty \subset \overline{R}^{\mathrm{w}}$, we assert by linearity that $\{\mathrm{Op}_S(\{0,0,l_n,0\})\}_1^\infty \subset \overline{R}^{\mathrm{w}}$, and, by Lemma 3.4, $\mathrm{Op}_S(\{0,0,l,0\}) = \mathrm{w}\text{-}\lim\limits_n \mathrm{Op}_S(\{0,0,l_n,0\}) \in \overline{R}^{\mathrm{w}}$. Therefore, for any $\lambda,\gamma \in \mathbb{C}$, $l \in L$ and $A(t) \in M(T)$, we have that $\mathrm{Op}_S(\{\lambda,0,0,0\})$, $\mathrm{Op}_S(\{0,\gamma,0,0\})$, $\mathrm{Op}_S(\{0,0,l,0\})$ and $\mathrm{Op}_S(\{0,0,0,\ A(t)\})$ belong to $\overline{R}^{\mathrm{w}}$, and so, by Lemma 3.4, we conclude that $\overline{R}^{\mathrm{w}} = \mathrm{Op}_S(G_S)$, where $G_S = \mathbb{C}\times\mathbb{C}\times L\times M(T)$. ∎

Now we formulate the fundamental result of this section. Let $\Pi_1^{'}$ be a two dimensional Π_1-space with the biorthogonal base $\{\xi_0,\eta_0\}$, T a compact with the second countability axiom, σ a regular Borel measure on T, $\{P(t), t \in T\}$ a family of separable anti-Hilbert spaces defined σ-a.e. on T, $\zeta(t)$ a measurable vector-function with values in $P(t)$, Q a non-zero separable anti-Hilbert space, L a closed subspace in Q and V an anti-isometric operator from L onto L such that $V^2 = 1$. Denote

$$P = \int\limits_T \oplus P(t)\,\mathrm{d}\sigma, \quad \Pi_1 = \Pi_1^{'} \oplus P \oplus Q,$$

$$\mathrm{d}\mu(t) = (\zeta(t),\zeta(t))\mathrm{d}\zeta(t),$$

$$M(T) = L_\infty(T,\sigma) \cap L_2(T,\mu)$$

and

$$G_S = \mathbb{C}\times\mathbb{C}\times L\times M(T).$$

The addition and scalar multiplication in $G_S = \{\omega = \{\lambda, \gamma, l; A(t)\}\}$ are defined to be componentwise while multiplication and involution — by (3.12). Let us define the mapping $\mathrm{Op}_S : G_S \to B(\Pi_1)$ such that, for any $\omega = \{\lambda, \gamma, l, A(t)\} \in G_S$, the operator $A = A(\omega) \in B(\Pi_1)$ acts in $\Pi_1 = \Pi_1' \oplus P \oplus Q$ by (3.11). Let $\mathcal{E}_S$ be a $*$-algebra in G_S with a unit such that the conditions (3.13) are satisfied.

THEOREM 3.12. *R is a singular c.s.o.a. with a unit in a separable space Π_1 iff $R = \mathrm{Op}_S(\mathcal{E}_S)$ for some manifold $\mathcal{E}_S$. Moreover, $\overline{R}^{\mathrm{w}} = \mathrm{Op}_S(G_S)$ and $\overline{R}^{\mathrm{u}} = \mathrm{Op}_S(G_{\mathrm{su}})$, where $G_{\mathrm{su}} = \mathbb{C} \times \mathbb{C} \times L \times (M(T) \cap \overline{M_1}^{\mathrm{u}}) \subset G_S$.*

Proof. We have already shown (see Lemmas 3.1, 3.2 and further on), that if R is a singular c.s.o.a. with unit in a separable space Π_1, then for some $*$-algebra $\mathcal{E}_S \subset G_S$ with unit, the equality $R = \mathrm{Op}_S(\mathcal{E}_S)$ holds and, in addition, for $\mathcal{E}_S$, the conditions (3.13) take place. Moreover, as it was previously obtained, for any $*$-algebra $\mathcal{E}_S \subset G_S$ with unit and satisfying the conditions (3.13), we have $\overline{\mathrm{Op}_S(\mathcal{E}_S)}^{\mathrm{w}} = \mathrm{Op}_S(G_S)$. Further, exactly by the scheme of [6] it can be proved that $\overline{\mathrm{Op}_S(\mathcal{E}_S)}^{\mathrm{u}} = \mathrm{Op}_S(G_{\mathrm{su}})$. Thus it remains only to prove that if $R = \mathrm{Op}_S(\mathcal{E}_S)$ for some manifold $\mathcal{E}_S$ described above, then R is a singular c.s.o.a. with unit and acting in the separable space Π_1.

From the definition of the mapping Op_S and the operations in G_S it follows immediately that R is a c.s.o.a. in Π_1. The unit is the element $\mathrm{Op}_S(\{1, 0, 0, 0\})$ and the separability of the space Π_1 follows from the separability of the spaces $\{P(t) : t \in T\}$, the second countability axiom for T and the separability of the space Q.

Let us show now that R is algebra of type III, i.e. that any of its invariant non-negative vectors is neutral and the corresponding e.f. is Hermitian. Denote $\lambda(A)$, $A \in R$, the e.f. of algebra R, corresponding to the invariant vector ξ_0. Let ξ_0' be a non-negative vector from Π_1 which is linearly independent with ξ_0 and invariant under R. The e.f. of R corresponding to ξ_0' is denoted $\mu(A)$. Then $(\xi_0', \xi_0) \neq 0$, otherwise a two-dimentional non-negative subspace would exist in Π_1 which is impossible, so we have $\mu(A)(\xi_0', \xi_0) = (A\xi_0', \xi_0) = (\xi_0', A^*\xi_0) = (\xi_0', \lambda(A^*)\xi_0) = \overline{\lambda(A^*)}(\xi_0', \xi_0)$ which implies $\mu(A) = \lambda(A)$. Further, let $(\xi_0', \xi_0') > 0$ and $\xi_0' = \alpha\xi_0 + \beta\eta_0 + p(t) + q$, with $\alpha, \beta \in \mathbb{C}$; $p(t) \in P$, $q \in Q$. Then $(\xi_0', \xi_0') = \alpha\overline{\beta} + \beta\overline{\alpha} + (p(t), p(t)) + (q, q)$, hence $\alpha \neq 0$, $\beta \neq 0$, otherwise $(\xi_0', \xi_0') = (p(t), p(t)) + (q, q) \leqslant 0$. Without loss of generality assume that $\beta = 1$. Since $\mu(A) = \lambda(A)$ we have $A'\xi_0' = 0$ for any $A' = \mathrm{Op}_S(\{0, \gamma, l, A(t)\}) \in M = \mathrm{Ker}\, \lambda(A)$. Therefore $0 = A'\xi_0' = A'(\alpha\xi_0 + \beta\eta_0 + p(t) + q) = \gamma\xi_0 + A(t)\zeta(t) + l + \int_T A(t)(p(t), \zeta(t))\, d\sigma\xi_0 + A(t)p(t) + (q, Vl)\xi_0$ (see (3.11)), and, in particular, it follows that $A(t)\zeta(t) = -A(t)p(t)$ for any $A(t) \in M_1 = \{A(t) : \omega \in \mathcal{E}_S\}$. Since, by our assumption $\overline{M_1}^{\mathrm{w}} = L_\infty(T, \sigma)$, we get

$\zeta(t) = -p(t)$, which contradicts the condition (ii) of Remark 3.8. Thus, $(\xi_0', \xi_0') = 0$ and $\mu(A) = \lambda(A)$ is a Hermitian functional, i.e. R is an algebra of class III. The singularity of R (see Definition 3.3) follows from $P \oplus Q \neq P$, since $Q \neq \{0\}$, and P is the main unit of M_1 because $\overline{M}_1^{\mathrm{w}} = L_\infty(T, \sigma)$. This completes the proof. ∎

Let us show now that the conditions (i) and (ii) of Remark 3.8 do not lead to the loss of generality in any of the previous propositions.

LEMMA 3.13. *If $R = \mathrm{Op}_S(\mathcal{E}_S)$ and $\zeta(t)$ is the corresponding measurable vector-function, $p_0(t) \in P = \int\limits_T \oplus P(t)\, \mathrm{d}\sigma$, then the algebra R is equivalent to the algebra $\widetilde{R} = \mathrm{Op}_S(\widetilde{\mathcal{E}}_S)$ with $\widetilde{\zeta}(t) = \zeta(t) + p_0(t)$.*

Proof. Consider the vector $\widetilde{\eta}_0 = \eta_0 - \frac{(p_0(t), p_0(t))}{2} \cdot \xi_0 + p_0(t) \in \Pi_1$. Then $(\widetilde{\eta}_0, \widetilde{\eta}_0) = 0$ and, consequently, $\widetilde{\eta}_0$ is a neutral skewly-linked vector with ξ_0. Construct a new realisation of the algebra R by changing the vector η_0 to the vector $\widetilde{\eta}_0$.

We get the algebra $\mathrm{Op}_S(\widetilde{\mathcal{E}}_S)$ which is equivalent to R (see [10]) and for any $A = \mathrm{Op}_S(\{0, \gamma, l, A(t)\}) \in \mathrm{Op}_S(\mathcal{E}_S) = \mathrm{Op}_S(\widetilde{\mathcal{E}}_S)$ we have

$$A\widetilde{\eta}_0 = \Big(\gamma + \int\limits_T A(t)(p_0(t), \zeta(t))\, \mathrm{d}\sigma\Big)\xi_0 + A(t)(\zeta(t) + p_0(t)) + l,$$

hence $\widetilde{\zeta}(t) = \zeta(t) + p_0(t)$. ∎

If $R = \mathrm{Op}_S(\mathcal{E}_S)$, then one of the two cases can occure: either $\zeta(t) \in P$ or $\zeta(t) \notin P$, i.e. $\int\limits_T \mathrm{d}\mu = -\infty$. In the first case, by Lemma 3.13 we can assume that $\zeta(t) = 0$ $(p_0(t) = -\zeta(t))$, and all the formulas and proofs become simplier while the general scheme remains the same. For example, proving Theorem 3.12 we get $0 = A'\zeta_0' = \gamma\zeta_0 + l + A(t)p(t) + (q, Vl)\zeta_0$ which implies $l = 0$ and, consequently, $\gamma = 0$, which would contradict (i) from (3.13). If $\int\limits_T \mathrm{d}\mu = -\infty$, then, introducing the set $T_0 = \{t \in T : \zeta(t) = 0\}$ and taking $p_0(t) \in P$ with restriction $p_0(t) \neq 0$, $t \in T_0$, by Lemma 3.13, we may assume that $\zeta(t) \neq 0$ σ-a.e. on T.

At the end of the section let us consider the regular i.e. such c.s.o.a. R of class III for which the main unit E of the algebra $M_1 = \pi M \pi$, $M = \mathrm{Ker}\, \lambda(A)$, coincides with $\mathbf{1}_{\mathfrak{H}}$ (see Definition 3.3).

Similarly to G_S, $\mathcal{E}_S$ and $\mathrm{Op}_S : G_S \to B(\Pi_1)$ we define $G_r, \mathcal{E}_r$ and $\mathrm{Op}_r : G_r \to B(\Pi_1)$ assuming additionally that $Q = \{0\}$.

THEOREM 3.14. *R is a regular c.s.o.a. with unit in a separable space Π_1 if and only if $R = \mathrm{Op}_r(\mathcal{E}_r)$ for some manifold $\mathcal{E}_r$. Moreover, $\overline{R}^{\mathrm{w}} = \mathrm{Op}_r(G_r)$, and $\overline{R}^{\mathrm{u}} = \mathrm{Op}_r(G_{r\mathrm{u}})$, where $G_{r\mathrm{u}} = \mathbb{C} \times \mathbb{C} \times (M(T) \cap \overline{M}_1^{\mathrm{u}}) \subset G_r$.*

The proof of this theorem follows the scheme of that of Theorem 3.13. We should only note that, instead of M^3 we consider M^2, and instead of M_1^3 and M_3 we take M_1^2 and M_2, respectively, wherever they appear.

4. THE THEOREM OF BICOMMUTANT FOR A C.S.O.A IN Π_1

In [13] it is shown that, for an arbitrary general complete symmetric operator algebra in Π_1, the theorem of bicommutant is not, in general, true. However, for a c.s.o.a. in Π_1, the following holds.

THEOREM 4.1. *Let R be a c.s.o.a. with a unit in a separable space Π_1. Then $\overline{R}^{\mathrm{w}} = R''$.*

Proof. Let R be a regular algebra of class III such that the corresponding function $\zeta(t) \in P$. Then, as we noted at the end of Section 3, we can assume that $\zeta(t) = 0$, and, consequently, by Theorem 3.14,

$$\overline{R}^{\mathrm{w}} = \left\{ \begin{pmatrix} \lambda & 0 \\ \gamma & \lambda \end{pmatrix} : \lambda, \gamma \in \mathbb{C} \right\} \oplus \mathcal{M}$$

in $\Pi_1' \oplus P$ (see 3.11), where $\mathcal{M} \cong L_\infty(T, \sigma)$, and therefore

$$R' = (\overline{R}^{\mathrm{w}})' = \left\{ \begin{pmatrix} \lambda & 0 \\ \gamma & \lambda \end{pmatrix} : \lambda, \gamma \in \mathbb{C} \right\}' \oplus \mathcal{M}' = \left\{ \begin{pmatrix} \lambda & 0 \\ \gamma & \lambda \end{pmatrix} : \lambda, \gamma \in \mathbb{C} \right\} \oplus \mathcal{M}'$$

and

$$R'' = \left\{ \begin{pmatrix} \lambda & 0 \\ \gamma & \lambda \end{pmatrix} : \lambda, \gamma \in \mathbb{C} \right\} \oplus \mathcal{M}'' = \left\{ \begin{pmatrix} \lambda & 0 \\ \gamma & \lambda \end{pmatrix} : \lambda, \gamma \in \mathbb{C} \right\} \oplus \mathcal{M} = \overline{R}^{\mathrm{w}}.$$

The theorem for the algebras I or II can be proved similarly on the base of results of Section 2.

Let R be an algebra of class III such that if $R = \mathrm{Op}_r(\mathcal{E}_r)$, then $\zeta(t) \in P$. Then, by Theorems 3.12 and 3.14, $\overline{R}^{\mathrm{w}} = \mathrm{Op}_S(G_S)$ (or $\mathrm{Op}_r(G_r)$) is a weakly closed general symmetric algebra in Π_1, with the property that for $A \in R$ from $A_1 = \pi A \pi$ it follows that $\lambda(A) = 0$; and, as it easy to verify, from $A_1 = 0$ does not follow $A = 0$. Therefore, R belongs to class I of [13], and do not coincide with its bicommutant iff $B(H_F) \notin M_1$, where $H_F = \{h_A = \pi A \eta_0 : A_1 = 0, h_{A^*} = 0\}$ ([13]). If $\overline{R}^{\mathrm{w}} = \mathrm{Op}_S(G_S)$, then $H_F = \{A(t)\zeta(t) + q_A : A(t) = -\lambda(A) = 0, \overline{A(t)}\zeta(t) + q_{A^*} = 0\} = \{q_A : V q_A = 0\} = \{0\}$. If $\overline{R}^{\mathrm{w}} = \mathrm{Op}_r(G_r)$, then $H_F = \{A(t)\zeta(t) : A(t) = 0\} = \{0\}$. Therefore, for $\overline{R}^{\mathrm{w}}$ the inclusion $B(H_F) \subset M_1$ always holds. Thus, $\overline{R}^{\mathrm{w}} = R''$ and the theorem is completely proved. ∎

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