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GLOBAL STRUCTURE IN THE SEMIGROUP OF ENDOMORPHISMS ON A VON NEUMANN ALGEBRA

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ABSTRACT. In [17], we studied the natural u-topology on the endomorphism semigroup End(M) of a von Neumann algebra M, given by pointwise convergence in the predual. For many purposes, however, the topological approach seems to be difficult. In this paper, the Borel structure induced by the u-topology on End(M) is then investigated. In particular a Borel implementation of endomorphisms is found and applied to prove that invariants such as the dimension mapping are Borel. Moreover we prove that End(M) is Borel isomorphic to the cartesian product of automorphisms and subfactors of M, and some related problems for quotient spaces of End(M) are discussed.

KEYWORDS: *Endomorphisms, von Neumann algebras, Borel structure.*

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1. INTRODUCTION

Over the last decade, the theory of endomorphisms on von Neumann algebras has received growing attention by many scientists. Powers' program for classification of (shift type and semigroups of) endomorphisms on factors of type I_∞ and type II_1 has created an area where many interesting results and techniques have been obtained (see e.g. [1] and references therein).

From a different point of view, endomorphisms of type III factors serve as models for superselection sectors in algebraic quantum field theory, and the stastistical dimension in that theory has been related to generalizations of Jones' index for subfactors (see [10] and its references). Moreover, the 'sector technique' has become an important tool in general subfactor theory and its applications to quantum groups, as a pleasant variation of the bimodule approach.

The theory of endomorphisms can (for the moment, in a non-formal sense) be considered as a 'cartesian product' of automorphism theory and subfactor theory, both of which are by now highly developped. We should thus expect techniques from both areas to play important roles; in fact, Popa's classification of subfactors does have some direct implications for endomorphisms (see [15]).

Nevertheless, although concrete examples of individual endomorphisms have been studied, very little is known about the semigroup of endomorphisms of a von Neumann algebra as a whole. In the view of automorphism theory, the first step is to study its natural topology; a first attempt in this direction was presented in [17]. The definition of this topology, called the u-topology, is given (in a more general context) in Section 2.

The main fact for automorphisms is the canonical implementation by unitaries on a standard Hilbert space for the algebra (cf. [6]), but for endomorphisms the continuous choice of range algebras causes severe troubles, since we cannot choose a common standard representation for all of them. However, another important fact about the automorphism group is that it is Polish, and its Borel structure (just as much as the topology) plays a vital role. The main purpose of this paper is to see what can be recovered in this weaker set-up for endomorphisms. In fact, most of the results in [17] required severe assumptions on the range algebras of the endomorphisms considered. The Borel structure approach is more compatible with algebraic manipulations, and we can disregard range algebras completely.

We first extend our study of the u-topology to all bounded normal operators on M in order to extract certain global properties of $\mathrm{End}(M)$ from properties in this bigger space (Section 2); a keypoint here is that although $\mathrm{End}(M)$ is not u-closed, it is anyway a Polish space (in u-topology, assuming M_* is separable). Instead of a canonical continuous implementation, we can then exhibit a unitary Borel implementation which at least in some sense is unique (Section 3).

A main motivation for considering the Borel structure rather than the topology on $\mathrm{End}(M)$ is the realization (cf. Example 5.1) that the dimension, one of the important invariants that do not show up when we consider automorphisms (or more generally, fixed range), is not compatible with a topological approach, but we shall see that it is in fact a Borel map (Theorem 5.3). To prove this, we use the Effros–Borel structure on intertwiner spaces; these technicalities are dealt with in Section 4.

It goes without saying that the idea of considering Borel structure rather than topology is inspired by Mackey's viewpoint in representation theory (cf. e.g. [12]). We end this paper with some remarks along this line, and use the Effros–Borel structure to give a precise meaning to the isomorphism

$$\mathrm{End}(M) \cong \mathrm{Aut}(M) \times \{N \subseteq M : N \cong M\}.$$

Throughout, we indicate important problems which are left open and that we think should be solved.

2. TOPOLOGICAL PRELIMINARIES

In this section, M will denote a von Neumann algebra. Let $\mathcal{B}_*(M)$ denote the set of linear, bounded, normal maps from M into itself, and $\mathcal{B}_*(M)_1$ its unit ball. On $\mathcal{B}_*(M)$, the u-topology is defined as the locally convex topology induced by the separating family of seminorms

$$T \mapsto \|\varphi \circ T\|, \quad \varphi \in M_*.$$

Convergence in u-topology of a net $(T_k) \subseteq \mathcal{B}_*(M)$ to $T \in \mathcal{B}_*(M)$ will be expressed as $T_k \xrightarrow{u} T$. The u-topology was introduced in [6] on the automorphism group $\mathrm{Aut}(M)$, where it has been studied extensively.

The following two lemmas are probably well known and may be hidden many places in the literature, but we include proofs for completeness' sake.

LEMMA 2.1. *Let $\mathcal{B}(M_*)_1$ denote the unit ball in the Banach space of bounded operators on M_*. Then $\mathcal{B}(M_*)_1 \cong \mathcal{B}_*(M)_1$, namely, for any $S \in \mathcal{B}(M_*)_1$ there is a unique $T \in \mathcal{B}_*(M)_1$ such that $S(\varphi) = \varphi \circ T$, $\varphi \in M_*$.*

Proof. Represent M on a some Hilbert space H; for any $\xi, \eta \in H$ we define $\omega_{\xi,\eta} \in M_*$ as the vector functional $x \mapsto \langle x\xi, \eta\rangle$. Let $S \in \mathcal{B}(M_*)_1$. Given $x \in M$, a sesquilinear form $H \times H \to \mathbb{C}$ is defined by $(\xi, \eta) \mapsto S(\omega_{\xi,\eta})(x)$, and this form is bounded with norm less than $\|x\|$. By Riesz representation there is a (clearly unique) $T(x) \in \mathcal{B}(H)$ with $\|T(x)\| \leqslant \|x\|$ such that

$$\langle T(x)\xi, \eta\rangle = S(\omega_{\xi,\eta})(x), \quad \xi, \eta \in H.$$

If $z \in M'$ then

$$\langle zT(x)\xi, \eta\rangle = \langle T(x)\xi, z^*\eta\rangle = S(\omega_{\xi,z^*\eta})(x) = S(\omega_{z\xi,\eta})(x) = \langle T(x)z\xi, \eta\rangle$$

for all $\xi, \eta \in H$, and thus $T(x) \in M$. Thus we have $T \in \mathcal{B}_*(M)_1$. As moreover $\omega_{\xi,\eta} \circ T = S(\omega_{\xi,\eta})$ for all $\xi, \eta \in H$ and S is bounded, in fact $\varphi \circ T = S(\varphi)$ for all $\varphi \in \overline{\mathrm{span}}\{\omega_{\xi,\eta} : \xi, \eta \in H\} = M_*$. ∎

If M_* is separable, its unit ball contains a separating sequence, i.e. a sequence such that any element of M_* can be approximated arbitrarily well by finite sums of elements from the sequence.

LEMMA 2.2. *Assume M_* is separable and fix a separating sequence (φ_n) in the unit ball of M_*. The u-topology on $\mathcal{B}_*(M)_1$ is induced by the metric d defined by*

$$d(S,T) = \sum_{n=1}^{\infty} 2^{-n} \|\varphi_n \circ S - \varphi_n \circ T\|, \quad S,T \in \mathcal{B}_*(M)_1.$$

Moreover $(\mathcal{B}_*(M)_1, d)$ is a complete metric space.

Proof. It is routine to check that d does indeed define a metric which induces the u-topology. Let (T_n) be a d-Cauchy sequence in $\mathcal{B}_*(M)_1$. As M_* is a Banach space, $(\varphi \circ T_n)$ will be convergent in M_* for any $\varphi \in M_*$. Call the limit $S(\varphi)$. Then clearly $S \in \mathcal{B}(M_*)_1$, so by the previous lemma there is $T \in \mathcal{B}_*(M)_1$ such that $\varphi \circ T = S(\varphi) = \lim_n \varphi \circ T_n$ for all $\varphi \in M_*$, and hence $d(T, T_n) \to 0$. ∎

In addition to the space of normal conditional expectations of M, the following two subsets of $\mathcal{B}_*(M)_1$ are particularly important: the group $\mathrm{Aut}(M)$ of automorphisms of M, and the semigroup $\mathrm{End}(M)$ of endomorphisms of M. It is immediate from 2.2 in [17] that $\mathrm{Aut}(M)$ is closed in u-topology as a subset of $\mathrm{End}(M)$. In general, however, $\mathrm{End}(M)$ is not closed in $\mathcal{B}_*(M)$ (cf. below and Lemma 3.3), but of course the above gives:

COROLLARY 2.3. *In the u-topology of $\mathcal{B}_*(M)$, the closure $\overline{\mathrm{End}(M)}$ of $\mathrm{End}(M)$ is a complete metric space.*

It is also straightforward to see that the limit of a net of endomorphisms is a unital $*$-mapping, so the problem is that it need not be multiplicative.

EXAMPLE 2.4. Let H be a separable Hilbert space, and u_k a sequence of unitaries converging strongly to $v \in \mathcal{B}(H)$. Then v is necessarily an isometry, but it might not be surjective as familiar examples show (take for instance $H = \ell^2$ with u_k given as a cyclic permutation of the first k elements in a sequence, converging strongly to the right 1-shift on ℓ^2). Assume this is the case. Let $\alpha_k = \mathrm{ad}(u_k^*) \in \mathrm{Int}(\mathcal{B}(H))$ and let $T \in \mathcal{B}_*(\mathcal{B}(H))_1$ be given by $T(x) = v^* x v$, $x \in \mathcal{B}(H)$. Clearly T is not multiplicative. If $\xi \in H$ and ω_ξ is the induced vector functional then (cf. [6])

$$\begin{aligned}\|\omega_\xi \circ T - \omega_\xi \circ \alpha_k\| &= \|\omega_{v\xi} - \omega_{u_k \xi}\| \\ &\leqslant \|v\xi + u_k\xi\| \, \|v\xi - u_k\xi\| \\ &\leqslant 2\|\xi\| \, \|v\xi - u_k\xi\| \to 0 \qquad (k \to \infty)\end{aligned}$$

hence $\alpha_k \xrightarrow{u} T$.

So, even the closure $\overline{\mathrm{Int}(M)}$ of the inner automorphisms need not be contained in End(M); by considering $\mathbf{1} \otimes T$ as above, this effect can be turned on in any properly infinite von Neumann algebra. To see that even in the finite case, End(M) is not closed, we include the following example, which can obviously be generalized to other infinite tensor products (cf. Example 5.1):

EXAMPLE 2.5. Consider the hyperfinite type II_1-factor realized as the infinite tensorpower of the full 2-by-2 matrix algebra with respect to its normalized trace tr:

$$R = M_2(\mathbb{C}) \otimes_{\mathrm{tr}} M_2(\mathbb{C}) \otimes_{\mathrm{tr}} \cdots .$$

Here the tracial state on R is obtained as $\tau = \mathrm{tr} \otimes \mathrm{tr} \otimes \cdots$. Let $R_0 \subseteq R$ be the subfactor

$$R_0 = \mathbb{C}\mathbf{1} \otimes_{\mathrm{tr}} M_2(\mathbb{C}) \otimes_{\mathrm{tr}} M_2(\mathbb{C}) \otimes_{\mathrm{tr}} \cdots$$

then the unique conditional expectation $E_0 : R \to R_0$ satisfying $\tau \circ E_0 = \tau$ is given by

$$E_0(x_1 \otimes x_2 \otimes \cdots) = \mathrm{tr}(x_1)\mathbf{1} \otimes x_2 \otimes x_3 \otimes \cdots$$

where each $x_j \in M_2(\mathbb{C})$. If we define $(\rho_k)_{k\in\mathbb{N}} \subseteq \mathrm{End}(R)$ by

$$\rho_k(x_1 \otimes x_2 \otimes \cdots) = \mathbf{1} \otimes x_2 \otimes \cdots x_k \otimes x_1 \otimes x_{k+1} \otimes x_{k+2} \otimes \cdots$$

then it is not difficult to see $\rho_k \xrightarrow{u} E_0$ in $\mathcal{B}_*(R)$; so the limit is not in End(R).

PROPOSITION 2.6. *Assuming M_* is separable, the space* End(M) *with u-topology is Polish.*

Proof. The idea is to consider the topology of pointwise σ-strong $*$-convergence on $\mathcal{B}_*(M)$, that we will refer to as the ps^*-topology. Namely, in this topology, End(M) is closed, because the algebra multiplication is strongly $*$-continuous on the unit ball M_1 of M. Moreover, the σ-strong $*$-topology on M_1 is induced by the complete norm

$$x \mapsto \|x\|_\varphi^\# := \varphi(x^*x + xx^*)^{\frac{1}{2}}$$

where φ is some fixed normal faithful state on M_1. Let (x_n) be a fixed σ-strongly $*$-dense sequence in M_1. The ps^*-topology on $\mathcal{B}_*(M)_1$ is then induced by the metric

$$d_1(S,T) = \sum_{n=1}^{\infty} 2^{-n} \|S(x_n) - T(x_n)\|_\varphi^\#.$$

Let d be as in Lemma 2.2, then $d + d_1$ is a metric inducing the u-topology on End(M), because the p-topology and the ps^*-topology coincide on End(M) (cf.

[17], 1.4) and the former is weaker than the u-topology. Moreover, the restriction of $d + d_1$ to End(M) is complete: if $(\rho_n) \subseteq \text{End}(M)$ is a Cauchy-sequence with respect to $d+d_1$, then by completeness of $\|\cdot\|_\varphi^\#$, there is $T \in \mathcal{B}(M)$ with $\rho_n \xrightarrow{ps^*} T$, and by Lemma 2.2, $\rho_n \xrightarrow{u} S$ for some $S \in \mathcal{B}_*(M)$. But then $\rho_n \xrightarrow{p} S$ and hence, for all $\psi \in M_*$ and all $x \in M$,

$$\psi \circ S(x) = \lim_{n\to\infty} \psi \circ \rho_n(x) = \psi \circ T(x)$$

so $S = T$ and hence $(d + d_1)(\rho_n, S) \to 0$. Finally as $\rho_n \xrightarrow{ps^*} T = S$ and End(M) is ps^*-closed, we have $S \in \text{End}(M)$. ∎

3. BOREL IMPLEMENTATION OF ENDOMORPHISMS

In this section we assume that M is a properly infinite von Neumann algebra with separable predual. On $\mathcal{B}_*(M)$ and its subspaces, we always consider the u-topology and its associated Borel structure, as defined in the previous section.

Consider M in a standard representation $M \subseteq \mathcal{B}(H)$. Let $\mathcal{N}(M)$ be the subset of the unitary group $\mathcal{U}(H)$ of H defined by

$$\mathcal{N}(M) = \{U \in \mathcal{U}(H) : \; UMU^* \subseteq M\}.$$

The interest of this semigroup in connection with the study of End(M) is the natural map $\sigma : \mathcal{N}(M) \to \text{End}(M)$ defined by

$$\sigma_U(x) = UxU^*, \quad x \in M, \; U \in \mathcal{N}(M).$$

We always regard $\mathcal{U}(H)$ in the strong (equivalently, weak) operator topology. We shall use standard terminology and results from the theory of Borel spaces. The main source for these is [12] and a very good introduction can be found in [16], Appendix.

The starting point for our discussion is the following:

LEMMA 3.1. *As a subset of $\mathcal{U}(H)$, the set $\mathcal{N}(M)$ is closed, hence it is a Polish semigroup.*

Proof. Let $(u_k) \subseteq \mathcal{N}(M)$ converge strongly to some $u \in \mathcal{U}(H)$. Then for any $x \in M$, a routine calculation shows $u_k x u_k^* \to u x u^*$ strongly, so $uxu^* \in M$. ∎

LEMMA 3.2. *The map σ is continuous and surjective.*

Proof. The map σ is continuous by the argument in Example 2.4, and it is surjective because, by 1.2 in [3], we may choose a joint separating cyclic vector for M and the image of any element of $\mathrm{End}(M)$, and thus obtain a unitary implementation for any endomorphism. ∎

LEMMA 3.3. *The semigroup* $\mathrm{End}(M)$ *is a standard Borel space which is Borel isomorphic to* $\mathcal{N}(M)/\mathcal{U}(M')$.

Proof. This is immediate from Proposition 2.6, Lemma 3.2 and 4.2 in [12]. ∎

NOTE 3.4. The last lemma being of vital importance here, we mention that a more direct proof goes as follows: Consider the equivalence relation $\sim$ on $\mathcal{N}(M)$ given by

$$u \sim v \Longleftrightarrow \exists\, z \in \mathcal{U}(M') : u = vz.$$

It is easy to see that the orbits of $\sim$ are closed in $\mathcal{N}(M)$ and that the saturation $A\mathcal{U}(M')$ of an open set A under $\sim$ is Borel (in fact, open). Hence by Lemme 2 in [2] there is a Borel set $S \subseteq \mathcal{N}(M)$ which meets each orbit at exactly one point; S is a standard Borel space by Lemma 3.1. Moreover σ maps S injectively onto $\mathrm{End}(M)$, so $\sigma|S$ is a Borel isomorphism according to 4.2 in [12].

THEOREM 3.5. (Borel implementation) *Let M be a properly infinite von Neumann algebra. Then there is a Borel map*

$$u : \mathrm{End}(M) \to \mathcal{N}(M)$$

such that

$$\rho = \sigma_{u(\rho)}, \quad \rho \in \mathrm{End}(M).$$

Proof. By the above, $\sigma : \mathcal{N}(M) \to \mathrm{End}(M)$ is a surjective Borel map between standard Borel spaces. In particular these spaces are Souslin and are equiped with σ-finite complete measures. Applying the Mackey-von Neumann measurable cross section theorem (see 6.3 in [12] or p. 384 in [16]) to σ we obtain a Borel map $u : \mathrm{End}(M) \to \mathcal{N}(M)$ such that $\sigma \circ u$ is the identity mapping. ∎

The above implementation is, in contrast with the standard implementation of automorphisms 3.6 in [6], not canonical. But by the same argument as above, we do have:

COROLLARY 3.6. *The inverse* $\underline{u}$: $\mathrm{End}(M) \to \mathcal{N}(M)/\mathcal{U}(M')$ *of the (continuous, injective) map* $\underline{\sigma} : \mathcal{N}(M)/\mathcal{U}(M') \to \mathrm{End}(M)$ *induced by* σ, *is a multiplicative Borel map.*

Assume now that M is represented on a Hilbert space H that contains a cyclic separating vector for M. Let J denote the corresponding modular conjugation. Define a map $\Gamma : \mathcal{U}(H) \to \mathcal{U}(H)$ by

$$\Gamma(u) = uJu^*J, \qquad u \in \mathcal{U}(H).$$

It is easy to see that Γ is continuous. Moreover $\Gamma(\mathcal{N}(M)) \subseteq \mathcal{N}(M)$ since, for any $u \in \mathcal{N}(M)$:

$$\begin{aligned}\Gamma(u)M\Gamma(u)^* &= uJu^*JMJuJu^* \\ &= uJu^*M'uJu^* \\ &\subseteq uJM'Ju^* \\ &= uMu^* \subseteq M.\end{aligned}$$

In fact, if $u \in \mathcal{N}(M)$ then $(uMu^*, H, u\xi, uJu^*)$ is a standard form. Hence by definition (cf. [10]), $\gamma_{uMu^*} = \sigma_{\Gamma(u)}$ is a canonical endomorphism for $M \supseteq uMu^*$. As Γ is continuous we thus have a continuous choice

$$\mathcal{N}(M) \ni u \mapsto \gamma_{uMu^*} \in \mathrm{End}(M)$$

of canonical endomorphisms. Combining with the previous theorem we conclude:

PROPOSITION 3.7. *Let* $\gamma = \sigma \circ \Gamma \circ u : \mathrm{End}(M) \to \mathrm{End}(M)$. *Then* γ *is a Borel map with the property that* $\gamma(\rho)$ *is a canonical endomorphism for* $M \supseteq \rho(M)$ *for all* $\rho \in \mathrm{End}(M)$. *In particular,* $\rho \mapsto \rho^{-1}\gamma(\rho)$ *is a Borel choice of conjugate endomorphisms (cf.* [11]).

We shall mostly use the more familiar notation γ_ρ for $\gamma(\rho)$ as above.

PROBLEM 3.8. Does there exist an involutive Borel choice of conjugates on $\mathrm{End}(M)$, i.e. a Borel map $c : \mathrm{End}(M) \to \mathrm{End}(M)$ such that $c \circ c = \mathrm{id}$ and $c(\rho)$ is a conjugate to ρ for each $\rho \in \mathrm{End}(M)$? Does such a mapping exist at least on irreducibles?

4. INTERTWINER SPACES

We prove some technical lemmas which will be needed in the next section. Readers may skip them and Proposition 4.3 first, and return when the lemmas are applied.

For any separable von Neumann algebra $M \subseteq \mathcal{B}(H)$, we let $\mathcal{W}(M)$ denote the set of σ-weakly closed subspaces of M. The *Effros–Borel structure* on $\mathcal{W}(M)$ is the weakest Borel structure on $\mathcal{W}(M)$ in which the map $F \mapsto \|\varphi|F\|$ is a Borel function on $\mathcal{W}(M)$ for all $\varphi \in M_*$. By Theorem 1 and Corollary 1 of [4], this Borel structure is standard. Moreover by Theorem 2 of [4], there are Borel choice maps $a_n : \mathcal{W}(M) \to M_1$ (where M_1 denotes the unit ball of M) such that $\{a_n(F) : n \in \mathbb{N}\}$ is σ-weakly dense in F_1, for all $F \in \mathcal{W}(M)$.

For all $\rho, \eta \in \mathrm{End}(M)$ we consider the space $H_{\rho,\eta}$ of intertwiners between ρ and η, i.e.

$$H_{\rho,\eta} = \{v \in M : \rho(x)v = v\eta(x) \quad \text{for all } x \in M\}.$$

Clearly $H_{\rho,\eta} \in \mathcal{W}(M)$. We write $H_\rho = H_{\rho,1}$; if M is a factor, then $v^*v \in [0,\infty)\cdot\mathbf{1}$ for all $v \in H_\rho$.

LEMMA 4.1. *The map $(\rho,\eta) \mapsto H_{\rho,\eta}$ is a Borel map of* $\mathrm{End}(M) \times \mathrm{End}(M)$ *into $\mathcal{W}(M)$.*

Proof. The method of the proof is taken from the proof of the second part of Theorem 3 in [4], so we only indicate how to apply it in the present situation.

We need to prove that the map $(\rho,\eta) \mapsto \|\varphi|H_{\rho,\eta}\|$ is a Borel function on $\mathrm{End}(M) \times \mathrm{End}(M)$ for every $\varphi \in M_*$. Let $(\rho,\eta) \in \mathrm{End}(M) \times \mathrm{End}(M)$. Note first that, with (b_n) a σ-weakly dense sequence in the unit ball of M, we have

$$H_{\rho,\eta} = \{v \in M : \rho(b_n)v - v\eta(b_n) = 0, \quad n = 1,2,\ldots\}.$$

Now let $\{\psi^j\}$ be a norm-dense sequence in $(\ell^\infty \otimes M)_* = \ell^1 \overline{\otimes} M_*$, so $\psi^j = \{\psi^j_k\}$ is a 1-summable sequence in M_* for each j. Then as in [4], for each $\varphi \in M_*$ we get

$$\|\varphi|H_{\rho,\eta}\| = \inf_j \sup_n \left|\varphi(b_n) - \sum_{k=1}^{\infty} \psi^j_k(\rho(b_k)b_n - b_n\eta(b_k))\right|,$$

and the right hand side is clearly a Borel function of (ρ,η). ∎

LEMMA 4.2. *Assume M is properly infinite. With $\rho \mapsto \gamma_\rho$ as in Proposition 3.7, the maps $\rho \mapsto H_{\gamma_\rho}$ and $\rho \mapsto H_{\gamma_\rho|\rho(M)}$ are both Borel.*

Proof. Since $\rho \mapsto \gamma_\rho$ is Borel, the map $\rho \mapsto H_{\gamma_\rho}$ is clearly Borel by Lemma 4.1. Similarly, the map $\rho \mapsto H_{\rho^{-1}\gamma_\rho\rho}$ is Borel. A simple calculation (cf. 5.1, [11]) shows that for any ρ,

$$H_{\gamma_\rho|\rho(M)} = \rho(H_{\rho^{-1}\gamma_\rho\rho})$$

(in fact, the right hand side may be taken as the definition).

Let $a_n : \mathcal{W}(M) \to M_1$ be as above, and let $b_n : \mathrm{End}(M) \times \mathcal{W}(M) \to M_1$ be given by

$$b_n(\rho, F) = \rho(a_n(F)), \quad \rho \in \mathrm{End}(M),\ F \in \mathcal{W}(M).$$

Then all b_n are Borel maps and $\{b_n(\rho, F) : n = 1, 2, \ldots\}$ is σ-weakly dense in $\rho(F_1) = \rho(F)_1$ for all (ρ, F). Hence, by p. 1158, [4] the map $(\rho, F) \mapsto \rho(F)$ is a Borel map of $\mathrm{End}(M) \times \mathcal{W}(M)$ into $\mathcal{W}(M)$. Now the lemma follows. ∎

We denote by $\mathcal{W}^*(M)$ the set of sub-von Neumann algebras of M. By Theorem 3 and Corollary 2 of [4] this is a Borel subset of $\mathcal{W}(M)$, and the map $(M, N) \mapsto M \cap N'$ is Borel on $\mathcal{W}^*(M) \times \mathcal{W}^*(M)$. The map $\rho \mapsto \rho(M)$ is Borel on $\mathrm{End}(M)$ because $\rho \mapsto \|\varphi|\rho(M)\|$ is a continuous function on $\mathrm{End}(M)$ for every $\varphi \in M_*$, by the definition of u-topology. Hence the map $\rho \mapsto M \cap \rho(M)'$ follows Borel. Thus we get:

PROPOSITION 4.3. *The set of irreducible endomorphisms is Borel in* $\mathrm{End}(M)$.

A similar argument, using Proposition 3.7, shows that the standard invariant of $M \supseteq \rho(M)$ is a Borel function of ρ.

5. APPLICATIONS TO INDEX THEORY

In index theory, one considers a σ-finite factor M and its subfactors. For technical convenience, we shall assume further that M is infinite and has separable predual (cf. Remark 5.5). If N is a subfactor of M, we denote by $\mathcal{E}(M, N)$ the set of normal conditional expectations of M onto N; each $E \in \mathcal{E}(M, N)$ has an index which can be defined or shown to be

$$\mathrm{Index}(E) = (\sup\{\lambda \geqslant 0 : E(x) \geqslant \lambda x \quad \text{for all } \in M_+\})^{-1},$$

cf. [14] and [8]. Either all, or none, of the elements of $\mathcal{E}(M, N)$ have finite index (see [9]), and in the first case there is a unique element of $\mathcal{E}(M, N)$ whose index is least possible (cf. [7]). This *minimal index* is denoted $[M : N]_0$. In the case no element of $\mathcal{E}(M,N)$ has finite index, or if the set is just empty, we decide to put $[M:N]_0 = \infty$.

The study of the minimal index as a map arises most naturally in the more 'dynamical' setting of endomorphisms. We define

$$\mathrm{End}_E(M) = \{\rho \in \mathrm{End}(M) : \mathcal{E}(M, \rho(M)) \neq \emptyset\}.$$

In the terminology of Longo ([10]), for any $\rho \in \mathrm{End}(M)$, the *dimension* is given by $d(\rho) = [M : \rho(M)]_0^{1/2}$ (so that, by definition, $d(\rho) = \infty$ if $\rho \notin \mathrm{End}_E(M)$). The natural question whether this map is continuous has a negative answer in general:

EXAMPLE 5.1. Let F be a factor with a fixed state ϕ, and let M be the inifinite tensor power of (F,ϕ). Its predual M_* contains, as a total subset M_*^0, functionals of the form $\phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_n \otimes \phi \otimes \phi \otimes \cdots$, where $n \in \mathbb{N}$ and $\phi_1, \ldots, \phi_n \in F_*$ have $\phi_j(\mathbf{1}) = 1$. Now consider the sequence $(\rho_n) \subseteq \mathrm{End}(M)$ defined by its action on elementary tensors as follows:

$$\rho_n(x_1 \otimes x_2 \otimes \cdots) = x_1 \otimes \cdots \otimes x_n \otimes \mathbf{1} \otimes x_{n+1} \otimes x_{n+2} \otimes \cdots.$$

Clearly

$$d(\rho_n) = \begin{cases} m & \text{if } F \text{ is type } \mathrm{I}_m; \\ \infty & \text{otherwise;} \end{cases}$$

so all ρ_n share the same dimension which, according to our choice of F, could be any number in $\mathbb{N} \cup \{\infty\}$. Nevertheless $\rho_n \xrightarrow{u} \mathbf{1}$ (the identity in $\mathrm{End}(M)$): considering the action of ρ_n on M_*^0, we find

$$(\phi_1 \otimes \phi_2 \otimes \cdots) \circ \rho_n = \phi_1 \otimes \phi_2 \otimes \cdots \otimes \phi_n \otimes \phi_{n+2} \otimes \phi_{n+3} \otimes \cdots$$

and hence the map $\psi \mapsto \psi \circ \rho_n$ converges to the identity mapping in M_*.

Thus, in general, the dimension function $d : \mathrm{End}(M) \to [1,\infty]$ is not continuous. However, using Lemma 4.2 together with the algebraic formulation (via intertwiners) of index theory due to R. Longo, we get nice properties of maps and sets in the Borel structure of $\mathrm{End}(M)$ studied above.

PROPOSITION 5.2. *The sets* $\mathrm{End}_0(M) = \{\rho \in \mathrm{End}(M) : d(\rho) < \infty\}$ *and* $\mathrm{End}_E(M)$, *defined above, are both Borel subsets of* $\mathrm{End}(M)$.

Proof. According to [10], 5.1,

$$\mathrm{End}_E(M) = \{\rho \in \mathrm{End}(M) : H_{\gamma_\rho|\rho(M)} \neq \{0\}\}$$

so the claim follows from Lemma 4.2. Similarly, 4.4 in [11] means that

$$\mathrm{End}_0(M) = \mathrm{End}_E(M) \cap \{\rho \in \mathrm{End}(M) : H_{\gamma_\rho} \neq \{0\} \text{ and } \dim(M \cap \rho(M)') < \infty\}$$

so this set is Borel by Lemma 4.2 and the fact that finite dimensional von Neumann algebras form a Borel subset of $\mathcal{W}^*(M)$ (see 2.4 in [5]). ∎

Pushing this argument further, we get:

THEOREM 5.3. *The dimension* $d : \mathrm{End}(M) \to [1, \infty]$ *is a Borel map.*

Proof. For brevity, we let

$$S_\rho = H_{\gamma_\rho|\rho(M)}, \quad \rho \in \mathrm{End}(M).$$

By the preceding proposition, we need only prove that the restriction of d to $\mathrm{End}_0(M)$ is Borel, in particular the above space will be non-zero for the endomorphisms under consideration, and we may then assume that the maps a_n, introduced in Section 4, satisfy $a_n(S_\rho) \neq 0$ for all such ρ and all $n \in \mathbb{N}$, upon replacing each a_n by the map $F \to a_n(F) + n^{-1}1_{\{0\}}(a_n(F))$. Further, by the method of 2.6 in [5] one can assume that $\{a_n(F) : n = 1, 2, \ldots\}$ is *strongly* dense in F_1 for all F. From 5.1, [10] the map $E : S_\rho \setminus \{0\} \to \mathcal{E}(M, \rho(M))$ given by

$$E_v(x) = (v^*v)^{-1}v^*\gamma_\rho(x)v, \quad x \in M, \; v \in S_\rho \setminus \{0\}$$

is surjective and strongly continuous when $\mathcal{E}(M, \rho(M))$ is considered in the p-topology (cf. Section 2). Note here that, as M is a factor, v^*v is a scalar for all $v \in S_\rho$ and $\rho \in \mathrm{End}(M)$. Moreover, the map $E \mapsto \mathrm{Index}(E)$ is a continuous function on $\mathcal{E}(M, N)$ in the same topology, for any subfactor of N, in fact more holds: whenever $(E_n) \subseteq \mathcal{E}(M, N)$ and $E \in \mathcal{E}(M, N)$ satisfies $E_n(x) \to E(x)$, σ-weakly for all $x \in M \cap N'$, one has $\mathrm{Index}(E_n) \to \mathrm{Index}(E)$ (cf. [9]). Hence the composed map $v \mapsto \mathrm{Index}(E_v)$ is continuous on $S_\rho \setminus \{0\}$. It follows that, for any $\rho \in \mathrm{End}_0(M)$, we have

$$d(\rho)^2 = \inf\{\mathrm{Index}(E_{a_n(S_\rho)}) : n = 1, 2, \ldots\}$$

and this also shows that d is Borel by the above, since $\mathcal{E}(M, N)$ is p-closed for each subfactor N of M. ∎

Note that, for an irreducible inclusion $M \supseteq N$ of factors, there is at most one expectation, so in this case we can simply speak of the index $[M : N]$, infinite if no expectation exists.

COROLLARY 5.4. *The set* $\mathcal{I}_{\mathcal{M}} = \{[\mathcal{M} : \mathcal{N}] : \mathcal{N}$ *is an irreducible subfactor of* $M\}$ *is an analytic subset of* $[1, \infty]$.

Proof. Immediate from Theorem 5.3 and Proposition 4.3. ∎

REMARK 5.5. In index theory, as well as its applications, the interesting case is the properly infinite one as treated here. Moreover, the finite case is usually trivial from the infinite one. In fact, the results of this section hold for finite factors as well, because the map $\rho \mapsto \rho \otimes 1$ is a u-u-continuous map from $\mathrm{End}(M)$ to $\mathrm{End}(M \otimes N)$ for any pair M, N of von Neumann algebras.

PROBLEMS 5.6. In view of earlier work of Mashhood and Taylor ([13]) (on a similar but different problem in the II_1-case), the following questions seem interesting: Is d continuous on the set of irreducible endomorphisms? Is d lower semicontinuous?

6. CARTESIAN PRODUCT OF TWO THEORIES

We end this paper by proving a rather precise version of our claim in the introduction, namely that the theory of endomorphisms in von Neumann algebra is essentially a cartesian product of subfactor (or, more precisely, subalgebra) theory and the theory of automorphisms.

In order for the subfactor part to materialize, we consider as in Section 4 the set $\mathcal{W}^*(M)$ of sub-von Neumann algebras of M, where M denotes throughout a fixed separable properly infinite von Neumann algebra. Moreover we let $\mathcal{W}_0^*(M)$ denote the subset of $\mathcal{W}^*(M)$ consisting of subalgebras which are algebraically isomorphic to M.

LEMMA 6.1. *The set* $\mathcal{W}_0^*(M) = \{N \in \mathcal{W}^*(M) : N \cong M\}$ *is a standard Borel space when equipped with the Effros–Borel structure.*

Proof. Represent M standardly in $\mathcal{B}(H)$. With $a_n : \mathcal{W}(\mathcal{B}(H)) \to \mathcal{B}(H)_1$ as in Theorem 2, [4], we let $f_{n,m} : \mathcal{W}^*(\mathcal{B}(H)) \to \mathcal{B}(H)$ be given by

$$f_{n,m}(N) = a_n(N)a_m(M') - a_m(M')a_n(N), \quad N \in \mathcal{W}^*(\mathcal{B}(H))$$

for all $n, m \in \mathbb{N}$. Then

$$\mathcal{W}^*(M) = \bigcap_{n,m=1}^{\infty} f_{n,m}^{-1}(\{0\})$$

so $\mathcal{W}^*(M)$ is a Borel subset of $\mathcal{W}^*(\mathcal{B}(H))$. Let $[M] = \{N \in \mathcal{W}^*(\mathcal{B}(H)) : N \cong M\}$. Then $[M]$ is Borel by 2.2 in [5] and hence $\mathcal{W}_0^*(M) = \mathcal{W}^*(M) \cap [M]$ is Borel in the standard Borel space $\mathcal{W}^*(\mathcal{B}(H))$. ∎

As all standard Borel spaces are either countable or isomorphic to $[0,1]$, it is now immediate that $\mathrm{End}(M)$ and $\mathcal{W}_0^*(M) \times \mathrm{Aut}(M)$ are Borel isomorphic. But we can make this more concrete:

THEOREM 6.2. *There is a Borel choice of endomorphism for subalgebras, namely a Borel map* $\varepsilon : \mathcal{W}_0^*(M) \to \mathrm{End}(M)$ *such that, with* $\varepsilon_N = \varepsilon(N)$, *we have* $\varepsilon_N(M) = N$ *for all* $N \in \mathcal{W}_0^*(M)$. *Moreover the map* $I : \mathrm{End}(M) \to \mathcal{W}_0^*(M) \times \mathrm{Aut}(M)$ *given by*

$$I(\rho) = (\rho(M), \rho^{-1}\varepsilon_{\rho(M)}), \quad \rho \in \mathrm{End}(M)$$

is a Borel isomorphism.

Proof. The map ε is obtained as a Borel section of the surjective Borel map $\rho \mapsto \rho(M)$ of $\mathrm{End}(M)$ onto $\mathcal{W}_0^*(M)$. The map I thus defined is then clearly Borel and it is straightforward to see that it is bijective. Hence also its inverse is Borel by 4.1 in [12]. ∎

As in Mackey's classical work on group representations (see [12] and references), a main goal for investigating the global structure of $\mathrm{End}(M)$ is to decide whether classification is possible in principle; so we wish to prove that relevant quotient spaces (with quotient Borel structure, cf. p. 137 in [12]) are countably separated. The main two equivalence relations on $\mathrm{End}(M)$ are *outer equivalence*:

$$\rho \sim \rho' \iff \exists\, \alpha \in \mathrm{Int}(M) : \rho' = \alpha \circ \rho$$

and *conjugacy*:

$$\rho \equiv \rho' \iff \exists\, \alpha \in \mathrm{Aut}(M) : \rho' = \alpha \circ \rho \circ \alpha^{-1}.$$

Here, the quotient space $\mathrm{End}(M)/\sim$ is usually denoted $\mathrm{Sect}(M)$, its elements being called sectors. The classification of (certain) sectors is particularly important to applications in quantum field theory (cf. e.g. [10]).

PROBLEMS 6.3. Is $\mathrm{Sect}(M)$ a standard Borel space? (A famous result due to Glimm suggests that the answer is usually no.) What about $\mathrm{End}(M)/\equiv$?

The equivalence relations defined above are of course just generalizations of the usual ones in automorphism theory; however, as $\mathrm{End}(M)$ is not a group there is a *right outer equivalence* which is strictly stronger than the usual outer equivalence:

$$\rho \sim_r \rho' \iff \exists, \alpha \in \mathrm{Int}(M) : \rho' = \rho \circ \alpha.$$

(A similar asymmetry is discussed in [1].) For this relation, the quotient space can be identified, using Theorem 6.2:

COROLLARY 6.4. *With* $\mathrm{Out}(M) = \mathrm{Aut}(M)/\sim$, *we have*

$$(\mathrm{End}(M)/\sim_r) \cong \mathcal{W}_0^*(M) \times \mathrm{Out}(M)$$

(Borel isomorphism).

Proof. For $u \in M$ unitary and $\rho \in \mathrm{End}(M)$, and with I as in the theorem, we have

$$I(\rho \circ \mathrm{ad}(u)) = (\rho(M), \mathrm{ad}(u^*)\rho^{-1}\varepsilon_{\rho(M)}),$$

so $\rho \sim_r \rho'$ means that $I(\rho) = I(\rho')$ in $\mathcal{W}_0^*(M) \times \mathrm{Out}(M)$. ∎

REMARK 6.5. If an involutive Borel choice c of conjugates as defined in Problem 3.8 existed, then the map $I \circ c$ would give a Borel isomorphism of $\mathrm{Sect}(M)$ with $\mathcal{W}_0^*(M) \times \mathrm{Out}(M)$ as in the proof of Corollary 6.4.

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