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SUBSPACES AND SUBALGEBRAS OF $K(H)$ WHOSE DUALS HAVE THE SCHUR PROPERTY

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ABSTRACT. Let $K(H)$ be the algebra of the compact operators on a Hilbert space H. Recently, S.W. Brown has shown that the dual A^* of a commutative closed subalgebra of $K(H)$ satisfying a very mild condition has the Schur property. In this paper, continuing this work of S.W. Brown, we characterize the closed subspaces and subalgebras of $K(H)$ whose duals have the Schur property.

KEYWORDS: *Schur property, algebras of compact operators.*
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INTRODUCTION

As is well known, weakly compact subsets of ℓ^1 are norm compact. A Banach space X sharing with ℓ^1 this property is said to have the Schur property. Now let H be a Hilbert space and $K(H)$ be the algebra of the compact operators on H. The space of null sequences c_o can be identified with a closed commutative subalgebra of $K(H)$ and $c_o^* = \ell^1$. Recently, S.W. Brown ([2]) in a beautifully written paper has shown that if A is any closed commutative subalgebra of $K(H)$ and the sets $\{a(x) : a \in A \text{ and } x \in H\}$ and $\{a^*(x) : a \in A \text{ and } x \in H\}$ are dense in H, then A^* has the Schur property. Actually he proves the following stronger result: if X is a closed subspace of $K(H)$ and the sets $X_1(x) = \{a(x) : a \in X, \|a\| \leqslant 1\}$ and $\widetilde{X}_1(x) = \{a^*(x) : a \in X, \|a\| \leqslant 1\}$ are relatively (norm) compact in H, then X^* has the Schur property. In this paper, continuing this work of Brown, we characterize the closed subspaces and subalgebras of $K(H)$ whose duals have the Schur property. The first main result of the paper says that compactness of the above two sets are also necessary for X^* to have the Schur property so that we have the following characterization: the dual X^* of a subspace X of $K(H)$ has the Schur property if and only if, for each x in H, the sets $X_1(x)$ and $\widetilde{X}_1(x)$ are relatively compact in H. Then we consider the Schur property on the duals of the closed subalgebras of $K(H)$. To this end, we first show that, for any reflexive Banach space Y, every closed commutative subalgebra A of $K(Y)$ is completely continuous. That is, for each a in A the left (and right) multiplication operator $L_a : A \longrightarrow A$, defined by $L_a(x) = ax$ ($R_a(x) = xa$), is compact. We also show that the complete continuity is a necessary condition for the dual A^* of a subalgebra A of $K(Y)$ to have the Schur property. Then, as the second main result, we show, under a very mild condition — the same condition as in Brown's paper ([2]) — that the dual A^* of a closed (not necessarily commutative) subalgebra A of $K(H)$ has the Schur property if and only if A is completely continuous. The paper also contains some corollaries of these results.

NOTATION AND TERMINOLOGY

Our notation and terminology are quite standard. For any Banach space X, we denote the dual of X by X^* and the closed unit ball of X by X_1. The natural duality between X and X^* is denoted as $\langle x, f\rangle$. By H we denote an arbitrary Hilbert space. The inner product of H is also denoted as $\langle \cdot, \cdot \rangle$. The reader can avoid any confusion by observing the context in which the symbols are used. For any Banach space X, by $K(X)$ and $B(X)$ we denote, respectively, the operator algebras of the compact and the bounded linear operators on X. The dual space of $K(H)$, the space of the trace class operators on H, is denoted by $C_1(H)$. The ultraweak topology of $B(H)$ is the weak* topology $\sigma(B(H), C_1(H))$. For x in a Banach space X and x^* in its dual, by $x \otimes x^*$ we denote the simple tensor, considered as an element of the dual space $K(X)^*$, that acts on $K(X)$ as well as on $B(X)$ through $\langle u, x \otimes x^*\rangle = \langle u(x), x^*\rangle$ for any $u \in K(H)$ (or $B(H)$). The composition of the two operators u and v in $B(X)$ is denoted as uv. For any Banach algebra A and a in A, by L_a and R_a we denote the left and right multiplication operators on A defined by $L_a(x) = ax$ and $R_a(x) = xa$, respectively. The algebra A is said to be completely continuous if, for each a in A, the multiplication operators L_a and R_a are compact. Finally, by a *weakly null sequence* in a Banach space we mean a sequence that converges to zero in the weak topology of the space.

MAIN RESULTS

In this section we present some results characterizing the closed subspaces and subalgebras of $K(H)$ whose duals have the Schur property. The first main result of the paper is the following theorem. The "if" part of this result is due to S.W. Brown ([2], Theorem 1.1 and Remark 3). We present only the proof of the "only if" part of it. For a in $K(H)$, a^* denotes the adjoint of a.

THEOREM 1. *Let X be a subspace of $K(H)$. Then X^* has the Schur property if and only if, for each x in H, the sets $X_1(x) = \{a(x) : a \text{ is in } X_1\}$ and $\widetilde{X}_1(x) = \{a^*(x) : a \text{ is in } X_1\}$ are relatively compact in H.*

Proof. Assume that X^* has the Schur property. Fix an x in H. In order to prove that the sets $X_1(x)$ and $\widetilde{X}_1(x)$ are relatively compact in H it is enough to show that any weakly null sequence (y_n) in H converges uniformly to zero on each of the sets $X_1(x)$ and $\widetilde{X}_1(x)$, see e.g. ([1], Proposition in p. 55). Now since the sequences $(x_n \otimes y)$ and $(y \otimes x_n)$ are in $C_1(H)$ and since, for u in $B(H)$, $\langle u, x_n \otimes y\rangle = \langle u(x_n), y\rangle$ and $\langle u, y_n \otimes x\rangle = \langle y_n, u^*(x)\rangle$, we see that the sequences $(x_n \otimes y)$ and $(y \otimes x_n)$ are weakly null in $C_1(H)$. As X^* is (linearly isometric to) $C_1(H)/X^{\perp}$, the sequences $(x_n \otimes y + X^{\perp})$ and $(y \otimes x_n + X^{\perp})$ are weakly null in X^*. Since X^* has the Schur property, these sequences converge to zero in the norm of X^*. It follows that

$$\sup\{|\langle a(x), y_n\rangle| : a \in X_1\} = \sup\{|\langle a, x \otimes y_n + X^{\perp}\rangle| : a \in X_1\} \to 0,$$

as n goes to infinity. Similarly, from

$$\begin{aligned}\sup\{|\langle a^*(x), y_n\rangle| : a \in X_1\} &= \sup\{|\langle x, a(y_n)\rangle| : a \in X_1\} \\ &= \sup\{|\langle a(y_n), x\rangle| : a \in X_1\} \\ &= \sup\{|\langle a, y_n \otimes x + X^{\perp}\rangle| : a \in X_1\} \to 0,\end{aligned}$$

as n goes to infinity, we conclude that the sets $X_1(x)$ and $\widetilde{X}_1(x)$ are relatively compact in H. ∎

The following corollary shows that (for $X \subset K(H)$) the Schur property on X^* can also be characterized in terms of X^{**}. For this, let B be the closed unit ball of X^{**}. Identify X^{**} with the ultraweak closure of X in $B(H)$ and let $\widetilde{B} = \{a^* : a \in B\}$. Then B and $\widetilde{B}$ are ultraweakly compact but need not be compact in the strong operator (= so) topology of $B(H)$. The next result shows that this is the case only if X^* has the Schur property. At this point we recall that for the von Neumann subalgebras of $B(H)$ this result is known, as proved by Hamana in Theorem 3, [6].

COROLLARY 2. *Let X be a subspace of $K(H)$. Then X^* has the Schur property if and only if the sets B and $\widetilde{B}$ are compact in* $(B(H), \text{so})$.

Proof. Assume first that the sets B and $\widetilde{B}$ are compact in $(B(H), \text{so})$. For each x in H, let $\tau_x : B(H) \to H$ be the evaluation mapping defined by $\tau_x(u) = u(x)$. This mapping is continuous from $(B(H), \text{so})$ into H. It follows that the sets $\tau_x(B) = \{u(x) : u \in B\}$ and $\tau_x(\widetilde{B}) = \{u^*(x) : u \in B\}$ are compact in H. As $X_1(x) \subseteq \tau_x(B)$ and $\widetilde{X}_1(x) \subseteq \tau_x(\widetilde{B})$, by the above theorem, we conclude that X^* has the Schur property.

Conversely, assume that X^* has the Schur property. Then, again by the above theorem, the sets $X_1(x)$ and $\widetilde{X}_1(x)$ are relatively (norm) compact in H. The sets B and $\widetilde{B}$ being convex and ultraweakly compact in $B(H)$ and τ_x being continuous from $(B(H), \text{ultraweak})$ into (H, weak), the sets $B(x) = \tau_x(B)$ and $\widetilde{B}(x) = \tau_x(\widetilde{B})$ are weakly (hence norm) closed in H. On the other hand, the set X_1 being dense in (B, weak^*) (by Goldstine Lemma, [4], p. 13]) and the set $\widetilde{X}_1$ being dense in $(\widetilde{B}, \text{weak}^*)$ (since the involution on $B(H)$ is continuous in the ultraweak topology), we conclude that the closures of the sets $X_1(x)$ and $\widetilde{X}_1(x)$ in H are equal to $B(x)$ and $\widetilde{B}(x)$, respectively. It follows that the sets $B(x)$ and $\widetilde{B}(x)$ are compact in H. This being true for each x in H, B and $\widetilde{B}$ are compact in $(B(H), \text{so})$. ∎

As a simple application of this result, let $H = \ell^2$ and (e_n) be the standard basis of ℓ^2. For $\lambda = (\lambda_n)$ in ℓ^∞, let $a_\lambda = \sum_{n \geqslant 1} \lambda_n e_n$ be the diagonal operator defined on ℓ^2 associated to λ. Then, for each x in H, the set $B(x) = \{a_\lambda(x) : \|\lambda\|_\infty \leqslant 1\}$ is compact in ℓ^2. From this it follows that the closed unit ball of c_0^{**}, considered as a subset of $B(H)$, is compact in the strong operator topology of this space. As we also have $B = \widetilde{B}$, we conclude that ℓ^1 has the Schur property.

As another application of the above theorem we have the following result. Since a reflexive Banach space has the Schur property only if it is finite dimensional, this result is immediate.

COROLLARY 3. *Let X be a reflexive subspace of $K(H)$. Then X is finite dimensional if and only if, for each x in H, the sets $X_1(x)$ and $\widetilde{X}_1(x)$ are relatively compact in H.*

We recall that a Banach space X is said to have the Dunford-Pettis property (= DPp) if, for each weakly null sequence (x_n) in X and (f_n) in X^*, we have that $\langle x_n, f_n \rangle \to 0$, as n goes to infinity. The DPp and the Schur property are connected through the following result ([5], Theorem 3): The dual X^* of a Banach space X has the Schur property if and only if X has the DPp and does not contain an isomorphic copy of ℓ^1. Since $K(H)$ does not contain an isomorphic copy of ℓ^1 (see e.g. [3], Corollary 1.12), as an immediate corollary of the above theorem we have the following result.

COROLLARY 4. *A closed subspace X of $K(H)$ has the* DPp *if and only if for each x in H, the sets $X_1(x)$ and $\widetilde{X}_1(x)$ are relatively compact in H.*

Now we shall consider the closed subalgebras of $K(H)$. The next two results, which will be used below, are of independent interest. In these results Y is an arbitrary reflexive Banach space. By S we denote the closed unit ball of Y endowed with the relative weak topology of Y. By $C(S,Y)$ we denote the space of the continuous functions $\varphi : S \to Y$ equipped with the supremum norm. The space $K(Y)$ embeds in a natural and isometric fashion into $C(S,Y)$. Below we will consider $K(Y)$ as a subspace of $C(S,Y)$.

THEOREM 5. *Any commutative closed subalgebra A of $K(Y)$ is completely continuous.*

Proof. Let A be a commutative closed subalgebra of $K(Y)$, and a be an arbitrary element of A. We have to show that the set $A_1 a$ is relatively compact in $C(S,Y)$. To show this, by the vector valued version of the Ascoli Theorem, it is enough to show that, for each x in S, the set $\{A_1 a\}(x) = \{ba(x) : b \in A_1\}$ is relatively compact in Y and the set $A_1 a$ is equicontinous on S. Since A is commutative and a is a compact operator, $\{ba(x) : b \in A_1\} = a(\{b(x) : b \in A_1\})$ is a relatively compact subset of Y. It also follows from the compactness of a that given $\varepsilon > 0$, there exists a weak neighborhood V of o such that $\|a(v)\| < \varepsilon$ for all $v \in V$. Therefore $\|ba(v)\| < \varepsilon$ for all $b \in A_1$, and equicontinuty of $A_1 a$ follows. This proves that the set $A_1 a$ is relatively compact, and the algebra A is completely continuous.

PROPOSITION 6. *Let A be a closed subalgebra of $K(Y)$. If A^* has the Schur property then A is completely continuous.*

Proof. Let us first show that, for each a in $K(Y)$, the multiplication operators L_a and R_a, as mappings from $K(Y)$ into itself, are weakly compact. To this purpose, fix an element a in $K(Y)$. Let (b_n) be a sequence in the unit ball of $K(Y)$. Since $K(Y)$ does not contain an isomorphic copy of ℓ^1 ([3], Corollary 1.12), by Rosenthal's ℓ^1-Theorem ([9]), (b_n) has a weakly Cauchy subsequence, which is denoted again by (b_n). It follows that, for y in Y and y^* in Y^*, the sequence $(\langle b_n, y \otimes y^* \rangle) = (\langle b_n(y), y^* \rangle)$ converges. This means that the sequence $(b_n(y))$ is weakly Cauchy in Y. Since the space Y is reflexive, the sequence $(b_n(y))$ converges weakly to some element $u(y)$ of Y. By the Uniform Boundedness Principle, the map $y \to u(y)$ defines a bounded linear operator on Y. As $K(Y)$ is an ideal in $B(Y)$, the operators au and ua are in $K(Y)$. Moreover, for each y in Y, $ab_n(y) \to au(y)$, and $b_n a(y) \to ua(y)$ weakly in Y. From this, by Corollary 1 in Kalton ([7], p. 268), we conclude that the sequences (ab_n) and $(b_n a)$ converge in the weak topology of the space $K(Y)$ to au and ua, respectively. This proves that the mappings L_a and R_a from $K(Y)$ into itself, are weakly compact. Now assume that a is in A. Then, by what precedes, the mappings L_a and R_a, from A into itself, are weakly compact. So their adjoints, L_a^* and R_a^*, are weakly compact mappings on A^*. Since A^* has the Schur property, L_a^* and R_a^* are compact. It follows that L_a and R_a are compact, and A is completely continuous. ∎

The next theorem, taking into consideration Theorem 5 above, is the non-commutative version of Theorem 1.1 of S.W. Brown ([2]). In this theorem A is a closed subalgebra of $K(H)$, $M = \text{Span}\{a(x) : a \text{ is in } A \text{ and } x \text{ in } H\}$ and $\widetilde{M} = \text{Span}\{a^*(x) : a \text{ is in } A \text{ and } x \text{ in } H\}$. We assume, as does Brown, that the sets M and $\widetilde{M}$ are dense in H.

THEOREM 7. *The dual A^* of the algebra A has the Schur property if and only if A is completely continuous.*

Proof. The direct implication is clear by the preceding proposition. To prove the backward implication, assume that A is completely continuous. Let us see that, for each x in H, the sets $A_1(x) = \{a(x) : a \in A_1\}$ and $\widetilde{A}_1(x) = \{a^*(x) : a \in A_1\}$ are relatively compact in H. To this end fix an x in H and $\varepsilon > 0$ arbitrarily. Since M is dense in H, there exists an element $y = \lambda_1 a_1(x_1) + \cdots + \lambda_n a_n(x_n)$ in M such that $\|x - y\| < \varepsilon$. As A is completely continuous, each of the sets $\lambda_1 A_1 a, \lambda_2 A_1 a_2, \ldots, \lambda_n A_1 a_n$ is relatively compact in $K(H)$. It follows that the sets $(\lambda_1 A_1 a_1)(x_1), (\lambda_2 A_1 a_2)(x_2), \ldots, (\lambda_n A_1 a_n)(x_n)$ are relatively compact in H. Let $K_\varepsilon = (\lambda_1 A_1 a_1)(x_1) + \cdots + (\lambda_n A_1 a_n)(x_n)$ be the sum of these sets. The set K_ε is relatively compact in H and we have $A_1(x) \subseteq K_\varepsilon + \varepsilon H_1$. This proves that the set $A_1(x)$ is relatively compact in H. Similarly, using the facts that $(ab)^* = b^* a^*$, $\widetilde{M}$ is dense in H and that aA_1 is relatively compact, we show as above that, for each x in H, the set $\widetilde{A}_1(x)$ is also relatively compact in H. Hence, by Theorem 1, A^* has the Schur property.

Needless to say that the preceding theorem is rather particular to the subalgebras of $K(H)$. The dual of an arbitrary completely continuous Banach algebra in general does not have the Schur property, as the following simple example shows. For $1 \leqslant p < \infty$, ℓ^p is a completely continuous Banach algebra with the coordinatewise multiplication; but the dual space of this algebra does not have the Schur property. We finish the paper with some remarks.

REMARK 8. (i) In Theorem 7 above, to pass from the relative compactness of the set $A_1 a$ to that of $A_1(x)$, the density of the set M in H seems to be necessary. However we do not know whether it is indispensable.

(ii) It is easy to see that, for a closed subalgebra A of $K(H)$, the spaces M and $\widetilde{M}$ are dense in H if and only if both A and $\widetilde{A} = \{a^* : a \in A\}$ separate the points of H.

(iii) For a closed subalgebra A of $K(H)$ and x in H, let $A(x) = \{a(x) : a \in A\}$. If, for each x in H, x is in the closure of the set $A(x)$ then, as one can see very easily, the sets M and $\widetilde{M}$ are dense in H.

(iv) If A is a closed self-adjoint completely continuous subalgebra of $K(H)$, then the subspace $E = \{x \in H : A_1(x)$ is relatively compact in $H\}$ of H is a reducing subspace for A and, considering A as a subalgebra of $K(E)$, from Theorem 7 above, one deduces, without any other hypothesis, that A^* has the Schur property.

(v) As is well-known, ℓ^∞ has the so-called Grothendieck property, i.e., in $(\ell^\infty)^*$ the weak* convergent sequences are weakly convergent. We wonder whether the second duals of other subspaces of $K(H)$ have the Grothendieck property. We recall that, as proved by H. Pfitzner ([8]), any von Neumann algebra, in particular $K(H)^{**} = B(H)$, has this property.

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