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STABLE RANK OF THE C^*-ALGEBRAS OF SOLVABLE LIE GROUPS OF TYPE I

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ABSTRACT. In this paper we show that the stable rank of the C^*-algebras of simply-connected connected solvable Lie groups of type I is estimated by the complex dimension of the fixed point subspaces of the real dual spaces of their Lie algebras under the coadjoint actions. This result generalizes the estimation in the case of simply-connected connected nilpotent Lie groups. As corollaries, we show that the product formula of the stable rank holds for the C^*-algebras of connected solvable Lie groups of type I, and estimate the real rank in the case of simply-connected connected solvable Lie groups of type I.

KEYWORDS: *Stable rank, solvable Lie group, coadjoint orbit space, polarization.*

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1. INTRODUCTION

M.A. Rieffel ([8]) introduced the notion of stable rank of C^*-algebras, i.e. non commutative complex dimension, and raised the problem of describing the stable rank of the C^*-algebras of Lie groups in terms of their geometry. First of all, A.J-L. Sheu ([9]) succeeded in the computation of the stable rank of the C^*-algebras of certain simply-connected connected nilpotent Lie groups. By different methods, we showed that the stable rank of the C^*-algebras of simply-connected connected nilpotent Lie groups is equal to the complex dimension of the fixed point subspace of the real dual spaces of their Lie algebras under the coadjoint actions ([11]). This formula is not valid in the case of exponential Lie groups in general, for example $ax + b$-groups.

In this paper we first analyze the spectra of simply-connected connected solvable Lie groups of type I. This is crucial to the computation of the stable rank of their C^*-algebras. Next we show that the stable rank of these algebras is estimated by the complex dimension of the fixed point subspaces of the real dual spaces of their Lie algebras under the coadjoint actions. As corollaries, we show that the product formula of the stable rank holds for the C^*-algebras of connected solvable Lie groups of type I, and compute the real rank in the case of simply-connected connected solvable Lie groups of type I using the estimation of the stable rank.

2. SPECTRUM OF SOLVABLE LIE GROUPS OF TYPE I

In this section we show that every irreducible representation of simply-connected connected solvable Lie groups of type I is either 1 or ∞ dimensional. This property is crucial to the estimation of the stable rank of the C^*-algebras of those groups. Also we show that 1-dimensional representations of such groups correspond naturally to the fixed points of the real dual spaces of their Lie algebras under the coadjoint actions.

Let G be a connected Lie group. We denote by $\widehat{G}$ the spectrum of G, i.e. the set of all continuous irreducible unitary representations of G up to unitary equivalence equipped with hull-kernel topology. Let $\widehat{G}_1$, $\widehat{G}_\infty$ be the set of all 1, ∞-dimensional irreducible representations of G respectively. We call $\widehat{G}_1$ the character space of G, which is a topological group with the pointwise multiplication. Let $C^*(G)$ be the C^*-algebra of G, which is generated by the image of the universal unitary representation of G. We identify $\widehat{G}$ with the spectrum $C^*(G)^\wedge$ of $C^*(G)$. Then we show in what follows that $\widehat{G} = \widehat{G}_1 \cup \widehat{G}_\infty$ if G is a simply-connected connected solvable Lie group of type I.

Let $\mathfrak{G}$ be the Lie algebra of G and $\mathfrak{G}^*$ the real dual space of $\mathfrak{G}$. We denote by Ad the adjoint action of G on $\mathfrak{G}$ and by Ad^* the coadjoint action of G on $\mathfrak{G}^*$ defined by $\mathrm{Ad}^*(g)\varphi(X) = \varphi(\mathrm{Ad}(g^{-1})(X))$ for g in G, X in $\mathfrak{G}$ and φ in $\mathfrak{G}^*$. We denote by $(\mathfrak{G}^*)^G$ the fixed point subspace of $\mathfrak{G}^*$ under Ad^*. Note that $(\mathfrak{G}^*)^G$ is isomorphic to a Euclidean space as a topological (vector) group. Then the following lemma holds:

LEMMA 2.1. *Let G be a simply-connected connected Lie group. Then $\widehat{G}_1$ is isomorphic to $(\mathfrak{G}^*)^G$ as a topological group.*

Proof. Let χ be an element of $\widehat{G}_1$, i.e. a Lie homomorphism from G to 1-torus $\mathbf{T}$. Then its differential $\mathrm{d}\chi$ defined by $\mathrm{d}\chi(X) = \frac{\mathrm{d}}{\mathrm{d}t}\chi(\exp tX)|_{t=0}$ for any X

in $\mathfrak{G}$ is a Lie homomorphism from $\mathfrak{G}$ to $\mathrm{i}\mathbb{R}$. Then the following diagram commutes (cf. [4]):

$$\begin{array}{ccc} G & \xrightarrow{\chi} & \mathbb{T} \\ \exp \uparrow & & \exp \uparrow \\ \mathfrak{G} & \xrightarrow{\mathrm{d}\chi} & \mathrm{i}\mathbb{R}. \end{array}$$

Then

$$\begin{aligned} \mathrm{Ad}^*(\exp(Y))\left(\frac{\mathrm{d}\chi}{2\pi\mathrm{i}}\right)(X) &= \left(\frac{\mathrm{d}\chi}{2\pi\mathrm{i}}\right)(\mathrm{Ad}(\exp(-Y))X) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\chi}{2\pi\mathrm{i}}\right)(\exp t(\mathrm{Ad}(\exp(-Y))X))|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\chi}{2\pi\mathrm{i}}\right)(\exp(-Y)\exp tX\exp(Y))|_{t=0} \\ &= \left(\frac{\mathrm{d}\chi}{2\pi\mathrm{i}}\right)(X) \end{aligned}$$

for every X, Y in $\mathfrak{G}$. Hence $\mathrm{d}\chi/2\pi\mathrm{i}$ is in $(\mathfrak{G}^*)^G$. Let Φ be the mapping from $\widehat{G}_1$ to $(\mathfrak{G}^*)^G$ defined by $\Phi(\chi) = \mathrm{d}\chi/2\pi\mathrm{i}$. It is clear that Φ is well defined.

Let χ_1, χ_2 be in $\widehat{G}_1$. Then by definition, $\mathrm{d}(\chi_1 \cdot \chi_2) = \mathrm{d}\chi_1 + \mathrm{d}\chi_2$. Thus $\Phi(\chi_1 \cdot \chi_2) = \Phi(\chi_1) + \Phi(\chi_2)$.

Suppose that $\Phi(\chi_1) = \Phi(\chi_2)$ for χ_1, χ_2 in $\widehat{G}_1$. Then $\mathrm{d}\chi_1 = \mathrm{d}\chi_2$. By the connectedness of G, every element g of G has the form $\exp X_1 \cdots \exp X_n$ for some $X_1, \ldots, X_n$ in $\mathfrak{G}$. By the above commutative diagram, we have $\chi_1 = \chi_2$.

Let φ be an element of $(\mathfrak{G}^*)^G$. Then $2\pi\mathrm{i}\varphi$ is a Lie homomorphism from $\mathfrak{G}$ to $\mathrm{i}\mathbb{R}$. By the simply-connectedness of G, there exists an element χ in $\widehat{G}_1$ such that $\mathrm{d}\chi = 2\pi\mathrm{i}\varphi$. Thus $\Phi(\chi) = \varphi$.

Next let $\{\chi_n\}$ be a sequence of $\widehat{G}_1$ converging to χ in $\widehat{G}_1$. By the above diagram, we have that $\chi_n(\exp tX) = \mathrm{e}^{t\mathrm{d}\chi_n(X)}$, $\chi(\exp tX) = \mathrm{e}^{t\mathrm{d}\chi(X)}$ for any X in $\mathfrak{G}$ and small t. Thus $\{\mathrm{Log}(\mathrm{e}^{t\mathrm{d}\chi_n(X)})\}$ converges to $\mathrm{Log}(\mathrm{e}^{t\mathrm{d}\chi(X)})$ where Log is the principal branch of log. Hence $\{\mathrm{d}\chi_n(X)\}$ converges to $\mathrm{d}\chi(X)$. Thus $\{\mathrm{d}\chi_n\}$ converges to $\mathrm{d}\chi$.

Conversely, let $\{\varphi_n\}$ be a sequence of $(\mathfrak{G}^*)^G$ converging to φ in $(\mathfrak{G}^*)^G$. Let χ_n and χ be in $\widehat{G}_1$ such that $\mathrm{d}\chi_n = 2\pi\mathrm{i}\varphi_n$ and $\mathrm{d}\chi = 2\pi\mathrm{i}\varphi$ respectively. Then $\{\chi_n\}$ converges to χ by the continuity of exponential map. ∎

REMARK 2.2. There exist some non simply-connected connected solvable Lie groups of type I, for which the above lemma is false. In fact, let G be the n-dimensional torus $\mathbb{T}^n$. Then $(\mathfrak{G}^*)^G = \mathbb{R}^n$. On the other hand, $\widehat{G} = \mathbb{Z}^n$.

LEMMA 2.3. *Let G be a connected Lie group. Then $\widehat{G}_1$ is isomorphic to $(G/[G,G])^\wedge$ as a topological group where $[G,G]$ is the commutator subgroup of G.*

Proof. We consider the mapping Φ from $(G/[G,G])^\wedge$ to $\widehat{G}_1$ defined by $\Phi(\chi) = \chi \circ q$ for χ in $\widehat{G}_1$ where q is the quotient mapping from G to $G/[G,G]$. Since $G/[G,G]$ is abelian, $(G/[G,G])^\wedge = (G/[G,G])^\wedge_1$. It is clear that Φ is an injective homomorphism. For η in $\widehat{G}_1$, let $\tilde{\eta}$ be a well-defined element of $(G/[G,G])^\wedge$ by $\tilde{\eta}(g[G,G]) = \eta(g)$ for $g[G,G]$ in $G/[G,G]$. Hence $\eta = \tilde{\eta} \circ q$. By definition, it is clear that Φ is homeomorphic. ∎

REMARK 2.4. Since $G/[G,G]$ is a connected commutative Lie group, it is isomorphic to $\mathbf{R}^k \times \mathbf{T}^{n-k}$ for some $k \geqslant 0$ where $n = \dim(G/[G,G])$. Thus, by Lemma 2.1,

$$\widehat{G}_1 \cong (G/[G,G])^\wedge \cong \mathbf{R}^k \times \mathbf{Z}^{n-k}$$

as a topological group. If G is a simply-connected connected Lie group, then it follows from Lemmas 2.1 and 2.3 that

$$(\mathfrak{G}^*)^G \cong \widehat{G}_1 \cong (G/[G,G])^\wedge \cong \mathbf{R}^n$$

as a topological group.

Next we recall briefly the representation theory of simply-connected connected solvable Lie groups of type I by Auslander and Kostant ([1]):

Let $\mathfrak{G}_{\mathbf{C}}$ be the complexification of $\mathfrak{G}$ and $\mathfrak{G}_{\mathbf{C}}^*$ its dual space. Let φ be an element of $\mathfrak{G}^*$. We denote by G_φ (resp. $[\varphi]$) the stabilizer (resp. the orbit) of φ with respect to the coadjoint action of G and by $\mathfrak{G}_\varphi$ its Lie algebra, which equals the radical of φ, i.e. $\{X \in \mathfrak{G} \mid \varphi([X,Y]) = 0 \text{ for every } Y \in \mathfrak{G}\}$. We extend φ to an element of $\mathfrak{G}_{\mathbf{C}}^*$ by $\varphi(X + \mathrm{i}Y) = \varphi(X) + \mathrm{i}\varphi(Y)$ for $X + \mathrm{i}Y$ in $\mathfrak{G}_{\mathbf{C}}$. Let $\mathfrak{H}$ be a polarization for φ, which satisfies the following conditions:

(i) $\mathfrak{H}$ is a Lie subalgebra of $\mathfrak{G}_{\mathbf{C}}$;
(ii) $\mathfrak{H}$ contains $\mathfrak{G}_\varphi$ and is stable under $\mathrm{Ad}(G_\varphi)$;
(iii) $\varphi([\mathfrak{H},\mathfrak{H}]) = \{0\}$;
(iv) $\dim_{\mathbf{C}}(\mathfrak{G}_{\mathbf{C}}/\mathfrak{H}) = \frac{1}{2}\dim_{\mathbf{R}}[\varphi]$;
(v) $\mathfrak{H} + \overline{\mathfrak{H}}$ is a Lie subalgebra of $\mathfrak{G}_{\mathbf{C}}$,

where $\overline{\mathfrak{H}}$ is the conjugate space of $\mathfrak{H}$ in $\mathfrak{G}_{\mathbf{C}}$.

Put $\mathfrak{H} \cap \mathfrak{G} = \mathfrak{D}$ and $(\mathfrak{H} + \overline{\mathfrak{H}}) \cap \mathfrak{G} = \mathcal{E}$. Then $\mathfrak{D}_{\mathbf{C}} = \mathfrak{H} \cap \overline{\mathfrak{H}}$ and $\mathcal{E}_{\mathbf{C}} = \mathfrak{H} + \overline{\mathfrak{H}}$. Let D_o and E_0 be the connected Lie subgroups of G corresponding to Lie algebras $\mathfrak{D}$ and $\mathcal{E}$ respectively. Put $D = G_\varphi D_0$ and $E = G_\varphi E_0$. Then it holds that $E = DE_0$. We have that $\mathrm{Ad}^*(D)\varphi$ is open in the affine subspace $\varphi + \mathcal{E}^\perp$ of $\mathfrak{G}^*$ where $\mathcal{E}^\perp$ is the annihilator of $\mathcal{E}$.

We define an alternating bilinear form $\overline{B}_\varphi$ on $\mathcal{E}/\mathfrak{D}$ by

$$\overline{B}_\varphi(\overline{X},\overline{Y}) = \varphi([Y,X])$$

for φ in $\mathfrak{G}^*$ and $\overline{X}, \overline{Y}$ in $\mathcal{E}/\mathfrak{D}$. Then it is a non-singular alternative form on $\mathcal{E}/\mathfrak{D}$. $(\mathcal{E}/\mathfrak{D})_{\mathbf{C}}$ is identified with $\mathcal{E}_{\mathbf{C}}/\mathfrak{D}_{\mathbf{C}}$. Then $(\mathcal{E}/\mathfrak{D})_{\mathbf{C}} = \mathfrak{H}/\mathfrak{D}_{\mathbf{C}} \oplus \overline{\mathfrak{H}}/\mathfrak{D}_{\mathbf{C}}$ where $\oplus$ is the direct sum. Let J be a linear mapping of $(\mathcal{E}/\mathfrak{D})_{\mathbf{C}}$ defined by $J = -\mathrm{i}I$ on $\mathfrak{H}/\mathfrak{D}_{\mathbf{C}}$ and $J = \mathrm{i}I$ on $\overline{\mathfrak{H}}/\mathfrak{D}_{\mathbf{C}}$. Then J maps $\mathcal{E}/\mathfrak{D}$ onto itself, and $J^2 = -I$ on $\mathcal{E}/\mathfrak{D}$. Let S_φ be the bilinear form on $\mathcal{E}/\mathfrak{D}$ defined by

$$S_\varphi(\overline{X}, \overline{Y}) = \overline{B}_\varphi(J\overline{X}, \overline{Y}).$$

Then it is a non-singular symmetric bilinear form on $\mathcal{E}/\mathfrak{D}$. We say that a polarization $\mathfrak{H}$ for φ is positive if S_φ is positive definite.

Let $\mathfrak{N}$ be the maximal nilpotent ideal of $\mathfrak{G}$. Since it is stable under $\mathrm{Ad}(G)$, so is $\mathfrak{N}^*$ under $\mathrm{Ad}^*(G)$. A polarization $\mathfrak{H}$ for φ is called strongly admissible if $\mathfrak{H} \cap \mathfrak{N}_{\mathbf{C}}$ is a polarization for $\varphi|\mathfrak{N}$ in $\mathfrak{N}^*$, which is stable under $G_{\varphi|\mathfrak{N}}$ where $\varphi|\mathfrak{N}$ is the restriction of φ to $\mathfrak{N}$.

We say that a polarization $\mathfrak{H}$ for φ satisfies Pukanszky condition if $\mathrm{Ad}^*(E)\varphi$ is closed in $\mathfrak{G}^*$. If this condition is satisfied, then $\mathrm{Ad}^*(D)\varphi = \varphi + \mathcal{E}^\perp$. Any strongly admissible positive polarization satisfies Pukanszky condition.

Every element φ in $\mathfrak{G}^*$ is integral, i.e. there exists a character η_φ of G_φ whose differential $\mathrm{d}\eta_\varphi$ is equal to the restriction of $2\pi\mathrm{i}\varphi$ to $\mathfrak{G}_\varphi$. More precisely, it is defined by $\eta_\varphi(\exp X) = \mathrm{e}^{2\pi\mathrm{i}\varphi(X)}$ for X in $\mathfrak{G}_\varphi$. If a polarization $\mathfrak{H}$ for φ satisfies Pukanszky condition, then η_φ extends uniquely to a character χ_φ of D.

Let $L^2(E/D, \chi_\varphi)$ be the Hilbert space of all complex valued μ_E-measurable functions f on E satisfying

$$\chi_\varphi(d)^{-1} f(e) = f(ed)$$

for d in D and e in E, where μ_E is the Haar measure on E, and

$$\int_{E/D} |f(\overline{e})|^2 \, \mathrm{d}\mu_{E/D}(\overline{e}) < \infty$$

where $\mu_{E/D}$ is the quotient measure of μ_E on E/D and $\overline{e} = eD$ in E/D. The inner product of $L^2(E/D, \chi_\varphi)$ is defined by

$$\langle f_1 | f_2 \rangle = \int_{E/D} f_1(\overline{e}) \overline{f_2(\overline{e})} \, \mathrm{d}\mu_{E/D}(\overline{e})$$

for f_1, f_2 in $L^2(E/D, \chi_\varphi)$. Then the induced representation $\mathrm{ind}_{D\uparrow E}\chi_\varphi$ of χ_φ to E on $L^2(E/D, \chi_\varphi)$ is defined by

$$(\mathrm{ind}_{D\uparrow E}\chi_\varphi)(h) f(e) = f(h^{-1}e)$$

for e, h in E.

Let $\mathfrak{H}$ be a strongly admissible positive polarization for φ. We denote by $L^2(E/D, \chi_\varphi, \mathfrak{H})$ the closed subspace of $L^2(E/D, \chi_\varphi)$ consisting of all smooth functions f on E with the property that

$$f \cdot Z = 2\pi \mathrm{i} \varphi(Z) f$$

for every Z in $\mathfrak{H}$ where $Z = X + \mathrm{i}Y$ for X, Y in $\mathcal{E}$, $f \cdot Z = f \cdot X + \mathrm{i} f \cdot Y$ and

$$f \cdot X(e) = \frac{\mathrm{d}}{\mathrm{d}t} f(e \exp(-tX))|_{t=0}$$

for e in E. In fact, differentiating both sides of the following equation:

$$\chi_\varphi(\exp tX)^{-1} f(e) = f(e \exp tX)$$

at $t = 0$ for X in $\mathcal{D}$, we have that $f \cdot X = 2\pi \mathrm{i} \varphi(X) f$. We denote by $\mathrm{ind}_{D\uparrow E}(\chi_\varphi, \mathfrak{H})$ the subrepresentation of $\mathrm{ind}_{D\uparrow E}\chi_\varphi$ corresponding to $L^2(E/D, \chi_\varphi, \mathfrak{H})$.

Let $L^2(G/E) \otimes L^2(E/D, \chi_\varphi, \mathfrak{H})$ be the Hilbert space of all $L^2(E/D, \chi_\varphi, \mathfrak{H})$-valued μ_G-measurable functions on G satisfying the similar conditions as above with respect to $\mathrm{ind}_{D\uparrow E}(\chi_\varphi, \mathfrak{H})$. We denote by $\mathrm{ind}_{E\uparrow G}(\mathrm{ind}_{D\uparrow E}(\chi_\varphi, \mathfrak{H}))$ the induced representation of $\mathrm{ind}_{D\uparrow E}(\chi_\varphi, \mathfrak{H})$ to G on $L^2(G/E) \otimes L^2(E/D, \chi_\varphi, \mathfrak{H})$. Let

$$\mathrm{ind}_{D\uparrow G}(\chi_\varphi, \mathfrak{H}) = \mathrm{ind}_{E\uparrow G}(\mathrm{ind}_{D\uparrow E}(\chi_\varphi, \mathfrak{H})).$$

Then we know that every element π in $\widehat{G}$ is equivalent to an induced representation $\mathrm{ind}_{D\uparrow G}(\chi_\varphi, \mathfrak{H})$ of G.

Note that E/D has a complex structure so that it is holomorphic to $\mathbf{C}^n$ for some $n \geqslant 0$. Let $\mathcal{A}(E/D)$ be the set of all holomorphic functions on E/D and $\tilde{\mathcal{A}}(E)$ the pull back of $\mathcal{A}(E/D)$ to E. We denote by $z_1^{k_1} \cdots z_n^{k_n}$ the functions of $\mathcal{A}(E/D)$ for $(k_1, \ldots, k_n)$ in $\mathbf{Z}_+^n$ with respect to a system of complex coordinates $(z_1, \ldots, z_n)$, where $\mathbf{Z}_+ = \{k \in \mathbf{Z} \mid k \geqslant 0\}$. Let $(z_1^{k_1} \cdots z_n^{k_n})^\sim$ be the pull back of $z_1^{k_1} \cdots z_n^{k_n}$ to E. Then there exists a nowhere vanishing smooth function f on E such that $\{(z_1^{k_1} \cdots z_n^{k_n})^\sim f\}$ for $\{(k_1, \ldots, k_n)\}$ in $\mathbf{Z}_+^n$ are in $L^2(E/D, \chi_\varphi, \mathfrak{H})$. Then

$$\langle (z_1^{k_1} \cdots z_n^{k_n})^\sim f | (z_1^{l_1}, \ldots, z_n^{l_n})^\sim f \rangle = 0, \quad \text{for } (k_1, \ldots, k_n) \neq (l_1, \ldots, l_n) \in \mathbf{Z}_+^n.$$

We now show the following lemma:

LEMMA 2.5. *Let G be a simply-connected connected solvable Lie group of type* I. *Then* $\widehat{G} = \widehat{G}_1 \cup \widehat{G}_\infty$.

Proof. We use the above observation. Let π be an element of $\widehat{G}$, which is equivalent to some $\mathrm{ind}_{D\uparrow G}\chi_\varphi$. If $\mathfrak{D} = \mathfrak{G}$, then $\mathfrak{H} \cap \overline{\mathfrak{H}} = \mathfrak{G}_{\mathbf{C}}$. Hence $\mathfrak{H} = \mathfrak{G}_{\mathbf{C}}$. It implies that $\varphi([\mathfrak{G}, \mathfrak{G}]) = 0$. Thus φ is in $(\mathfrak{G}^*)^G$. Therefore $\pi = \chi_\varphi$.

Next suppose that $\mathfrak{D} \neq \mathfrak{G}$. If $\dim(E/D) > 0$, then $L^2(E/D, \chi_\varphi, \mathfrak{H})$ is infinite dimensional. If $\dim(E/D) = 0$, then $E_0 = \{1\}$, namely $E = D$. Since D_0 contains $(G_\varphi)_0$ which is the connected component of G_φ containing the unit, $D/D_0 = G_\varphi D_0/D_0$ is diffeomorphic to $G_\varphi/(D_0 \cap G_\varphi) = G_\varphi/(G_\varphi)_0$. Thus $\dim D = \dim D_0$, which implies $\dim(G/E) > 0$. Hence, $\mathrm{ind}_{D\uparrow G}\chi_\varphi$ is infinite dimensional. ∎

Moreover, the following lemma holds:

LEMMA 2.6. *Let G be a connected Lie group. Then $\widehat{G}_1$ is closed in $\widehat{G}$.*

Proof. Let π be in the closure of $\widehat{G}_1$. Let $\varphi_{\pi,\xi}$ be the state of $C^*(G)$ defined by $\varphi_{\pi,\xi}(a) = \langle \pi(a)\xi | \xi \rangle$ for a in $C^*(G)$ and ξ in the representation space H_π of π with $\|\xi\| = 1$ where $\langle \cdot | \cdot \rangle$ means the inner product of H_π. By Theorem 3.4.10 in [3], we have that for any a, b in $C^*(G)$, there exist $\{\chi_i\}$ in $\widehat{G}_1$ and $\{\alpha_i\}$ in $\mathbf{C}$ such that $\varphi_{\pi,\xi}(ab - ba) = \sum_{i=1}^{\infty} \alpha_i \chi_i(ab - ba) = 0$. Since ξ is arbitrary, $\pi(ab) = \pi(ba)$. From the irreducibility of π, it belongs to $\widehat{G}_1$. Therefore $\widehat{G}_1$ is closed in $\widehat{G}$. ∎

REMARK 2.7. The similar result also holds for arbitrary C^*-algebras.

Combining Lemmas 2.5 and 2.6, we have the following:

LEMMA 2.8. *Let G be a simply-connected connected solvable Lie group of type* I *and $C^*(G)$ its C^*-algebra. Let $\mathfrak{J}$ be the closed ideal of $C^*(G)$ corresponding to $\widehat{G}_\infty$ and $C_0(\widehat{G}_1)$ the C^*-algebra of all continuous functions on $\widehat{G}_1$ vanishing at infinity. Then the following exact sequence is obtained:*

$$0 \to \mathfrak{J} \to C^*(G) \to C_0(\widehat{G}_1) \to 0.$$

REMARK 2.9. The similar result also holds for connected solvable Lie groups where $\mathfrak{J}$ corresponds to $\widehat{G} \setminus \widehat{G}_1$.

3. MAIN THEOREMS

In this section we prove that the stable rank of the C^*-algebras of simply-connected connected solvable Lie groups of type I is estimated by the complex dimension of the fixed point subspaces of the real dual spaces of their Lie algebras under the coadjoint actions. Before doing this, we prove some useful propositions for computation of the stable rank. As corollaries, we show that the product formula of stable rank holds for the C^*-algebras of connected solvable Lie groups of type I, and estimate the real rank in the case of simply-connected connected solvable Lie groups of type I. First of all, we recall the definitions of stable rank and real rank respectively.

Let $\mathfrak{A}$ be a unital C^*-algebra. We denote by $\mathrm{sr}(\mathfrak{A})$ the stable rank of $\mathfrak{A}$. Then $\mathrm{sr}(\mathfrak{A}) \leqslant n$ if every element $(a_i)_{i=1}^n$ of the n-direct sum $\mathfrak{A}^n$ of $\mathfrak{A}$ can be approximated by the element $(b_i)_{i=1}^n$ of $\mathfrak{A}^n$ such that $\sum_{i=1}^n b_i^* b_i$ is invertible in $\mathfrak{A}$. If there exists no such n, then we let $\mathrm{sr}(\mathfrak{A}) = \infty$. If $\mathfrak{A}$ is non unital, then the stable rank of $\mathfrak{A}$ is defined by $\mathrm{sr}(\tilde{\mathfrak{A}})$ where $\tilde{\mathfrak{A}}$ means the unitization of $\mathfrak{A}$. We use later the basic results of stable rank in [8].

Let $\mathfrak{A}_{\mathrm{sa}}$ be the set of all self-adjoint elements of $\mathfrak{A}$. We denote by $\mathrm{rr}(\mathfrak{A})$ the real rank of $\mathfrak{A}$. Then $\mathrm{rr}(\mathfrak{A}) \leqslant n$ means that every element $(a_i)_{i=0}^n$ of $\mathfrak{A}_{\mathrm{sa}}^{n+1}$ can be approximated by elements $(b_i)_{i=0}^n$ such that $\sum_{i=0}^n b_i^2$ is invertible in $\mathfrak{A}$. If there exists no such n, then we let $\mathrm{rr}(\mathfrak{A}) = \infty$. If $\mathfrak{A}$ is non unital, then the real rank of $\mathfrak{A}$ is defined by $\mathrm{rr}(\tilde{\mathfrak{A}})$ (cf. [2]).

The next result is related in a certain sense with the formula such that $\mathrm{sr}(\mathfrak{A} \otimes \mathbf{K}) \leqslant 2$ for arbitrary C^*-algebra $\mathfrak{A}$ where $\mathbf{K}$ is the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space.

PROPOSITION 3.1. *Let $\mathfrak{A}$ be a separable C^*-algebra of type* I *such that every element of $\hat{\mathfrak{A}}$ is infinite dimensional. Then* $\mathrm{sr}(\mathfrak{A}) \leqslant 2$.

Proof. Let $\{\mathcal{I}_n\}_{n=1}^\infty$ be a composition series of $\mathfrak{A}$ with $\mathcal{I}_0 = 0$ such that $\{\mathcal{I}_n/\mathcal{I}_{n-1}\}_{n=1}^\infty$ are of continuous trace. Consider the following exact sequences:

$$0 \to \mathcal{I}_n/\mathcal{I}_{n-1} \to (\mathcal{I}_n/\mathcal{I}_{n-1})^{\sim} \to \mathbb{C} \to 0$$

for every n. By Nistor's result ([6], Lemma 2),

$$\mathrm{sr}((\mathcal{I}_n/\mathcal{I}_{n-1})^{\sim}) \leqslant 2 \vee \mathrm{sr}(\mathbb{C}) = 2,$$

where $\vee$ means maximum. Hence $\mathrm{sr}(\mathcal{I}_n/\mathcal{I}_{n-1}) \leqslant 2$ for every n. Next consider the following exact sequences:

$$0 \to \mathcal{I}_k/\mathcal{I}_{k-1} \to \mathcal{I}_n/\mathcal{I}_{k-1} \to \mathcal{I}_n/\mathcal{I}_k \to 0$$

for $1 \leqslant k \leqslant n-1$. Again by Nistor's result,

$$\mathrm{sr}(\mathcal{I}_n/\mathcal{I}_{k-1}) \leqslant 2 \vee \mathrm{sr}(\mathcal{I}_n/\mathcal{I}_k)$$

for $1 \leqslant k \leqslant n-1$. It follows that $\mathrm{sr}(\mathcal{I}_n) \leqslant 2$ for every n. By the density of $\bigcup_{n=1}^{\infty} \mathcal{I}_n$ in $\mathfrak{A}$, we conclude that $\mathrm{sr}(\mathfrak{A}) \leqslant 2$. ∎

As a first step of the computation of the stable rank of the C^*-algebras of simply-connected connected solvable Lie groups of type I, we have the following:

LEMMA 3.2. *Let G be a simply-connected connected solvable Lie group of type* I, *$\widehat{G}_1$ its character space and $C^*(G)$ its C^*-algebra. Then*

$$\mathrm{sr}(C^*(G)) \begin{cases} \leqslant 2 & \text{if } \dim \widehat{G}_1 = 1; \\ = \dim_{\mathbb{C}}(\widehat{G}_1) & \text{if } \dim \widehat{G}_1 \geqslant 2 \end{cases}$$

where $\dim_{\mathbb{C}}(\cdot) = [\dim(\cdot)/2] + 1$ *and* $[\,\cdot\,]$ *is the Gauss symbol.*

Proof. Put $\mathfrak{A} = C^*(G)$. Let $\{\mathfrak{I}_k\}_{k=1}^{\infty}$ be a composition series of $\mathfrak{A}$ with $\mathfrak{I}_0 = \{0\}$ such that $\{\mathfrak{I}_k/\mathfrak{I}_{k-1}\}_{k=1}^{\infty}$ are of continuous trace. We consider the following exact sequences:

$$0 \to \mathfrak{I} \cap \mathfrak{I}_k \to \mathfrak{I}_k \to C_0(\widehat{G}_1 \cap (\widehat{\mathfrak{I}_k} \setminus (\mathfrak{I} \cap \mathfrak{I}_k)^\wedge)) \to 0$$

for every k, where $\mathfrak{I}$ is the closed ideal of $\mathfrak{A}$ as in Lemma 2.8. Then $\{\mathfrak{I} \cap \mathfrak{I}_s\}_{s=1}^{k}$ is the finite composition series of $\mathfrak{I} \cap \mathfrak{I}_k$. Put $\mathfrak{D}_s = \mathfrak{I} \cap \mathfrak{I}_s$ for $1 \leqslant s \leqslant k$ with $\mathfrak{D}_0 = \{0\}$. Next we consider the following exact sequences:

$$0 \to \mathfrak{D}_s/\mathfrak{D}_{s-1} \to \mathfrak{I}_k/\mathfrak{D}_{s-1} \to \mathfrak{I}_k/\mathfrak{D}_s \to 0$$

for $1 \leqslant s \leqslant k$. Note that $\{\mathfrak{D}_s/\mathfrak{D}_{s-1}\}_{s=1}^{k}$ are of continuous trace, and every element of $(\mathfrak{D}_s/\mathfrak{D}_{s-1})^\wedge$ is infinite dimensional. Then applying Nistor's result ([6], Lemma 2),

$$\mathrm{sr}(\mathfrak{I}_k/\mathfrak{D}_{s-1}) \leqslant 2 \vee \mathrm{sr}(\mathfrak{I}_k/\mathfrak{D}_s)$$

for $1 \leqslant s \leqslant k$. By repetition, $\mathrm{sr}(\mathfrak{I}_k) \leqslant 2 \vee \mathrm{sr}(C_0(\widehat{G}_1 \cap (\widehat{\mathfrak{I}_k} \setminus (\mathfrak{I} \cap \mathfrak{I}_k)^\wedge)))$. Hence, we obtain $\mathrm{sr}(\mathfrak{I}_k) \leqslant 2 \vee \dim_{\mathbb{C}}(\widehat{G}_1)$ for every k. By the density of $\bigcup_{k=1}^{\infty} \mathfrak{I}_k$ in $\mathfrak{A}$ or by Theorem 5.1 in [8], we conclude $\mathrm{sr}(\mathfrak{A}) \leqslant 2 \vee \dim_{\mathbb{C}}(\widehat{G}_1)$. ∎

We now show that Lemma 3.2 extends to the case of connected solvable Lie groups of type I.

PROPOSITION 3.3. *Let G be a connected solvable Lie group of type* I, $\widehat{G}_1$ *its character space and* $C^*(G)$ *its* C^**-algebra. Then*

$$\mathrm{sr}(C^*(G)) \begin{cases} \leqslant 2 & \textit{if } \dim \widehat{G}_1 = 0 \textit{ or } 1; \\ = \dim_{\mathbf{C}} \widehat{G}_1 & \textit{if } \dim \widehat{G}_1 \geqslant 2. \end{cases}$$

Proof. Let G be a connected Lie group of type I and $\tilde{G}$ its universal covering group. We denote by q the quotient map from $\tilde{G}$ to G and by Γ the kernel of q. Then we define the map Φ from $\widehat{G}$ to $(\tilde{G})^\wedge$ by $\Phi(\pi)(g) = \pi(g\Gamma)$ for π in $\widehat{G}$ and g in $\tilde{G}$. It follows from Lemma 3.2 in [10], that $\widehat{G} = \widehat{G}_\infty \cup \widehat{G}_1$. Therefore Lemma 3.2 holds for connected solvable Lie groups of type I. ∎

REMARK 3.4. This result suggests that the stable rank of $C^*(G)$ is controlled by the character space $\widehat{G}_1$ of G. By Remark 2.2, $\widehat{G}_1$ is not replaced by $(\mathfrak{G}^*)^G$ in general. T. Nomura informed us about Dixmier's example of a non type I simply-connected solvable Lie group which is locally isomorphic to a connected solvable Lie group of type I. But the first author generalized Lemma 2.5 to general simply-connected solvable Lie groups using the Pukanzsky's results ([7]).

We give an application of Proposition 3.3 to show the product formula of stable rank in the case of the C^*-algebras of connected solvable Lie groups of type I as follows:

COROLLARY 3.5. *Let G, H be two connected solvable Lie groups of type* I, *and* $C^*(G)$, $C^*(H)$ *their* C^**-algebras respectively. Then*

$$\mathrm{sr}(C^*(G) \otimes C^*(H)) \leqslant \mathrm{sr}(C^*(G)) + \mathrm{sr}(C^*(H)).$$

Proof. First of all, note that $C^*(G) \otimes C^*(H)$ is isomorphic to $C^*(G \times H)$. We also have $(G \times H)_1^\wedge = \widehat{G}_1 \times \widehat{H}_1$ and $\mathrm{sr}(C^*(G)) + \mathrm{sr}(C^*(H)) \geqslant 2$.

If $\dim(G \times H)_1^\wedge = 0$ or 1, then, by Proposition 3.3, $\mathrm{sr}(C^*(G \times H)) \leqslant 2$. Thus the product formula holds in the case of the group C^*-algebras under consideration here.

Next we consider the case $\dim(G \times H)_1^\wedge \geqslant 2$. If $\dim \widehat{G}_1 = 1$ and $\dim \widehat{H}_1 = 1$, then $\dim_{\mathbf{C}}(G \times H)_1^\wedge = 2$. Thus by Proposition 3.3, $\mathrm{sr}(C^*(G \times H)) = 2$. Hence the product formula holds in this case. If $\dim \widehat{G}_1 = 2m \geqslant 2$ and $\dim \widehat{H}_1 = 0$ or 1, then

$$\dim_{\mathbf{C}}(G \times H)_1^\wedge \leqslant \left[\frac{2m+1}{2}\right] + 1 = m + 1 = \left[\frac{2m}{2}\right] + 1 = \dim_{\mathbf{C}} \widehat{G}_1.$$

By Proposition 3.3, the product formula holds in this case. If $\dim \widehat{G}_1 = 2m+1 \geqslant 3$ and $\dim \widehat{H}_1 = 0$ or 1, then

$$\dim_{\mathbf{C}}(G \times H)_1^\wedge \leqslant \left[\frac{2m+2}{2}\right] + 1 = m+2 = \left[\frac{2m+1}{2}\right] + 1 + 1 = \dim_{\mathbf{C}} \widehat{G}_1 + 1.$$

By Proposition 3.3, the product formula holds in this case. In the case $\dim \widehat{G}_1 = 0$ or 1 and $\dim \widehat{H}_1 \geqslant 2$, the product formula holds similarly.

Finally we consider the case $\dim \widehat{G}_1 \geqslant 2$ and $\dim \widehat{H}_1 \geqslant 2$. Note that $\dim_{\mathbf{C}}(G \times H)_1^\wedge \leqslant \dim_{\mathbf{C}} \widehat{G}_1 + \dim_{\mathbf{C}} \widehat{H}_1$. By Proposition 3.3, the product formula holds in this case. ∎

REMARK 3.6. The above product formula gives a partial answer to a question raised by M.A. Rieffel ([8]), whether for any two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$,

$$\mathrm{sr}(\mathfrak{A} \otimes \mathfrak{B}) \leqslant \mathrm{sr}(\mathfrak{A}) + \mathrm{sr}(\mathfrak{B}).$$

We proceed to refine Lemma 3.2. Next lemma is useful in computation of the stable rank. To prove it we use the basic results of K-theory and a generalized index theory (refer to [12]).

LEMMA 3.7. *Let G be a simply-connected connected solvable Lie group and $C^*(G)$ its C^*-algebra. Then* $\mathrm{sr}(C^*(G)) = 1$ *if and only if* $G \cong \mathbf{R}$.

Proof. If $G \cong \mathbf{R}$, then by Fourier transform, $C^*(G) \cong C_0(\mathbf{R})$. Hence $\mathrm{sr}(C^*(G)) = 1$.

Conversely, let $\dim G = m+1 \geqslant 2$. Then G is considered as a semi-direct product $N \rtimes \mathbf{R}$ where N is a simply-connected connected solvable Lie subgroup of G and $\dim N = m$. By Lemma 2.8, the following exact sequence is obtained:

$$0 \to \mathfrak{I}_N \to C^*(N) \to C_0(\widehat{N}_1) \to 0$$

where $\mathfrak{I}_N$ is the ideal corresponding to the open subset $\widehat{N} \setminus \widehat{N}_1$ of $\widehat{N}$. Moreover, since $\widehat{N}_1$ is $\mathbf{R}$-invariant closed, the following exact sequence is obtained:

$$0 \to \mathfrak{I}_N \rtimes \mathbf{R} \to C^*(N) \rtimes \mathbf{R} \to C_0(\widehat{N}_1) \rtimes \mathbf{R} \to 0.$$

Note that $\widehat{N}_1$ is homeomorphic to a Euclidean space $\mathbf{R}^n$, for $n = \dim(\widehat{N}_1) \geqslant 1$.

Denote by $\mathbf{R}_1^n$ the set of all φ in $\mathbf{R}^n$ such that $\mathbf{R}_\varphi = \mathbf{R}$ where $\mathbf{R}_\varphi$ means the stabilizer of φ under the coadjoint action of $\mathbf{R}$. Since $\mathbf{R}_1^n$ is $\mathbf{R}$-invariant, we have the following exact sequence:

$$0 \to C_0(\mathbf{R}^n \setminus \mathbf{R}_1^n) \rtimes \mathbf{R} \to C_0(\mathbf{R}^n) \rtimes \mathbf{R} \to C_0(\mathbf{R}_1^n \times \mathbf{R}) \to 0.$$

If $\mathbf{R}_1^n \neq \{0\}$, then $\mathrm{sr}(C_0(\mathbf{R}_1^n \times \mathbf{R})) \geqslant 2$. It implies that $\mathrm{sr}(C^*(G)) \geqslant 2$.

Next consider the case $\mathbf{R}_1^n = \{0\}$. Then we have the following six-term exact sequence:

$$\begin{array}{ccccc} \mathrm{K}_0(C_0(\mathbf{R}^n \setminus \{0\}) \rtimes \mathbf{R}) & \longrightarrow & \mathrm{K}_0((C_0(\mathbf{R}^n) \rtimes \mathbf{R})) & \longrightarrow & \mathrm{K}_0(C_0(\mathbf{R})) \\ \delta \uparrow & & & & \downarrow \\ \mathrm{K}_1(C_0(\mathbf{R})) & \longleftarrow & \mathrm{K}_1((C_0(\mathbf{R}^n) \rtimes \mathbf{R})) & \longleftarrow & \mathrm{K}_1(C_0(\mathbf{R}^n \setminus \{0\}) \rtimes \mathbf{R}). \end{array}$$

Using Connes' Thom isomorphism, if n is even, say $n = 2m \geqslant 2$, then

$$\mathrm{K}_i((C_0(\mathbf{R}^{2m}) \rtimes \mathbf{R})) \cong \mathrm{K}_{i+1}(C_0(\mathbf{R}^{2m})) \cong \begin{cases} \mathrm{K}_{1+2m}(\mathbf{C}) = 0 & \text{if } i = 0; \\ \mathrm{K}_{2+2m}(\mathbf{C}) = \mathbf{Z} & \text{if } i = 1. \end{cases}$$

If n is odd, say $n = 2m+1 \geqslant 1$, then $\mathrm{K}_i((C_0(\mathbf{R}^{2m+1}) \rtimes \mathbf{R})) \cong \mathbf{Z}$ if $i = 0$, 0 if $i = 1$.

Again, using Connes' Thom isomorphism,

$$\mathrm{K}_i(C_0(\mathbf{R}^n \setminus \{0\}) \rtimes \mathbf{R}) \cong \mathrm{K}_{i+1}(C_0(\mathbf{R}^n \setminus \{0\})) \cong \mathrm{K}_{i+1}(C_0(S^{n-1} \times \mathbf{R})) \cong \mathrm{K}_i(C(S^{n-1})).$$

Note that

$$\begin{aligned} \mathrm{K}_i(C(S^{n-1})) &\cong \mathrm{K}_i(C_0(\mathbf{R}^{n-1}) \oplus \mathbf{C}) \\ &\cong \begin{cases} \mathrm{K}_0(C_0(\mathbf{R}^{n-1})) \oplus \mathbf{Z} \cong \mathrm{K}_{n-1}(\mathbf{C}) \oplus \mathbf{Z} & \text{if } i = 0 \\ \mathrm{K}_1(C_0(\mathbf{R}^{n-1})) \cong \mathrm{K}_n(\mathbf{C}) & \text{if } i = 1. \end{cases} \end{aligned}$$

Hence, if $n = 2m \geqslant 2$, then $\mathrm{K}_i(C_0(\mathbf{R}^{2m} \setminus \{0\}) \rtimes \mathbf{R}) \cong \mathbf{Z}$ if $i = 0$ or 1. If $n = 2m+1 \geqslant 1$, then $\mathrm{K}_i(C_0(\mathbf{R}^{2m+1} \setminus \{0\}) \rtimes \mathbf{R}) \cong \mathbf{Z} \oplus \mathbf{Z}$ if $i = 0$, 0 if $i = 1$. Thus, if n is even, the above six-term exact sequence is equal to the following diagram:

$$\begin{array}{ccccc} \mathbf{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\ \delta \uparrow & & & & \downarrow \\ \mathbf{Z} & \longleftarrow & \mathbf{Z} & \longleftarrow & \mathbf{Z}, \end{array}$$

and if n is odd, then

$$\begin{array}{ccccc} \mathbf{Z} \oplus \mathbf{Z} & \longrightarrow & \mathbf{Z} & \longrightarrow & 0 \\ \delta \uparrow & & & & \downarrow \\ \mathbf{Z} & \longleftarrow & 0 & \longleftarrow & 0. \end{array}$$

Note that the index map δ from $\mathrm{K}_1(C_0(\mathbf{R}))$ ($\cong \mathrm{K}_1(C(S^1))$) to $\mathrm{K}_0(C_0(\mathbf{R}^n \setminus \{0\}) \rtimes \mathbf{R})$ is non zero in both cases.

Putting $\mathfrak{J} = (C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R}) \otimes \mathbf{K}$, we have the following exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{J} & \longrightarrow & ((C_0(\mathbb{R}^n) \rtimes \mathbb{R})^\sim) \otimes \mathbf{K} & \xrightarrow{\sigma} & C(S^1) \otimes \mathbf{K} & \longrightarrow & 0 \\
 & & \| & & \mu\downarrow & & \tau\downarrow & & \\
0 & \longrightarrow & \mathfrak{J} & \longrightarrow & M(\mathfrak{J}) & \xrightarrow{q} & M(\mathfrak{J})/\mathfrak{J} & \longrightarrow & 0
\end{array}$$

where $M(\mathfrak{J})$ is the multiplier algebra of $\mathfrak{J}$. Then the following six-term exact sequence is obtained:

$$\begin{array}{ccccc}
\mathrm{K}_0(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R}) & \longrightarrow & \mathrm{K}_0(M(\mathfrak{J})) & \longrightarrow & \mathrm{K}_0(M(\mathfrak{J})/\mathfrak{J}) \\
\eta\uparrow & & & & \downarrow \\
\mathrm{K}_1(M(\mathfrak{J})/\mathfrak{J}) & \longleftarrow & \mathrm{K}_1(M(\mathfrak{J})) & \longleftarrow & \mathrm{K}_1(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R}).
\end{array}$$

From the fact that $\mathrm{K}_i(M(\mathfrak{A} \otimes \mathbf{K}) \otimes \mathfrak{B}) = 0$ for $i = 0, 1$ where $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras and $\mathfrak{B}$ is unital ([12], Theorem 10.2), we have $\mathrm{K}_i(M(\mathfrak{J})) = 0$ for $i = 0, 1$. Thus,

$$\mathrm{K}_i(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R}) \cong \mathrm{K}_{i+1}(M(\mathfrak{J})/\mathfrak{J}), \quad \text{for } i = 0, 1 \,(\mathrm{mod}\, 2).$$

If n is even, then the above six-term exact sequence is equal to the following diagram:

$$\begin{array}{ccccc}
\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \\
\eta\uparrow & & & & \downarrow \\
\mathbb{Z} & \longleftarrow & 0 & \longleftarrow & \mathbb{Z}
\end{array}$$

and if n is odd, then

$$\begin{array}{ccccc}
\mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
\eta\uparrow & & & & \downarrow \\
\mathbb{Z} \oplus \mathbb{Z} & \longleftarrow & 0 & \longleftarrow & 0.
\end{array}$$

Let D be an element of $(C_0(\mathbb{R}^n) \rtimes \mathbb{R})^\sim$ such that $\sigma(D) = \mathrm{id}$ where $\mathrm{id}(z) = z$ for z in S^1, which is identified with a diagonal matrix in $M_\infty((C_0(\mathbb{R}^n) \rtimes \mathbb{R})^\sim)$ having the diagonal entries $(D, 0, \ldots)$. Then the class $[\sigma(D)]$ in $\mathrm{K}_1(C(S^1))$ is a generator. By a generalized index theory, the index of $\mu(D)$ is defined by

$$\mathrm{index}(\mu(D)) = \eta([q(\mu(D))]) \quad \text{in } \mathrm{K}_0(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R})$$

where $[q(\mu(D))]$ is in $\mathrm{K}_1(M(\mathfrak{J})/\mathfrak{J})$. We take a unitary w in $M_2((C_0(\mathbb{R}^n) \rtimes \mathbb{R})^\sim)$ such that $\sigma(D) \oplus \sigma(D)^* = \sigma(w)$. Then $\tau(\sigma(D)) \oplus \tau(\sigma(D)^*) = \tau(\sigma(w))$. It follows that $q(\mu(D)) \oplus q(\mu(D))^* = q(\mu(w))$ and $\mu(w)$ is a unitary in $M_2(M(\mathfrak{J}))$. By the definition of the index map,

$$\delta([\sigma(D)]) = [wp_1w^*] - [p_1] \neq 0$$

where p_1 is a rank 1 projection in $(C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R})^\sim \otimes \mathbb{K}$, which is identified with a diagonal matrix in $M_\infty((C_0(\mathbb{R}^n \setminus \{0\}) \rtimes \mathbb{R})^\sim)$ having the diagonal entries $(1, 0, \ldots)$. On the other hand,

$$\eta([q(\mu(D))]) = [\mu(w)p_1\mu(w)^*] - [p_1] = [wp_1w^*] - [p_1].$$

If $\mathrm{sr}(C^*(G)) = 1$, then $\mathrm{sr}(C_0(\mathbb{R}^n) \rtimes \mathbb{R}) = 1$. Hence, $\mathrm{sr}((C_0(\mathbb{R}^n) \rtimes \mathbb{R})^\sim \otimes \mathbb{K}) = 1$. It follows that the invertible elements of $M(\mathfrak{J})$ are dense in $\mu((C_0(\mathbb{R}^n) \rtimes \mathbb{R})^\sim \otimes \mathbb{K})$. By the property of the generalized index, we deduce that $\mathrm{index}(\mu(D)) = 0$ which is a contradiction. Therefore $\mathrm{sr}(C^*(G)) \geqslant 2$. ∎

REMARK 3.8. Let G be as in Lemma 3.7. If $\dim G = 2$, then $\mathrm{sr}(C^*(G)) = 2$. In fact, it is known that G is isomorphic to $\mathbb{R}^2$ or the real $ax + b$-group which is treated in Example 4.1 later. If G is the real $ax+b$-group, then $\mathrm{sr}(C^*(G)) = 2$ (cf. the remark before 4.14 in [8]). However, the converse of the implication is false in general. For example, $\mathbb{R}^3$ is a counterexample.

Combining Lemma 2.1, 3.2 and 3.7, we obtain the following main result.

THEOREM 3.9. *Let G be a simply-connected connected solvable Lie group of type* I, *$C^*(G)$ its C^*-algebra and $(\mathfrak{G}^*)^G$ the fixed point subspace under its coadjoint action. Then*

$$\mathrm{sr}(C^*(G)) = (\dim_{\mathbb{C}}(\mathfrak{G}^*)^G \vee 2) \wedge \dim G.$$

Proof. By Lemma 3.7, we know that $\mathrm{sr}(C^*(G)) = 1$ if and only if $\dim G = 1$. By Lemma 2.1, we replace $\widehat{G}_1$ in Lemma 3.2 with $(\mathfrak{G}^*)^G$. By Lemma 3.2 and 3.7, if $\dim G \geqslant 2$ and $\dim(\mathfrak{G}^*)^G = 1$, then

$$\mathrm{sr}(C^*(G)) = 2 = (\dim_{\mathbb{C}}(\mathfrak{G}^*)^G \vee 2) \wedge \dim G.$$

By Lemma 3.7, if $\dim G \geqslant 2$ and $\dim(\mathfrak{G}^*)^G \geqslant 2$, then

$$\mathrm{sr}(C^*(G)) = \dim_{\mathbb{C}}(\mathfrak{G}^*)^G = (\dim_{\mathbb{C}}(\mathfrak{G}^*)^G \vee 2) \wedge \dim G. \quad ∎$$

REMARK 3.10. This result extends our estimation in the case that G is a simply-connected connected nilpotent Lie group. It also suggests that the stable rank of $C^*(G)$ is controlled by the geometrical structure of G. If G is abelian, then $C^*(G) \cong C_0(\widehat{G})$. Thus $\mathrm{sr}(C^*(G)) = \dim_{\mathbb{C}} \widehat{G}$. By Lemma 2.3, the formula in Theorem 3.9 is replaced by

$$\mathrm{sr}(C^*(G)) = (\dim_{\mathbb{C}}(G/[G,G])^\wedge \vee 2) \wedge \dim G.$$

Therefore, Theorem 3.9 extends naturally the abelian case.

Next, we apply Theorem 3.9 to compute the real rank as follows:

COROLLARY 3.11. *Let G be a simply-connected connected solvable Lie group of type* I, *$C^*(G)$ its C^*-algebra and $(\mathfrak{G}^*)^G$ the fixed point subspace under its coadjoint action. Then*

$$\mathrm{rr}(C^*(G)) = 1 \quad \text{if } \dim G = 1,$$

$$\dim(\mathfrak{G}^*)^G \leqslant \mathrm{rr}(C^*(G)) \leqslant \begin{cases} \dim(\mathfrak{G}^*)^G + 1 & \text{if } \dim(\mathfrak{G}^*)^G \text{ is even;} \\ \dim(\mathfrak{G}^*)^G \vee 3 & \text{if } \dim(\mathfrak{G}^*)^G \text{ is odd;} \end{cases} \qquad \text{if } \dim G \geqslant 2.$$

Proof. We first use the following inequality:

$$\mathrm{rr}(C_0((\mathfrak{G}^*)^G)) \leqslant \mathrm{rr}(C^*(G)) \leqslant 2\,\mathrm{sr}(C^*(G)) - 1.$$

See [2] for the second inequality. If $\dim G = 1$, then $\mathrm{rr}(C^*(G)) = \dim(\mathfrak{G}^*)^G = 1$.

Suppose that $\dim G \geqslant 2$. If $\dim(\mathfrak{G}^*)^G = 2m$ ($m \geqslant 1$), then it follows from Theorem 3.9 that

$$\begin{aligned} 2\,\mathrm{sr}(C^*(G)) - 1 &= 2(\dim_{\mathbf{C}}(\mathfrak{G}^*)^G \vee 2) - 1 \\ &= 2(([2m/2] + 1) \vee 2) - 1 = (2m+1) \vee 3 = 2m + 1. \end{aligned}$$

Therefore we have that

$$\dim(\mathfrak{G}^*)^G \leqslant \mathrm{rr}(C^*(G)) \leqslant \dim(\mathfrak{G}^*)^G + 1.$$

If $\dim(\mathfrak{G}^*)^G = 2m + 1 (m \geqslant 0)$, then it again follows from Theorem 3.9 that

$$\begin{aligned} 2\,\mathrm{sr}(C^*(G)) - 1 &= 2((\dim_{\mathbf{C}}(\mathfrak{G}^*)^G \vee 2) - 1 \\ &= 2(([(2m+1)/2] + 1) \vee 2) - 1 = (2m+1) \vee 3. \end{aligned}$$

Therefore we conclude that

$$\dim(\mathfrak{G}^*)^G \leqslant \mathrm{rr}(C^*(G)) \leqslant \dim(\mathfrak{G}^*)^G \vee 3. \quad \blacksquare$$

4. EXAMPLES

In this section we give several examples which support Theorem 3.9.

EXAMPLE 4.1. Let G be the extended real $ax+b$-group, i.e. the semi-direct product $\mathbf{R}^n \rtimes \mathbf{R}$ defined by all $(n+1)\times(n+1)$ matrices of the following form:

$$g = \begin{pmatrix} \alpha(t) & a \\ 0 & 1 \end{pmatrix}, \quad \alpha(t) = \begin{pmatrix} e^t & & 0 \\ & \ddots & \\ 0 & & e^t \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

for each $t, a_1, \ldots, a_n$ in $\mathbf{R}$. Put $g = (t, a_1, \ldots, a_n)$. If $n = 1$, then G is the real $ax+b$-group. The Lie algebra $\mathfrak{G}$ of G is defined by all $(n+1)\times(n+1)$ matrices of the following form:

$$X = \begin{pmatrix} tI_n & x \\ 0 & 0 \end{pmatrix}, \quad I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

for each $t, x_1, \ldots, x_n$ in $\mathbf{R}$. The real dual space $\mathfrak{G}^*$ of $\mathfrak{G}$ is defined by all $(n+1)\times(n+1)$ matrices of the following form:

$$\varphi = \begin{pmatrix} lI_n & 0 \\ m & 0 \end{pmatrix}, \quad m = \begin{pmatrix} m_1 & \cdots & m_n \end{pmatrix}$$

for each $l, m_1, \ldots, m_n$ in $\mathbf{R}$. We let $\varphi = (l, m_1, \ldots, m_n)$. The duality is defined by $\varphi(X) = \mathrm{tr}(X\varphi)$ for X in $\mathfrak{G}$ and φ in $\mathfrak{G}^*$ where tr is the natural trace of $M_{n+1}(\mathbf{R})$. By using the formula $\mathrm{Ad}(\exp X) = \exp(\mathrm{ad}(X))$ where $\mathrm{ad}(X)(Y) = [X,Y] = XY - YX$ for $X, Y \in \mathfrak{G}$, the coadjoint action of G is given by

$$\mathrm{Ad}^*(\exp X)\varphi = (l - (nt)^{-1}(e^{-t} - 1)\sum_{i=1}^{n} x_i m_i, e^{-t}m_1, \ldots, e^{-t}m_n).$$

Thus $(\mathfrak{G}^*)^G$ consists of all matrices of the form $(l, 0, \ldots, 0)$. Hence $\dim_{\mathbf{C}}(\mathfrak{G}^*)^G = 1$. By Theorem 3.9, we conclude that $\mathrm{sr}(C^*(G)) = 2$.

On the other hand, let $g = (t, a_1, \ldots, a_n), h = (s, b_1, \ldots, b_n)$ be in G. Then

$$ghg^{-1}h^{-1} = (0, (1-e^s)a_1 - (1-e^t)b_1, \cdots, (1-e^s)a_n - (1-e^t)b_n).$$

It follows that $[G,G]$ contains all matrices of the form $(0, a_1, \ldots, a_n)$. Thus we see $G/[G,G] \cong \mathbf{R}$. Hence $(G/[G,G])^\wedge \cong \mathbf{R}$.

Next we consider the structure of $C^*(G)$. The following exact sequence is obtained:

$$0 \to C_0(\mathbf{R}^n \setminus \{0\}) \rtimes \mathbf{R} \to C^*(G) \to C_0(\mathbf{R}) \to 0.$$

Then $C_0(\mathbf{R}^n \setminus \{0\}) \rtimes \mathbf{R} \cong C(S^{n-1}) \otimes \mathbf{K}$ where S^{n-1} is the $(n-1)$-dimensional sphere and $S^0 = \{-1, +1\}$. If $n \geqslant 3$, then $\mathrm{sr}(C^*(G)) = 2$. In the case of $n = 1$ or 2, we have $\mathrm{sr}(C^*(G))$ is either 1 or 2. By Theorem 3.9, we conclude that $\mathrm{sr}(C^*(G)) = 2$.

From the above observation,

$$\dim_{\mathbf{C}}(\mathfrak{G}^* \oplus \mathfrak{G}^*)^{G \times G} = 2, \quad (G \times G/[G \times G, G \times G])^\wedge \cong \mathbf{R}^2.$$

Applying Theorem 3.9, we obtain $\mathrm{sr}(C^*(G \times G)) = 2$.

EXAMPLE 4.2. Let G be the split oscillator group, i.e. the semi-direct product $H \rtimes \mathbf{R}$ defined by all 3×3 matrices of the following form:

$$g = \begin{pmatrix} 1 & a & b \\ 0 & e^t & c \\ 0 & 0 & 1 \end{pmatrix}$$

for t, a, b, c in $\mathbf{R}$ where H is the 3-dimensional Heisenberg group. Put $g = (t, a, b, c)$. Then G is a simply-connected connected exponential solvable Lie group. The Lie algebra $\mathfrak{G}$ of G is defined by all 3×3 matrices of the following form:

$$X = \begin{pmatrix} 0 & x & y \\ 0 & t & z \\ 0 & 0 & 0 \end{pmatrix}$$

for t, x, y, z in $\mathbf{R}$. The real dual space $\mathfrak{G}^*$ of $\mathfrak{G}$ is defined by all 3×3 matrices of the following form:

$$\varphi = \begin{pmatrix} 0 & 0 & 0 \\ l & u & 0 \\ m & n & 0 \end{pmatrix}$$

for u, l, m, n in $\mathbf{R}$. We let $\varphi = (u, l, m, n)$. The duality is the same as in Example 4.1. Then the coadjoint action of G is given by

$$\mathrm{Ad}^*(\exp X)\varphi = (u', e^t l + t^{-1}(e^t - 1)zm, m, e^{-t}n + t^{-1}(e^{-t} - 1)xm)$$

where $u' = t^{-1}(e^t - 1)xl - t^{-1}(e^{-t} - 1)zn - 2xym + u$. Thus $(\mathfrak{G}^*)^G$ consists of all matrices of the form $(u, 0, 0, 0)$. Hence $\dim_{\mathbf{C}}(\mathfrak{G}^*)^G = 1$. By Theorem 3.9, we conclude that $\mathrm{sr}(C^*(G)) = 2$.

On the other hand, let $g = (t, a_1, 0, 0), h = (s, a_2, 0, 0)$ be in G. Then

$$ghg^{-1}h^{-1} = (0, e^{-t}(1 - e^{-s})a_1 + e^{-s}(e^{-t} - 1)a_2, 0, 0).$$

Let $g = (t, 0, 0, c_1), h = (s, 0, 0, c_2)$ be in G. Then

$$ghg^{-1}h^{-1} = (0, 0, 0, (1 - e^s)c_1 + (e^t - 1)c_2).$$

It follows that $[G, G]$ contains all matrices of the form $(0, a, 0, c)$. Let $g = (0, a_1, b_1, c_1)$, $h = (0, a_2, b_2, c_2)$ be in G. Then

$$ghg^{-1}h^{-1} = (0, 0, a_1c_2 - a_2c_1, 0).$$

Note that $(0, a, b, c) = (0, a, 0, c)(0, 0, b, 0)$. Since $[G, G]$ is a subgroup of G, it contains all matrices of the form $(0, a, b, c)$. It follows that $[G, G] \cong H$. Thus $G/[G, G] \cong \mathbf{R}$. Hence $(G/[G, G])^\wedge \cong \mathbf{R}$.

From the above observation,

$$\dim_{\mathbf{C}}(\mathfrak{G}^* \oplus \mathfrak{G}^*)^{G\times G} = 2, \quad (G \times G/[G \times G, G \times G])^\wedge \cong \mathbf{R}^2.$$

Applying Theorem 3.9, we obtain $\mathrm{sr}(C^*(G \times G)) = 2$.

EXAMPLE 4.3. Let G be the semi-direct product $\mathbf{R}^2 \rtimes \mathbf{R}$ defined by all 3×3 matrices of the following form:

$$g = \begin{pmatrix} \alpha(t) & a \\ 0 & 1 \end{pmatrix}, \quad \alpha(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \quad a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

for each t, a_1, a_2 in $\mathbf{R}$. Put $g = (t, a_1, a_2)$. Then G is the only non exponential simply-connected connected solvable Lie group with dimension $\leqslant 3$, up to isomorphisms (cf. [5]). Actually, the Lie algebra $\mathfrak{G}$ of G is defined by all 3×3 matrices of the following form:

$$X = \begin{pmatrix} 0 & -t & x_1 \\ t & 0 & x_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

The real dual space $\mathfrak{G}^*$ of $\mathfrak{G}$ is defined by all 3×3 matrices of the following form:

$$\varphi = \begin{pmatrix} 0 & m & 0 \\ -m & 0 & 0 \\ l_1 & l_2 & 0 \end{pmatrix}.$$

Put $\varphi = (m, l_1, l_2)$. The duality is the same as in Example 4.1. Then the coadjoint action of G is given by

$$\mathrm{Ad}^*(\exp X)\varphi = (m', l_1 \cos(-t) + l_2 \sin(-t), -l_1 \sin(-t) + l_2 \cos(-t))$$

where $m' = m + (2t)^{-1}(\sin(-t)(x_2 l_1 - x_1 l_2) + (1 - \cos(-t))(x_1 l_1 + x_2 l_2))$. Note that $G_\varphi = \mathbb{R}^2 \rtimes \mathbb{Z}$ for $\varphi = (0, l_1, l_2)$ with non zero l_1, l_2. It is known that if G is an exponential Lie group, then G_φ is connected for every φ in $\mathfrak{G}^*$ (cf. [5]). Thus G is non exponential. Then $(\mathfrak{G}^*)^G$ consists of all matrices of the form $(m, 0, 0)$. Hence $\dim_{\mathbb{C}}(\mathfrak{G}^*)^G = 1$. By Theorem 3.9, we conclude that $\mathrm{sr}(C^*(G)) = 2$.

On the other hand, let $g = (t, a_1, a_2), h = (s, b_1, b_2)$ be in G. Then

$$ghg^{-1}h^{-1} = \begin{pmatrix} \alpha(0) & (1_2 - \alpha(s))a + (\alpha(t) - 1_2)b \\ 0 & 1 \end{pmatrix}, \quad 1_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $[G, G]$ consists of all matrices of the form $(0, a_1, a_2)$. Hence $G/[G,G] \cong \mathbb{R}$. Thus $(G/[G,G])^\wedge \cong \mathbb{R}$.

From the above observation,

$$\dim_{\mathbb{C}}(\mathfrak{G}^* \oplus \mathfrak{G}^*)^{G\times G} = 2, \quad (G \times G/[G \times G, G \times G])^\wedge \cong \mathbb{R}^2.$$

Applying Theorem 3.9, we obtain $\mathrm{sr}(C^*(G \times G)) = 2$.

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REFERENCES

1. L. AUSLANDER, B. KOSTANT, Polarization and unitary representations of solvable Lie groups, *Invent. Math.* **14**(1971), 255–354.
2. L.G. BROWN, G.K. PEDERSEN, C^*-algebras of real rank zero, *J. Funct. Anal.* **99** (1991), 131–149.
3. J. DIXMIRE, *C^*-Algebras*, North-Holland, Amsterdam – New York – Oxford 1962.
4. G. HOCHSCHILD, *The Structure of Lie Groups*, Holden-Day, San Francisco 1965.
5. H. LEPTIN, J. LUDWIG, *Unitary Representation Theory of Exponential Lie Groups*, Walter de Gruyter, Berlin – New York 1994.
6. V. NISTOR, Stable rank for a certain class of type I C^*-algebras, *J. Operator Theory* **17**(1987), 365–373.
7. L. PUKANSZKY, Characters of connected Lie groups, *Acta Math.* **133**(1974), 81–137.
8. M.A. RIEFFEL, Dimension and stable rank in the K-theory of C^*-algebras, *Proc. London Math. Soc. (3)* **46**(1983), 301–333.
9. A.J.-L. SHEU, A cancellation theorem for projective modules over the group C^*-algebras of certain nilpotent Lie groups, *Canad. J. Math.* **39**(1987), 365–427.
10. T. SUDO, Stable rank of the C^*-algebras of amenable Lie groups of type I, preprint, 1996.
11. T. SUDO, H. TAKAI, Stable rank of the C^*-algebras of nilpotent Lie groups, *Internat. J. Math.* **6**(1995), 439–446.

12. N.E. WEGGE-OLSEN, *K-Theory and C^*-Algebras*, Oxford Univ. Press, Oxford – New York – Tokyo, 1993.

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