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AN EXCISION THEOREM FOR THE K-THEORY OF C^*-ALGEBRAS

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ABSTRACT. We consider a pair of C^*-algebras $A' \subseteq A$. The K-theory of the mapping cone for this inclusion can be regarded as a relative K-group. We describe a situation where two such pairs have isomorphic relative groups.

KEYWORDS: *C^*-algebra, K-theory.*

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1. INTRODUCTION

This paper is concerned with a certain excision result for K-theory of C^*-algebras.

Let us begin by setting out some notation. Let A be any C^*-algebra. We let $A^\sim$ be the C^*-algebra obtained by adjoining a unit to A (even if A is already unital). Let $M_n(A)$ denote the C^*-algebra of $n \times n$ matrices with entries from A. For any a in $A^\sim$ (respectively, $M_n(A^\sim)$), let $\dot{a}$ denote its image in $\mathbb{C}$, the complex numbers, (respectively, $M_n(\mathbb{C})$), under the map moding out by A. We also regard $\mathbb{C}$ and $M_n(\mathbb{C})$ implicitly as subalgebras of $A^\sim$ and $M_n(A^\sim)$, respectively.

Suppose A' is a C^*-subalgebra of A. We regard $A'^\sim \subseteq A^\sim$ as the natural unital inclusion. Recall [6], [7], [1] that the mapping cone for the inclusion $A' \subseteq A$ is

$$C(A';A) = \{f : [0,1] \longrightarrow A \mid f \text{ is continuous}, \ f(0) = 0, \ f(1) \in A'\}.$$

It is a C^*-algebra with pointwise operations and

$$\|f\| = \sup\{\|f(t)\| \mid 0 \leqslant t \leqslant 1\}$$

for f in $C(A';A)$. There is a natural short exact sequence

$$0 \longrightarrow C_0(0,1)\otimes A \xrightarrow[i]{} C(A';A) \xrightarrow[ev]{} A' \longrightarrow 0,$$

where

$$ev(f) = f(1), \quad f \in C(A';A)$$

$$i(g\otimes a)(t) = g(t)a, \quad g \in C_0(0,1),\ a \in A,\ 0 \leqslant t \leqslant 1.$$

Let $b : \mathrm{K}_i(A) \to \mathrm{K}_{i+1}(C_0(0,1)\otimes A)$ denote the usual isomorphism ([1]). After using b to replace the terms involving $\mathrm{K}_*(C_0(0,1)\otimes A)$, the six-term exact squence for K-groups associated with the sequence above becomes

$$\begin{array}{ccccc}
\mathrm{K}_1(A) & \xrightarrow{i_*b} & \mathrm{K}_0(C(A';A)) & \xrightarrow{ev_*} & \mathrm{K}_0(A') \\
\uparrow{\scriptstyle j_*} & & & & \downarrow{\scriptstyle j_*} \\
\mathrm{K}_1(A') & \xleftarrow[ev_*]{} & \mathrm{K}_1(C(A';A)) & \xleftarrow[i_*b]{} & \mathrm{K}_0(A)
\end{array}$$

where $j : A' \to A$ denotes the inclusion map. We regard $\mathrm{K}_*(C(A';A))$ as a "relative group" for the C^*-algebra inclusion $A' \subseteq A$. Indeed, if A' is actually an ideal in A, then there is a natural isomorphism

$$\mathrm{K}_*(C(A';A)) \cong \mathrm{K}_*(A/A').$$

To see this, let

$$J = \{f \in C(A';A) \mid f(t) \in A' \text{ for all } 0 \leqslant t \leqslant 1\},$$

which is an ideal in $C(A';A)$. Moreover, $J \cong C_0(0,1]\otimes A'$ and so $\mathrm{K}_*(J) = 0$, since $C_0(0,1]$ is contractible ([7], [1]). We also have a short exact sequence

$$0 \to J \to C(A,A') \to C_0(0,1)\otimes(A/A') \to 0.$$

Taking the six-term exact sequence for K-groups and noting $\mathrm{K}_*(J) = 0$ yields the result. Thus, if A' is an ideal, $\mathrm{K}_*(C(A';A))$ depends only on A/A'.

Our goal is to describe two pairs of inclusions $A' \subseteq A$ and $B' \subseteq B$ which are related in a specific way that we may conclude that there is an isomorphism

$$\mathrm{K}_*(C(A';A)) \cong \mathrm{K}_*(C(B';B)),$$

which is natural in some sense. The roles of A and B here will not be symmetric. In some sense, the inclusion $A' \subseteq A$ will be the more tractible. We suppose that A and B are both C^*-algebras of operators acting on the Hilbert space $\mathcal{H}$. We suppose that z is a selfadjoint unitary on $\mathcal{H}$ and that the following conditions are satisfied. First, B should lie in the multiplier algebra of A. We should have $zAz = A$ and, for all b in B, $zbz - b$ lies in A. One interesting case where this occurs is when $(\mathcal{H}, z)$ is a Fredholm module for B ([1]). Let A be the C^*-algebra of compact operators on $\mathcal{H}$. These conditions are satisfied. Returning to the general situation, we let A' and B' be those operators in A and B, respectively, which commute with z. We require three more technical assumptions on A, B and z (given as conditions 4, 5, 6 in Section 3). Under these hypotheses, we construct a homomorphism

$$\alpha : \mathrm{K}_* (C(B'; B)) \longrightarrow \mathrm{K}_* (C(A'; A))$$

and prove that it is an isomorphism.

The main applications of this result are in various situations arising from dynamical systems where B, B', A and A' can all be described as groupoid C^*-algebras. For example, $B = C(X) \times_{\varphi} \mathbb{Z}$ and $B' = A_Y$ of [4], where φ is a minimal homeomorphism of a Cantor set X, can be described in this way. Here, A is the compact operators on $\ell^2(\mathbb{Z})$ and A' is the direct sum of compact operators on two orthogonal subspaces. More applications can be found in [5]. (Also, see [2].)

In Section 2, we provide a description of $\mathrm{K}_0 (C(A'; A))$ which will be useful. In Section 3, we state and prove the main results (Theorems 3.1 and 3.7).

2. K-THEORY OF MAPPING CONES

Our aim in this section is to provide a natural description of $\mathrm{K}_0 (C(A'; A))$.

We begin, as in Section 1, with C^*-algebras $A' \subseteq A$. For each $n = 1, 2, 3, \ldots$, we let $V_n(A'; A)$, or simply V_n, denote the set of elements v in $M_n(A^\sim)$ such that:

(i) v is a partial isometry;
(ii) v^*v is in $M_n(\mathbb{C})$;
(iii) vv^* is in $M_n(A'^\sim)$.

(In some ways, it would be more natural to require v^*v to be in $M_n(A'^\sim)$; our definition will be more convenient, however.) We regard $V_n \subseteq V_{n+1}$ by identifying v and $v \oplus 0$, for all v in V_n. We let

$$V(A'; A) = \bigcup_n V_n(A'; A).$$

We will make use of the following two facts:

1. If h is a selfadjoint element of a C^*-algebra and $\|h - h^2\| < \delta < 1/2$, then the spectrum of h is contained in $(-2\delta, 2\delta) \cup (1 - 2\delta, 1 + 2\delta)$. The proof is an easy application of the spectral theorem.

2. If p_1 and p_2 are projections in a C^*-algebra with $\|p_1 - p_2\| < \delta < 1/2$, then there is a unitary u in the C^*-algebra such that $up_1u^* = p_2$ and $\|u - 1\| < \pi\delta$. For a proof, see 4.3.2, 4.6.5 of [1].

LEMMA 2.1. *Suppose* $0 < \varepsilon < 100^{-1}$ *and* v *in* $M_n(A^\sim)$ *satisfies* (i) *and* (ii) *above and there exists* q *in* $M_n(A'^\sim)$ *such that* $\|vv^* - q\| < \varepsilon$. *Then there exists a unitary* u *in* $M_n(A^\sim)$ *such that* $\|u - 1\| < 30\varepsilon$ *and* uv *is in* $V_n(A'; A)$.

Proof. First replace q by $(q + q^*)/2$ so we may assume it is selfadjoint. Since v is a partial isometry, vv^* is a projection and so

$$\|q^2 - q\| < 4\varepsilon.$$

Then, using the first fact above, $q_1 = \chi_{(1/2,\infty)}(q)$ is a projection and $\|q_1 - q\| < 8\varepsilon$ hence

$$\|q_1 - vv^*\| < 9\varepsilon.$$

The second fact above then gives the desired u. ∎

We define a map

$$\kappa : V(A'; A) \longrightarrow \mathrm{K}_0\left(C(A'; A)\right).$$

Begin with v in $V_n(A'; A)$. Consider

$$v_1 = \begin{bmatrix} 1 - v^*v & v^* \\ v & 1 - vv^* \end{bmatrix}$$

in $M_{2n}(A^\sim)$. It is easily verified that v_1 is a selfadjoint unitary. We define a path of selfadjoint unitaries in $M_{2n}(A^\sim)$ by

$$v_2(t) = \left[\dot{v}_1 + 1 + e^{i\pi t}(1 - \dot{v}_1)\right]^{-1}\left[v_1 + 1 + e^{i\pi t}(1 - v_1)\right],$$

for $0 \leqslant t \leqslant 1$. Notice that v_2 satisfies:

(i) $v_2(t)$ is unitary for all t,
(ii) v_2 is in $C[0,1] \otimes M_{2n}(A^\sim)$,
(iii) $\dot{v}_2(t) = 1$, for all t,
(iv) $v_2(0) = 1$,
(v) $v_2(1) = \dot{v}_1^{-1} v_1$.

Together, (ii), (iii) and (iv) imply that v_2 may be regarded as an element of

$$[C_0(0,1]\otimes M_{2n}]^{\sim}.$$

Finally, we define

$$p_v(t) = v_2(t)\, e_{11}\, v_2(t)^*,$$

for $0 \leqslant t \leqslant 1$, where e_{11} denotes $1_n \oplus 0$ in $M_{2n}(A^{\sim})$. It is easy to verify that

(i) $p_v(0) = e_{11}$;
(ii) $p_v(1) = (1_n - v^*v) \oplus vv^* \in M_{2n}(A'^{\sim})$;
(iii) $\dot{p}_v(t) = e_{11}$, for all $0 \leqslant t \leqslant 1$.

Thus, p_v is in $M_{2n}\left(C(A';A)^{\sim}\right)$ and $[p_v]-[e_{11}]$ is in $\mathrm{K}_0\left(C(A';A)\right)$. We denote this element by $\kappa(v)$. We summarize the properties of κ.

LEMMA 2.2. (i) *For v, w in $V(A';A)$,*

$$\kappa(v \oplus w) = \kappa(v) + \kappa(w).$$

(ii) *If v, w are in $V_n(A';A)$ and $\|v - w\| < 200^{-1}$, then $\kappa(v) = \kappa(w)$.*

(iii) *For v in $V_n(A';A)$, w_1 in $U_n(A'^{\sim})$ and w_2 in $U_n(\mathbb{C})$, then w_1vw_2 is in $V_n(A';A)$ and*

$$\kappa(w_1) = \kappa(w_2) = 0$$

$$\kappa(w_1vw_2) = \kappa(v).$$

(iv) *For any projection p in $M_n(\mathbb{C})$, $\kappa(p) = 0$.*

(v) *If v is a partial isometry in $M_n(A'^{\sim})$, then $\kappa(v) = 0$.*

Proof. Parts (i) and (iv) are verified by direct computations, which we omit.

In proving (ii), one notes that the construction of p_v depends continuously on v. In fact, $\|v - w\| < 200^{-1}$ implies $\|p_v - p_w\| < 1/2$ (we omit the details), which implies $[p_v] = [p_w]$ and the conclusion. As a consequence of (ii), if v and w are homotopic in $V_n(A';A)$ then $\kappa(v) = \kappa(w)$.

In part (iii), we begin by considering $v\oplus 0$, $w_1 \oplus w_1^*$ and $w_2 \oplus w_2^*$. By standard methods (see 4.2.9 of [7]), $w_1 \oplus w_1^*$ and $w_2 \oplus w_2^*$ are both homotopic to the identity in $U_{2n}(A'^{\sim})$ and $U_{2n}(\mathbb{C})$ respectively. Thus, $w_1vw_2 \oplus 0$ is homotopic to $v \oplus 0$ in $V_{2n}(A';A)$, so $\kappa(v) = \kappa(w_1vw_2)$ by (ii) and (i). Finally, $\kappa(w_1) = \kappa(w_2) = 0$ both following as special cases ($v = w_2 = 1$, $w_1 = v = 1$) of (iii) and (iv). As for (v), writing

$$v \oplus 0 = \begin{bmatrix} v & 1 - vv^* \\ 1 - v^*v & v^* \end{bmatrix} \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}$$

the conclusion follows from (iii) and (iv). ∎

We now want to see how this map κ relates to the six-term exact sequence for K-groups appearing in Section 1.

LEMMA 2.3. (i) *For v in $V_n(A';A)$,*

$$ev_*(\kappa(v)) = [vv^*] - [v^*v].$$

(ii) *For v in $U_n(A^\sim)$*

$$i_*b[v] = \kappa(v).$$

Proof. (i) We compute

$$ev_*(\kappa(v)) = [p_v(1)] - [e_{11}] = [(1_n - v^*v) \oplus vv^*] - [e_{11}] = [vv^*] - [v^*v].$$

(ii) In the construction of $\kappa(v)$, v_2 is a path of unitaries in $M_{2n}(A^\sim)$ from 1 to $\dot{v}_1^{-1}v_1$. Let $v_3(t)$ be any path of unitaries in $M_{2n}(\mathbb{C})$ from 1 to $\dot{v} \oplus \dot{v}^*$. Then $v_3(t)v_2(t)$ is a path from 1 to $v \oplus v^*$. By the definition of b

$$b[v] = [v_3v_2e_{11}v_2v_3^*] - [e_{11}] = [v_3p_vv_3^*] - [e_{11}] = [p_v] - [e_{11}] = \kappa(v),$$

since $v_3(t)$ is in $M_{2n}(\mathbb{C})$. ∎

LEMMA 2.4. *$\kappa : V(A';A) \to K_0(C(A';A))$ is onto.*

Proof. Let p, q be projections in $M_m(C(A';A)^\sim)$ with $[\dot{p}] = [\dot{q}]$ in $K_0(\mathbb{C})$; i.e. $[p]-[q]$ is in $K_0(C(A';A))$. By exactness, $j_*ev_*([p]-[q]) = 0$ in $K_0(A)$. This means $[p(1)] = [q(1)]$ in $K_0(A)$. So there exists positive integers k, $n = 2m+k$ and a partial isometry v in $M_n(A^\sim)$ such that

$$v^*v = 1_m \oplus 0_m \oplus 1_k$$

$$vv^* = p(1) \oplus (1_m - q(1)) \oplus 1_k.$$

Then v is in $V_n(A';A)$ and by Lemma 2.3 (i), we have

$$ev_*([p]-[q]) = ev_*(\kappa(v)).$$

Hence, $\kappa(v) - [p] + [q]$ is in the kernel of ev_* which is the image of i_*. For some unitary w in $M_\ell(A'^\sim)$, $i_*(w) = \kappa(v) - [p] + [q]$. Using Lemma 2.3 (ii), we have

$$\kappa(v \oplus w^*) = \kappa(v) + \kappa(w^*) = \kappa(v) - i_*(w) = [p] - [q]. \quad ∎$$

LEMMA 2.5. *Let $\approx$ denote the equivalence relation on $V(A';A)$ generated by:*

(i) *$v \approx v \oplus p$, $v \in V(A';A)$, p a projection in $M_n(\mathbb{C})$;*

(ii) *if $v(t)$ is a continuous path in $V_n(A';A)$, then $v(0) \approx v(1)$.*

Then $\kappa : V(A';A)/\approx \longrightarrow K_0(C(A';A))$ is a well-defined bijection.

Proof. It follows from Lemma 2.2 (i), (ii) and (iv) that κ is well-defined. From Lemma 2.4, we see that κ is onto. It remains to show that if v_1, v_2 are in $V_n(A'; A)$ and $\kappa(v_1) = \kappa(v_2)$, then $v_1 \approx v_2$.

First, note that if v, w_1 and w_2 are as in Lemma 2.2 (iii), then

$$w_1 v w_2 = w_1 \tilde{v} w_2 \oplus 0 = (w_1 \oplus w_1^*)(v \oplus 0)(w_2 \oplus w_2^*).$$

By homotoping the first and third terms of the right hand side, we see that $w_1 v w_2 \approx v$.

Returning to v_1 and v_2 with $\kappa(v_1) = \kappa(v_2)$, we may first assume that by taking direct sums with (different) scalar projections that the ranks of $v_1^* v_1$ and $v_2^* v_2$ are equal. We can then right multiply v_1 by a scalar unitary — without changing its $\approx$-equivalence class — to obtain $v_1^* v_1 = v_2^* v_2$.

From $\kappa(v_1) = \kappa(v_2)$, we apply ev_* to both sides, use Lemma 2.3 (i) and $v_1^* v_1 = v_2^* v_2$ to conclude that $[v_1 v_1^*] = [v_2 v_2^*]$ in $K_0(A'^{\sim})$. Again we may take direct sum with a scalar projection and reduce to the case $v_1 v_1^*$ and $v_2 v_2^*$ are unitarily equivalent. By left multiplying v_1 by a unitary in $M_n(A'^{\sim})$, we obtain $v_1 v_1^* = v_2 v_2^*$, $v_1^* v_1 = v_2^* v_2$, without changing the $\approx$-equivalence class of v_1 or v_2.

Let

$$R_n(t) = \begin{bmatrix} t & -\sqrt{1-t^2} \\ \sqrt{1-t^2} & t \end{bmatrix}, \qquad 0 \leqslant t \leqslant 1$$

be in $M_{2n}(\mathbb{C})$ and define the path in $M_{2n}(A^{\sim})$

$$v(t) = R_n(t)\,[v_1 \oplus v_1^* v_1]\; R_n(t)^{-1}\,[(v_1^* v_2 + 1 - v_1^* v_1) \oplus 1]$$

for $0 \leqslant t \leqslant 1$. Observe that for all t, $v(t)$ is in $V_{2n}(A'; A)$, $v(0) = v_1^* v_2 \oplus v_1$ and $v(1) = v_2 \oplus v_1^* v_1$. We have $v_1^* v_2$ is in $V_n(A'; A)$ and

$$\kappa\left(v_1^* v_2\right) = \kappa\left(v(0)\right) - \kappa(v_1) = \kappa\left(v(1)\right) - \kappa(v_1) = \kappa(v_2) - \kappa(v_1) = 0.$$

Now, consider the unitary $v = v_1^* v_2 + (1 - v_1^* v_1)$ in $M_n(A^{\sim})$. We have

$$i_* b[v] = \kappa(v) = \kappa(v_1^* v_2) = 0,$$

which implies $[v]$ is in the image of j_*. That is, v is homotopic (after direct summing with the identity) to a unitary in $M_n(A'^{\sim})$. Let $v'(t)$ be any path of unitaries in $M_n(A^{\sim})$ with $v'(0) = v$ and $v'(1) \in M_n(A'^{\sim})$.

Now define a path in $M_{4n}(A^{\sim})$

$$w(t) = \begin{bmatrix} v'(t)v_1 & v'(t)(1 - v_1v_1^*) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 - v_1^*v_1 & 0 & 0 & 0 \\ 0 & v_1v_1^* & 0 & 0 \end{bmatrix}.$$

It is straightforward to verify that, for all $0 \leqslant t \leqslant 1$,

$$w(t)^*w(t) = 1_n \oplus 1_n \oplus 0_n \oplus 0_n$$

$$w(t)w(t)^* = 1_n \oplus 0 \oplus (1 - v_1^*v_1) \oplus v_1v_1^*$$

and so $w(t)$ is a path in $V_{4n}(A'; A)$. Evaluating at $t = 0$, we see

$$w(0) = \begin{bmatrix} v_2 & 1 - v_1v_1^* & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 - v_1^*v_1 & 0 & 0 & 0 \\ 0 & v_1v_1^* & 0 & 0 \end{bmatrix} = \begin{bmatrix} v_1v_1^* & 1 - v_1v_1^* & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 - v_1v_1^* & v_1v_1^* & 0 & 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} v_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 - v_2^*v_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_2^*v_2 & 0 & 1 - v_2^*v_2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 - v_2^*v_2 & 0 & v_2^*v_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The first matrix in this product is a unitary in $M_{4n}(A'^{\sim})$, the last in $M_{4n}(\mathbb{C})$ and so

$$w(0) \approx v_2 \oplus 1 \oplus (1 - v_2^*v_2) \oplus 0 \approx v_2.$$

A similar calculation shows $w(1) \approx v_1$ and we are done. ∎

Regarding the relation $\approx$, it is clear that if v_0 and v_1 are homotopic, then for any scalar projection p, $v_0 \oplus p$ and $v_1 \oplus p$ are homotopic. Therefore, if $v_0 \approx v_1$ then there are scalar projections p_0 and p_1 such that $v_0 \oplus p_0$ and $v_1 \oplus p_1$ are homotopic.

A few other remarks are in order. Following exactly as in the beginning of the proof (before $\kappa(v_1) = \kappa(v_2)$ was used), given any v_1 and v_2 in $V(A'; A)$ we may direct sum scalar projections and right multiply one by a scalar unitary to get $v_1^*v_1 = v_2^*v_2$. Finally, if $v(r)$ is a path in $V_n(A'; A)$, we may right multiply by a path of scalar unitaries so that $v(r)^*v(r) = v(0)^*v(0)$, for all r.

For each $0 < \varepsilon < 400^{-1}$, we let $V_n^\varepsilon(A';A)$ denote the set of v in $M_n(A^\sim)$ such that:

(i) v is a partial isometry,
(ii) v^*v is in $M_n(\mathbb{C})$,
(iii) $\|vv^* - q\| < \varepsilon$, for some q in $M_n(A'^\sim)$.

We let $V^\varepsilon(A';A)$ denote the union of the $V_n^\varepsilon(A';A)$, with the usual inclusion of V_n^ε in V_{n+1}^ε. For any a in $V^\varepsilon(A';A)$, let v be as in Lemma 2.1. We define $\kappa(a) = \kappa(v)$. This is independent of the choice of v by Lemma 2.2 (ii). It is also easy to see that Lemma 2.2 is valid if we replace $V(A';A)$ by $V^\varepsilon(A';A)$. We extend the definition of $\approx$ to $V^\varepsilon(A';A)$ in the obvious way.

LEMMA 2.6. *Suppose A has a countable approximate unit $\{e_n\}_1^\infty$ contained in A'. Then for every v in $V_n(A';A)$ and $0 < \varepsilon < 400^{-1}$, $v \approx w$, for some w in $V_{2n}^\varepsilon(A';A)$ such that*

$$w = \begin{bmatrix} w_0 & 0 \\ (p - w_0^* w_0)^{\frac{1}{2}} & 0 \end{bmatrix},$$

where w_0 is in $M_n(A)$, p is a projection in $M_n(\mathbb{C})$ and $0 \leqslant w_0^* w_0 \leqslant p$. Moreover if

$$w = \begin{bmatrix} w_0 & 0 \\ (p - w_0^* w_0)^{\frac{1}{2}} & 0 \end{bmatrix}, \qquad w' = \begin{bmatrix} w_0' & 0 \\ (p - w_0'^* w_0')^{\frac{1}{2}} & 0 \end{bmatrix}$$

are homotopic in $V_{2n}^\varepsilon(A';A)$ then there is a path

$$w(t) = \begin{bmatrix} w_0(t) & 0 \\ (p - w_0(t)^* w_0(t))^{\frac{1}{2}} & 0 \end{bmatrix}$$

joining them.
(The point here is that w_0 lies in $M_n(A)$ and not just $M_n(A^\sim)$.)

Proof. Notice that $v \approx \dot{v}^* v$ — see the proof of Lemma 2.5 — and $(\dot{v}^* v)^\cdot = \dot{v}^* \dot{v} = p$ is a projection in $M_n(\mathbb{C})$. Thus, we may assume $\dot{v} = p$. Using e_m to denote $1_n \otimes e_m$ in $M_n(A)$, notice that

$$e_m' = \begin{bmatrix} e_m & -(1 - e_m^2)^{\frac{1}{2}} \\ (1 - e_m^2)^{\frac{1}{2}} & e_m \end{bmatrix}$$

is a unitary in $M_{2n}(A'^\sim)$ so

$$v \approx e_m' \, (v \oplus 0) = \begin{bmatrix} e_m v & 0 \\ (1 - e_m^2)^{\frac{1}{2}} v & 0 \end{bmatrix}.$$

We will let $w_0 = e_m v$, for some sufficiently large m, which is in $M_n(A)$. It is clear that $w_0^* w_0 \leqslant p$. Consider

$$\|(1-e_m^2)^{\frac{1}{2}} v - (p - w_0^* w_0)^{\frac{1}{2}}\| \leqslant \|(1-e_m^2)^{\frac{1}{2}} (v-p)\| + \|(1-e_m^2)^{\frac{1}{2}} p - (p - w_0^* w_0)^{\frac{1}{2}}\|.$$

The first term tends to zero since $v - p$ is in $M_n(A)$ and e_m is an approximate unit. As for the second, since $(1-e_m^2)$ and p commute, their product is positive and

$$\|(1-e_m^2)^{\frac{1}{2}} p - (p - w_0^* w_0)^{\frac{1}{2}}\| \leqslant \|(1-e_m^2) p - (p - w_0^* w_0)\|^{\frac{1}{2}} = \|(p-v)^*(1-e_m^2)(p-v)\|^{\frac{1}{2}}$$

which tends to zero as m goes to infinity. Therefore, we may choose m so that $e'_m(v \oplus 0)$ and

$$\begin{bmatrix} w_0 & 0 \\ (p - w_0^* w_0)^{\frac{1}{2}} & 0 \end{bmatrix}$$

are sufficiently close so that the latter is in $V_{2n}^{\varepsilon}(A'; A)$ and is $\approx$-equivalent to the former.

For the final part, consider the C^*-algebra $C[0,1] \otimes A$. We omit the details. ∎

3. THE EXCISION THEOREM

Here, we state and prove our main results (Theorems 3.1–3.7). We describe the hypotheses. We suppose that A and B are C^*-algebras acting on the Hilbert space $\mathcal{H}$. We also suppose that z is a selfadjoint unitary operator on $\mathcal{H}$. Note that we regard $M_n(A)$ and $M_n(B)$ as acting on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$, the n-fold direct sum. We also let z denote the operator $z \oplus \cdots \oplus z$ on $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$. We let $[a,b] = ab - ba$ for any operators a, b on $\mathcal{H}$.

We will assume conditions 1–6 hold.

1. For all a in A, b in B, ab is in A; i.e. B acts as multipliers of A.
2. $zAz = A$.
3. For all b in B, $zbz - b$ is in A.
4. There is a continuous path $\{e_t \mid t \geqslant 0\}$ in A such that:
   (i) $0 \leqslant e_t \leqslant e_s \leqslant 1$, for $t \leqslant s$;
   (ii) $e_s e_t = e_t$ for $s \geqslant t + 2$;
   (iii) for all a in A,

$$\lim_{t\to\infty} \|e_t a - a\| = 0 = \lim_{t\to\infty} \|a e_t - a\|;$$

(iv) $[e_t, z] = 0$, for all t.

We define C^*-subalgebras

$$A' = \{a \in A \mid [a, z] = 0\}$$

$$B' = \{b \in B \mid [b, z] = 0\}.$$

5. For all b in B, there exists b' in B' such that

$$\|b - b'\| \leqslant 2\|[b, z]\|.$$

(In the terminology of M.-D. Choi, almost commuting with z implies nearly commuting with z.)

6. There is a dense $*$-subalgebra $\mathcal{A} \subseteq A$ such that for a in $\mathcal{A}$, there is $t_0 \geqslant 1$ such that:

(i) $ae_t = e_t a = a$, for all $t \geqslant t_0$;

and, for any such t_0 as above, there is b in B such that:

(ii) $be_t = e_t b = a$, $t_0 \leqslant t \leqslant t_0 + 2$;

(iii) $[b, z] = [a, z]$;

(iv) $\|b\| \leqslant \|a\|$.

(The choice of b will depend on t_0 as well as a.)

Note that the condition on A analogous to 5 is valid; let $a' = (a + zaz)/2$.

Many examples are found in the theory of C^*-algebras associated to dynamical systems via the crossed product or groupoid C^*-algebra constructions. Let us mention one explicit example.

Fix an irrational number θ, $0 < \theta < 1$. Let $\mathcal{H} = \ell^2(\mathbb{Z})$ and let u and v denote the unitary operators

$$(u\xi)(n) = \xi(n-1)$$

$$(v\xi)(n) = \exp(2\pi i\theta)\xi(n),$$

for ξ in $\ell^2(\mathbb{Z})$, n in $\mathbb{Z}$. Then u and v satisfy the relation $uv = \exp(2\pi i\theta)\, vu$ and generate a C^*-algebra, B, isomorphic to the irrational rotation C^*-algebra, A_θ. We let $A = K(\mathcal{H})$, the compact operators, and

$$(z\xi)(n) = \begin{cases} \xi(n) & n \geqslant 1; \\ -\xi(n) & n \leqslant 0. \end{cases}$$

It is easy to verify 1, 2 and 3. It is also easy to see that

$$A' = K\left(\ell^2\{n \mid n \leqslant 0\}\right) \oplus K\left(\ell^2\{n \mid n \geqslant 1\}\right).$$

The proofs that 4, 5 and 6 hold can be found in [5]. Also the techniques of [5] show that B' is the C^*-subalgebra of B generated by v and $u(v-1)$. (See Example 2.6 of [5].)

THEOREM 3.1. *Let A, B, z satisfy 1–6 as above. Then there is an isomorphism*

$$\alpha : \mathrm{K}_0\left(C(B'; B)\right) \to \mathrm{K}_0\left(C(A'; A)\right)$$

which is natural in a sense to be described.

Let us take a moment to try to justify our description of Theorem 3.1 as an "excision" theorem. Section 2 describes the K-theory of the mapping cone $C(A'; A)$ as partial isometries in A with initial and final projection in A'. The extent to which an element a lies in A' can be measured by $zaz - a = z[a, z]$. A similar remark applies to B' and B. Conditions 2, 3 and 6 (iii) essentially mean that the sets

$$\left\{zaz - a \mid a \in A\right\}$$

$$\left\{zbz - b \mid b \in B\right\}$$

"agree". The conclusion is then that the corresponding "relative K-groups" are isomorphic.

We begin by describing the map α. We use e_t to also denote the element $1_n \otimes e_t$ in $M_n(A)$, for any $n = 1, 2, 3, \ldots$. We will use the description of $\mathrm{K}_0\left(C(B'; B)\right)$ provided by Lemma 2.5 and the discussion following it. Let v be in $V_n^{\varepsilon}(B'; B)$. For all $t \geqslant 1$, we define $\alpha(v)_t$ by

$$\alpha(v)_t = \begin{bmatrix} ve_t & 0 \\ (v^*v - e_t v^* v e_t))^{\frac{1}{2}} & 0 \end{bmatrix}.$$

Since B acts as multipliers of A, ve_t is in $M_n(A)$. Also, v^*v is a projection in $M_n(\mathbb{C})$ and it follows that $\alpha(v)_t$ lies in $M_{2n}(A^{\sim})$. It is also worth noting that e_t and v^*v commute so that

$$(v^*v - e_t v^* v e_t)^{\frac{1}{2}} = v^*v\left(1 - e_t^2\right)^{\frac{1}{2}}.$$

It is easy to check that

$$\alpha(v)_t^* \, \alpha(v)_t = v^*v \oplus 0,$$

which is in $M_{2n}(\mathbb{C})$ and is a projection.

LEMMA 3.2. *For v in $V_n^\varepsilon(B';B)$ and $0<\varepsilon<400^{-1}$, there is $t\geqslant 1$ such that $\alpha(v)_s$ is in $V_{2n}^\varepsilon(A';A)$ for all $s\geqslant t$.*

Proof. We claim that

$$\limsup_{t\to\infty} \|[\alpha(v)_t\,\alpha(v)_t^*,\, z]\| \leqslant \varepsilon.$$

To see this,

$$\alpha(v)_t\,\alpha(v)_t^* = \begin{bmatrix} ve_t^2v^* & ve_t\left(1-e_t^2\right)^{\frac{1}{2}} \\ \left(1-e_t^2\right)^{\frac{1}{2}} e_tv^* & v^*v\left(1-e_t^2\right) \end{bmatrix}$$

and we will check the commutators of the four entries with z separately. The lower right entry actually commutes with z since e_t does and v^*v is in $M_n(\mathbb{C})$. As for the upper right (or lower left)

$$\lim_{t\to\infty}\left[ve_t\left(1-e_t^2\right)^{half},\, z\right] = \lim_{t\to\infty}[v,z]\,e_t\left(1-e_t^2\right)^{\frac{1}{2}} = 0$$

since $z[v,z]$ is in $M_n(A)$ and e_t is an approximate unit for A. For the upper left entry, we have

$$\limsup_{t\to\infty}\left\|\left[ve_t^2v^*,\, z\right]\right\| = \limsup_{t\to\infty}\left\|[v,z]\,e_t^2v^* + ve_t^2[v^*,z]\right\|.$$

Since $z[v,z]$ and $z[v^*,z]$ are both in A, e_t will asymptotically commute both, so this equals

$$\limsup_{t\to\infty}\left\|e_t^2[v,z]\,v^* + v[v^*,z]\,e_t^2\right\|.$$

Applying the same argument and noting $[v,z]\,v^*$ is in $M_n(A)$ since v^* is in the multiplier algebra of $M_n(A)$, this equals

$$\limsup_{t\to\infty}\left\|([v,z]\,v^* + v[v^*,z])\,e_t^2\right\| = \limsup_{t\to\infty}\left\|[vv^*,\, z]\,e_t^2\right\| \leqslant \varepsilon$$

since vv^* is within ε of an element of in $M_n(A'^\sim)$. The claim is established.

To see the conclusion, let

$$q = \frac{z\,\alpha(v)_t\,\alpha(v)_t^*\,z + \alpha(v)_t\,\alpha(v)_t^*}{2}.$$

Now, (iii) follows from the claim and it is clear that q is in $M_{2n}(A'^\sim)$. ∎

Notice that

$$\alpha(v \oplus w)_t = \alpha(v)_t \oplus \alpha(w)_t$$

(at least after a change of basis which we will suppress). It follows from Lemma 3.2 that letting

$$\alpha\left(\kappa(v)\right) = \kappa(\alpha(v)_s),$$

for any sufficiently large s defines an element in $\mathrm{K}_0\left(C(A';A)\right)$. To see that α is well-defined it suffices to apply Lemma 2.5 and observe the following. If p is a projection in $M_n(\mathbb{C})$ then

$$\alpha(p)_t = e'_t(p \oplus 0),$$

where e'_t is as in Lemma 2.6. So then $\kappa\left(\alpha(p)_t\right) = 0$ by Lemma 2.2 (ii), (iii).

Also observe that if $v(r)$, $0 \leqslant r \leqslant 1$ is a path in $V_n^\varepsilon(B';B)$ then the limit in Lemma 3.2 can be made uniform over r, and, hence, for s large $\alpha\left(v(r)\right)_s$ will be a homotopy in $V_{2n}^{2\varepsilon}(A';A)$.

The proof of Theorem 3.1 will require several technical lemmas.

LEMMA 3.3. *Let w_0 be in $M_n(A)$ and p be a projection in $M_n(\mathbb{C})$ such that $p \geqslant w_0^* w_0$. Then there is $t_0 \geqslant 1$ and v_0 in $M_n(B)$ with $v_0^* v_0 \leqslant p$ such that:*

(i) $w_0 e_s = e_s w_0 = w_0$, *for* $s \geqslant t_0$;

(ii) $v_0 e_s = e_s v_0 = w_0$, *for* $t_0 + 2 \geqslant s \geqslant t_0$;

(iii) $[v_0, z] = [w_0, z]$;

(iv) $[v_0^* v_0,\, z] = [w_0^* w_0,\, z]$;

(v) $[v_0 v_0^*,\, z] = [w_0 w_0^*,\, z]$;

(vi) $[(p - v_0^* v_0)^{\frac{1}{2}},\, z] = [(p - w_0^* w_0)^{\frac{1}{2}},\, z]$.

Proof. Choose any t_0 and b as in hypothesis 6 for $a = w_0$. Then let

$$v_0 = bp \quad \text{so} \quad v_0^* v_0 = p\, b^* b\, p \leqslant p \|b\|^2 p \leqslant p.$$

Conditions (i), (ii) and (iii) follow at once from hypothesis 6.

We have

$$[v_0^* v_0,\, z] = [v_0^*,\, z]\, v_0 + v_0^* [v_0,\, z] = [w_0^*,\, z]\, v_0 + v_0^* [w_0,\, z]$$

$$= [w_0^* e_t,\, z]\, v_0 + v_0^* [e_t w_0,\, z],$$

for $t_0 \leqslant t \leqslant t_0 + 2$,

$$= [w_0^*,\, z]\, e_t v_0 + v_0^* e_t [w_0,\, z]$$
$$= [w_0^*,\, z]\, w_0 + w_0^* [w_0,\, z],$$

by (ii),

$$= [w_0^* w_0, z]$$

and so (iv) holds. A similar argument establishes (v). As for (vi), it follows from (iv) that

$$[f(p - v_0^* v_0), z] = [f(p - w_0^* w_0), z]$$

for any polynomial f. By standard approximation arguments, the same holds for $f(t) = t^{\frac{1}{2}}$. ∎

LEMMA 3.4. *Let w_0, p, t_0, v_0 be as in Lemma 3.3. Define w in $M_{2n}(A^\sim)$ and v in $M_{2n}(B^\sim)$ by*

$$w = \begin{bmatrix} w_0 & 0 \\ (p - w_0^* w_0)^{\frac{1}{2}} & 0 \end{bmatrix}$$

$$v = \begin{bmatrix} v_0 & 0 \\ (p - v_0^* v_0)^{\frac{1}{2}} & 0 \end{bmatrix}.$$

Then:

(i) $w^* w = v^* v = p \oplus 0$,

(ii) $e_s[v, z] = [v, z]\, e_s = [v, z] = [w, z]$ *for* $s \geqslant t_0$,

(iii) $[ww^*, z] = [vv^*, z]$.

The proof is an easy consequence of Lemma 3.3; we omit the details.

LEMMA 3.5. *Let w_0 be in $M_n(A^\sim)$, p a projection in $M_n(\mathbf{C})$ with $p \geqslant w_0^* w_0$. Let t_0, v_0 be as in Lemma 3.3, w, v as in Lemma 3.4 and assume w is in $V_{2n}^{\varepsilon}(A'; A)$ for some $0 < \varepsilon < 400^{-1}$. Then:*

(i) *v is in $V_{2n}^{4\varepsilon}(B'; B)$,*

(ii) *$\alpha(v)_s$ is in $V_{4n}^{4\varepsilon}(A'; A)$, for all $s \geqslant t_0$,*

(iii) *$\kappa(\alpha(v)_s) = \kappa(w)$, for $t_0 \leqslant s \leqslant t_0 + 2$.*

Proof. (i) From Lemma 3.4 (i), $v^* v = p \oplus 0$ and we must check only that vv^* is close to an element of $M_{2n}(B'^\sim)$. From Lemma 3.4 (iii)

$$\|[vv^*, z]\| = \|[ww^*, z]\| \leqslant 2\varepsilon$$

since w is in $V_{2n}^{\varepsilon}(A'; A)$. Apply hypothesis 5 to find q in $M_{2n}(B'^\sim)$ so that $\|q - vv^*\| \leqslant 4\varepsilon$, and (i) is complete.

(ii) As before, we must compute

$$\|[\alpha(v)_s\, \alpha(v)_s^*, z]\|.$$

Now, for $s \geqslant t_0$,

$$\alpha(v)_s\,\alpha(v)_s^* = \begin{bmatrix} ve_s^2 v^* & ve_t\left(1-e_t^2\right)^{\frac{1}{2}} v^*v \\ v^*v\left(1-e_t^2\right)^{\frac{1}{2}} e_t v^* & v^*v\left(1-e_t^2\right) \end{bmatrix}$$

and commutators with z for each of the entries is done separately. The off-diagonal entries commute with z because $v^*v = p$ and by condition (ii) of Lemma 3.4, so $(1-e_t)[v,z] = 0$. The lower right entry also commutes with z while

$$\left[ve_s^2 v^*,\, z\right] = [ww^*,\, z] \qquad \text{for } s \geqslant t_0.$$

This completes the proof of (ii).

(iii) By direct computation

$$\alpha(v)_s = \begin{bmatrix} v_0 e_s & 0 & 0 & 0 \\ (p - v_0^* v_0)^{\frac{1}{2}} e_s & 0 & 0 & 0 \\ p(1-e_2^2)^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e_s & -\left(1-e_s^2\right)^{\frac{1}{2}} & 0 \\ 0 & \left(1-e_s^2\right)^{\frac{1}{2}} & e_s & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\cdot \begin{bmatrix} w_0 & 0 & 0 & 0 \\ (p - w_0^* w_0)^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for $t_0 \leqslant s \leqslant t_0 + 2$, using Lemma 3.2. The first matrix above is in $M_{4n}\left(A'^{\sim}\right)$ and so the result follows from Lemma 2.2 (iii). ∎

LEMMA 3.6. *Suppose v is in $V_n(B';B)$ and $\|[v,z]\| \leqslant \varepsilon \leqslant 10^{-6}$. Then* $\kappa(v) = 0$.

Proof. By hypothesis 5, there is a v' in $M_n(B'^{\sim})$ such that $\|v'\| \leqslant 1$ and $\|v - v'\| \leqslant 2\varepsilon$. Let

$$w = \begin{bmatrix} v'p & 0 \\ (p - pv'^* v'p)^{\frac{1}{2}} & 0 \end{bmatrix},$$

where $p = v^*v$, so w is in $V_{2n}(B';B)$ and in $M_{2n}(B'^{\sim})$ and

$$\|v \oplus 0 - w\| \leqslant 4\varepsilon^{\frac{1}{2}}.$$

Moreover, $\kappa(w) = 0$ by Lemma 2.2 (v) and $\kappa(v) = \kappa(w)$ by Lemma 2.2 (ii). ∎

Let us describe the naturality of the isomorphism described in Theorem 3.1. Suppose $(A_1, B_1, z_1, \{e_t^{(1)}\})$ and $(A_2, B_2, z_2, \{e_t^{(2)}\})$ are two systems satisfying 1–6. Also suppose

$$\sigma : A_1 \longrightarrow A_2$$
$$\pi : B_1 \longrightarrow B_2$$

be $*$-homomorphisms such that

$$\begin{aligned}
&\sigma(ab) = \sigma(a)\pi(b), && a \in A_1,\ b \in B_1 \\
&\sigma(z_1 a z_1) = z_2\, \sigma(a)\, z_2, && a \in A_1 \\
&\pi(z_1 b z_1) = z_2\, \pi(b)\, z_2, && b \in B_1 \\
&\sigma(z_1 b z_1 - b) = z_2\, \pi(b)\, z_2 - \pi(b), && b \in B_1 \\
&\sigma(e_t^{(1)}) = e_t^{(2)}, && \text{for all } t.
\end{aligned}$$

It is easy to see that σ and π induce $*$-homomorphisms

$$\tilde{\sigma} : C\,(A_1';\, A_1) \longrightarrow C\,(A_2';\, A_2)$$

$$\tilde{\pi} : C\,(B_1';\, B_1) \longrightarrow C\,(B_2';\, B_2)\,.$$

The map α is natural in the sense that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{K}_0\,(C(B_1';\, B_1)) & \xrightarrow{\ \alpha\ } & \mathrm{K}_0\,(C(A_1';\, A_1)) \\
\big\downarrow \tilde{\pi}_* & & \big\downarrow \tilde{\sigma}_* \\
\mathrm{K}_0\,(C(B_2';\, B_2)) & \xrightarrow{\ \alpha\ } & \mathrm{K}_0\,(C(A_2';\, A_2))\,.
\end{array}$$

The proof of this is immediate. We omit the details.

As an application, suppose (A, B, z, e_t) satisfies 1–6 and suppose X is a compact second countable Hausdorff space. Fix some regular Borel measure μ on X with full support. Then we can regard $A \otimes C(X)$, $B \otimes C(X)$ and $z \otimes 1$ as operators on $\mathcal{H} \otimes L^2(X, \mu)$. Hypotheses 1–3 are easily checked and $e_t \otimes 1$ satisfies 4. We also have

$$(A \otimes C(X))' = A' \otimes C(X)$$

$$(B \otimes C(X))' = B' \otimes C(X)$$

and 5 follows. The algebraic tensor product of $\mathcal{A}$ and $C(X)$ can be seen to satisfy 6.

Proof of Theorem 3.1. First of all, it is fairly clear that α is additive. The surjectivity of α follows at once from Lemmas 2.6 and 3.5.

Suppose v is in $V_n(B';B)$ and $\alpha(\kappa(v)) = 0$ in $K_0(C(A';A))$. Let $p = v^*v$ which is a projection in $M_n(\mathbb{C})$. Fix $\varepsilon = 10^{-7}$. Choose $t_1 \geqslant 1$ such that

$$\|[v,z]\,e_t - [v,z]\| \leqslant \varepsilon \tag{3.1}$$

$$\|[v,z] - [v,z]\,e_t\| \leqslant \varepsilon, \quad t \geqslant t_1$$

and such that

$$\alpha(v)_t \in V_{2n}^{\varepsilon}(A';A), \quad t \geqslant t_1. \tag{3.2}$$

Since $\kappa(\alpha(v)) = 0$, we may direct sum $\alpha(v)_{t_1}$ with a scalar projection q so that the result is homotopic to a scalar projection in $V^{\varepsilon}(A';A)$. By replacing v by $v \oplus q$, we may assume simply that $\alpha(v)_{t_1}$ is homotopic to $\begin{bmatrix} 0 & 0 \\ p & 0 \end{bmatrix}$, which is homotopic to $p \oplus 0$. We apply Lemma 2.6 to obtain a path as described there. We may then approximate the "w_0" part of this path by a path in $M_n(\mathcal{A})$. We right multiply this path by p and we obtain a path $a(s)$, $0 \leqslant s \leqslant 1$, such that a is in the algebraic tensor product of $C[0,1]$ and $M_n(\mathcal{A})$,

$$w(s) = \begin{bmatrix} a(s) & 0 \\ (p - a(s)^* a(s))^{\frac{1}{2}} & 0 \end{bmatrix}, \quad 0 \leqslant s \leqslant 1, \ \in V_{2n}^{2\varepsilon}(A';A) \tag{3.3}$$

$$a(1) = 0 \tag{3.4}$$

$$\|w(0) - \alpha(v)_{t_1}\| \leqslant 2\varepsilon,$$

hence,

$$\|a(0) - v e_{t_1}\| \leqslant 2\varepsilon, \tag{3.5}$$

$$\left\|(p - a(0)^* a(0))^{\frac{1}{2}} - p\left(1 - e_{t_1}^2\right)^{\frac{1}{2}}\right\| \leqslant 2\varepsilon.$$

We may apply the sequence of Lemmas 3.3, 3.4 and 3.5 to the element a in $M_n(\mathbb{C}[0,1] \odot \mathcal{A})$ (algebraic tensor product) and p in $M_n(\mathbb{C})$ to obtain a path $b(s)$, $0 \leqslant s \leqslant 1$

$$v_1(s) = \begin{bmatrix} b(s) & 0 \\ (p - b(s)^* b(s))^{\frac{1}{2}} & 0 \end{bmatrix}$$

$0 \leqslant s \leqslant 1$ and $t_2 \geqslant t_1 + 2$ such that

$$[b(s), z] = [a(s), z], \tag{3.6}$$

(3.7) $$b(s)\, e_t = e_t\, b(s), \quad t_2 \leqslant t \leqslant t_2 + 2,$$

(3.8) $$a(s)\, e_t = e_t\, a(s) = a(s), \quad t \geqslant t_2,$$

(3.9) $$[b(s)^* b(s),\, z] = [a(s)^* a(s),\, z]$$

(3.10) $$[b(s)b(s)^*,\, z] = [b(s)b(s)^*,\, z]$$

(3.11) $$\left[(p - b(s)^* b(s))^{\frac{1}{2}},\, z\right] = \left[(p - a(s)^* a(s))^{\frac{1}{2}},\, z\right],$$

$$v_1(s) \text{ is in } V_{2n}^{4\varepsilon}(B'; B)$$

$$\alpha\,(v_1(s))_t \text{ is in } V_{4n}^{4\varepsilon}(A'; A), \quad t \geqslant t_2.$$

Let us evaluate v_1 at $s = 1$. Making use of (3.4), (3.6) and (3.9), we see that

(3.12) $$[v_1(1),\, z] = 0$$

and so $v_1(1)$ is in $M_n(B'^{\sim})$. Next, we claim that

(3.13) $$\|[v\, b(0)^*,\, z]\| \leqslant 3\varepsilon,$$

(3.14) $$\left\| \left[v\,(p - b(0)^* b(0))^{\frac{1}{2}},\, z\right] \right\| \leqslant 3\varepsilon.$$

To see the first, we have

$$\begin{aligned} \|[v\, b(0)^*,\, z]\| &= \|[v, z]\, b(0)^* + v\, [b(0)^*,\, z]\| \\ &\leqslant \|[v, z]\, e_{t_1}\, b(0)^* + v\, [a(0)^*,\, z]\| + \varepsilon \end{aligned}$$

by (3.1) and (3.6),

$$\leqslant \|[v, z]\, e_{t_1} e_{t_2}\, b(0)^* + v\, [e_{t_1}\, v^*,\, z]\| + \varepsilon$$

by hypothesis 4 (ii) and (3.5),

$$= \|[v, z]\, e_{t_1}\, a(0)^* + v e_{t_1} [v^*, z]\| + \varepsilon$$

by (3.7)

$$\leqslant \left\|[v, z]\, e_{t_1}^2\, v^* + v e_{t_1}^2 [v^*, z]\right\| + 2\varepsilon$$

by (3.5) and (3.1)

$$= \left\| \left[v e_{t_1}^2 v^*, z \right] \right\| + 2\varepsilon \leqslant 3\varepsilon$$

because of (3.2). To see the second, there is a similar computation which we omit.

Now consider

$$v_2(s) = (v \oplus 0)\, v_1(s)^*, \qquad 0 \leqslant s \leqslant 1.$$

This is a path of partial isometries in $M_{2n}(B^\sim)$. For each s, its range projection is the range projection of v which is in $M_{2n}(B'^\sim)$. Its initial projection is the range projection of $v_1(s)$ which is in $M_{2n}(B'^\sim)$, for all s. As noted in (3.12), when $s = 1$, this projection is actually Murray-von Neumann equivalent to $p \oplus 0$ in $M_{2n}(B'^\sim)$. So we may find a path of unitaries $u(s)$, $0 \leqslant s \leqslant 1$ in $M_{2n}(B'^\sim)$ (actually, it may be necessary to pass to $M_{4n}(B'^\sim)$) such that

$$v_1(1)^*\, u(1) = p \oplus 0$$

$$v_1(s)^*\, u(s) \text{ has initial projection } p \oplus 0,$$

$$0 \leqslant s \leqslant 1.$$

Now, consider the path

$$v_3(s) = (v \oplus 0)\, v_1(s)^*\, u(s), \qquad 0 \leqslant s \leqslant 1.$$

It is a path in $V_{2n}(B'; B)$. Moreover, for $s = 1$,

$$v_3(1) = v \oplus 0$$

while for $s = 0$,

$$v_3(0) = \begin{bmatrix} v\, b(0)^* & v\, (p - b(0)^*\, b(0))^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix} u(0)$$

which commutes with z, to within 3ε, by (3.13) and (3.14). By Lemma 2.2 (v) and the homotopy invariance of κ,

$$\kappa(v) = \kappa\,(v_3(1)) = \kappa\,(v_3(0)) = 0.$$

This proves that α is injective and we are done. ∎

THEOREM 3.7. *Let A, B, z satisfy 1–6 as before. Then there are isomorphisms*

$$\alpha : \mathrm{K}_i\,(C(B';B)) \longrightarrow \mathrm{K}_i\,(C(A';A)),$$

which are natural, for $i = 0, 1$.

Proof. The case $i = 0$ is done. For the other case, let $B_1 = C(S^1) \otimes B$, $A_1 = C(S^1) \otimes A$, $z_1 = 1 \otimes z$ and $\sigma : A_1 \to A$, $\pi : B_1 \to B$ be given by evaluation at some fixed point of the circle, S^1. There is a split exact sequence

$$0 \to C_0(0,1) \otimes C(B';B) \to C(B_1';B_1) \xrightarrow{\pi} C(B';B) \to 0$$

and a corresponding one for A and A_1. Using the naturality of α on K_0 and the usual isomorphism

$$\mathrm{K}_1\,(C(B';B)) \cong \mathrm{K}_0\,(C_0(0,1) \otimes C(B';B))$$

and the usual techniques, one obtains the result for K_1 groups as well. ∎

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