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MORPHISMS OF MULTIPLICATIVE UNITARIES

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ABSTRACT. In this paper we will give a natural definition for morphisms between multiplicative unitaries. We will then discuss some equivalences of this definition and some interesting properties of them. Moreover, we will define normal sub-multiplicative unitaries for multiplicative unitaries of discrete type and prove an imprimitivity type theorem for discrete multiplicative unitaries.

KEYWORDS: *Hopf C^*-algebras, multiplicative unitaries, morphisms.*

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0. INTRODUCTION

In [2], Baaj and Skandalis defined multiplicative unitaries and showed that they are nice generalisation of locally compact groups. They also showed that the Woronowicz C^*-algebras (which can also be considered as compact quantum groups) can be included in the consideration of multiplicative unitaries of compact type. However, there is not any notion of morphism being defined so far. In [8], Wang define morphisms between compact quantum groups as Hopf $*$-homomorphisms between the underlying Woronowicz C^*-algebras. However, for a given multiplicative unitary V, we can associate with it four Hopf C^*-algebras (if the multiplicative unitary is good), namely, $S_V, \widehat{S}_V, (S_V)_p$ and $(\widehat{S}_V)_p$. It is not clear which of the Hopf $*$-homomorphisms between these Hopf C^*-algebras should be used as a candidate for the morphisms.

In this paper we will investigate a natural notion called *birepresentation* and show that it is a good candidate for the morphisms between multiplicative unitaries. More precisely, given two multiplicative unitaries U and V, we define morphisms from V to U to be the collection of all U-V-birepresentations. Some of the Hopf $*$-homomorphisms between the Hopf C^*-algebras defined by U and V are equivalent to U-V-birepresentations (see Theorem 4.9). We also find another equivalence that birepresentations are in one to one correspondence with the *mutual coactions* (see Definition 3.13) between those Hopf C^*-algebras. We also investigate the crossed products of the different coactions arising from a *morphism* and show that $(S_V)_p \times_{\delta,\max} (\widehat{S}_V)_p \cong (\widehat{S}_V)_p \times_{\widehat{\delta}^{\rm op},\max} (S_V)_p^{\rm op}$.

Now we obtain a category of multiplicative unitaries (those satisfing some good property). It contains the locally compact groups as a full subcategory. However, it seem not easy to define kernels of these morphisms. By looking at the case of discrete groups, we can define kernels of morphisms between multiplicative unitaries of discrete type (which is really a kernel in the categorical sense).

Finally, from the definition of a kernel, we can define normal sub-multiplicative unitaries of multiplicative unitaries of discrete type. We then prove an imprimitivity type theorem for this setting.

1. PRELIMINARY AND NOTATION

The notation in this paper mainly follows from those of [2] and [5]. We also assume the basic definitions and the results from these two papers.

DEFINITION 1.1. Let (A, δ) be a Hopf C^*-algebra.

(a) A $*$-subalgebra B of $M(A)$ is called a *Hopf C^*-subalgebra* of A if:

(i) there exists an approximate unit $\{e_i\}$ of B such that e_i converge strictly to 1 in $M(A)$;

(ii) $\delta(B) \subseteq M(B \otimes B)$;

(iii) the restriction, ε, of δ in B is a co-multiplication on B.

(b) Let $A^{\rm op}$ be the C^*-algebra A with a co-multiplication $\delta^{\rm op}$ defined by $\delta^{\rm op} = \sigma \circ \delta$ (where σ is the flip of variables).

Note that condition (iii) in Definition 1.1 (a) means that $\varepsilon(B)(1 \otimes B) \subseteq B \otimes B$ and condition (ii) makes sense because of (i). It is easy to see that $\delta^{\rm op}$ is a co-multiplication on A and $(A^{\rm op})^* = (A^*)^{\rm op}$ (algebra with an opposite multiplication). We recall that a *Hopf $*$-homomorphism* φ from a Hopf C^*-algebra (A, δ) to another Hopf C^*-algebra (B, ε) is a non-degenerate $*$-homomorphism from A to $M(B)$ such that $(\varphi \otimes \varphi) \circ \delta = \varepsilon \circ \varphi$.

LEMMA 1.2. *Let (A, δ) and (B, ε) be two Hopf C^*-algebras and φ be a Hopf $*$-homomorphism from A to $M(B)$. Then $B_0 = \varphi(A)$ is a Hopf C^*-subalgebra of B.*

LEMMA 1.3. *Let A and B be C^*-algebras. If φ and ψ are non-degenerate $*$-homomorphisms from A to $M(B)$, and from B to $M(A)$, respectively such that $\varphi \circ \psi$ and $\psi \circ \varphi$ are identity maps, then φ is an isomorphism from A to B.*

Proof. We first note that $\varphi(A)$ is an ideal of $M(B)$ since $\psi(\varphi(a)m) = a\psi(m) \in A$ for any $a \in A$ and $m \in M(B)$. Let $\{e_i\}$ be an approximate unit of A. Then for any $b \in B$, $\varphi(e_i)b \in \varphi(A)$ will converge to b and hence $B \subseteq \varphi(A)$. ∎

DEFINITION 1.4. Let (R, ε) and (S, δ) be Hopf C^*-algebras. A unitary $U \in M(R \otimes S)$ is said to be a *unitary R-S-birepresentation* if $(\mathrm{id} \otimes \delta)(U) = U_{12}U_{13}$ and $(\varepsilon \otimes \mathrm{id})(U) = U_{13}U_{23}$.

LEMMA 1.5. *Let (S, δ) and (T, ε) be Hopf C^*-algebras. Let w and v be unitary co-representations of S and T respectively on the same Hilbert space H and let $u = w^\sigma \in M(S \otimes \mathcal{K}(H))$ (where σ means the flip of the two variables). If X is a unitary in $M(S \otimes T)$ such that $u_{12}X_{13}v_{23} = v_{23}u_{12}$, then X is a unitary S-T-birepresentation.*

Proof. Applying $(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)$ to the equation we have that

$$u_{12}(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(X_{13})v_{23}v_{24} = v_{23}v_{24}u_{12} = v_{23}u_{12}X_{14}v_{24}.$$

Thus,

$$(\mathrm{id} \otimes \mathrm{id} \otimes \varepsilon)(X_{13}) = u_{12}^*v_{23}u_{12}X_{14}v_{23}^* = X_{13}v_{23}X_{14}v_{23}^* = X_{13}X_{14}.$$

Similarly, $(\delta \otimes \mathrm{id})(X) = X_{13}X_{23}$.

DEFINITION 1.6. Let X be a unitary S-T-birepresentation. Let w and v be unitary co-representations of S and T respectively on the same Hilbert space and let $u = w^\sigma$. Then (u, v) is said to be a *covariant pair for X* if u and v satisfy the condition in the previous lemma.

We now recall the following definitions from [1].

DEFINITION 1.7. Let V be a multiplicative unitary on a Hilbert space H. Then:

(a) V is said to be *semi-regular* if the norm closure of the set $\{(\mathrm{id} \otimes \omega)(\Sigma V) : \omega \in \mathcal{L}(H)_*\}$ contains the set of all compact operators $\mathcal{K}(H)$. Moreover, V is said to be *semi-biregular* if it is regular and the norm closure of the set $\{(\omega \otimes \mathrm{id})(\Sigma V) : \omega \in \mathcal{L}(H)_*\}$ contains $\mathcal{K}(H)$ as well;

(b) V is said to be *balanced* if there exists a unitary $U \in \mathcal{L}(H)$ such that

(i) $U^2 = I_H$;

(ii) the unitary $\widehat{V} = \Sigma(U \otimes 1)V(U \otimes 1)\Sigma$ is multiplicative.

REMARK 1.8. For simplicity, we will call a multiplicative unitary *semi-irreducible* if it is both semi-regular and balanced.

PROPOSITION 1.9. *Let S and T be Hopf C^*-algebras and φ be a Hopf $*$-homomorphism from S to $M(T)$. If ε is a coaction on a C^*-algebra A by S, then $\delta = (\mathrm{id} \otimes \varphi) \circ \varepsilon$ is a coaction on A by T.*

Proof. The coaction identity follows easily from the fact that φ respects the co-multiplications. It remains to show that $\delta(A) \subseteq \widetilde{M}(A \otimes T)$. Let (u_i) be an approximate unit of S. For any $a \in A$ and $t \in T$, $t_i = \varphi(u_i)t$ converges to t in norm and so $\delta(a)(1 \otimes t_i)$ converges to $\delta(a)(1 \otimes t)$ in norm. Now, $\delta(a) \cdot (1 \otimes t_i) = (\mathrm{id} \otimes \varphi)[\varepsilon(a)(1 \otimes u_i)] \cdot (1 \otimes t) \in A \otimes T$. ∎

PROPOSITION 1.10. *Let $A, S, T, \varphi, \varepsilon$ and δ be the same as in Proposition 1.9. Suppose that the crossed products $A \times_{\varepsilon,\max} \widehat{S}$ and $A \times_{\delta,\max} \widehat{T}$ exist. Then there exists a $*$-homomorphism Φ from $A \times_{\varepsilon,\max} \widehat{T}$ to $M(A \times_{\delta,\max} \widehat{S})$.*

Proof. Let (B, ψ, u) be a covariant pair for (A, S, ε) and let $v = (\mathrm{id} \otimes \varphi)(u)$. Then (B, ψ, v) is a covariant pair for (A, T, δ) and the proposition follows from the definition of crossed product (see [5], 2.11 (b)). ∎

2. BASIC MULTIPLICATIVE UNITARIES

The aim of this section is to find some basic assumptions on the multiplicative unitaries such that the results in this paper holds. We will show that the semi-irreducible multiplicative unitaries and regular multiplicative unitaries both satisfy these basic assumptions (the manageable multiplicative unitaries "almost" satisfy these assumptions, at least when they are either amenable or co-amenable).

DEFINITION 2.1. Let V be a multiplicative unitary. Then V is called a *C^*-multiplicative unitary* if for any representation X and co-representation Y of V on K and L respectively:

(i) $\widehat{S}_X = \overline{\{(\mathrm{id} \otimes \omega)(X) : \omega \in \mathcal{L}(H)_*\}}$ and $S_Y = \overline{\{(\omega \otimes \mathrm{id})(Y) : \omega \in \mathcal{L}(H)_*\}}$ are both C^*-algebras;

(ii) $X \in M(\widehat{S}_X \otimes S_V)$ and $Y \in M(\widehat{S}_V \otimes S_Y)$.

Basic examples of C^*-multiplicative unitaries are regular multiplicative unitaries, semi-irreducible multiplicative unitaries (see Remark 1.8) and manageable multiplicative unitaries.

REMARK 2.2. (a) By the argument in [11], Section 5, if V is a C^*-multiplicative unitary, then S_V and $\widehat{S}_V$ are both Hopf C^*-algebras with coactions δ and $\widehat{\delta}$, respectively. Moreover, $(\mathrm{id} \otimes \delta)(X) = X_{12}X_{13}$ and $(\widehat{\delta} \otimes \mathrm{id})(Y) = Y_{13}Y_{23}$ if X and Y are representation and co-representation of V, respectively. Furthermore, by using the argument in that section, we can also show that the closure of $\{(\omega \otimes \mathrm{id} \otimes \mathrm{id})(Y_{12}Y_{13})(1 \otimes s) : \omega \in \mathcal{L}(H)_*, s \in S_Y\} = S_Y \otimes S_Y$ for any co-representation Y.

(b) If V is a C^*-multiplicative unitary, then by the same argument as in [2], A6, ([2], A6 (a)–(d)) hold for V. Moreover by part (a) above, ([2], A6 (e)) holds as well.

(c) We also note that all the main results in [5] hold for C^*-multiplicative unitaries (actually, except [5], 3.7 and 3.15 which involve the Takesaki-Takai type duality).

Let V' and V'' be as defined in [2], A6. We call V' and V'' the universal representation and the universal co-representation of V, respectively. We also need the following technical assumption.

DEFINITION 2.3. Let V be a C^*-multiplicative unitary.

(a) Let V_p be a unitary in $M((\widehat{S}_V)_p \otimes (S_V)_p)$. Then V_p is said to be a *universal birepresentation* of V if $V'_{12}(V_p)_{13}V''_{23} = V''_{23}V'_{12}$ in $M((\widehat{S}_V)_p \otimes \mathcal{K}(H) \otimes (S_V)_p)$.

(b) A C^*-multiplicative unitary V is said to be *basic* if there exists a universal birepresentation for V.

REMARK 2.4. (a) It is clear that if a C^*-multiplicative unitary V is amenable (respectively, co-amenable), then $V_p = V''$ (respectively, $V_p = V'$) exists and V is basic.

(b) By Lemma 1.5, if the universal birepresentation of V exists, it is a unitary $(\widehat{S}_V)_p$-$(S_V)_p$-birepresentation. Moreover, it is clear that $(\mathrm{id} \otimes L_V)(V_p) = V'$ and $(\rho_V \otimes \mathrm{id})(V_p) = V''$.

We are going to show that V_p exists in good case. Note that we can also deduce the existence of V_p from [2], A8, but irreducibility is required there. We first recall the set $\mathcal{C}(V) = \{(\mathrm{id} \otimes \omega)(\Sigma V) : \omega \in \mathcal{L}(H)_*\}$ from [2]. Note that the idea of the proof of the following lemma is from [2], 3.6 (c).

LEMMA 2.5. *Let $V \in \mathcal{L}(H \otimes H)$ be a multiplicative unitary and X and Y are a representation and a co-represention of V on K and L respectively. Let $W = X^*_{12}Y_{23}X_{12}Y^*_{23} \in \mathcal{L}(K \otimes H \otimes L)$. Then $(1 \otimes c \otimes 1)W = W(1 \otimes c \otimes 1)$ for any $c \in \mathcal{C}(V)$. Consequently, if $\overline{\mathcal{C}(V)}^{\mathrm{weak}} = \mathcal{L}(H)$, then W is of the form $W = Z_{13}$ for some unitary $Z \in \mathcal{L}(K \otimes L)$.*

Proof. We first note that

$$\begin{aligned} X_{12}^*Y_{24}X_{12}Y_{24}^*\Sigma_{23}V_{23} &= \Sigma_{23}X_{13}^*Y_{34}X_{13}Y_{34}^*V_{23} = \Sigma_{23}X_{13}^*Y_{34}X_{13}V_{23}Y_{34}^*Y_{24}^* \\ &= \Sigma_{23}X_{13}^*Y_{34}X_{12}^*V_{23}X_{12}Y_{34}^*Y_{24}^* \\ &= \Sigma_{23}(X_{12}X_{13})^*V_{23}Y_{24}Y_{34}X_{12}Y_{34}^*Y_{24}^* \\ &= \Sigma_{23}V_{23}X_{12}^*Y_{24}X_{12}Y_{24}^*. \end{aligned}$$

Now let $c = (\mathrm{id} \otimes \omega)(\Sigma V)$. Then

$$\begin{aligned} W(1 \otimes c \otimes 1) &= (\mathrm{id} \otimes \mathrm{id} \otimes \omega \otimes \mathrm{id})(X_{12}^*Y_{24}X_{12}Y_{24}^*\Sigma_{23}V_{23}) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \omega \otimes \mathrm{id})(\Sigma_{23}V_{23}X_{12}^*Y_{24}X_{12}Y_{24}^*) = (1 \otimes c \otimes 1)W. \end{aligned}$$

The final part of the proposition is clear. ∎

PROPOSITION 2.6. *If V is a C^*-multiplicative unitary such that $\overline{C(V)}^{\mathrm{weak}} = \mathcal{L}(H)$, then V is basic.*

Proof. This proposition is clear by putting $X = V'$ and $Y = V''$ into Lemma 2.5. ∎

COROLLARY 2.7. *Every semi-irreducible (respectively, regular) multiplicative unitary is basic.*

REMARK 2.8. It is natural to ask whether we can use similar arguments as in Lemma 2.5 to prove that manageable multiplicative unitaries are basic as well. However, we encounter a difficulty in doing so. The difficulty comes from the unboundedness of Q. It is not hard to show that if $(V, Q, \widetilde{V})$ is a manageable multiplicative unitary such that $(f \otimes \mathrm{id} \otimes g)(W)(\mathrm{Dom}\, Q) \subseteq \mathrm{Dom}\, Q$ (where $W = V_{12}'^*V_{23}''V_{12}'V_{23}''^*$) for any $f \in (\widehat{S}_V)_p^*$ and $g \in (S_V)_p^*$, then V is basic. Note that if Q is bounded, then V is regular and $\kappa_{V'}$ is bounded.

In the rest of this section, we assume that the multiplicative unitary $V \in \mathcal{L}(H \otimes H)$ is basic. Let $V^\top = \Sigma V^* \Sigma \in \mathcal{L}(H \otimes H)$ and $V_p \in M[(\widehat{S}_V)_p \otimes (S_V)_p]$ be the universal birepresentation of V.

LEMMA 2.9. (i) *$S_V = \widehat{S}_{V^\top}$ and $\widehat{S}_V = S_{V^\top}$ as $*$-subalgebras of $\mathcal{L}(H)$ (in fact, $S_V^{\mathrm{op}} = \widehat{S}_{V^\top}$ and $\widehat{S}_V^{\mathrm{op}} = S_{V^\top}$ as Hopf C^*-algebras);*

(ii) *If W is a representation (respectively, co-representation) of V, then $W^\top = \Sigma W^* \Sigma$ is a co-representation (respectively, representation) of $V^\top$;*

(iii) *$(S_V)_p^{\mathrm{op}} \cong (\widehat{S}_{V^\top})_p$ and $(\widehat{S}_V)_p^{\mathrm{op}} \cong (S_{V^\top})_p$ (as Hopf C^*-algebras);*

(iv) *$(V^\top)' = (V''^*)^\sigma$, $(V^\top)'' = (V'^*)^\sigma$ and $V_p^\top = (V_p^*)^\sigma$ (where σ means the flip of variables).*

We recall the antipode κ_V from A_V to S_V defined by $\kappa_V((\omega \otimes \mathrm{id})(V)) = (\omega \otimes \mathrm{id})(V^*)$. In the same way, we define the antipode j_V from $(A_V)_p = \{(\omega \otimes \mathrm{id})(V'') : \omega \in \mathcal{L}(H)_*\}$ to $(S_V)_p$ by $j_V((\omega \otimes \mathrm{id})(V'')) = (\omega \otimes \mathrm{id})(V''^*)$. Note that j_V is well defined since $(\omega \otimes \mathrm{id})(V'') = 0$ implies $\omega = 0$ on $\widehat{S}$ and hence $\omega^* = 0$ on $M(\widehat{S})$ (which implies that $\omega^*((\mathrm{id} \otimes f)(V'')) = 0$ for all $f \in (S_V)_p^*$). Moreover, we can extend κ_V and j_V as follow.

LEMMA 2.10. *κ_V and j_V can be extended to* $\widetilde{A_V} = \{(f \otimes \mathrm{id})(V') : f \in (\widehat{S}_V)_p^*\}$ *and* $(\widetilde{A_V})_p = \{(f \otimes \mathrm{id})(V_p) : f \in (\widehat{S}_V)_p^*\}$, *respectively.*

Proof. Since the map that send $f \in (\widehat{S}_V)_p^*$ to $(f \otimes \mathrm{id})(V')$ is injective (see [5], A6), the map κ_V that send $(f \otimes \mathrm{id})(V')$ to $(f \otimes \mathrm{id})(V'^*)$ is well defined and is clearly an extension of the κ_V above. Similarly, since the map that send $f \in (\widehat{S}_V)_p^*$ to $(f \otimes \mathrm{id})(V_p)$ is injective, the extension of j_V is also well defined. ∎

PROPOSITION 2.11. *There is a one to one correspondence between unitary co-representations of S_V and those of $(S_V)_p$.*

Proof. If w is a unitary co-representation of S_V, then $w_p = (\rho_w \otimes \mathrm{id})(V_p)$ is a unitary co-representation of $(S_V)_p$ (by Remark 2.4 (b)). On the other hand, if u is a unitary co-representation of $(S_V)_p$, then $u_0 = (\mathrm{id} \otimes L_V)(u)$ is a unitary co-representation of S_V. Moreover, it is clear that $(\mathrm{id} \otimes L_V)(\rho_w \otimes \mathrm{id})(V_p) = (\rho_w \otimes \mathrm{id})(V') = w$. It remains to show that if $(\mathrm{id} \otimes L_V)(u_1) = (\mathrm{id} \otimes L_V)(u_2)$, then $u_1 = u_2$. It follows from exactly the same argument as in [5], 2.7. ∎

COROLLARY 2.12. *If ε' is a coaction of a C^*-algebra A by the Hopf C^*-algebra $(S_V)_p$, then the full crossed product $A \times_{\varepsilon',\max} (\widehat{S}_V)_p$ exists and is a quotient of $A \times_{\varepsilon,\max} \widehat{S}_V$ (where ε is the reduced coaction that corresponds to ε' as defined in the paragraph before* ([5], 2.14)*).*

Proof. Using Proposition 2.11 and the same argument as in [5], 2.12 (a), we can reformulate the full crossed product of (A, ε') as in [5], 2.12 (c). Now by a similar argument as in [5], 2.13, the full crossed product exists. Since any covariant representation of (A, ε') is a covariant representation of (A, ε), it is clear that $A \times_{\varepsilon',\max} (\widehat{S}_V)_p$ is a quotient of $A \times_{\varepsilon,\max} \widehat{S}_V$. ∎

PROPOSITION 2.13. *Let A be a C^*-algebra and ε a coaction on A by S_V. Let (B, φ, μ) be the full crossed product. Then there is a dual coaction $\bar{\varepsilon}$ on B by $(\widehat{S}_V)_p$ such the μ is equivariant.*

Proof. Let $v = (\mu \otimes \mathrm{id})(V')$. Then $(\varphi \otimes \mathrm{id})\varepsilon(a) \cdot v = v \cdot (\varphi(a) \otimes 1)$ for any $a \in A$. Now define the $*$-homomorphisms $\psi = (\mathrm{id} \otimes 1) \circ \varphi$ and $\nu = (\mu \otimes \mathrm{id}) \circ \widehat{\delta}_V$ from A and $(\widehat{S}_V)_p$ respectively to $M(B \otimes (\widehat{S}_V)_p)$. We first show that (ψ, ν) is a covariant pair for (A, S_V, ε). In fact, for any $a \in A$,

$$(\psi \otimes \mathrm{id})\varepsilon(a) = (\varphi \otimes \mathrm{id})\varepsilon(a)_{13}$$

and

$$(\nu \otimes \mathrm{id})(V') = (\mu \otimes \mathrm{id} \otimes \mathrm{id})(\widehat{\delta}_V \otimes \mathrm{id})(V') = v_{13}V'_{23}.$$

Hence

$$\begin{aligned}(\psi \otimes \mathrm{id})\varepsilon(a) \cdot (\nu \otimes \mathrm{id})(V') &= [(\varphi \otimes \mathrm{id})\varepsilon(a) \cdot v]_{13}V'_{23} = v_{13} \cdot (\varphi(a) \otimes 1 \otimes 1) \cdot V'_{23} \\ &= (\nu \otimes \mathrm{id})(V') \cdot (\psi(a) \otimes 1).\end{aligned}$$

Thus, we have a map $\bar{\varepsilon}$ from B to $M(B \otimes (\widehat{S}_V)_p)$ such that $\psi = \bar{\varepsilon} \circ \varphi$ and $\nu = \bar{\varepsilon} \circ \mu$. Now for any $a \in A$ and $s, t \in (\widehat{S}_V)_p$,

$$\bar{\varepsilon}[\varphi(a)\mu(s)] \cdot (1 \otimes t) = (\varphi(a) \otimes 1) \cdot (\mu \otimes \mathrm{id})[\widehat{\delta}_V(s) \cdot (1 \otimes t)] \in B \otimes (\widehat{S}_V)_p$$

(as $\widehat{\delta}_V(s) \cdot (1 \otimes t) \in (\widehat{S}_V)_p \otimes (\widehat{S}_V)_p$ and $\varphi(a)\mu(u) \in B$ for all $u \in (\widehat{S}_V)_p$). Since $\{\varphi(a)\mu(s) : a \in A, s \in (\widehat{S}_V)_p\}$ generates B (by [5], 2.12 (b)(3)]), we have $\bar{\varepsilon}(B) \subseteq \widetilde{M}(B \otimes (\widehat{S}_V)_p)$. It remains to show the coaction identity. For $a \in A$ and $s \in (\widehat{S}_V)_p$,

$$\begin{aligned}(\bar{\varepsilon} \otimes \mathrm{id})\bar{\varepsilon}[\varphi(a)\mu(s)] &= (\bar{\varepsilon} \otimes \mathrm{id})[(\varphi(a) \otimes 1) \cdot (\mu \otimes \mathrm{id})\widehat{\delta}_V(s)] \\ &= (\varphi(a) \otimes 1 \otimes 1) \cdot [(\mu \otimes \mathrm{id}) \circ \widehat{\delta}_V \otimes \mathrm{id}]\widehat{\delta}_V(s) \\ &= (\varphi(a) \otimes 1 \otimes 1) \cdot (\mu \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \widehat{\delta}_V)\widehat{\delta}_V(s).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}(\mathrm{id} \otimes \widehat{\delta}_V)\bar{\varepsilon}[\varphi(a)\mu(s)] &= (\mathrm{id} \otimes \widehat{\delta}_V)[(\varphi(a) \otimes 1) \cdot (\mu \otimes \mathrm{id})\widehat{\delta}_V(s)] \\ &= (\varphi(a) \otimes 1 \otimes 1) \cdot (\mu \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \widehat{\delta}_V)\widehat{\delta}_V(s).\end{aligned}$$

Finally, μ is equivariant by the definition of $\bar{\varepsilon}$. ∎

3. U-V-BIREPRESENTATIONS

In this section we mainly deal with C^*-multiplicative unitaries. We will discuss basic multiplicative unitaries in the next section. We will define and study the birepresentation of two C^*-multiplicative unitaries. Let (K,U) and (H,V) be C^*-multiplicative unitaries and $X \in \mathcal{L}(K \otimes H)$ be a unitary operator.

DEFINITION 3.1. X is said to be a *U-V-birepresentation* if X is a representation of V as well as a co-representation of U.

Let X be a U-V-birepresentation. Then there are $*$-representations L_X and ρ_X of $(S_U)_p$ and $(\widehat{S}_V)_p$ on H and K respectively. Moreover, we have:

PROPOSITION 3.2. *L_X is a Hopf $*$-homomorphism from $(S_U)_p$ to $M(S_V)$. Consequently, $S_X = L_X(S_U)_p$ is a Hopf C^*-subalgebra of S_V. Moreover, L_X preserves antipodes.*

Proof. The first statement is clear from the fact that $(\rho_X \otimes \mathrm{id})(V') = X$ and $V' \in M((\widehat{S}_V)_p \otimes S_V)$. The second one follows from Lemma 1.2. Finally, we need to show that $L_X \circ j_U = \kappa_V \circ L_X$. Note that $L_X \circ j_U((\omega \otimes \mathrm{id})(U'')) = (\omega \otimes \mathrm{id})(X^*)$. Now since $(\rho_X \otimes \mathrm{id})(V') = X$, $L_X((\omega \otimes \mathrm{id})(U'')) = (\omega \circ \rho_X \otimes \mathrm{id})(V') \in \widetilde{A}_V$ (see Lemma 2.10). Moreover, $\kappa_V \circ L_X((\omega \otimes \mathrm{id})(U'')) = (\omega \circ \rho_X \otimes \mathrm{id})(V'^*) = (\omega \otimes \mathrm{id})(X^*)$. ∎

REMARK 3.3. (a) Similar things hold for ρ_X and $\widehat{S}_X$.

(b) It is clear that X is a birepresentation if and only if X is a unitary $\widehat{S}_U$-S_V-birepresentation in $M(\widehat{S}_U \otimes S_V)$ (see Definition 1.4).

(c) Let X be a U-V-birepresentation. Then $X^\top = \Sigma X^* \Sigma$ is a $V^\top$-$U^\top$-birepresentation. Moreover, $L_{X^\top} = \rho_X$ and $\rho_{X^\top} = L_X$.

(d) If we borrow Proposition 3.5 below, we have the following: *Any Hopf $*$-homomorphism from $(S_U)_p$ to $M(S_V)$ preserves antipodes.* ∎

We are going to give a converse to Proposition 3.2. Let us first investigate under what condition a co-representation will be a birepresentation.

LEMMA 3.4. *Let X be a co-representation of U on H. If L_X is Hopf $*$-homomorphism from $(S_U)_p$ to $M(S_V)$, then X is a U-V-birepresentation.*

Proof. It is required to show that X is a representation of V. For any $\omega \in \mathcal{L}(K)_*$, we have

$$(L_X \otimes L_X)(\delta_U(\omega \otimes \mathrm{id})(U'')) = (\omega \otimes \mathrm{id} \otimes \mathrm{id})(X_{12}X_{13})$$

and

$$\delta_V(L_X(\omega \otimes \mathrm{id})(U'')) = V[(\omega \otimes \mathrm{id})(X) \otimes 1]V^* = (\omega \otimes \mathrm{id} \otimes \mathrm{id})(V_{23}X_{12}V_{23}^*).$$

Since $\mathcal{L}(K)_*$ separates points of $\mathcal{L}(K)$, $X_{12}X_{13}V_{23} = V_{23}X_{12}$. ∎

PROPOSITION 3.5. *Any Hopf $*$-homomorphism π from $(S_U)_p$ to $M(S_V)$ induces a unique U-V-birepresentation X such that $\pi = L_X$. Similarly, any Hopf $*$-homomorphism from $(\widehat{S}_V)_p$ to $M(\widehat{S}_U)$ also induces a U-V-birepresentation.*

THEOREM 3.6. *There are one to one correspondences between the followings:*
(a) *U-V-birepresentations;*
(b) *Hopf $*$-homomorphisms from $(S_U)_p$ to $M(S_V)$;*
(c) *Hopf $*$-homomorphisms from $(\widehat{S}_V)_p$ to $M(\widehat{S}_U)$.*

COROLLARY 3.7. *Let φ be a non-degenerate representation of $(S_U)_p$ on $\mathcal{L}(H)$. Then $\varphi((S_U)_p) \subseteq M(S_V)$ and is a Hopf $*$-homomorphism if and only if there exists a non-degenerate representation ψ of $(\widehat{S}_U)_p$ on $\mathcal{L}(H)$ such that $(\mathrm{id} \otimes \varphi)(U'') = (\psi \otimes \mathrm{id})(V')$.*

LEMMA 3.8. *There is only one V-$I_{\mathbb{C}}$-birepresentation.*

Proof. Let φ be a Hopf $*$-homomorphism from $(S_V)_p$ to $\mathbb{C}$. Then $\varphi \in (S_V)_p^*$ is an idempotent. Suppose that χ is the homomorphism from $(S_V)_p^*$ to $M(\widehat{S}_V)$ as defined in [5], A6. Then $\chi(\varphi) = (\mathrm{id} \otimes \varphi)(V'')$ is both an idempotent and a unitary. Hence $\chi(\varphi) = 1$. But since χ is unital and injective (see [5], A6), φ is the co-identity of $(S_V)_p$. ∎

DEFINITION 3.9. Let W be a C^*-multiplicative unitary. Let X be a U-V-birepresentation and Y a V-W-birepresentation. Then the unitary Z as given in Lemma 2.5 (if exists) is called the *composition* of X and Y.

REMARK 3.10. (a) By Lemma 1.5, the composition Z (if exists) is a U-W-birepresentation.

(b) In the above setting, if V is basic, then $Z = (\rho_X \otimes L_Y)(V_p)$ exists.

It is not clear for the moment how to relate L_Z to L_X and L_Y, i.e. how to define "composition" of L_X and L_Y. We will deal with this in the next section.

EXAMPLE 3.11. (a) $X = V$ is a V-V-birepresentation.

(b) $X = I_{H \otimes K}$ is a U-V-birepresentation. Hence the collection of U-V-birepresentations is non-empty.

(c) Let G and H be locally compact groups and $U = V_H$ and $V = V_G$ (where $V_G\xi(s,t) = \xi(ts,t)$ for all $s,t \in G$ and $\xi \in L^2(G \times G)$). If φ is a group homomorphism from G to H, then $X \in L^2(H \times G)$ defined by $X\eta(r,s) = \eta(\varphi(s)r, s)$ is a U-V-birepresentation such that the map L_X from $(S_U)_p = C_0(H)$ to $M(S_V) = C_b(G)$ is the $*$-homomorphism defined by φ. In fact, for any $\xi, \eta \in L^2(H)$, the map g defined by $g(s) = (\omega_{\xi,\eta} \otimes \mathrm{id})(U)(s) = \int \xi(t)\overline{\eta(st)}\,dt$ is in $C_0(H)$ and $L_X(g)(r) =$

$(\omega_{\xi,\eta} \otimes \mathrm{id})(X)(r) = \int \xi(t)\overline{\eta(\varphi(r)t)}\,dt = g(\varphi(r))$. Note that by Theorem 3.6, U-V-birepresentations are precisely group homomorphisms in this case.

As a corollary of Proposition 1.9, we have the following:

LEMMA 3.12. *Let ψ and φ be Hopf $*$-homomorphisms from $(S_U)_p$ to $M(S_V)$ and from $(\widehat{S}_V)_p$ to $M(\widehat{S}_U)$ respectively. Let δ_U and $\widehat{\delta}_V$ be the co-multiplications on $(S_U)_p$ and $(\widehat{S}_V)_p$ respectively. Then $\varepsilon = (\mathrm{id} \otimes \psi) \circ \delta_U$ is a coaction on $(S_U)_p$ by S_V and $\widehat{\varepsilon} = (\mathrm{id} \otimes \varphi)\widehat{\delta}_V$ is a coaction on $(\widehat{S}_V)_p$ by $\widehat{S}_U$.*

Let X be a U-V-birepresentation. Then, by Lemma 3.12, X induces coactions ε_X and $\widehat{\varepsilon}_X$ on $(S_U)_p$ and $(\widehat{S}_V)_p$ by S_V and $\widehat{S}_U$ respectively. Moreover, these coactions are "mutual" in the following sense:

DEFINITION 3.13. Let (S, δ_S) and (T, δ_T) be two Hopf C^*-algebras. A coaction ε on S by T is said to be a *mutual coaction* if $(\delta_S \otimes \mathrm{id}) \circ \varepsilon = (\mathrm{id} \otimes \varepsilon) \circ \delta_S$.

We can now add one more equivalence to Theorem 3.6.

PROPOSITION 3.14. *Let ε be a mutual coaction of S by T. If S has a co-identity E, then $\psi = (E \otimes \mathrm{id}) \circ \varepsilon$ is a Hopf $*$-homomorphism from S to $M(T)$ such that $\varepsilon = (\mathrm{id} \otimes \psi) \circ \delta_S$. Hence, U-V-birepresentations are in one to one correspondence with mutual coactions of $(S_U)_p$ by S_V (and also with mutual coactions of $(\widehat{S}_V)_p$ by $\widehat{S}_U$).*

Proof. First note that $\delta_T \circ \psi = (E \otimes \mathrm{id} \otimes \mathrm{id})(\varepsilon \otimes \mathrm{id})\varepsilon = (\psi \otimes \mathrm{id}) \circ \varepsilon$. Moreover, $(\varepsilon \otimes \varepsilon) \circ \delta_S = (\varepsilon \otimes \mathrm{id} \otimes \mathrm{id})(\delta_S \otimes \mathrm{id})\varepsilon$. Hence, $(\psi \otimes \psi) \circ \delta_S = (E \otimes \mathrm{id} \otimes \mathrm{id})(\varepsilon \otimes \mathrm{id})(\mathrm{id} \otimes E \otimes \mathrm{id})(\delta_S \otimes \mathrm{id})\varepsilon = (\psi \otimes \mathrm{id}) \circ \varepsilon$. Finally, $(\mathrm{id} \otimes \psi) \circ \delta_S = (\mathrm{id} \otimes E \otimes \mathrm{id})(\delta_S \otimes \mathrm{id})\varepsilon = \varepsilon$. Now by [5], 2.1, $(S_U)_p$ has a co-identity and the second part follows from Theorem 3.6. ∎

It is natural to ask what is the relation between the crossed product of ε_X and that of $\widehat{\varepsilon}_X$. Before comparing these two crossed products, let us first give the following lemmas.

LEMMA 3.15. *Let B be a C^*-algebra and let φ and μ be $*$-homomorphisms from $(S_U)_p$ and $(\widehat{S}_V)_p$ respectively to $M(B)$. Then (φ, μ) is a covariant pair for $((S_U)_p, S_V, \varepsilon_X)$ if and only if $((\mathrm{id} \otimes \varphi)(U''), (\mu \otimes \mathrm{id})(V'))$ is a covariant pair for X in the sense of Definition* 1.6.

Proof. Let $u = (\mathrm{id} \otimes \varphi)(U'')$ and $v = (\mu \otimes \mathrm{id})(V')$. Then, by definition, (φ, μ) is a covariant pair for $((S_U)_p, S_V, \varepsilon_X)$ if and only if for any $\omega \in \mathcal{L}(K)_*$,

$$(\varphi \otimes \mathrm{id})(\mathrm{id} \otimes L_X)\delta_U[(\omega \otimes \mathrm{id})(U'')] = (\mu \otimes \mathrm{id})(V')[\varphi((\omega \otimes \mathrm{id})(U'')) \otimes 1](\mu \otimes \mathrm{id})(V')^*.$$

This is the case if and only if

$$(\varphi \otimes \mathrm{id})(\omega \otimes \mathrm{id} \otimes \mathrm{id})(U''_{12} X_{13}) \cdot v = v \cdot (\omega \otimes \mathrm{id} \otimes \mathrm{id})(u_{12}).$$

Now the left hand side equals $(\omega \otimes \mathrm{id} \otimes \mathrm{id})(u_{12} X_{13} v_{23})$ while the right hand side is $(\omega \otimes \mathrm{id} \otimes \mathrm{id})(v_{23} u_{12})$. Therefore, the lemma follows from the fact that $\mathcal{L}(H)_*$ separates the points of S_U. ∎

LEMMA 3.16. *Let B, φ and μ be the same as in the previous lemma. Then (φ, μ) is a covariant pair for $((S_U)_p, S_V, \varepsilon_X)$ if and only if (μ, φ) is a covariant pair for $((S_{V^\top})_p, S_{U^\top}, \varepsilon_{X^\top})$.*

Proof. Let $u = (\mathrm{id} \otimes \varphi)(U'')$ and $v = (\mu \otimes \mathrm{id})(V')$ as in the previous lemma. Suppose that (φ, μ) is a covariant pair for $((S_U)_p, S_V, \varepsilon_X)$. Then $u_{12} X_{13} v_{23} = v_{23} u_{12}$. Now let $y = (\mathrm{id} \otimes \mu)(V^{\top\prime\prime})$ and $z = (\varphi \otimes \mathrm{id})(U^{\top\prime})$. It is required to show that $y_{12} X^\top_{13} z_{23} = z_{23} y_{12}$. In fact, $y = v^{*\sigma}$ and $z = u^{*\sigma}$ (where σ is the flip of variables). Thus,

$$y_{12} X^\top_{13} z_{23} = (\mu \otimes \mathrm{id})(V'^*)_{21} X^*_{31} (\mathrm{id} \otimes \varphi)(U''^*)_{32} = (u_{32} X_{31} v_{21})^* = (v_{21} u_{32})^*$$

(this is true by flipping the first and the third variables) and so $y_{12} X^\top_{13} z_{23} = z_{23} y_{12}$. The proof for the converse is the same. ∎

Actually, the crossed product of ε_X is the same as that of the opposite of $\widehat{\varepsilon}_X$ i.e. $\varepsilon_{X^\top}$. (Note that $S_{U^\top} = (\widehat{S})^{\mathrm{op}}_U$).

PROPOSITION 3.17. $(S_U)_p \times_{\varepsilon_X, \max} \widehat{S}_V \cong (S_{V^\top})_p \times_{\varepsilon_{X^\top}, \max} \widehat{S}_{U^\top}$.

By this proposition and Corollary 2.12, we have:

COROLLARY 3.18. *If δ, $\delta^\top$ and $\widehat{\delta}$ are co-multiplications on $(S_V)_p$, $(S_{V^\top})_p$ and $(\widehat{S}_V)_p$ respectively, then $(S_V)_p \times_{\delta, \max} (\widehat{S}_V)_p \cong (S_{V^\top})_p \times_{\delta^\top, \max} (\widehat{S}_{V^\top})_p \cong (\widehat{S}_V)_p \times_{\widehat{\delta}^{\mathrm{op}}, \max} (S_V)^{\mathrm{op}}_p$.*

Now for any U-V-birepresentation X, we obtain a C^*-algebra $C^*(X) = (S_U)_p \times_{\varepsilon_X, \max} \widehat{S}_V = (\widehat{S}_V)_p \times_{\widehat{\varepsilon}_{X^\top}, \max} S^{\mathrm{op}}_U$ which has coactions by $(\widehat{S}_V)_p$ and by $(S_U)_p$ respectively (see Proposition 2.13) such that the canonical maps μ and φ from $(\widehat{S}_V)_p$ and $(S_U)_p$ respectively to $M(C^*(X))$ are equivariant (see Proposition 2.13).

REMARK 3.19. By Proposition 1.10, we obtain a map π_0 from $C^*(X)$ to $M(\widehat{S}_V \times_{\delta_V, \max} S_V)$ and hence a representation π of $C^*(X)$ on H. Similarly, we have a representation τ of $C^*(X)$ on K. Moreover, $L_X = \pi \circ \varphi$ and $\rho_X = \tau \circ \mu$. In fact, if ψ is the canonical map from S_V to $M(S_V \times_{\delta_V, r} \widehat{S}_V)$ (which equals $\mathcal{L}(H)$), then $\pi \circ \varphi = \psi \circ L_X = L_X$ (since $\psi = L_V$).

4. LIFTING OF BIREPRESENTATIONS

In this section, we assume that all multiplicative unitaries are basic. We will show that any birepresentation $X \in M(\widehat{S}_U \otimes S_V)$ is the image of a unitary $(\widehat{S}_U)_p$-$(S_V)_p$-birepresentation X_p. (see Definition 1.4). Consequently, we can lift any Hopf $*$-homomorphism $\varphi : (S_U)_p \to M(S_V)$ to a Hopf $*$-homomorphism $\varphi' : (S_U)_p \to M(S_V)_p$. First of all, let X be a U-V-birepresentation and define $X' = (\mathrm{id} \otimes L_X)(U_p)$ and $X'' = (\rho_X \otimes \mathrm{id})(V_p)$. Then:

LEMMA 4.1. *X' and X'' are unitary $(\widehat{S}_U)_p$-S_V-birepresentation and unitary $\widehat{S}_U$-$(S_V)_p$-birepresentation, respectively, such that $(\rho_U \otimes \mathrm{id})(X') = X = (\mathrm{id} \otimes L_V)(X'')$.*

Proof. Since $U'_{12}(U_p)_{13}U''_{23} = U''_{23}U'_{12}$, we have $U'_{12}X'_{13}X_{23} = X_{23}U'_{12}$. Thus, by Lemma 1.5 and the fact X is a representation of V, X' is a unitary $(\widehat{S}_U)_p$-S_V-birepresentation. The second part of the lemma is clear. ∎

REMARK 4.2. (a) Note that by Proposition 2.11, X' and X'' are uniquely determined by X.

(b) Since X' is a representation of V, we can define a map $\rho_{X'}$ from $(\widehat{S}_V)_p$ to $M((\widehat{S}_U)_p)$ such that $X' = (\rho_{X'} \otimes \mathrm{id})(V')$. Moreover, by a similar argument as the proof of Proposition 3.2, $\rho_{X'}$ is a Hopf $*$-homomorphism. Similarly, we have a Hopf $*$-homomorphism $L_{X''}$ from $(S_U)_p$ to $M((S_V)_p)$ such that $X'' = (\mathrm{id} \otimes L_{X''})(V'')$.

(c) Both $L_{X''}$ and $\rho_{X'}$ preserve antipodes (by a similar argument as in the proof of Proposition 3.2). Hence *any Hopf $*$-homomorphism from $(S_U)_p$ to $M((S_V)_p)$ preserves antipodes.*

(d) Both $L_{X''}$ and $\rho_{X'}$ preserve co-identities in the following sense: if E_V is the co-identity of $(S_V)_p$, then $E_V \circ L_{X''}$ is the co-identity of $(S_U)_p$. This follows directly from Lemma 3.8. Thus *any Hopf $*$-homomorphism from $(S_U)_p$ to $M((S_V)_p)$ preserves co-identities.*

LEMMA 4.3. *Let X and Y be the unitaries as in Definition 3.9 and let $Z = X \circ Y$ (see Definition 3.9). Then $L_Z = L_Y \circ L_{X''}$ and $\rho_Z = \rho_X \circ \rho_{Y'}$.*

Proof. We first note that $L_{X''}((\omega \otimes \mathrm{id})(U'')) = (\omega \otimes \mathrm{id})(X'') = (\omega \circ \rho_X \otimes \mathrm{id})(V_p)$. Thus, $L_Y \circ L_{X''}((\omega \otimes \mathrm{id})(U'')) = (\omega \otimes \mathrm{id})(\rho_X \otimes L_Y)(V_p) = (\omega \otimes \mathrm{id})(Z)$. The proof for $\rho_Z = \rho_X \circ \rho_{Y'}$ is the same. ∎

LEMMA 4.4. $(\mathrm{id}\otimes L_{X''})(U_p) = (\rho_{X'}\otimes \mathrm{id})(V_p)$.

Proof. Let $X_1 = (\mathrm{id}\otimes L_{X''})(U_p)$ and $X_2 = (\rho_{X'}\otimes \mathrm{id})(V_p)$. First notice that both X_1 and X_2 are unitary co-representations of $(S_V)_p$ (as $L_{X''}$ is a Hopf $*$-homomorphism). Moreover, $(\mathrm{id}\otimes L_V)(X_1) = (\mathrm{id}\otimes L_X)(U_p) = X'$ and $(\mathrm{id}\otimes L_V)(X_2) = (\rho_{X'}\otimes \mathrm{id})(V') = X'$. Hence, the lemma follows from Proposition 2.11. ∎

DEFINITION 4.5. Let X be a U-V-birepresentation. Then $X_p = (\mathrm{id}\otimes L_{X''})(U_p) = (\rho_{X'}\otimes \mathrm{id})(V_p)$ is called the *lifting* of X in $M((\widehat{S}_U)_p\otimes (S_V)_p)$.

REMARK 4.6. X_p is a unitary $(\widehat{S}_U)_p$-$(S_V)_p$-birepresentation by Remark 2.4 (b). Moreover, $(\mathrm{id}\otimes L_V)(X_p) = X'$ and $(\rho_U\otimes \mathrm{id})(X_p) = X''$.

LEMMA 4.7. *$L_{Z''} = L_{Y''}\circ L_{X''}$ and $\rho_{Z'} = \rho_{X'}\circ\rho_{Y'}$.*

Proof. For any $\omega\in\mathcal{L}(H)_*$, we have

$$\begin{aligned} L_{Y''}\circ L_{X''}[(\omega\otimes \mathrm{id})(U'')] &= (\omega\otimes L_{Y''})(X'') = (\omega\otimes \mathrm{id})(\rho_X\otimes L_{Y''})(V_p)\\ &= (\omega\otimes \mathrm{id})(\rho_X\circ\rho_{Y'}\otimes \mathrm{id})(W_p)\\ &= (\omega\otimes \mathrm{id})(\rho_Z\otimes \mathrm{id})(W_p) = (\omega\otimes \mathrm{id})(Z''). \end{aligned}$$

The proof for $\rho_{Z'} = \rho_{X'}\circ\rho_{Y'}$ is the same. ∎

LEMMA 4.8. *Let X be a U-V-birepresentation and Y a V-W-birepresentation. Let $Z = X\circ Y$. Then there are equivariant maps from $C^*(Z)$ to $M(C^*(X))$ and to $M(C^*(Y))$ (see Remark 3.19 for the definition of $C^*(X)$).*

Proof. We first show that $L_{X''}$ from $(S_U)_p$ to $M[(S_V)_p]$ is equivariant with respect to the coactions ε_Z and ε_Y respectively. In fact, $\varepsilon_Y\circ L_{X''} = (\mathrm{id}\otimes L_Y)\circ\delta_V\circ L_{X''} = (\mathrm{id}\otimes L_Y)\circ(L_{X''}\otimes L_{X''})\circ\delta_U = (L_{X''}\otimes\mathrm{id})\circ(\mathrm{id}\otimes L_Z)\circ\delta_U = (L_{X''}\otimes\mathrm{id})\circ\varepsilon_Z$. Hence, we obtain a non-degenerate map Ψ from $C^*(Z)$ to $M(C^*(Y))$ (by [5], 3.9). Now let φ,μ,φ' and μ' be the canonical maps from $(S_U)_p, (\widehat{S}_W)_p, (S_V)_p$ and $(\widehat{S}_W)_p$ to $C^*(Z)$ and $C^*(Y)$ respectively. Then we have

$$\begin{aligned} (\Psi\otimes \mathrm{id})\overline{\varepsilon}_Z(\varphi(s)\mu(t)) &= (\Psi\otimes \mathrm{id})[(\varphi(s)\otimes 1)(\mu\otimes \mathrm{id})\widehat{\delta}_W(t)]\\ &= (\varphi'(L_{X''}(s))\otimes 1)\cdot(\mu'\otimes \mathrm{id})\widehat{\delta}_W(t)\\ &= \overline{\varepsilon}_Y(\Psi\otimes \mathrm{id})(\varphi(s)\mu(t)) \end{aligned}$$

for any $s\in(S_U)_p$ and $t\in(\widehat{S}_W)_p$ (where $\overline{\varepsilon}_Y$ and $\overline{\varepsilon}_Z$ are the dual coactions as defined in Proposition 2.13). The map from $C^*(Z)$ to $M(C^*(X))$ is defined similarly by considering $C^*(Z) = (\widehat{S}_V)_p\times_{\widehat{\varepsilon}_X,\max} S_U$. ∎

We summarise the equivalences of U-V-birepresentations as follows:

THEOREM 4.9. *There are one to one correspondences between the collections of the following objects:*

(a) *U-V-birepresentations;*

(b) *Hopf $*$-homomorphisms from $(S_U)_p$ to $M(S_V)$ (respectively, from $(\widehat{S}_V)_p$ to $M(\widehat{S}_U)$);*

(c) *mutual coactions on $(\widehat{S}_V)_p$ by $(\widehat{S}_U)$ (respectively, on $(S_U)_p$ by (S_V));*

(a′) *unitary $(\widehat{S}_U)_p$-$(S_V)_p$-birepresentations;*

(b′) *Hopf $*$-homomorphisms from $(S_U)_p$ to $M[(S_V)_p]$ (respectively, from $(\widehat{S}_V)_p$ to $M[(\widehat{S}_U)_p]$);*

(c′) *mutual coactions on $(\widehat{S}_V)_p$ by $(\widehat{S}_U)_p$ (respectively, on $(S_U)_p$ by $(S_V)_p$).*

In this case, the Hopf $*$-homomorphisms in (b) *and* (b′) *preserve antipodes automatically. Moreover, Hopf-$*$-homomorphisms in* (b′) *preserve co-identity.*

REMARK 4.10. (a) It is also clear from the above results that unitary $\widehat{S}_U$-S_V-, $(\widehat{S}_U)_p$-S_V-, $\widehat{S}_U$-$(S_V)_p$- and $(\widehat{S}_U)_p$-$(S_V)_p$- birepresentations are all the same.

(b) Note that there may not be a one to one correspondence between the set of Hopf $*$-homomorphisms from S_U to $M(S_V)$ and the sets in Theorem 4.9. In fact, they are in one to one correspondence if and only if U is co-amenable. Note that if the trivial birepresentation induces a Hopf $*$-homomorphism from S_U to $M(S_V)$, then S_U is co-unital (by Lemma 3.8) which implies that $S_U = (S_U)_p$ (by [5], A4).

5. THE CATEGORY OF BASIC MULTIPLICATIVE UNITARIES

Let $\mathcal{M}$ be the metagraph with the collection of all basic multiplicative unitaries as its objects and birepresentations as arrows such that given a U-V-birepresentation X, we denote $\mathrm{dom}(X) = V$ and $\mathrm{cod}(X) = U$. Then by the results in Sections 3 and 4, we have the following:

PROPOSITION 5.1. *$\mathcal{M}$ is a category with null object $I_{\mathbf{C}}$. It contains the category of all locally compact groups as a full subcategory.*

More generally, $\mathcal{M}$ also contains the category $\mathcal{M}_{\mathrm{ca}}$ (respectively, $\mathcal{M}_{\mathrm{a}}$) of co-amenable (respectively, amenable) multiplicative unitaries in $\mathcal{M}$ as a full subcategory. Moreover, $\mathcal{M}_{\mathrm{ca}}$ is a strict monoidal category (and so is $\mathcal{M}_{\mathrm{a}}$) as shown by the following lemma.

LEMMA 5.2. *Let U and V be co-amenable (respectively, amenable) C^*-multiplicative unitaries. Then $W = U \otimes V$ is also a co-amenable (respectively, amenable) C^*-multiplicative unitary (hence is also basic by Remark 2.4).*

Proof. We first show that W is a C^*-multiplicative unitary. In fact, by [5], 2.4 and 2.5, any representation X of W is of the form $X = Y_{13}Z_{24}$ for some representations Y and Z of U and V respectively. Hence $\widehat{S}_X$ is a C^*-algebra which equals $\widehat{S}_Y \otimes \widehat{S}_Z$. Hence condition (ii) of Definition 2.1 holds as well. Now we will show that W is co-amenable. Let E_U, E_V and E_W be the co-identities of $(S_U)_p, (S_V)_p$ and $(S_W)_p$ respectively. Since U and V are co-amenable, $(S_U)_p = S_U$ and $(S_V)_p = S_V$ are nuclear (see [5], 3.6). Let

$$E = E_U \otimes E_V \in [(S_U)_p \otimes (S_V)_p]^* = S_W^*.$$

Then

$$\begin{aligned}(E \circ L_W)(\omega \otimes \nu) &= (E_U \otimes E_V)[L_U(\omega) \otimes L_V(\nu)] \\ &= (\mathrm{id} \otimes \omega)(I_K) \otimes (\mathrm{id} \otimes \nu)(I_H) = E_W(\omega \otimes \nu).\end{aligned}$$

Now $L_W^*(S_W^*)$ is a right ideal of $(S_W)_p^*$ (see [5], A4) containing the identity and therefore $L_W^*(S_W^*) = (S_W)_p^*$. ∎

Since Woronowicz C^*-algebras will give multiplicative unitaries of compact type, we can roughly say that $\mathcal{M}_a$ contains all Woronowicz C^*-algebras if we identify all Woronowicz C^*-algebras that give the same multiplicative unitaries. Now we turn to subobjects and quotients.

DEFINITION 5.3. Let U and V be basic multiplicative unitaries.

(a) V is said to be a *sub-multiplicative unitary* of U if there exists a Hopf $*$-homomorphism $L_{X''}$ from $((S_U)_p)$ onto $(S_V)_p$.

(b) U is said to be a *quotient* of V if there exists a Hopf $*$-homomorphism $\rho_{X'}$ from $((\widehat{S}_V)_p)$ onto $(\widehat{S}_U)_p$.

(c) A U-V-birepresentation X is said to be an *isomorphism* if there exists a V-U-birepresentation Y such that $X \circ Y = U$ and $Y \circ X = V$.

REMARK 5.4. Let U and V be basic multiplicative unitaries. Then an isomorphism between U and V is equivalent to the existence of two Hopf $*$-homomorphisms ψ and φ from $(S_U)_p$ to $M((S_V)_p)$ and from $(S_V)_p$ to $M((S_U)_p)$ respectively such that $\psi \circ \varphi = \mathrm{id}$ and $\varphi \circ \psi = \mathrm{id}$. Hence, by Lemma 1.3, isomorphisms between U and V are equivalent to Hopf $*$-isomorphisms between $(S_U)_p$ and $(S_V)_p$. Moreover, if $S_U \cong S_V$ as Hopf C^*-algebras, then U is isomorphic to V.

LEMMA 5.5. *Let U, V and W be basic multiplicative unitaries. Let X and Y be a U-V-birepresentation and a V-W-representation respectively. Then $X \circ Y = I$ if and only if $\rho_{Y'}$ map $(\widehat{S}_W)_p$ into the fixed point algebra of $\widehat{\varepsilon}_X$ in $M((\widehat{S}_V)_p)$.*

Proof. Let $Z = X \circ Y$. Note that $\widehat{\varepsilon}_X \circ \rho_{Y'} = (\mathrm{id} \otimes \rho_{X'}) \circ \widehat{\delta}_V \circ \rho_{Y'}$. If $Z = I$, then $\widehat{\varepsilon}_X \circ \rho_{Y'} = (\rho_{Y'} \otimes \rho_{Z'}) \circ \widehat{\delta}_W = (\rho_{Y'} \otimes E \cdot 1) \circ \widehat{\delta}_W = (\rho_{Y'} \otimes 1)$ (where E is the co-identity of $(\widehat{S}_W)_p$). Conversely, suppose that $\rho_{Y'}(\widehat{S}_W)_p \subseteq M((\widehat{S}_V)_p)^{\widehat{\varepsilon}_X}$. Then for any $\omega \in \mathcal{L}(K)_*$ (where K is the underlying Hilbert space for W),

$$(\mathrm{id} \otimes \rho_{X'}) \circ \widehat{\delta}_V \circ \rho_{Y'}((\mathrm{id} \otimes \omega)(W')) = \rho_{Y'}((\mathrm{id} \otimes \omega)(W')) \otimes 1.$$

Now the right hand side of the equation equals

$$(\mathrm{id} \otimes \rho_{X'})((\mathrm{id} \otimes \mathrm{id} \otimes \omega)(Y'_{13}Y'_{23})) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega)(Y'_{13}Z'_{23})$$

while the left hand side is $(\mathrm{id} \otimes \mathrm{id} \otimes \omega)(Y'_{13})$. Hence, $(\mathrm{id} \otimes \mathrm{id} \otimes \omega)(Y'_{13}Z'_{23} - Y'_{13}) = 0$ for all $\omega \in \mathcal{L}(K)_*$. Since $\mathcal{L}(K)_*$ separates points of $\mathcal{L}(K)$, we have $Y'_{13}Z'_{23} = Y'_{13}$ and thus $Z = I$ (as Y' is a unitary). ∎

It is natural to ask whether we can define the kernel of a morphism. We do not know how to define it in general. However, by examining normal subgroups of discrete groups (see the Appendix) and suggested by the above lemma, we try to define kernels of morphisms between basic multiplicative unitaries of discrete type as follows.

DEFINITION 5.6. Let U, V and W be regular multiplicative unitaries of discrete type. Let X be a U-V-birepresentation and Y be a V-W-birepresentation.

(a) Y is said to be a *kernel* of X if $\rho_{Y'}$ is an isomorphism from $(\widehat{S}_W)_p$ to the fixed point algebra of $\widehat{\varepsilon}_X$ in $(\widehat{S}_V)_p$.

(b) If W is a submultiplicative unitary of V through Y, then W is said to be *normal* if Y is a kernel of a morphism.

PROPOSITION 5.7. *If the kernel of a morphism X exists, then it is unique up to isomorphism.*

Proof. Let (W_1, Y_1) and (W_2, Y_2) be kernels of X. Then there exists a Hopf $*$-isomorphism between $(\widehat{S}_{W_1})_p$ and $(\widehat{S}_{W_2})_p$ (by definition). Now the proposition follows from Remark 5.4.

PROPOSITION 5.8. *Suppose that the kernel (W, Y) of a U-V-birepresentation X exists. Let Y_1 be a V-W_1-birepresentation such that $X \circ Y_1 = I$. Then there exists a unique W-W_1-birepresentation Z such that $Y_1 = Y \circ Z$.*

Proof. By Lemma 5.5, $\rho_{Y_1'}$ induces a unique Hopf $*$-homomorphism φ from $(\widehat{S}_{W_1})_p$ to $(\widehat{S}_W)_p$ such that $\rho_{Y'} \circ \varphi = \rho_{Y_1'}$ (φ is unique since $\rho_{Y'}$ is injective). Now the proposition follows from Theorem 4.9. ∎

REMARK 5.9. (a) Proposition 5.8 justifies the use of the term "kernel".

(b) The kernel of a morphism need not exist in general, e.g. if H is a closed subgroup of a compact group G, then the fixed point algebra $C(G)^{\alpha_H}$ (which equals $C(G/H)$) is not a Hopf C^*-subalgebra of $C(G)$ unless H is normal (where α_H is the action of H on $C(G)$ induced from the canonical action of G on itself).

EXAMPLE 5.10. The only example about normal submultiplicative unitaries that we have, for the moment, is the following very simple one. Let V be the product of U and W, then W is a normal submultiplicative unitary of V.

6. AN IMPRIMITIVITY TYPE THEOREM FOR MULTIPLICATIVE UNITARIES OF DISCRETE TYPE

Let U be a regular multiplicative unitary of discrete type. U is clearly co-amenable. Let φ_U be the Haar state on $\widehat{S}_U$. If ε is a coaction on A by $\widehat{S}_U$ with fixed point algebra A^ε, then $E = (\mathrm{id} \otimes \varphi_U) \circ \varepsilon$ is a conditional expectation from A onto A^ε.

In this section we will give an imprimitivity type theorem for discrete type multiplicative unitaries. On our way to this, we found the following interesting fact from Lemma 6.5: the set $\{(\omega_{e,\xi} \otimes \mathrm{id})(U) : \xi \in H\}$ generates S_U if U is of discrete type and e is the co-fixed vector of U (see [2], 1.8).

Stimulated by [9], 2.2.16, we are going to use Watatani's C^*-basic construction (see [9], Sections 2.1 and 2.2) to prove the imprimitivity type theorem. We recall that if A is a C^*-subalgebra of B with a common unit and E is a faithful conditional expectation from B to A, then the C^*-basic construction $C^*\langle B, e_A\rangle$ is equal to $\mathcal{K}(\mathcal{F})$ where $\mathcal{F}$ is the completion of B with respect to the norm defined by E (see [9], 2.1.3 and 2.2.10). Moreover, we recall the following result:

PROPOSITION 6.1. (Watatani) *Let B be a unital C^*-algebra and A be a C^*-subalgebra of B that contains the unit of B. Let E be a faithful conditional expectation from B to A. If B acts on a Hilbert space H faithfully and e is a projection on H such that*

(i) *$ebe = E(b)e$ for all $b \in B$ and*

(ii) *the map that sends $a \in A$ to $ae \in \mathcal{L}(H)$ is injective,* *then the norm closure of BeB is isomorphic to $C^*\langle B, e_A\rangle$ canonically.*

We now state the main theorem of this section. Let U, V and W be multiplicative unitaries of discrete type such that W is a normal sub-multiplicative unitary of V with quotient U. Let ε' be the coaction on $(\widehat{S}_V)_p$ by $(\widehat{S}_U)_p$ as defined in Section 3. For technical reasons, we assume that U is amenable.

THEOREM 6.2 . *$(\widehat{S}_W)_p$ is strongly Morita equivalence to $(\widehat{S}_V)_p \times_{\varepsilon',r} S_U$.*

Note that there exists a faithful Haar state for $\widehat{S}_U$ if U is of discrete type. Hence U is biregular and irreducible (up to multiplicity). Since it is more convenient for us to work with the reduced crossed product of the form $A \times_{\varepsilon,r} \widehat{S}_U$, we will consider $\widehat{U}$ instead of U. Let $\widehat{U}$ be as defined in [2], 6.1. By [2], 6.8, $\widehat{S}_U \cong S_{\widehat{U}}$ as Hopf C^*-algebras. Let ε be the coaction on $(\widehat{S}_V)_p$ by $S_{\widehat{U}}$ induced by ε' and let ψ_U be the corresponding Haar state on $S_{\widehat{U}}$. Then $(\widehat{S}_V)_p \times_{\varepsilon',r} S_U = (\widehat{S}_V)_p \times_{\varepsilon,r} \widehat{S}_{\widehat{U}}$. Let $E = (\mathrm{id} \otimes \psi_U) \circ \varepsilon$ be the conditional expectation from $(\widehat{S}_V)_p$ to $(\widehat{S}_W)_p$ as defined by the first paragraph of this section. Since ψ_U is faithful, E is faithful (ε is injective since it is defined by a Hopf $*$-homomorphism from $(\widehat{S}_V)_p$ to $\widehat{S}_U$ and $\widehat{S}_U$ has a co-identity). We first give the following lemmas.

LEMMA 6.3. $(\mathrm{id} \otimes \mathrm{id} \otimes \psi_U)(\widehat{U}_{12}\widehat{U}_{13}) = (\mathrm{id} \otimes \mathrm{id} \otimes \psi_U)(\widehat{U}_{13})$.

Proof. Since ψ_U is the Haar state, $(\mathrm{id} \otimes \psi_U)\delta_{\widehat{U}}(x) = \psi_U(x) \cdot 1$ for all $x \in S_{\widehat{U}}$ (where $\delta_{\widehat{U}}$ is the co-multiplication on $S_{\widehat{U}}$). Hence

$$(\omega \otimes \mathrm{id} \otimes \psi_U)(\widehat{U}_{12}\widehat{U}_{13}) = (\omega \otimes \mathrm{id} \otimes \psi_U)(\widehat{U}_{13})$$

for all $\omega \in \mathcal{L}(H_U)_*$ and the lemma follows immediately. ∎

LEMMA 6.4. *Let $(\widehat{S}_V)_p$ be faithfully represented on a Hilbert space H. Regard ε as an injective map from $(\widehat{S}_V)_p$ to $\mathcal{L}(H \otimes H_U)$, and let $e = 1 \otimes p$ (where $p = (\varphi_U \otimes \mathrm{id})(U) = (\mathrm{id} \otimes \psi_U)(\widehat{U}) \in \mathcal{L}(H_U)$). Then ε and e will satisfy the two conditions on Proposition* 6.1.

Proof. Since for any $a \in (\widehat{S}_W)_p$, $\varepsilon(a) = a \otimes 1$, the map in (ii) of Proposition 6.1 will send a to $a \otimes p$ and so is injective. We can formulate condition (i) in the following way: $(1 \otimes p)\varepsilon(b)(1 \otimes p) = (\mathrm{id} \otimes \psi_U)\varepsilon(b) \otimes p$ for any $b \in (\widehat{S}_V)_p$. Now

$$\begin{aligned}(1 \otimes p)\varepsilon(b)(1 \otimes p) &= (\mathrm{id} \otimes \mathrm{id} \otimes \psi_U \otimes \psi_U)(\widehat{U}_{23}(\varepsilon(b) \otimes 1 \otimes 1)\widehat{U}_{24}) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \psi_U \otimes \psi_U)(((\mathrm{id} \otimes \delta_{\widehat{U}})\varepsilon(b) \otimes 1)\widehat{U}_{23}\widehat{U}_{34})\end{aligned}$$

(since ε satisfies the coaction identity). Thus, using Lemma 6.3,

$$\begin{aligned}(1 \otimes p)\varepsilon(b)(1 \otimes p) &= (\mathrm{id} \otimes \mathrm{id} \otimes \psi_U \otimes \psi_U)(((\mathrm{id} \otimes \delta_{\widehat{U}})\varepsilon(b) \otimes 1)\widehat{U}_{24}) \\ &= [(\mathrm{id} \otimes \mathrm{id} \otimes \psi_U)((\mathrm{id} \otimes \delta_{\widehat{U}})\varepsilon(b))](1 \otimes p) \\ &= (\mathrm{id} \otimes \psi_U)(\varepsilon(b)) \otimes p.\end{aligned}$$

This proves the lemma. ∎

LEMMA 6.5. *The set* $P = \{(\varphi_U \cdot s \otimes \mathrm{id})(U) : s \in \widehat{A}_U\}$ *is dense in* S_U. *Equivalently,* $\{(\mathrm{id} \otimes \psi_U \cdot s)(\widehat{U}) : s \in A_{\widehat{U}}\}$ *is dense in* $\widehat{S}_{\widehat{U}}$.

Proof. We first note that because $p = (\varphi_U \otimes \mathrm{id})(U)$ is a minimal central projection, $p \in S_U$ ($p \cdot S_U = \mathbb{C} \cdot p$). Moreover, if $s = (\mathrm{id} \otimes \omega)(U)$ then

$$\begin{aligned}(\varphi_U \otimes \mathrm{id})((s \otimes 1)U) &= (\varphi_U \otimes \mathrm{id})(\mathrm{id} \otimes \omega \otimes \mathrm{id})(U_{12}U_{13}) \\ &= (\varphi_U \otimes \mathrm{id})(\mathrm{id} \otimes \omega \otimes \mathrm{id})((\mathrm{id} \otimes \delta_U)U) = (\omega \otimes \mathrm{id})\delta_U(p).\end{aligned}$$

Note that $\delta_U(p)(x \otimes 1) \in S_U \otimes S_U$ (for any $x \in S_U$) and so $(\omega \otimes \mathrm{id})\delta_U(p) \in S_U$. Thus P is a subset of S_U. Let $t \in \widehat{S}_U$ be such that $(\varphi_U \cdot s)(t) = 0$ for all $s \in \widehat{A}_U$. Then $\varphi_U(t^*t) = 0$ (as $\widehat{A}_U$ is dense in $\widehat{S}_U$). Because φ_U is faithful, P separates points of $\widehat{S}_U$. Hence P is $\sigma(\widehat{S}_U^*, \widehat{S}_U)$-dense in $\widehat{S}_U^*$. Therefore, for any $f \in \widehat{S}_U^*$, there exists a net s_i in $\widehat{A}_U$ such that $\varphi_U \cdot s_i$ converges to f weakly. Note that $g(L_U(h)) = h(\rho_U(g))$ for all $g \in S_U^*$ and $h \in \widehat{S}_U^*$ and that $\rho_U(S_U^*)$ is a dense subset of $\widehat{S}_U$ (because $1 \in \widehat{S}_U$). Hence for any $\nu \in \mathcal{L}(H_U)_*$, there exists a net a_i in $L_U(P)$ such that $g(a_i)$ converges to $g(L_U(\nu))$ for any $g \in S_U^*$. Therefore, the $\sigma(S_U, S_U^*)$-closure of $L_U(P)$ will contain S_U and so $L_U(P)$ is norm dense in S_U (because $L_U(P)$ is a convex subset, in fact a vector subspace, of S_U). ∎

LEMMA 6.6. *Let the notation be the same as in Lemma* 6.4. *Then the linear span,* T*, of* $\{\varepsilon(a)(1 \otimes p)\varepsilon(b) : a, b \in (\widehat{S}_V)_p\}$ *is norm dense in* $(\widehat{S}_V)_p \times_{\varepsilon,r} \widehat{S}_{\widehat{U}} = (\widehat{S}_V)_p \times_{\varepsilon',r} S_U$.

Proof. We first note that T is a subset of $(\widehat{S}_V)_p \times_{\varepsilon,r} \widehat{S}_{\widehat{U}}$. Since ε is a coaction,

$$(1 \otimes p)\varepsilon(b) = (\mathrm{id} \otimes \mathrm{id} \otimes \psi_U)((\varepsilon \otimes \mathrm{id})\varepsilon(b)\widehat{U}_{23}).$$

Therefore,

$$\varepsilon(a)(1 \otimes p)\varepsilon(b) = (\mathrm{id} \otimes \mathrm{id} \otimes \psi_U)((\varepsilon \otimes \mathrm{id})((a \otimes 1)\varepsilon(b))\widehat{U}_{23}).$$

Now

$$((\widehat{S}_V)_p \otimes 1)\varepsilon(\widehat{S}_V)_p = (\mathrm{id} \otimes \Phi)(((\widehat{S}_V)_p \otimes 1)\delta_V(\widehat{S}_V)_p) = (\widehat{S}_V)_p \otimes S_{\widehat{U}}$$

(where δ_V is the co-multiplication on $(\widehat{S}_V)_p$ which is non-degenerate and Φ is the map from $(\widehat{S}_V)_p$ to $S_{\widehat{U}}$ that define ε, Φ is surjective since U is a quotient of V). Thus element of the form

$$(\mathrm{id} \otimes \mathrm{id} \otimes \psi_U)((\varepsilon \otimes \mathrm{id})(c \otimes s)\widehat{U}_{23})$$

($c \in (\widehat{S}_V)_p$ and $s \in S_{\widehat{U}}$) can be approximated in norm by elements in T. Note that

$$(\mathrm{id} \otimes \mathrm{id} \otimes \psi_U)((\varepsilon \otimes \mathrm{id})(c \otimes s)\widehat{U}_{23}) = \varepsilon(c)(1 \otimes (\mathrm{id} \otimes \psi_U \cdot s)(\widehat{U})).$$

Hence by Lemma 6.5, T is norm dense in $(\widehat{S}_V)_p \times_{\varepsilon,r} \widehat{S}_{\widehat{U}}$. ∎

We can now prove the main theorem in this section very easily.

Proof of Theorem 6.2. By Lemma 6.4 and Proposition 6.1 (see the paragraph before Proposition 6.1 as well), $(\widehat{S}_W)_p$ is strongly Morita equivalent to the closure of the linear span of the set $\{\varepsilon(a)(1 \otimes p)\varepsilon(b) : a, b \in (\widehat{S}_V)_p\}$ which, by Lemma 6.6, equals $(\widehat{S}_V)_p \times_{\varepsilon, r} S_U$. ∎

REMARK 6.7. It is believed that the amenability of U can be removed.

APPENDIX

The aim of this appendix is to give a C^*-algebraic characterisation of normal subgroups of discrete groups. Let H be a discrete group and let Ψ_H be the canonical tracial state on $C^*(H)$. We first recall a well known fact about the fixed point algebra of a discrete coaction.

LEMMA A1. *Let B be a C^*-algebra with a coaction ε by $C^*(H)$ and let $\Phi = (\mathrm{id} \otimes \Psi_H) \circ \varepsilon$. Then $\Phi(B)$ is the fixed point algebra B^ε.*

THEOREM A2. *Let φ be a homomorphism from a discrete group G to a discrete group H. Let $N = \mathrm{Ker}(\varphi)$ and ε_H be the coaction on $C^*(G)$ by $C^*(H)$ as given in Theorem* 4.9. *Then $C^*(N)$ is isomorphic to the fixed point algebra $C^*(G)^{\varepsilon_H}$ of the coaction ε_H.*

Proof. By [7], 4.1, the canonical map j from $C^*(N)$ to $C^*(G)$ is injective. Therefore, we need only to show that $j(C^*(N)) = C^*(G)^{\varepsilon_H}$. It is clear that $j(C^*(N)) \subseteq C^*(G)^{\varepsilon_H}$. Let Φ be the map as defined in Lemma A1 with $B = C^*(G)$ and $\varepsilon = \varepsilon_H$. For any $t \in G$, $\Phi(u_t) = (\mathrm{id} \otimes \Psi_H)(u_t \otimes u_{\hat{t}})$ (where $\hat{t} = \varphi(t)$). Hence, $\Phi(u_t) = 0$ if $t \notin N$ and $\Phi(u_t) = u_t$ if $t \in N$. Now it is clear that $C^*(G)^{\varepsilon_H} \subseteq j(C^*(N))$ since $\bigoplus_{t \in G} \mathbb{C} \cdot u_t$ is a dense subspace of $C^*(G)$. ∎

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