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NONCOMPACTNESS MEASURE INVARIANCE OF THE INDEX

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ABSTRACT. We state some new results of perturbations with respect to the noncompactness measure, concerning Fredholm-type objects in Banach spaces. In particular, we solve the problem of the invariance of the index in this context. We generalize some related results to the case of the variable domains.

KEYWORDS: *Noncompactness measure, complex, Fredholm index.*

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It was proved in [13] that if a complex $(\beta^i)_i$ is sufficiently close to a Fredholm complex $(\alpha^i)_i$ in the sense of the noncompactness measures $\|\beta^i - \alpha^i\|_q$, then $(\beta^i)_i$ is Fredholm, too. It was natural to suppose the equality of their indices, but no proof was known except for certain particular cases. The operator case had been stated in [16]. In [14] the invariance of the index was proved for short complexes, using techniques which could not be extended to the general case. In the particular case of the small perturbations $\|\beta^i - \alpha^i\|$, various results were stated ([2], [12], [25]). In the case $\|\beta^i - \alpha^i\|_q = 0$ (i.e. for compact perturbations), the problem was solved in [6] using some results from [4] and [5]. By Corollary 14, we solve now this problem in the general case. The main results Theorems 9, 12, 13, 17 are stated in the more general context of the Fredholm pairs with variable domains. In this case the noncompactness of the perturbations is measured by the quantity $\widehat{q}$ which we define in (2). Theorem 17 generalizes some small perturbations results from [3], [4], [5] to the case of unbounded operators between quotient Banach spaces. It generalizes the corresponding statement concerning Fredholm complexes ([1], [2]), too. See [1], [7], [10], [11], [12], [23] for other related results.

We denote by $D(S)$, $N(S)$, $R(S)$, $G(S)$ and $\gamma(S)$ the domain, null-space, range, graph and reduced minimum modulus of a closed operator S, respectively ([2]). The product of two Banach spaces X, Y is endowed with the norm $\|(x,y)\|^2 = \|x\|^2 + \|y\|^2$. Let $\mathcal{C}(X,Y)$, $\mathcal{B}(X,Y)$, resp. $\mathcal{K}(X,Y)$ be the set of all closed, bounded, resp. compact operators from X to Y. We denote respectively by X_1, X^* and $\mathcal{G}(X)$ the unit ball, dual and set of closed linear subspaces of X. Let $d(a,B)$ be the distance from $a \in M$ to $B \subset M$, where M is a metric space. The *measure of noncompactness* of $A \subset M$ is defined by $q(A) = \inf_{B \text{ finite}} \sup_{a\in A} d(a,B)$ ([20]). Then $\|S\|_q := q(S(X_1))$ is called the *noncompactness measure* of $S \in \mathcal{B}(X,Y)$ ([16]). We have $\|S\|_q = 0 \Leftrightarrow S$ is compact. Set $\|S\|_c := d(S, \mathcal{K}(X,Y))$. If Y has the bounded approximation property, then $\|\cdot\|_q$ and $\|\cdot\|_c$ are equivalent seminorms ([21]). If Y, $Z \in \mathcal{G}(X)$, we set $\delta(Y,Z) := \sup_{y\in Y_1} d(y,Z)$ ([18]) and $\widehat{\delta}(Y,Z) := \max\{\delta(Y,Z), \delta(Z,Y)\}$ ([18], [19]). If $Y_i \in \mathcal{G}(X)$, $i = 1,2,3$, then ([17])

$$\delta(Y_1,Y_3) \leqslant \delta(Y_1,Y_2) + \delta(Y_2,Y_3) + \delta(Y_1,Y_2)\delta(Y_2,Y_3). \tag{1}$$

For S, $\widetilde{S} \in \mathcal{C}(X,Y)$, set $\delta(S,\widetilde{S}) := \delta(G(S), G(\widetilde{S}))$ and $\widehat{\delta}(S,\widetilde{S}) := \max\{\delta(S,\widetilde{S}), \delta(\widetilde{S},S)\}$ ([2]). If $D(S) = D(\widetilde{S})$, then $\widehat{\delta}(S,\widetilde{S}) \leqslant \|S - \widetilde{S}\|$ ([2]). Thus $\widehat{\delta}(S,\widetilde{S})$ measures the size of the perturbation when $D(S) \neq D(\widetilde{S})$, while $\|S - \widetilde{S}\|_q$ measures its noncompactness when $D(S) = D(\widetilde{S})$. For arbitrary S, $\widetilde{S} \in \mathcal{C}(X,Y)$, let us define

$$\begin{aligned} q(S,\widetilde{S}) &:= \inf_{Q=\text{compact}\subset Y} \sup_{x\in D(S)_1} d((x,Sx), G(\widetilde{S}) + \{0\}\times Q), \\ \widehat{q}(S,\widetilde{S}) &:= \max\{q(S,\widetilde{S}), q(\widetilde{S},S)\}. \end{aligned} \tag{2}$$

We obtain results of invariance of the index with respect to $\widehat{q}$. They generalize the corresponding statements concerning perturbations which are either compact, or small (with respect to $\widehat{\delta}$ or $\|\cdot\|$).

PROPOSITION 1. (i) *Let S, $\widetilde{S} \in \mathcal{C}(X,Y)$ with $\widehat{q}(S,\widetilde{S}) < \infty$. If $A \in \mathcal{K}(D(S),Y)$ and $\widetilde{A} \in \mathcal{K}(D(\widetilde{S}),Y)$, then $q(S + A, \widetilde{S} + \widetilde{A}) = q(S,\widetilde{S})$.*

(ii) *If $S_i \in \mathcal{C}(X,Y)$, $i = 1,2,3$, then we have the inequality $q(S_1,S_3) \leqslant q(S_1,S_2) + q(S_2,S_3) + q(S_1,S_2)q(S_2,S_3)$.*

(iii) *If S, $\widetilde{S} \in \mathcal{C}(X,Y)$ with $\|S\| < \infty$ and $\delta(\widetilde{S},S) < (2\|S\| + 2)^{-1}$, then $\|\widetilde{S}\| \leqslant 2\|S\| + 1$ and $\widehat{q}(S,\widetilde{S}) \leqslant (1 + \max\{\|S\|^2, \|\widetilde{S}\|^2\|)^{1/2}\widehat{\delta}(S,\widetilde{S})$.*

(iv) *Let* $S, \widetilde{S} \in \mathcal{C}(X,Y)$ *with* $D(S) = D(\widetilde{S})$. *If* $\|S - \widetilde{S}\|_q < \infty$, *then* $\widehat{q}(S,\widetilde{S}) \leqslant \|S - \widetilde{S}\|_q$. *If either* S *or* $\widetilde{S}$ *is bounded, then*

$$\|S - \widetilde{S}\|_q \leqslant (1 + \min\{\|S\|^2, \|\widetilde{S}\|^2\})^{\frac{1}{2}} \widehat{q}(S,\widetilde{S}).$$

If $S - \widetilde{S}$ *is bounded, then* $\widehat{q}(S,\widetilde{S}) \leqslant \|S - \widetilde{S}\|_q \leqslant \|S - \widetilde{S}\|_c \leqslant \|S - \widetilde{S}\|$.

Proof. (i) Set $K := A(D(S)_1) + (2 + q(S,\widetilde{S}))\widetilde{A}(D(\widetilde{S})_1)$. By (2), for any $\varepsilon \in (0,1]$ there exists a compact set $Q \subset Y$ such that for each $x \in D(S)_1$ there are $\widetilde{x} \in D(\widetilde{S})$ and $y \in Q$ with $\|(x - \widetilde{x}, Sx - \widetilde{S}\widetilde{x} - y)\| < q(S,\widetilde{S}) + \varepsilon$. Since $\|x\| \leqslant 1$ and $\|\widetilde{x}\| \leqslant \|x - \widetilde{x}\| + \|x\|$, it follows $y' := y + Ax - \widetilde{A}\widetilde{x} \in Q + K$. Then

$$\begin{aligned} &d((x,(S+A)x), G(\widetilde{S}+\widetilde{A}) + \{0\} \times (Q+K)) \\ &\qquad \leqslant \|(x,(S+A)x) - (\widetilde{x},(\widetilde{S}+\widetilde{A})\widetilde{x} + y')\| < q(S,\widetilde{S}) + \varepsilon. \end{aligned}$$

We take the supremum over x. Since $Q + K$ is precompact, then we infer the inequality $q(S + A, \widetilde{S} + \widetilde{A}) \leqslant q(S,\widetilde{S}) + \varepsilon$. Now let $\varepsilon \to 0$. Then (i) holds due to the symmetry (apply the previous step to $S + A$, $\widetilde{S} + \widetilde{A}$, write $S = (S + A) - A$ etc.).

(ii) Let $\varepsilon \in (0,1]$ be arbitrary. Set $a := q(S_1,S_2) + \varepsilon + 1$. By (2), for some compact sets Q, $Q' \subset Y$ we have the following. For any $x_1 \in D(S_1)_1$ there exist $x_2 \in D(S_2)$ and $y \in Q$ such that

$$\|u\| < q(S_1,S_2) + \varepsilon,\ u := (x_1, S_1x_1) - (x_2, S_2x_2) - (0,y).$$

Hence $\|x_2\| \leqslant \|x_1 - x_2\| + \|x_1\| \leqslant a$. For $a^{-1}x_2 \in D(S_2)_1$, there exist $x_3 \in D(S_3)$ and $y' \in Q'$ such that

$$\|v\| < q(S_2,S_3) + \varepsilon,\ v := (a^{-1}x_2, S_2a^{-1}x_2) - (x_3, S_3x_3) - (0,y').$$

Set $x_3' := ax_3$ and $Q'' := Q + \{tq'\}_{t,q'}$, where $0 \leqslant t \leqslant q(S_1,S_2) + 2$ and $q' \in Q'$. Hence $y + ay' \in Q''$ and $u + av = (x_1, S_1x_1) - (x_3', S_3x_3' + y + ay')$. We obtain the inequality

$$\begin{aligned} d((x_1,Sx_1), G(S_3) + \{0\} \times Q'') &\leqslant \|u + av\| \leqslant \|u\| + a\|v\| \\ &\leqslant q(S_1,S_2) + \varepsilon + a(q(S_2,S_3) + \varepsilon), \end{aligned}$$

in which we take then the supremum over x_1. This provides an estimate for $q(S_1,S_3)$, via (2), since Q'' is precompact. By letting $\varepsilon \to 0$, we obtain (ii).

(iii) Let $\|S\| < \infty$. For the uniform boundedness of $\widetilde{S}$, see Lemma II.3.18 from [7], which shows that $\delta(\widetilde{S},S) < (1 + \|S\|)^{-1}$ implies

$$\|\widetilde{S}\| \leqslant ((1 + \|S\|)\delta(\widetilde{S},S) + \|S\|)(1 - (1 + \|S\|)\delta(\widetilde{S},S))^{-1}.$$

If $x \in D(S)_1$ and $x = (1+\|S\|^2)^{1/2}x'$, then $(x', Sx') \in G(S)_1$ and

$$d((x,Sx),G(\widetilde{S})) = (1+\|S\|^2)^{\frac{1}{2}} d((x',Sx'),G(\widetilde{S})) \leqslant (1+\|S\|^2)^{\frac{1}{2}}\delta(S,\widetilde{S}).$$

We take the supremum over x to obtain $q(S,\widetilde{S}) \leqslant (1+\|S\|^2)^{1/2}\delta(S,\widetilde{S})$. Similarly, $q(\widetilde{S},S) \leqslant (1+\|\widetilde{S}\|^2)^{1/2}\delta(\widetilde{S},S)$ and the inequality (iii) is proved.

(iv) If $\|S-\widetilde{S}\|_q < \infty$, then for any $\varepsilon > 0$ there exists $Q \subset Y$ finite such that

$$q(S,\widetilde{S}) \leqslant \sup_{x\in D(S)_1} d((x,Sx),G(\widetilde{S})+\{0\}\times Q) \leqslant \sup_{x\in D(S)_1} d((S-\widetilde{S})x,Q) < \|S-\widetilde{S}\|_q+\varepsilon$$

(note that for each x, the left side distance does not change if we substract $(x,\widetilde{S}x) \in G(\widetilde{S})$). Since ε is arbitrary, then we obtain $q(S,\widetilde{S}) \leqslant \|S-\widetilde{S}\|_q$ (as well as $q(\widetilde{S},S) \leqslant \|\widetilde{S}-S\|_q$).

Now assume $\|\widetilde{S}\| < \infty$ and take again an arbitrary $\varepsilon > 0$. There exists $Q \subset Y$ compact such that for any $x \in D(S)$ with $\|x\| \leqslant 1$ there are $\widetilde{x} \in D(\widetilde{S})$ and $y \in Q$ with $\|(x-\widetilde{x}, Sx-\widetilde{S}\widetilde{x}-y)\| < q(S,\widetilde{S})+\varepsilon$. There exists an ε-covering of Q with balls of centers in a finite set Q'. For any $x \in D(S)$ etc., let $y' \in Q'$ such that $\|y-y'\| < \varepsilon$. By an elementary Cauchy–Schwarz inequality we obtain

$$\begin{aligned} d((S-\widetilde{S})x,Q') &\leqslant \|(S-\widetilde{S})x-y'\| \leqslant \|Sx-\widetilde{S}\widetilde{x}-y\| + \|\widetilde{S}(\widetilde{x}-x)\| + \|y-y'\| \\ &\leqslant (1+\|\widetilde{S}\|^2)^{\frac{1}{2}}\|(x-\widetilde{x},Sx-\widetilde{S}\widetilde{x}-y)\|+\varepsilon \\ &< (1+\|\widetilde{S}\|^2)^{\frac{1}{2}}(q(S,\widetilde{S})+\varepsilon)+\varepsilon. \end{aligned}$$

We take the supremum over $x \in D(S)_1$. This provides an estimate for $\|S-\widetilde{S}\|_q$, in which we let $\varepsilon \to 0$. When $\|S\| < \infty$, we simply change the roles of S, $\widetilde{S}$. We obviously have $\|\cdot\|_q \leqslant \|\cdot\|_c$. Thus (iv) holds, and Proposition 1 is proved. ∎

Let $F(K)$ denote the space of all bounded real-valued functions on a set K, endowed with the sup-norm.

LEMMA 2. *Let S, $\widetilde{S} \in C(X,F(K))$ with $\|S\|$, $\widehat{q}(S,\widetilde{S}) < \infty$, where X is a real Banach space. Then there exists $\widetilde{S}_1 \in C(X,F(K))$ such that $D(\widetilde{S}_1) = D(\widetilde{S})$, $\dim R(\widetilde{S}_1-\widetilde{S}) < \infty$ and $\widehat{\delta}(S,\widetilde{S}_1) \leqslant (\|S\|^2+5)^{1/2}\widehat{q}(S,\widetilde{S})$.*

Proof. Let $\varepsilon > 0$ be arbitrary. By (2), there exist some compact sets Q, $Q' \subset F(K)$ such that

$$\begin{aligned} &\sup_{x\in D(S)_1} d((x,Sx),G(\widetilde{S})+\{0\}\times Q) < q(S,\widetilde{S})+\varepsilon, \\ &\sup_{\widetilde{x}\in D(\widetilde{S})_1} d((\widetilde{x},\widetilde{S}\widetilde{x}),G(S)+\{0\}\times Q') < q(\widetilde{S},S)+\varepsilon. \end{aligned} \tag{3}$$

The closure of the finite-rank contractions on $F(K)$ in the operator strong topology contains the identity (see for instance [15]). Hence there is an $A \in \mathcal{B}(F(K))$ such that $\|A\| \leqslant 1$, $\dim R(A) < \infty$, and $\|y - Ay\| < \varepsilon$ for all y in the compact set $Q \cup (-Q')$. Namely, we let K be endowed with the topology of all subsets and apply Proposition 41.1 from [15] for the complexified of $F(K)$, to obtain an A which we replace then by $f \mapsto 2^{-1}(Af + \overline{A\overline{f}})$, $f \in F(K)$. By Theorem 2.9.2 from [22], there exists $B \in \mathcal{B}(X, F(K))$ such that $B|D(S) = AS$ and $\|B\| = \|AS\| \leqslant \|S\|$ (essentially because $F(K)$ is a complete lattice with respect to the usual order, and so one can follow the lines of the proof of the Hahn-Banach theorem). Since $Q'' := (\widehat{q}(S, \widetilde{S}) + \varepsilon + 1)AS(D(S)_1)$ is compact, then there exists $C \in \mathcal{B}(F(K))$ with $\|C\| \leqslant 1$, $\dim R(C) < \infty$, and $\|y - Cy\| < \varepsilon$ for all $y \in Q''$ ([15]). We set $\widetilde{S}_1 := \widetilde{S} - A\widetilde{S} + CB|D(\widetilde{S})$.

Let $g := (x, Sx) \in G(S)_1$ (resp. $\widetilde{g} := (\widetilde{x}, \widetilde{S}_1\widetilde{x}) \in G(\widetilde{S}_1)_1$) be arbitrary. Hence $x \in D(S)_1$ (resp. $\widetilde{x} \in D(\widetilde{S})_1$). By (3), there exist $\widetilde{x} \in D(\widetilde{S})$ and $f \in Q$ (resp. there exist $x \in D(S)$ and $f' \in Q'$) such that $\|g - (\widetilde{x}, \widetilde{S}\widetilde{x}) - (0, f)\| < q(S, \widetilde{S}) + \varepsilon$, resp. $\|(\widetilde{x}, \widetilde{S}\widetilde{x}) - g - (0, f')\| < q(\widetilde{S}, S) + \varepsilon$. Therefore, in each of these cases we can find an y ($= f$ or $-f'$) in $Q \cup (-Q')$ such that

$$\|(x - \widetilde{x}, Sx - \widetilde{S}\widetilde{x} - y)\| < \widehat{q}(S, \widetilde{S}) + \varepsilon. \tag{4}$$

Moreover, we have $\|x\| \leqslant \widehat{q}(S, \widetilde{S}) + \varepsilon + 1$ (since either $\|x\| \leqslant 1$, or $\|\widetilde{x}\| \leqslant 1$ in which case $\|x\| \leqslant \|x - \widetilde{x}\| + \|\widetilde{x}\|$). Hence $ASx \in Q''$, and so $\|ASx - CASx\| < \varepsilon$.

Set $a := ((\|Sx - \widetilde{S}\widetilde{x} - y\| + \varepsilon)^2 + \|x - \widetilde{x}\|^2)^{1/2}$. We compute a^2, then use the estimates (4) and $\|Sx - \widetilde{S}\widetilde{x} - y\| \leqslant \widehat{q}(S, \widetilde{S}) + \varepsilon$ to obtain

$$a \leqslant \widehat{q}(S, \widetilde{S}) + 2\varepsilon. \tag{5}$$

Note that $\|ASx - B\widetilde{x}\| = \|B(x - \widetilde{x})\| \leqslant \|B\|\,\|x - \widetilde{x}\| \leqslant \|S\|\,\|x - \widetilde{x}\|$. By (4) and the previous estimates, we obtain

$$\begin{aligned}
\|Sx - \widetilde{S}_1\widetilde{x}\| &= \|Sx - \widetilde{S}\widetilde{x} + A\widetilde{S}\widetilde{x} - CB\widetilde{x}\| \\
&\leqslant \|Sx - \widetilde{S}\widetilde{x} - y\| + \|y - Ay\| + \|A(y + \widetilde{S}\widetilde{x} - Sx)\| \\
&\qquad + \|ASx - CASx\| + \|C(ASx - B\widetilde{x})\| \\
&\leqslant 2(\|Sx - \widetilde{S}\widetilde{x} - y\| + \varepsilon) + \|S\|\,\|x - \widetilde{x}\| \\
&\leqslant (4 + \|S\|^2)^{\frac{1}{2}} a.
\end{aligned}$$

By using $\|x - \widetilde{x}\| \leqslant a$ also, the above inequality leads to

$$\|(x, Sx) - (\widetilde{x}, \widetilde{S}_1\widetilde{x})\| \leqslant (5 + \|S\|^2)^{\frac{1}{2}} a.$$

Thus for any $g \in G(S)_1$ (resp. $\widetilde{g} \in G(\widetilde{S}_1)_1$) we can find $\widetilde{g} \in G(\widetilde{S}_1)$ (resp. $g \in G(S)$) such that

$$\|g - \widetilde{g}\| \leqslant (5 + \|S\|^2)^{\frac{1}{2}}(\widehat{q}(S, \widetilde{S}) + 2\varepsilon)$$

(see (5)). It follows $\widehat{\delta}(S, \widetilde{S}_1) \leqslant (5 + \|S\|^2)^{1/2}(\widehat{q}(S, \widetilde{S}) + 2\varepsilon)$. Now let $\varepsilon \to 0$, and Lemma 2 is proved. ∎

LEMMA 3. *If S, $\widetilde{S} \in \mathcal{C}(X, Y)$, then we have the following.*

(i) *If $q(\widetilde{S}, S) < \infty$ and $\|S\| < \infty$, then $\|\widetilde{S}\| < \infty$.*

(ii) *For any $\lambda \geqslant 1$ we have $q(\lambda S, \lambda \widetilde{S}) \leqslant \lambda q(S, \widetilde{S})$.*

(iii) *Let $\mathcal{X}$, $\mathcal{Y}$, F be Banach spaces such that $F \times X \in \mathcal{G}(\mathcal{X})$, $Y \in \mathcal{G}(\mathcal{Y})$ and $\dim F < \infty$. Set $S_1(f, x) := Sx$ and $\widetilde{S}_1(f, \widetilde{x}) := \widetilde{S}\widetilde{x}$ for $f \in F$, $x \in D(S)$ and $\widetilde{x} \in D(\widetilde{S})$. Then S_1, $\widetilde{S}_1 \in \mathcal{C}(\mathcal{X}, \mathcal{Y})$ and $q(S_1, \widetilde{S}_1) \leqslant q(S, \widetilde{S})$.*

Proof. (i) Let $r > q(\widetilde{S}, S)$ be finite. There exists $Q \subset Y$ compact such that for any $\widetilde{x} \in D(\widetilde{S})_1$ there are $x \in D(S)$ and $y \in Q$ with $\|\widetilde{x} - x\|$, $\|\widetilde{S}\widetilde{x} - Sx - y\| < r$. We take the supremum over $\widetilde{x}$ in the inequality

$$\begin{aligned}\|\widetilde{S}\widetilde{x}\| &\leqslant \|\widetilde{S}\widetilde{x} - Sx - y\| + \|Sx\| + \|y\| \\ &\leqslant r + \|S\|(\|x - \widetilde{x}\| + \|\widetilde{x}\|) + \|y\| \\ &\leqslant r + \|S\|(r + 1) + \sup_{y \in Q} \|y\| < \infty.\end{aligned}$$

(ii) Let Q denote any compact subset of Y. Then take successively the infimum over $\widetilde{x} \in D(\widetilde{S})$, $y \in Q$, the supremum over $x \in D(S)_1$, and the infimum over $Q \subset Y$ in the inequality

$$\|(x, \lambda Sx) - (\widetilde{x}, \lambda \widetilde{S}\widetilde{x} - \lambda y)\| \leqslant \lambda \|(x, Sx) - (\widetilde{x}, \widetilde{S}\widetilde{x} - y)\|.$$

(iii) We have $d(((f, x), Sx), G(\widetilde{S}_1) + \{0\} \times Q) \leqslant d((x, Sx), G(\widetilde{S}) + \{0\} \times Q)$ for $f \in F$, $x \in D(S)$ and $Q \subset Y$. Take the supremum over all $(f, x) \in D(S)_1$, note that $\|(f, x)\| \leqslant 1$ implies $\|x\| \leqslant 1$, and then take the infimum over all Q compact in Y. Thus Lemma 3 is proved. ∎

Set $a = a(S, T) := \dim N(S)/N(S) \cap R(T)$, $b = b(S, T) := \dim R(T)/N(S) \cap R(T)$, $c := a(T, S)$ and $d := b(T, S)$ for $S \in \mathcal{C}(X, Y)$, $T \in \mathcal{C}(Y, X)$. If a, b, c, d, $\|S\|$, $\|T\| < \infty$, then (S, T) is called a *Fredholm pair*. In this case $\gamma(S)$, $\gamma(T) > 0$. The pair is said to be *exact* if $N(S) = R(T)$, $N(T) = R(S)$. Let $\mathcal{F}(X, Y)$ denote the set of all Fredholm pairs. We define the *index* of (S, T) by $\operatorname{ind}(S, T) = a - b - c + d$ ([4], [5], [6], [7]). See Definition 16 for a generalization to the unbounded case.

Theorem 4 and Lemmas 5–8 are proved in [5], [7] (and [4] in the case $D(S) = X$, $D(T) = Y$). See [6] also for Theorem 4. For the explicit estimate in Lemma 6, see for instance (3.16) from [5], or (II.4.15) from [7]. Set $G'(T) := \{(x, y);\ (y, x) \in G(T)\}$.

THEOREM 4. *If* $(S,T) \in \mathcal{F}(X,Y)$ *and* $A \in \mathcal{B}(D(S),Y)$, $B \in \mathcal{B}(D(T),X)$ *have finite rank, then* $(S+A,T+B) \in \mathcal{F}(X,Y)$ *and* $\operatorname{ind}(S+A,T+B) = \operatorname{ind}(S,T)$.

LEMMA 5. *Let* $(S,T) \in \mathcal{F}(X,Y)$ *such that* $\operatorname{ind}(S,T) = 0$. *Then there are two finite rank operators* $A \in \mathcal{B}(D(S),Y)$, $B \in \mathcal{B}(D(T),X)$ *such that* $N(S+A) = R(T+B)$ *and* $N(T+B) = R(S+A)$.

LEMMA 6. *Let* $(S,T) \in \mathcal{F}(X,Y)$ *such that* $N(S) = R(T)$ *and* $N(T) = R(S)$. *Set* $\gamma := \min\{\gamma(S), \gamma(T)\}$. *Then* $\delta(G(S), G'(T))^{-1} \geqslant \gamma^2(2\gamma+1)^{-1} - 1$.

LEMMA 7. *Let* $(S,T) \in \mathcal{F}(X,Y)$ *such that* $\dim Y/R(S)$, $\dim X/R(T) = \infty$. *Then for any* $\varepsilon > 0$ *there exists* $(\widetilde{S},\widetilde{T}) \in \mathcal{F}(X,Y)$ *with* $D(\widetilde{S}) = D(S)$, $D(\widetilde{T}) = D(T)$ *such that:* $\|\widetilde{S} - S\|$, $\|\widetilde{T} - T\| < \varepsilon$, $\operatorname{ind}(\widetilde{S},\widetilde{T}) = \operatorname{ind}(S,T)$ *and* $N(\widetilde{S}) \subset R(\widetilde{T})$, $N(\widetilde{T}) \subset R(\widetilde{S})$.

LEMMA 8. *Let* X, Y *be real Banach spaces. Let* $(S,T) \in \mathcal{F}(X,Y)$ *such that* $N(S) \subset R(T)$ *and* $N(T) \subset R(S)$. *If* $\operatorname{ind}(S,T) > 0$, *then there exist* $x \in D(S)$, $x^* \in X^*$ *and* $y^* \in Y^*$ *such that* $y^*|D(T) = x^*T$ *and* $16(x^*(x) - y^*(Sx))^2 > (\|x^*\|^2 + \|y^*\|^2)(\|x\|^2 + \|Sx\|^2)$.

In the proof of Theorem 9, the first part concerning the inequality (6) is stated without details (we follow some ideas from [4], [5], [7], where similar facts are proved with respect to $\|\cdot\|$ or $\widehat{\delta}$ instead of $\widehat{q}$).

THEOREM 9. *For any* $(S,T) \in \mathcal{F}(X,Y)$ *there exists* $\varepsilon > 0$ *such that if* $(\widetilde{S},\widetilde{T}) \in \mathcal{F}(X,Y)$ *and* $\widehat{q}(S,\widetilde{S})$, $\widehat{q}(T,\widetilde{T}) < \varepsilon$, *then* $\operatorname{ind}(\widetilde{S},\widetilde{T}) = \operatorname{ind}(S,T)$.

Proof. By definition $\|S\|$, $\|T\| < \infty$. Following [7], we make several successive assumptions without any loss of generality at each step, the various changes of (S,T) being accompanied by corresponding changes of $(\widetilde{S},\widetilde{T})$. We may assume that X, Y are real Banach spaces. We may also assume $\operatorname{ind}(S,T) \leqslant 0$ (since $\operatorname{ind}(T,S) = -\operatorname{ind}(S,T)$, and so we can change the roles of S, T if necessary). Now if we replace (S,T) by (S_0,T_0), where $S_0(t,x) := Sx$, $t \in \mathbf{R}^m$, $x \in D(S)$, $T_0 y := (0, Ty)$, $y \in D(T)$ and $m := -\operatorname{ind}(S,T)$, then we may assume $\operatorname{ind}(S,T) = 0$ (see Lemma 3 (iii)). By Lemma 5 and Proposition 1 (i), we may assume now $N(S) = R(T)$, $N(T) = R(S)$, since we can replace (S,T) by a certain $(S+A, T+B)$ with $N(S+A) = R(T+B)$, $N(T+B) = R(S+A)$, which we perturb to $(\widetilde{S},\widetilde{T})$. By Lemma 6 and Lemma 3 (ii), we may assume

$$\delta := \widehat{\delta}(G(S), G'(T)) < 4^{-1}, \tag{6}$$

since we can replace (S,T) by $(\lambda S, \lambda T)$ with a fixed large λ such that $\lambda^2\gamma^2(2\lambda\gamma + 1)^{-1} - 1 > 4$, where $\gamma := \min\{\gamma(S), \gamma(T)\}$ (note that $\gamma(\lambda S) = \lambda\gamma(S)$, $\gamma(\lambda T) =$

$\lambda\gamma(T))$. Finally, by Lemma 3 (iii) (with $F := \{0\}$) we may assume $X = F(L)$ and $Y = F(K)$, since we can replace X, resp. Y by $\mathcal{X} := F((X^*)_1)$, resp. $\mathcal{Y} := F((Y^*)_1)$ (we have isometric embeddings of the form $X \ni x \mapsto f_x \in \mathcal{X}$, $f_x(x^*) := x^*(x)$, $x^* \in X^*$, $\|x^*\| \leqslant 1$). Thus all the involved operators will have infinite codimensional ranges. Note the symmetry of the above hypothesis with respect to S, T.

Now let $\varepsilon > \widehat{q}(S, \widetilde{S})$, $\widehat{q}(T, \widetilde{T})$. By Lemma 2 and Theorem 4, there exists $(\widetilde{S}_1, \widetilde{T}_1) \in \mathcal{F}(X, Y)$ such that

$$\delta(\widetilde{S}_1, S), \delta(T, \widetilde{T}_1) = \mathrm{O}(\varepsilon), \quad \mathrm{ind}(\widetilde{S}_1, \widetilde{T}_1) = \mathrm{ind}(\widetilde{S}, \widetilde{T}). \tag{7}$$

Here the classical notation $\mathrm{O}(\varepsilon)$ stands for a quantity $f = f(\widetilde{S}, \widetilde{T})$ such that $|f| \leqslant C\varepsilon$ with a finite constant $C = C(S, T)$ independent of the perturbed data $\widetilde{S}$, $\widetilde{T}$ (see Proposition 10 for a more precise estimate on ε). By Lemma 7, there exists $(\widetilde{S}_2, \widetilde{T}_2) \in \mathcal{F}(X, Y)$ such that

$$\delta(\widetilde{S}_2, \widetilde{S}_1), \delta(\widetilde{T}_1, \widetilde{T}_2) < \varepsilon, \quad \mathrm{ind}(\widetilde{S}_2, \widetilde{T}_2) = \mathrm{ind}(\widetilde{S}_1, \widetilde{T}_1) \tag{8}$$

and

$$N(\widetilde{S}_2) \subset R(\widetilde{T}_2), \quad N(\widetilde{T}_2) \subset R(\widetilde{S}_2). \tag{9}$$

We apply the estimate (1) to the graphs of T, $\widetilde{T}_1$ and $\widetilde{T}_2$, instead of Y_1, Y_2 and Y_3 respectively. By (7) and (8), we obtain $\delta(T, \widetilde{T}_2) = \mathrm{O}(\varepsilon)$. By using again (1), now for the spaces $G(S)$, $G'(T)$ and $G'(\widetilde{T}_2)$, we obtain $\delta(G(S), G'(\widetilde{T}_2)) \leqslant \delta + \mathrm{O}(\varepsilon)$. By (7) and using (1) for $G(\widetilde{S}_1)$, $G(S)$, $G'(\widetilde{T}_2)$, we obtain $\delta(G(\widetilde{S}_1), G'(\widetilde{T}_2)) \leqslant \delta + \mathrm{O}(\varepsilon)$. Finally, by (8) and using (1) for $G(\widetilde{S}_2)$, $G(\widetilde{S}_1)$, $G'(\widetilde{T}_2)$, it follows $\delta(G(\widetilde{S}_2), G'(\widetilde{T}_2)) \leqslant \delta + \mathrm{O}(\varepsilon)$. By (6), this implies

$$\delta(G(\widetilde{S}_2), G'(\widetilde{T}_2)) < 4^{-1} \tag{10}$$

for a sufficiently small ε.

Suppose now that $\mathrm{ind}(\widetilde{S}_2, \widetilde{T}_2) > 0$. By (9) and Lemma 8, there are nonnull vectors $u = (x^*, -y^*)$ in $X^* \times Y^* \equiv (X \times Y)^*$ and $v = (x, \widetilde{S}_2 x)$ in $D(\widetilde{S}_2) \times Y$ such that $y^*|D(\widetilde{T}_2) = x^*\widetilde{T}_2$ and $4|u(v)| > \|u\|\,\|v\|$ (note that the isomorphism $\equiv$ is isometric). Since $v \in G(\widetilde{S}_2)$, then by (10) there exists $w \in G(\widetilde{T}_2)$ such that $\|v - w\| \leqslant 4^{-1}\|v\|$. Since $u(w) = 0$, it follows

$$\|u\|\,\|v\| < 4|u(v)| = 4|u(v - w)| \leqslant 4\|u\|\,\|v - w\| \leqslant \|u\|\,\|v\|,$$

which is false.

Therefore, we must have $\operatorname{ind}(\widetilde{S}_2, \widetilde{T}_2) \leqslant 0$. By (7) and (8), it follows $\operatorname{ind}(\widetilde{S}, \widetilde{T}) \leqslant 0$.

Due to the previously mentioned symmetry of the hypothesis, we may change now the roles of S and T. If ε is sufficiently small, then we obtain $\operatorname{ind}(\widetilde{T}, \widetilde{S}) \leqslant 0$. But $\operatorname{ind}(\widetilde{T}, \widetilde{S}) = -\operatorname{ind}(\widetilde{S}, \widetilde{T}) \geqslant 0$. Hence $\operatorname{ind}(\widetilde{S}, \widetilde{T}) = 0$ $(= \operatorname{ind}(S, T))$, and Theorem 9 is proved. ∎

PROPOSITION 10. *Let (S, T) be an exact pair on X, Y. Set $\nu := \max\{\|S\|, \|T\|\}$, $\gamma := \min\{\gamma(S), \gamma(T)\}$ and $\lambda := \max\{1, 11\gamma^{-1}\}$. For $(\widetilde{S}, \widetilde{T}) \in \mathcal{F}(X, Y)$, let $\varepsilon := \max\{\widehat{q}(S, \widetilde{S}), \widehat{q}(T, \widetilde{T})\}$. If $3\lambda(\lambda^2\nu^2+5)^{1/2}\varepsilon < 10^{-2}$, then $\operatorname{ind}(\widetilde{S}, \widetilde{T}) = \operatorname{ind}(S, T)$ $(= 0)$.*

Proof. We simply follow the proof of Theorem 9. We have to specify the estimates (7), (8) etc., for $(\lambda S, \lambda T)$ instead of (S, T) (see Lemma 3 (ii)). Note that the reduced minimum modulus and the norms are not modified by the embeddings $X \hookrightarrow \mathcal{X}$, $Y \hookrightarrow \mathcal{Y}$. Take an arbitrary $\varepsilon' > 0$, independent of $\widetilde{S}$, $\widetilde{T}$. Set $\delta := f(\lambda\gamma)^{-1}$ for $f(t) = t^2(2t+1)^{-1} - 1$. By Lemma 6 and Lemma 3 (ii), we have

$$\widehat{\delta}(G(\lambda S), G'(\lambda T)) \leqslant \delta,$$

as well as $\delta + 10^{-2} < 4^{-1}$ (use $\lambda\gamma \geqslant 11$ and $f(11) = 98/23$). Set $a := (\lambda^2\nu^2+5)^{1/2}\lambda$ and $b := \delta + a\varepsilon(a\varepsilon + 2)(\delta + 1)$. We keep the other notations in the proof of Theorem 9. Then Lemma 2 and Lemma 3 (ii) provide the following equivalent of (7):

$$\delta(\widetilde{S}_1, \lambda S), \delta(\lambda T, \widetilde{T}_1) \leqslant a\varepsilon.$$

By applying Lemma 7 for ε', we obtain the next version of (8):

$$\delta(\widetilde{S}_2, \widetilde{S}_1), \delta(\widetilde{T}_1, \widetilde{T}_2) < \varepsilon'.$$

The corresponding estimates obtained by successively applying (1) lead to

$$\delta(\lambda T, \widetilde{T}_2) \leqslant a\varepsilon + \varepsilon' + a\varepsilon\varepsilon',$$

$$\delta(G(\lambda S), G'(\widetilde{T}_2)) \leqslant \delta + a\varepsilon + \mathrm{O}(\varepsilon') + \delta(a\varepsilon + \mathrm{O}(\varepsilon'))$$

and

$$\delta(G(\widetilde{S}_2), G'(\widetilde{T}_2)) \leqslant a\varepsilon + \delta + a\varepsilon + \delta a\varepsilon + a\varepsilon(\delta + a\varepsilon + \delta a\varepsilon) + \mathrm{O}(\varepsilon') = b + \mathrm{O}(\varepsilon').$$

We have to verify (10), and then the proof will be completed as in Theorem 9. The above estimate shows that it suffices to prove $b < 4^{-1}$, since ε' is arbitrary. We use $\delta + 10^{-2} < 4^{-1}$, as well as $3a\varepsilon < 10^{-2}$ which we have by hypothesis. Namely, we have

$$a\varepsilon(a\varepsilon + 2)(\delta + 1) \leqslant 3^{-1} \cdot 10^{-2}(3^{-1} \cdot 10^{-2} + 2)(4^{-1} - 10^{-2} + 1) < 10^{-2},$$

and so $b \leqslant \delta + 10^{-2} < 4^{-1}$. Proposition 10 is proved. ∎

With some minor modifications in the statement, we can give an estimate on ε as in Proposition 10 without the hypothesis that (S,T) is exact. In this case we have to set $\gamma := \min\{\gamma(S_0 + A), \gamma(T_0 + B)\}$, where S_0, T_0 are the operators defined in the proof of Theorem 9, and A, B are given by Lemma 5. However, in this general case the estimate is not so explicit in terms of S, T.

If $S \in \mathcal{C}(X,Y)$ with $\|S\| < \infty$, then let $\overline{S}$ denote the bounded extension of S to the closure of $D(S)$. Since S is closed, then $N(\overline{S}) = N(S)$, $R(\overline{S}) = R(S)$. Hence for $S \in \mathcal{C}(X,Y)$, $T \in \mathcal{C}(Y,X)$ with $\|S\|$, $\|T\| < \infty$, we have $(S,T) \in \mathcal{F}(X,Y) \Leftrightarrow (\overline{S},\overline{T}) \in \mathcal{F}(X,Y)$, in which case

$$\operatorname{ind}(\overline{S},\overline{T}) = \operatorname{ind}(S,T). \tag{11}$$

If $\widetilde{S} \in \mathcal{C}(X,Y)$ also is bounded, then

$$q(\overline{S},\overline{\widetilde{S}}) \leqslant q(S,\widetilde{S}), \tag{12}$$

which holds as follows: for any $\varepsilon > 0$, Q compact in Y and $\overline{x} \in D(\overline{S})_1$, we take $x \in D(S)_1$ with $\|\overline{x} - x\| < \varepsilon(1 + \|S\|^2)^{-1/2}$, and note that we have the inequality

$$\begin{aligned} d((\overline{x},\overline{S}\overline{x}), G(\overline{\widetilde{S}}) + \{0\} \times Q) &\leqslant d((x,Sx), G(\widetilde{S}) + \{0\} \times Q) + \|(\overline{x},\overline{S}\overline{x}) - (x,Sx)\| \\ &\leqslant \sup_{x' \in D(S)_1} d((x',Sx'), G(\widetilde{S}) + \{0\} \times Q) + \varepsilon, \end{aligned}$$

in which we take then successively the supremum over $\overline{x}$ and the infimum over Q and ε.

Following now a classical construction ([9], [24], see also [13]), we let $l^\infty(X)$ (resp. $\tau(X)$) denote the space of all bounded (resp. precompact) sequences $(x_n)_n$ in X, endowed with the sup-norm, $\|(x_n)_n\| := \sup\limits_n \|x_n\|$. Then $\tau(X) \in \mathcal{G}(l^\infty(X))$. Set $X^{\cdot} := l^\infty(X)/\tau(X)$. If $S \in \mathcal{C}(X,Y)$ with $D(S) \in \mathcal{G}(X)$ (and hence $\|S\| < \infty$), then we define $S^{\cdot}$ by $(x_n)_n + \tau(X) \mapsto (Sx_n)_n + \tau(Y)$, $x_n \in D(S)$. Thus by definition $D(S^{\cdot}) \subset X^{\cdot}$. Moreover, $\|S^{\cdot}\| \leqslant \|S\|$.

PROPOSITION 11. (i) *If $S \in \mathcal{C}(X,Y)$ and $D(S) \in \mathcal{G}(X)$, then $S^{\cdot} \in \mathcal{C}(X^{\cdot},Y^{\cdot})$ and $D(S^{\cdot}) \in \mathcal{G}(X^{\cdot})$.*

(ii) *If $\widetilde{S} \in \mathcal{C}(X,Y)$ also with $D(\widetilde{S}) \in \mathcal{G}(X)$, then $\delta(S^{\cdot},\widetilde{S}^{\cdot}) \leqslant 2\,q(S,\widetilde{S})$.*

Proof. (i) If $a : D(S) \to X$ is the inclusion, then $R(a^{\cdot}) = D(S^{\cdot})$. Let $\varepsilon > 0$ be arbitrary. Let $P : X \to D(S)$ be an ε-projection on $D(S)$ ([8]). Then P is continuous (not necessarily linear), and $\|x - P(x)\| \leqslant (1+\varepsilon)d(x, D(S))$ for all

$x \in X$. For any $\xi = (x_n)_n \in l^\infty(D(S))$ there exists $(c_n)_n \in \tau(X)$ such that $\|(x_n - c_n)_n\| < d(\xi, \tau(X)) + \varepsilon$. Then $(P(c_n))_n \in \tau(D(S))$ and

$$\|c_n - P(c_n)\| \leqslant (1+\varepsilon)d(c_n, D(S)) \leqslant (1+\varepsilon)\|x_n - c_n\|,\ n \geqslant 1.$$

Hence

$$\begin{aligned} d(\xi, \tau(D(S))) &\leqslant \|(x_n - P(c_n))_n\| \leqslant \|(x_n - c_n)_n\| + \|(c_n - P(c_n))_n\| \\ &\leqslant d(\xi, \tau(X)) + \varepsilon + (1+\varepsilon)\sup_n \|x_n - c_n\| \\ &\leqslant (2+\varepsilon)d(\xi, \tau(X)) + \varepsilon^2 + 2\varepsilon. \end{aligned}$$

Then for any $\xi + \tau(D(S)) \in D(a^\cdot)$ and $\varepsilon > 0$ we have

$$\|\xi + \tau(D(S))\| \leqslant (2+\varepsilon)\|a^\cdot(\xi + \tau(D(S)))\| + \varepsilon^2 + 2\varepsilon.$$

Hence $\gamma(a^\cdot) \geqslant 2^{-1}$, and so $R(a^\cdot)$ $(= D(S^\cdot))$ is closed. Since $\|S^\cdot\| < \infty$, then $S^\cdot \in \mathcal{C}(X^\cdot, Y^\cdot)$.

(ii) Let $\varepsilon > 0$ be arbitrary. Let $Q \subset Y$ be compact such that

$$\sup_{x \in D(S)_1} d((x, Sx), G(\widetilde{S}) + \{0\} \times Q) < q(S, \widetilde{S}) + \varepsilon. \tag{13}$$

Let $P : X \to D(S)$ be an ε-projection onto $D(S)$ ([8]). Let $u \in G(S^\cdot)_1$ be arbitrary, $u = (\xi + \tau(X), S^\cdot(\xi + \tau(X)))$ with $\xi = (x_n)_n$ and $x_n \in D(S)$, $n \geqslant 1$. Since $\|u\| \leqslant 1$, then there are $(c_n)_n \in \tau(X)$ and $(d_n)_n \in \tau(Y)$ such that

$$\sup_n \|x_n - c_n\|^2 + \sup_n \|Sx_n - d_n\|^2 < 1 + \varepsilon.$$

Hence $x := (2+\varepsilon)^{-1}(1+\varepsilon)^{-1}(x_n - P(c_n)) \in D(S)$, and $\|x\| \leqslant 1$ because

$$\begin{aligned} \|x\| = \|x_n - P(c_n)\| &\leqslant \|x_n - c_n\| + \|c_n - P(c_n)\| \\ &\leqslant \|x_n - c_n\| + (1+\varepsilon)d(c_n, D(S)) \\ &\leqslant (2+\varepsilon)\|x_n - c_n\| < (2+\varepsilon)(1+\varepsilon). \end{aligned}$$

Set $y'_n := (2+\varepsilon)(1+\varepsilon)y_n$. By (13), there are $\widetilde{x}_n \in D(\widetilde{S})$ and $y_n \in Q$ such that

$$\|(x_n - P(c_n) - \widetilde{x}_n, S(x_n - P(c_n)) - \widetilde{S}\widetilde{x}_n - y'_n)\| < (q(S, \widetilde{S}) + \varepsilon)(2+\varepsilon)(1+\varepsilon).$$

If $\widetilde{\xi} := (\widetilde{x}_n)_n$ and $\widetilde{u} := (\widetilde{\xi} + \tau(X), \widetilde{S}^\cdot(\widetilde{\xi} + \tau(X)))$, then $\widetilde{u} \in G(\widetilde{S}^\cdot)$. Note that $(P(c_n))_n \in \tau(X)$ and $(SP(c_n))_n$, $(y'_n)_n \in \tau(Y)$. Then the above estimate provides $d(u, G(\widetilde{S}^\cdot)) \leqslant \|u - \widetilde{u}\| \leqslant 2q(S, \widetilde{S}) + \mathrm{O}(\varepsilon)$. Since u is arbitrary, it follows $\delta(G(S^\cdot), G(\widetilde{S}^\cdot)) \leqslant 2q(S, \widetilde{S}) + \mathrm{O}(\varepsilon)$. If we let $\varepsilon \to 0$, then we obtain (ii). Proposition 11 is proved. ∎

From the previous proof, it follows that $S^{\cdot} \in \mathcal{C}(X^{\cdot}, Y^{\cdot})$ and $S^{\cdot} \in \mathcal{B}(D(S)^{\cdot}, Y^{\cdot})$ are similar via the obvious map

$$(14)\quad D(S)^{\cdot} = l^{\infty}(D(S))/\tau(D(S)) \to (l^{\infty}(D(S)) + \tau(X))/\tau(X) = D(S^{\cdot}) \subset X^{\cdot}.$$

If $\widetilde{S} \in \mathcal{C}(X, Y)$ also with $D(\widetilde{S}) \in \mathcal{G}(X)$, then both $S^{\cdot}$, $\widetilde{S}^{\cdot} \in \mathcal{C}(X^{\cdot}, Y^{\cdot})$ have the graphs in the same space $X^{\cdot} \times Y^{\cdot}$, which was not the case for $S^{\cdot} \in \mathcal{B}(D(S)^{\cdot}, Y^{\cdot})$, $\widetilde{S}^{\cdot} \in \mathcal{B}(D(\widetilde{S})^{\cdot}, Y^{\cdot})$.

THEOREM 12. *Let $(S, T) \in \mathcal{F}(X, Y)$ with $R(S) \subset N(T)$ and $R(T) \subset N(S)$. Then there exists $\varepsilon > 0$ such that for any $\widetilde{S} \in \mathcal{C}(X, Y)$, $\widetilde{T} \in \mathcal{C}(Y, X)$ with $R(\widetilde{S}) \subset N(\widetilde{T})$, $R(\widetilde{T}) \subset N(\widetilde{S})$ and $\widehat{q}(S, \widetilde{S})$, $\widehat{q}(T, \widetilde{T}) < \varepsilon$, it follows $(\widetilde{S}, \widetilde{T}) \in \mathcal{F}(X, Y)$ and* $\operatorname{ind}(\widetilde{S}, \widetilde{T}) = \operatorname{ind}(S, T)$.

Proof. The perturbation $(\widetilde{S}, \widetilde{T})$ of (S, T) remains Fredholm if ε is sufficiently small. Essentially, this was established in [13] (with respect to $\|\cdot\|_q$, for fixed domains). For the sake of completness, note some details in our context. By Lemma 3 (i) we have $\|\widetilde{S}\|$, $\|\widetilde{T}\| < \infty$, and so we may assume that all domains are closed (see (11), (12)). For any such pair $(\widetilde{S}, \widetilde{T})$ (as well as for $(\widetilde{S}, \widetilde{T}) := (S, T)$), we may consider $\widetilde{S} \in \mathcal{B}(D(\widetilde{S}), D(\widetilde{T}))$, $\widetilde{T} \in \mathcal{B}(D(\widetilde{T}), D(\widetilde{S}))$ with $\widetilde{S}\widetilde{T} = 0$, $\widetilde{T}\widetilde{S} = 0$. By [13], we have $(\widetilde{S}, \widetilde{T})$ Fredholm if and only if $(\widetilde{S}^{\cdot}, \widetilde{T}^{\cdot})$ is exact in $\mathcal{F}(D(\widetilde{S})^{\cdot}, D(\widetilde{T})^{\cdot})$ (i.e. $N(\widetilde{S}^{\cdot}) = R(\widetilde{T}^{\cdot})$ and $N(\widetilde{T}^{\cdot}) = R(\widetilde{S}^{\cdot})$). Via (14), this exactness is now equivalent with the exactness of $(\widetilde{S}^{\cdot}, \widetilde{T}^{\cdot})$ in $\mathcal{F}(X^{\cdot}, Y^{\cdot})$. But the exactness in $\mathcal{F}(X^{\cdot}, Y^{\cdot})$ is preserved under small perturbations with respect to $\widehat{\delta}$, by Proposition 2.10 from [2] (which deals with closed operators and may be applied because of Proposition 11 (i)). Thus we obtain the desired conclusion via Proposition 11 (ii). Now the invariance of the index holds by Theorem 9, and Theorem 12 is proved. ∎

For a family $a_i \in \mathcal{C}(X_i)$, $i \in I$ with $\sup_i \|a_i\| < \infty$, let X be the space of all $(x_i)_i$ with $\sum_i \|x_i\|^2 < \infty$ endowed with the ℓ^2-norm, and let $\bigoplus_i a_i \in \mathcal{C}(X)$ be defined by $(x_i)_i \mapsto (a_i x_i)_i$ for $x_i \in D(a_i)$. Then $\|\bigoplus_i a_i\| = \sup_i \|a_i\|$. If $\widetilde{a}_i \in \mathcal{C}(X_i)$ also with $\sup_i \|\widetilde{a}_i\| < \infty$, then

$$(15)\qquad \delta\Big(\bigoplus_i a_i, \bigoplus_i \widetilde{a}_i\Big) \leqslant \sup_i \delta(a_i, \widetilde{a}_i).$$

If in addition I is finite, then

$$(16)\qquad q\Big(\bigoplus_i a_i, \bigoplus_i \widetilde{a}_i\Big) \leqslant \max_i q(a_i, \widetilde{a}_i).$$

Let us prove (16) for instance. Let I be finite. Set $a := \bigoplus_i a_i$, $\widetilde{a} := \bigoplus_i \widetilde{a}_i$ and $q := \max_i q(a_i, \widetilde{a}_i)$. Let $\varepsilon > 0$ be arbitrary. Since $q(a_i, \widetilde{a}_i) < q + \varepsilon$, then there are some compact sets $Q_i \subset X_i$ such that $\sup_{x \in D(a_i)_1} d((x, a_i x), G(\widetilde{a}_i) + \{0\} \times Q_i) < q + \varepsilon$, $i \in I$. Let Q be the closed convex hull of $\{0\} \cup \prod_i Q_i$. Let $x = (x_i)_i \in D(a)$ be arbitrary with $\|x\| \leqslant 1$. The previous estimates provide some $\widetilde{x}_i \in D(\widetilde{a}_i)$ and $y_i \in Q_i$ such that $\|(x_i - \widetilde{x}_i, a_i x_i - \widetilde{a}_i \widetilde{x}_i - \|x_i\| y_i)\| \leqslant (q + \varepsilon)\|x_i\|$, $i \in I$. Set $\widetilde{x} := (\widetilde{x}_i)_i$ and $y := (\|x_i\| y_i)_i$. Since $\|x_i\| \leqslant 1$, then $y \in Q$, and by the above inequalities we have

$$
\begin{aligned}
d((x, ax), G(\widetilde{a}) + \{0\} \times Q) &\leqslant \|(x - \widetilde{x}, ax - \widetilde{a}\widetilde{x} - y)\| \\
&\leqslant \Big(\sum_i \|x_i - \widetilde{x}_i\|^2 + \sum_i \|a_i x_i - \widetilde{a}_i \widetilde{x}_i - \|x_i\| y_i\|^2 \Big)^{\frac{1}{2}} \\
&\leqslant (q + \varepsilon) \Big(\sum_i \|x_i\|^2 \Big)^{\frac{1}{2}} \leqslant q + \varepsilon.
\end{aligned}
$$

Now take the supremum over all $x \in D(a)_1$. Since Q is compact, then we obtain $q(a, \widetilde{a}) \leqslant q + \varepsilon$. Hence (16) is proved by letting $\varepsilon \to 0$. The estimate (15) holds similarly.

For various definitions of the notion of (Fredholm) complex, see [2], [7], [10], [12], [25]. We state here a particular case. A family $\alpha = (\alpha^i)_{i \in \mathbf{Z}}$ with $\alpha^i \in \mathcal{C}(X^i, X^{i+1})$ and $X^i \in \mathcal{G}(\mathcal{X})$, where $\mathcal{X}$ is a fixed Banach space, will be called a *complex* in $\mathcal{X}$ if $R(\alpha^{i-1}) \subset N(\alpha^i)$ for all i. Let $\partial(\mathcal{X})$ denote the set of all complexes in $\mathcal{X}$. Set $H^i(\alpha) := N(\alpha^i)/R(\alpha^{i-1})$. The complex α is called *Fredholm* if $\inf_i \gamma(\alpha^i) > 0$, $\sup_i \|\alpha^i\| < \infty$ and the function $i \mapsto \dim H^i(\alpha)$ is finite and has finite support. In this case, we define the *index* of α by $\operatorname{ind}\alpha = \sum_i (-1)^i \dim H^i(\alpha)$.

THEOREM 13. *If $\alpha = (\alpha^i)_i \in \partial(\mathcal{X})$ is a Fredholm complex, then there exists an $\varepsilon > 0$ such that for any complex $\widetilde{\alpha} = (\widetilde{\alpha}^i)_i \in \partial(\mathcal{X})$ with $\sup_i \widehat{q}(\alpha^i, \widetilde{\alpha}^i) < \varepsilon$ and $\overline{\lim}_{|i| \to \infty} \widehat{\delta}(\alpha^i, \widetilde{\alpha}^i) < \varepsilon$, it follows $\widetilde{\alpha}$ Fredholm and $\operatorname{ind}\widetilde{\alpha} = \operatorname{ind}\alpha$.*

Proof. For any complex $\widetilde{\alpha}$ (as well as for $\widetilde{\alpha} := \alpha$), we construct two closed operators $\widetilde{S}$, $\widetilde{T}$ (resp. S, T) in $\ell^2(\mathcal{X})$ as follows. Set

$$
\begin{aligned}
D(\widetilde{S}) &:= \{(x_i)_i \in \ell^2(\mathcal{X});\ x_{2k} \in D(\widetilde{\alpha}^{2k}),\ x_{2k+1} = 0\}, \\
D(\widetilde{T}) &:= \{(x_i)_i \in \ell^2(\mathcal{X});\ x_{2k} = 0,\ x_{2k+1} \in D(\widetilde{\alpha}^{2k+1})\}
\end{aligned}
$$

and define $\widetilde{S}$ (resp. $\widetilde{T}$) on the components by $x_{2k} \mapsto \widetilde{\alpha}^{2k} x_{2k}$ (resp. $x_{2k+1} \mapsto \widetilde{\alpha}^{2k+1} x_{2k+1}$). Then $\widetilde{\alpha}$ is Fredholm if and only if $(\widetilde{S}, \widetilde{T})$ is Fredholm, in which case

$\operatorname{ind}\widetilde{\alpha} = \operatorname{ind}(\widetilde{S}, \widetilde{T})$. Note that $R(\widetilde{S}) \subset N(\widetilde{T})$ and $R(\widetilde{T}) \subset N(\widetilde{S})$. This construction was stated in [1]. We omit the details.

Then (S,T) is a Fredholm pair, $\operatorname{ind}\alpha = \operatorname{ind}(S,T)$ and $R(S) \subset N(T)$, $R(T) \subset N(S)$. Let $\varepsilon = \varepsilon(S,T) > 0$ as stated in Theorem 12. Set $s := \sup_i \|\alpha^i\|$. Let $\varepsilon_1 > 0$ be such that $(1 + (2s+1)^2)^{1/2}\varepsilon_1 < \varepsilon$ and $(2s+2)\varepsilon_1 < 1$. Let $\widetilde{\alpha} \in \partial(\mathcal{X})$ such that $\sup_i \widehat{q}(\alpha^i, \widetilde{\alpha}^i) < \varepsilon$ and $\overline{\lim\limits_{|i|\to\infty}}\, \widehat{\delta}(\alpha^i, \widetilde{\alpha}^i) < \varepsilon_1$. Then there exists $i_0 \geqslant 1$ such that $\sup_{|k|>i_0} \widehat{\delta}(\alpha^{2k}, \widetilde{\alpha}^{2k}) < \varepsilon_1$. By (15), we have

$$\widehat{\delta}\Big(\bigoplus_{|k|>i_0} \alpha^{2k}, \bigoplus_{|k|>i_0} \widetilde{\alpha}^{2k}\Big) \leqslant \sup_{|k|>i_0} \widehat{\delta}(\alpha^{2k}, \widetilde{\alpha}^{2k}) < \varepsilon_1. \tag{17}$$

Since $\Big\| \bigoplus_{|k|>i_0} \alpha^{2k} \Big\| \leqslant s$ and $\varepsilon_1 < (2s+2)^{-1}$, then (17) shows that we can apply Proposition 1 (iii) for $\bigoplus_{|k|>i_0} \alpha^{2k}$ and $\bigoplus_{|k|>i_0} \widetilde{\alpha}^{2k}$. It follows $\Big\| \bigoplus_{|k|>i_0} \widetilde{\alpha}^{2k} \Big\| \leqslant 2s+1$, and (using again (17))

$$\begin{aligned} q &:= \widehat{q}\Big(\bigoplus_{|k|>i_0} \alpha^{2k}, \bigoplus_{|k|>i_0} \widetilde{\alpha}^{2k}\Big) \\ &\leqslant \left(1 + \max\left\{\Big\|\bigoplus_{|k|>i_0} \alpha^{2k}\Big\|^2, \Big\|\bigoplus_{|k|>i_0} \widetilde{\alpha}^{2k}\Big\|^2\right\}\right)^{\frac{1}{2}} \widehat{\delta}\Big(\bigoplus_{|k|>i_0} \alpha^{2k}, \bigoplus_{|k|>i_0} \widetilde{\alpha}^{2k}\Big) \\ &\leqslant (1 + (2s+1)^2)^{\frac{1}{2}}\varepsilon_1 < \varepsilon. \end{aligned} \tag{18}$$

By (16), we have

$$q' := \widehat{q}\Big(\bigoplus_{|k|\leqslant i_0} \alpha^{2k}, \bigoplus_{|k|\leqslant i_0} \widetilde{\alpha}^{2k}\Big) \leqslant \max_{|k|\leqslant i_0} \widehat{q}(\alpha^{2k}, \widetilde{\alpha}^{2k}) < \varepsilon \tag{19}$$

and $\widehat{q}(S, \widetilde{S}) \leqslant \max\{q, q'\}$. By (18) and (19), we obtain $\widehat{q}(S, \widetilde{S}) < \varepsilon$. Similar estimates hold for the odd indices, and we have $\widehat{q}(T, \widetilde{T}) < \varepsilon$ also. By Theorem 12 and the previous construction of $(\widetilde{S}, \widetilde{T})$, it follows $\widetilde{\alpha}$ Fredholm and $\operatorname{ind}\widetilde{\alpha} = \operatorname{ind}\alpha$. Theorem 13 is proved. ∎

COROLLARY 14. *Let X^i, $i = 0, \dots, n$ be Banach spaces, and let*

$$\alpha: \ 0 \to X^0 \xrightarrow{\alpha^0} X^1 \xrightarrow{\alpha^1} \cdots \xrightarrow{\alpha^{n-1}} X^n \to 0$$

be a Fredholm complex. Then there exists an $\varepsilon > 0$ such that for any complex

$$\widetilde{\alpha}: \ 0 \to X^0 \xrightarrow{\widetilde{\alpha}^0} X^1 \xrightarrow{\widetilde{\alpha}^1} \cdots \xrightarrow{\widetilde{\alpha}^{n-1}} X^n \to 0$$

with $\|\widetilde{\alpha}^i - \alpha^i\|_q < \varepsilon$, $i = 0, \dots, n$ *we have* $\widetilde{\alpha}$ *Fredholm and* $\operatorname{ind}\widetilde{\alpha} = \operatorname{ind}\alpha$.

COROLLARY 15. *If* $T = (T_1, \dots, T_n)$ *is a Fredholm commuting multioperator in* $\mathcal{B}(X)^n$, *then there exists an* $\varepsilon > 0$ *such that for any commuting multioperator* $\widetilde{T} = (\widetilde{T}_1, \dots, \widetilde{T}_n)$ *in* $\mathcal{B}(X)^n$ *with* $\|\widetilde{T}_j - T_j\|_q < \varepsilon$, $j = 1, \dots, n$ *it follows* $\widetilde{T}$ *Fredholm and* $\operatorname{ind}\widetilde{T} = \operatorname{ind}T$.

Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces. If X, $X_0 \in \mathcal{G}(\mathcal{X})$ and Y, $Y_0 \in \mathcal{G}(\mathcal{Y})$ with $X_0 \subset X$, $Y_0 \subset Y$, then for $S \in \mathcal{C}(X/X_0, Y/Y_0)$ we set ([2])

$$N_0(S) := \{x \in X; x + X_0 \in N(S)\},\ R_0(S) := \{y \in Y\,;\, y + Y_0 \in R(S)\},$$
$$G_0(S) := \{(x, y) \in X \times Y; (x + X_0, y + Y_0) \in G(S)\}.$$

Hence $N(S) = N_0(S)/X_0$ and $R(S) = R_0(S)/Y_0$. If in addition $\widetilde{S} \in \mathcal{C}(\widetilde{X}/\widetilde{X}_0, \widetilde{Y}/\widetilde{Y}_0)$ with $\widetilde{X}$, $\widetilde{X}_0 \in \mathcal{G}(\mathcal{X})$ and $\widetilde{Y}$, $\widetilde{Y}_0 \in \mathcal{G}(\mathcal{Y})$, then we set $\widehat{\delta}_0(\widetilde{S}, S) := \widehat{\delta}(G_0(\widetilde{S}), G_0(S))$ ([2]).

Now let $S \in \mathcal{C}(X/X_0, Y/Y_0)$ and $T \in \mathcal{C}(Y'/Y_0', X'/X_0')$, where X, X', X_0, $X_0' \in \mathcal{G}(\mathcal{X})$ and Y, Y', Y_0, $Y_0' \in \mathcal{G}(\mathcal{Y})$. Set $a_0 = a_0(S, T) := \dim N_0(S)/N_0(S) \cap R_0(T)$, $b_0 = b_0(S, T) := \dim R_0(T)/R_0(T) \cap N_0(S)$, $c_0 := a_0(T, S)$ and $d_0 := b_0(T, S)$. With this notation, we state the following.

DEFINITION 16. Let $S \in \mathcal{C}(X/X_0, Y/Y_0)$ and $T \in \mathcal{C}(Y'/Y_0', X'/X_0')$. Then (S, T) is called a *Fredholm pair* if a_0, b_0, c_0, $d_0 < \infty$. In this case, we define the *index* of (S, T) by $\operatorname{ind}(S, T) = a_0 - b_0 - c_0 + d_0$.

Let $\mathcal{F}(\mathcal{X}, \mathcal{Y})$ denote the set of all such Fredholm pairs. Definition 16 generalizes the previously defined notion of Fredholm pair, since for $X' = X$ etc. we have $a_0(S, T) = a(S, T)$ etc. The results of stability of the index under small perturbations, concerning Fredholm complexes of closed operators between quotient Banach spaces ([1], [2]) are generalized by Theorem 17. This holds via the construction from [1] briefly stated in the proof of Theorem 13.

THEOREM 17. *If* $(S, T) \in \mathcal{F}(\mathcal{X}, \mathcal{Y})$, *then there exists an* $\varepsilon > 0$ *such that for any* $(\widetilde{S}, \widetilde{T}) \in \mathcal{F}(\mathcal{X}, \mathcal{Y})$ *with* $\widehat{\delta}_0(\widetilde{S}, S)$, $\widehat{\delta}_0(\widetilde{T}, T) < \varepsilon$, *we have* $\operatorname{ind}(\widetilde{S}, \widetilde{T}) = \operatorname{ind}(S, T)$.

Proof. For any $(\widetilde{S}, \widetilde{T})$ (as well as for $(\widetilde{S}, \widetilde{T}) := (S, T)$), we define the contractions $\widetilde{S}_1$ and $\widetilde{T}_1$ by $D(\widetilde{S}_1) := G_0(\widetilde{S})$, resp. $D(\widetilde{T}_1) := \{(\widetilde{x}, \widetilde{y})\,;\, (\widetilde{y}, \widetilde{x}) \in G_0(\widetilde{T})\}$ and $\widetilde{S}_1(\widetilde{x}, \widetilde{y}) := (0, \widetilde{y})$, resp. $\widetilde{T}_1(\widetilde{x}, \widetilde{y}) := (\widetilde{x}, 0)$. Since $N(\widetilde{S}_1) = N_0(\widetilde{S}) \times \{0\}$ etc., then $(\widetilde{S}_1, \widetilde{T}_1)$ (as well as (S_1, T_1)) is a Fredholm pair and

$$\operatorname{ind}(\widetilde{S}_1, \widetilde{T}_1) = \operatorname{ind}(\widetilde{S}, \widetilde{T}),\ \operatorname{ind}(S_1, T_1) = \operatorname{ind}(S, T). \tag{20}$$

Set $\delta_t := \delta(G_0(\widetilde{S}), G_0(S)) + t$ for $t > 0$. Let $\widetilde{g} := ((\widetilde{x}, \widetilde{y}), (0, \widetilde{y})) \in G(\widetilde{S}_1)_1$ be arbitrary. Then for $(\widetilde{x}, \widetilde{y}) \in G_0(\widetilde{S})_1$ there exists $(x, y) \in G_0(S)$ such that $\|(x - \widetilde{x}, y - \widetilde{y})\| < \delta_t$. Hence $((x, y), (0, y)) \in G(S_1)$ and $d(\widetilde{g}, G(S_1)) \leqslant \|\widetilde{g} - g\| < 2^{1/2}\delta_t$. We take the supremum over $\widetilde{g}$ and let $t \to 0$. Hence $\delta(\widetilde{S}_1, S_1) \leqslant 2^{1/2}\delta_0$. By similar computations, we finally obtain

$$\widehat{\delta}(\widetilde{S}_1, S_1) \leqslant 2^{\frac{1}{2}}\widehat{\delta}_0(\widetilde{S}, S),\ \widehat{\delta}(\widetilde{T}_1, T_1) \leqslant 2^{\frac{1}{2}}\widehat{\delta}_0(\widetilde{T}, T). \tag{21}$$

By Proposition 1 (iii), we have also

$$\widehat{q}(\widetilde{S}_1, S_1) \leqslant 2^{\frac{1}{2}}\widehat{\delta}(\widetilde{S}_1, S_1),\ \widehat{q}(\widetilde{T}_1, T_1) \leqslant 2^{\frac{1}{2}}\widehat{\delta}(\widetilde{T}_1, T_1). \tag{22}$$

Now let $\varepsilon = \varepsilon(S_1, T_1) > 0$ be as in Theorem 9. Let $(\widetilde{S}, \widetilde{T}) \in \mathcal{F}(\mathcal{X}, \mathcal{Y})$ such that $\widehat{\delta}_0(\widetilde{S}, S)$, $\widehat{\delta}_0(\widetilde{T}, T) < \varepsilon/2$. By (21), (22) and Theorem 9, we obtain $\operatorname{ind}(\widetilde{S}_1, \widetilde{T}_1) = \operatorname{ind}(S_1, T_1)$. Via (20), this implies $\operatorname{ind}(\widetilde{S}, \widetilde{T}) = \operatorname{ind}(S, T)$. Theorem 17 is proved. ∎

PROPOSITION 18. *Let* $(S, T) \in \mathcal{F}(\mathcal{X}, \mathcal{Y})$ *as in Definition* 16. *Set* $\gamma := \min\{\gamma(S), \gamma(T)\}$ *and* $\lambda := \max\{1, 11(1 + \gamma^{-2})^{1/2}\}$. *For* $(\widetilde{S}, \widetilde{T}) \in \mathcal{F}(\mathcal{X}, \mathcal{Y})$, *let* $\varepsilon := \max\{\widehat{\delta}_0(S, \widetilde{S}), \widehat{\delta}_0(T, \widetilde{T})\}$. *If the pair* (S, T) *is exact and* $6\lambda(\lambda^2 + 5)^{1/2}\varepsilon < 10^{-2}$, *then* $\operatorname{ind}(\widetilde{S}, \widetilde{T}) = \operatorname{ind}(S, T)$ $(= 0)$.

Proof. We apply Proposition 10 for the Fredholm pairs (S_1, T_1) and $(\widetilde{S}_1, \widetilde{T}_1)$ defined in the proof of Theorem 17. Then use the estimates (21) and (22). Note that S_1 etc. are contractions. We need only to estimate $\gamma(S_1)$, $\gamma(T_1)$ in terms of $\gamma(S)$, $\gamma(T)$. For instance, we have $\gamma(S_1) \geqslant (1 + \gamma(S)^{-2})^{-1/2}$, which holds as follows. Let $\varepsilon > 0$ and $(0, y) \in R(S_1)$ be arbitrary. Then $(0, y) = S_1(x, y)$ with $y + Y_0 = S(x + X_0)$. There are $x' \in N_0(S)$ and $x_0 \in X_0$ such that

$$\|x - x' + X_0\| < d(x + X_0, N(S)) + \varepsilon$$

and

$$\|x - x' - x_0\| < \|x - x' + X_0\| + \varepsilon.$$

Then $x' + x_0 \in N_0(S)$, and so $(x' + x_0, 0) \in N(S_1)$. Since

$$\gamma(S)d(x + X_0, N(S)) \leqslant \|S(x + X_0)\| \leqslant \|y\|,$$

it follows

$$\begin{aligned} d((x, y), N(S_1)) &\leqslant \|(x, y) - (x' + x_0, 0)\| \leqslant (\|x - x' - x_0\|^2 + \|y\|^2)^{\frac{1}{2}} \\ &\leqslant (1 + \gamma(S)^{-2})^{\frac{1}{2}}\|(0, y)\| + \mathrm{O}(\varepsilon), \end{aligned}$$

etc. We omit the details. ∎

Theorems 12 and 13 deal with pairs (S,T) with $ST = 0$, $TS = 0$ (or with complexes $(\alpha^i)_i$, $\alpha^i\alpha^{i-1} = 0$)). In these cases, the results from [1], [2] on the $\widehat{\delta}$-invariance of the exactness provide a better approach to quantitative estimates on ε. Namely, we apply the functor $S \mapsto S^{\cdot}$, $T \mapsto T^{\cdot}$, and then proceed as indicated in the proof of Theorem 12.

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