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INVERSE PROBLEM FOR A SMOOTH STRING WITH DAMPING AT ONE END

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ABSTRACT. Direct and inverse spectral problems for a smooth string with massive end (with concentrated mass at the end) moving with damping is considered. By means of Liouville transformation the problem has been reduced to the Sturm-Liouville equation with boundary conditions depending on the spectral parameter. So the V.A. Marchenko formalism proves to be a powerful tool of investigation. It is shown in this paper that the spectrum of vibrations and the length of the string are sufficient for finding all the parameters of the problem.

KEYWORDS: *Inverse problem, Sturm-Liouville equation.*

AMS SUBJECT CLASSIFICATION: Primary 34A55; Secondary 34B24, 34L20.

1. INTRODUCTION

In many areas of physics it is of interest to determine the characteristics of a sound source from frequencies of vibrations. In case when the source is a smooth string of finite length, this inverse problem may be reduced by means of Liouville transformation ([3]) to an inverse problem for the Sturm-Liouville equation with boundary conditions depending on the spectral parameter. Thus we may use the results of V.A. Marchenko ([17]). The input data necessary for this investigation are all the values of frequencies of small transversal vibrations. These frequencies are complex numbers due to absorption. We consider a nonhomogeneous string the left end of which is fixed and the right one equipped with a massive ring moving with damping in the direction orthogonal to the length of the string. The problem in linear approach has the following form

$$\frac{\partial^2 v}{\partial s^2} - B(s)\frac{\partial^2 v}{\partial t^2} = 0, \tag{1.1}$$

$$v(0,t) = 0, \tag{1.2}$$

$$\left(\frac{\partial v}{\partial s} + \nu\frac{\partial v}{\partial t} + M\frac{\partial^2 v}{\partial t^2}\right)\Big|_{s=l} = 0. \tag{1.3}$$

Here $B(s) > 0$ is the density of the string, $v(s,t)$ the transversal displacement, ν the coefficient of damping, M the mass of the ring, l the length of the string, s the coordinate along the longitudinal axis of the string, t the time. A similar problem has been considered in [1] for a class of strings much wider than that of ours (so called regular strings ([8]), i.e. strings of finite mass and length but possibly having no density in usual meaning). The boundary condition at the left end in [1] was of the form $\frac{\partial v}{\partial s}\big|_{s=0} = 0$. It was proved in [1] that a sequence of complex numbers located in the upper half-plane and symmetric with respect to the imaginary axis coincides with the spectrum of a regular string if and only if this sequence coincides with the set of zeroes of an integer function of exponential type satisfying some integral conditions. In [11] and [12] by extension of the class of strings (the class includes infinite strings and strings of infinite mass) an explicit description of spectra was obtained. We consider a much narrower class of strings (smooth strings) and obtain explicit description of the spectra. The method of construction of the parameters $B(s), \nu, M$ from the length l of the string and the spectrum Λ of the problem has been obtained. We have to remark that it is impossible to find all the parameters $B(s), \nu, M, l$ from the spectrum because the problem (1.1)–(1.3) is invariant under the transformation $s' = rs, l' = rl, B'(s') = r^{-2}B(s), \nu' = r^{-1}\nu, M' = r^{-1}M$, where r is an arbitrary positive number. But knowing the spectrum and one of the parameters, for example l, it is possible to find a unique set $\{B(s), \nu, M\}$. Substituting $v(s,t) = u(\lambda, s)\mathrm{e}^{\mathrm{i}\lambda t}$ into (1.1)–(1.3) we obtain

$$u''_{ss} + \lambda^2 B(s)u = 0, \tag{1.4}$$

$$u(\lambda, 0) = 0, \tag{1.5}$$

$$u'_s(\lambda, l) + (\mathrm{i}\nu\lambda - \lambda^2 M)u(\lambda, l) = 0. \tag{1.6}$$

Let us assume that $B(s) > 0$ for all $s \in [0, l]$ and $B(s) \in W_2^2(0, l)$. Then the Liouville transform

$$x(s) = \int_0^s B^{\frac{1}{2}}(s')\,\mathrm{d}s', \tag{1.7}$$

$$y(\lambda, x) = B^{\frac{1}{4}}(s(x))\,u(\lambda, s(x)) \tag{1.8}$$

makes it possible to reduce the problem (1.4)–(1.6) to the following one

$$y''_{xx} + \lambda^2 y - q(x)y = 0, \tag{1.9}$$

$$y(\lambda, 0) = 0, \tag{1.10}$$

$$y'_x(\lambda, a) + (-\lambda^2 m + \mathrm{i}\alpha\lambda + \beta)y(\lambda, a) = 0, \tag{1.11}$$

where $B(x) = B(s(x))$,

$$q(x) = B^{-\frac{1}{4}}(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2}B^{\frac{1}{4}}(x), \tag{1.12}$$

$$m = B^{-\frac{1}{2}}(a)M > 0, \tag{1.13}$$

$$\alpha = B^{-\frac{1}{2}}(a)\nu > 0, \tag{1.14}$$

$$\beta = -\frac{1}{4}B^{-1}(a)\frac{\mathrm{d}B(x)}{\mathrm{d}x}\bigg|_{x=a}, \tag{1.15}$$

$$a = \int_0^l B^{\frac{1}{2}}(s)\,\mathrm{d}s > 0. \tag{1.16}$$

The spectrum of (1.9)–(1.11) consists of normal ([6]) eigenvalues accumulating to infinity. The main results of the present paper are stated in Section 2. Section 3 contains the proofs and the description of the inversion procedure. It is necessary to remark that we prove here uniqueness of the solution of the inverse problem in the class of smooth strings with positive density. Using the results of [1] and

[2] it is possible to prove uniqueness of the solution also in the class of regular strings. The problem (1.9)–(1.11) with $M = 0$ has been considered in, [4], [24], [25] and [7]. In [4], for the case of $B(s) = \text{const} \neq 1$, it was shown that the eigenvectors form a Riesz basis of the corresponding double space. For the case where $B(s) \neq \text{const}$ and $B(l) \neq 1$ the precise asymptotics of eigenvalues were derived also. Under some additional conditions it was shown in [4] that $\inf_k \operatorname{Im} \lambda_k > 0$. The asymptotics were obtained in [24] for a more general class of $B(s)$. In [25] the basic properties of eigen- and associated functions were considered for $B(s) \neq \text{const}$. The inverse problem for this case ($M = 0$) was considered in [7]. Another approach to inverse problems with boundary conditions depending on the spectral parameter was developed in [23], [22], [16], [10] and [26].

2. THE MAIN RESULTS

The results consist of two theorems.

THEOREM 2.1. *If $B(s) \in W_2^4(0,l), B(s) > 0, M > 0, \nu > 0$, then the spectrum $\Lambda = \{\lambda_n\}$ of the problem* (1.4)–(1.6) *satisfies the following conditions:*

(i) $\operatorname{Im} \lambda_n > 0$,

(ii) *the number of pure imaginary λ_n is even,*

(iii) *the set Λ is symmetric with respect to the imaginary axis and the multiplicities of the symmetric points coincide,*

(iv) *being enumerated in the proper way (see below) the spectrum admits the following asymptotics*

$$\lambda_n \underset{n\to+\infty}{=} \frac{\pi n}{a} + \frac{P_1}{n} + \frac{\mathrm{i}P_2}{n^2} + \frac{P_3}{n^3} + \frac{b_n}{n^3}, \tag{2.1}$$

where $a > 0$ is defined by (1.16), *$P_1 \in \mathbb{R}, P_2 > 0, P_3 \in \mathbb{R}, \{b_n\}_1^\infty \in l_2$.*

An enumeration is called *proper* if: (1) the multiplicities are taken into account, (2) for complex eigenvalues $\lambda_{-n} = -\overline{\lambda}_n$, (3) $\operatorname{Re} \lambda_{n+1} \geqslant \operatorname{Re} \lambda_n$ and (4) there are two points of zero index (λ_{+0} and λ_{-0}).

Denote by $\mathcal{B}_l$ the class of sets $\{B(s), M, \nu\}$ satisfying the following conditions: $B(s) \in W_2^2(0,l), B(s) > 0$ for all $s \in [0,l], M > 0, \nu > 0$.

THEOREM 2.2. *If $\Lambda = \lambda_n$ satisfies the conditions* (1)–(4), *then for arbitrary $l > 0$ there exists a unique set $\{B(s), M, \nu\} \in \mathcal{B}_l$ such that the spectrum of the problem* (1.4)–(1.6) *coincides with Λ.*

For the proofs of the theorems see below (Section 3), where we describe also the algorithm of inversion, i.e. of constructing the set $\{B(s), M, \nu\}$.

3. THE PROOFS AND THE INVERSION FORMALISM

3.1. THE PROOF OF THEOREM 2.1. If λ_n is an eigenvalue of (1.4)–(1.6) and $u(\lambda_n, s)$ is the corresponding eigenfunction, then (1.4) implies

$$\int_0^l u''_{ss}\overline{u}\,\mathrm{d}s + \lambda_n^2 \int_0^l B(s)|u|^2\,\mathrm{d}s = 0. \tag{3.1}$$

Taking to account (1.5) and (1.6) we obtain

$$-\mathrm{i}\lambda_n\nu|u(l)|^2 - \int_0^l |u'_s|^2\,\mathrm{d}s + \lambda_n^2 \int_0^l B(s)|u|^2\,\mathrm{d}s + M\lambda_n^2|u(l)|^2 = 0. \tag{3.2}$$

The imaginary part of this equation is

$$-\mathrm{Re}\,\lambda_n|u(l)|^2 + 2\mathrm{Re}\,\lambda_n\,\mathrm{Im}\,\lambda_n\left(\int_0^l B(s)|u|^2\,\mathrm{d}s + M|u(l)|^2\right) = 0.$$

If $u(l) = 0$, then due to (1.6) $u'_s(l) = 0$ what is impossible ([8], p. 658). So either $\mathrm{Re}\,\lambda_n = 0$ or $\mathrm{Im}\,\lambda_n > 0$. Let $\mathrm{Re}\,\lambda_n = 0$, then taking the real part of (3.2) we obtain

$$\nu\mathrm{Im}\,\lambda_n|u(l)|^2 - \int_0^l |u'_s|^2\,\mathrm{d}s - (\mathrm{Im}\,\lambda_n)^2\left(\int_0^l B(s)|u|^2\,\mathrm{d}s + M|u(l)|^2\right) = 0$$

and thus $\mathrm{Im}\,\lambda_n > 0$. So we have proved assertion (i) of the Theorem 2.1. The assertion (iii) follows from symmetry of the problem $\left(u(-\overline{\lambda}_n, s) = \overline{u(\lambda_n, s)}\right)$.

In what follows we use the equivalence of the spectra of problems (1.4)–(1.6) and (1.9)–(1.11). Denote by $S(\lambda, x)$ the solution of (1.9) satisfying the conditions

$$S(\lambda, 0) = S'_x(\lambda, 0) - 1 = 0.$$

This solution has the form ([17], p. 23)

$$S(\lambda, x) = \frac{\sin\lambda x}{\lambda} + \int_0^x K(x,t)\frac{\sin\lambda t}{\lambda}\,\mathrm{d}t, \tag{3.3}$$

where $K(x,t) = \tilde{K}(x,t) - \tilde{K}(x,-t)$ and $\tilde{K}(x,t)$ is the solution of the integral equation

$$\tilde{K}(x,t) = \frac{1}{2}\int_0^{\frac{x+t}{2}} q(s)\,\mathrm{d}s + \int_0^{\frac{x+t}{2}} \mathrm{d}s \int_0^{\frac{x-t}{2}} q(s+p)\tilde{K}(s+p, s-p)\,\mathrm{d}p,$$

and $\tilde{K}(x,t) \equiv 0$ for $|t| > |x|$. Hence $K(x,0) = 0$. The eigenvalues of (1.9)–(1.11) coincide with the zeroes of the characteristic function

$$\tilde{\chi}(\lambda) = s'(\lambda, a) + (-\lambda^2 m + \mathrm{i}\alpha\lambda + \beta) S(\lambda, a). \tag{3.4}$$

As $B(s) \in W_2^4(0,l)$ and $B(s) > 0$ we obtain from (1.12) that $q(x) \in W_2^2(0,a)$. So according to p. 23 in [17] there exist all partial derivatives of $K(x,t)$ of the third order and these derivatives (among them $K''_{xt}(a,t)$ and $K'''_{ttt}(a,t)$) belong to $L_2(0,a)$. Substituting (3.3) into (3.4) and integrating by parts we obtain

$$\begin{aligned}\tilde{\chi}(\lambda) = {} & -\lambda m \sin\lambda a + \mathrm{i}\alpha \sin\lambda a + (1 + mK(a,a))\cos\lambda a \\ & + (\beta + K(a,a) - mK'_t(a,a))\frac{\sin\lambda a}{\lambda} - \mathrm{i}\alpha K(a,a)\frac{\cos\lambda a}{\lambda} \\ & - (\beta K(a,a) + K'_x(a,a) + mK''_{tt}(a,a))\frac{\cos\lambda a}{\lambda^2} \\ & + \frac{\mathrm{i}\alpha K'_t(a,a)\sin\lambda a}{\lambda^2} + \beta K'_t(a,a)\frac{\sin\lambda a}{\lambda^3} + \mathrm{i}\alpha K''_{tt}(a,a)\frac{\cos\lambda a}{\lambda^3} \\ & + \beta K''_{tt}(a,a)\frac{\cos\lambda a}{\lambda^4} + \left(m - \frac{\mathrm{i}\alpha}{\lambda} - \frac{\beta}{\lambda^2}\right)\int_0^a K'''_{ttt}(a,t)\frac{\cos\lambda t}{\lambda^2}\,\mathrm{d}t \\ & + \int_0^a K''_{xt}(a,t)\frac{\cos\lambda t}{\lambda^2}\,\mathrm{d}t.\end{aligned} \tag{3.5}$$

Using (3.5) we obtain

$$\tilde{\chi}(\mathrm{i}\tau) \underset{\tau\to+\infty}{=} \frac{1}{2}\tau m \mathrm{e}^{\tau a} + \mathrm{O}(\mathrm{e}^{\tau a}) \tag{3.6}$$

$$\tilde{\chi}(-\mathrm{i}\tau) \underset{\tau\to+\infty}{=} -\frac{1}{2}\tau m \mathrm{e}^{\tau a} + \mathrm{O}(\mathrm{e}^{\tau a}). \tag{3.7}$$

Now we prove the assertion (ii). If we insert $\alpha = 0$ into (1.11) then the total algebraic multiplicity of the pure imaginary spectrum of (1.9)–(1.11) is even because the spectral parameter is present only in the second power. The eigenvalues, i.e. the zeroes of $\tilde{\chi}(\lambda)$, being piecewise analytic and continuous functions of α (see for example [18]), cannot come from $\pm\mathrm{i}\infty$ because of asymptotics (3.6), (3.7). Thus, due to the symmetry of Λ with respect to the imaginary axis, the eigenvalues occur in pairs. The assertion (ii) follows. Thus it is possible to enumerate the eigenvalues as above. To prove assertion (iv) we need the following lemma.

LEMMA 3.1.1. *Let $\varphi(\lambda)$ be an entire function of the form*

$$\varphi(\lambda) = \sin\lambda a + \mathrm{i}A\frac{\sin\lambda a}{\lambda} + B\frac{\cos\lambda a}{\lambda} + D\frac{\sin\lambda a}{\lambda^2} + \mathrm{i}E\frac{\cos\lambda a}{\lambda^2} + \mathrm{i}F\frac{\sin\lambda a}{\lambda^3} + G\frac{\cos\lambda a}{\lambda^3} + \frac{\psi(\lambda)}{\lambda^3},$$

where A, B, D, E, F, G are real constants, $\psi(\lambda)$ is an entire function of exponential type $\leqslant a$ belonging to $L_2(-\infty,\infty)$ such that $\psi(-\lambda) = \overline{\psi(\lambda)}$, $\psi(0) = -G$, $\psi'(0) = -\mathrm{i}(E + Fa)$, $\psi''(0) = a^2G - 2aD - 2B$. Then the zeroes λ_n of $\varphi(\lambda)$ satisfy the following formula (for all $n \neq 0$)

$$\lambda_n = \frac{\pi n}{a} + \frac{P_1}{n} + \frac{\mathrm{i}P_2}{n^2} + \frac{P_3}{n^3} + \frac{b_n}{n^3}, \tag{3.8}$$

where

$$P_1 = -\frac{B}{\pi}, \quad P_2 = -\frac{a}{\pi^2}(E - AB), \tag{3.9}$$

$$P_3 = -\frac{a^2}{\pi^3}\left(BD - AE - G + B\left(A^2 + \frac{B^2}{3}\right) - \frac{B^2}{a^2}\right), \tag{3.10}$$

and $\{b_n\}_{-\infty}^{\infty} \in l_2$.

Proof. Due to p. 225 in [17], $\lambda_n = \frac{\pi n}{a} + \Delta_n$, where $\Delta_n = O(1)$. Then from $\varphi(\lambda_n) = 0$ we obtain

$$2\mathrm{i}\Delta_n a = \ln\left(\frac{\mathrm{i}\psi(\lambda_n)}{\lambda_n^3} + \left(\frac{-\psi^2(\lambda_n)}{\lambda_n^6} + \left(1 + \frac{\mathrm{i}(A-B)}{\lambda_n} + \frac{D+E}{\lambda_n^2} + \frac{\mathrm{i}(F-G)}{\lambda_n^3}\right)\right.\right.$$
$$\left.\left.\cdot\left(1 + \frac{\mathrm{i}(A+B)}{\lambda_n} + \frac{D-E}{\lambda_n^2} + \frac{\mathrm{i}(F+G)}{\lambda_n^3}\right)\right)^{\frac{1}{2}}\right)^2$$
$$- 2\ln\left(1 + \frac{\mathrm{i}(A+B)}{\lambda_n} + \frac{D-E}{\lambda_n^2} + \frac{\mathrm{i}(F+G)}{\lambda_n^3}\right).$$

We obtain the assertion of the Lemma 3.1.1 by expressing the right part into power series.

The assertion (iv) of Theorem 2.1 is a consequence of Lemma 3.2.1. ∎

3.2. THE PROOF OF THEOREM 2.2 AND THE ALGORITHM OF INVERSION. Let $\Lambda = \{\lambda_n\}$ satisfy the conditions of Theorem 2.2. Let us construct the set $\{q(x), a, \alpha, \beta, m\}$. There exists the limit

$$a = \lim_{n\to\infty}\frac{\pi n}{\lambda_n}. \tag{3.11}$$

Let us construct

$$\chi(\lambda)=\lim_{n\to\infty}\prod_{0}^{n}\left(1-\frac{\lambda}{\lambda_k}\right)\prod_{-n}^{-0}\left(1-\frac{\lambda}{\lambda_k}\right). \tag{3.12}$$

LEMMA 3.2.1. *The function $\chi(\lambda)$, where the set $\Lambda=\{\lambda_n\}$ satisfies the conditions of Theorem 2.2 may be expressed in the following form*

$$\begin{aligned}\chi(\lambda)&=-\lambda K_1\sin\lambda a+\mathrm{i}K_2\sin\lambda a+K_3\cos\lambda a\\&+K_4\frac{\sin\lambda a}{\lambda}+\mathrm{i}K_5\frac{\cos\lambda a}{\lambda}+\mathrm{i}K_6\frac{\sin\lambda a}{\lambda^2}+K_7\frac{\cos\lambda a}{\lambda^2}+\frac{\psi_1(\lambda)}{\lambda^2},\end{aligned} \tag{3.13}$$

where $\psi_1(\lambda)$ is an entire function of exponential type $\leqslant a$ belonging to $L_2(-\infty,\infty)$, $K_1\neq 0$, $K_i\in\mathbb{R}$ $(i=\overline{1,7})$.

Proof. Consider the auxiliary entire function

$$\varphi_1(\lambda)=-\sin\lambda a+\mathrm{i}A_1\frac{\sin\lambda a}{\lambda}+B_1\frac{\cos\lambda a}{\lambda}+D_1\frac{\sin\lambda a}{\lambda^2}+\mathrm{i}E_1\frac{\cos\lambda a}{\lambda^2}+\mathrm{i}F\frac{\sin\lambda a}{\lambda^3}, \tag{3.14}$$

where

$$A_1=\frac{\pi P_3}{aP_2}+\frac{3P_1^2}{P_2}+\frac{\pi aP_1^3}{3P_2},\quad B_1=\pi P_1,$$

$$D_1=\frac{2\pi P_1}{a},\quad E_1=\frac{\pi^2P_2}{a}+A_1B_1,\quad F_1=\frac{E_1}{a}.$$

According to Lemma 3.1.1 the zeroes $\lambda_n^{(1)}$ of the function $\varphi_1(\lambda)$ satisfy the following asymptotics

$$\lambda_n^{(1)}=\frac{\pi n}{a}+\frac{P_1}{n}+\frac{\mathrm{i}P_2}{n^2}+\frac{P_3}{n^3}+\frac{b_n^{(1)}}{n^3},\quad (n\neq 0),$$

where $\{b_n^{(1)}\}\in l_2$. The function $\chi(\lambda)\left(1-\frac{\lambda}{\lambda_{+0}}\right)^{-1}$ is an entire function of sinus-type ([13]) and

$$\lambda_n=\lambda_n^{(1)}+\tilde{b}_n\left(\lambda_n^{(1)}\right)^{-3},$$

where $\{\tilde{b}_n\}\in l_2$. Using Lemma 5 of [13] we obtain

$$\chi(\lambda)\left(1-\frac{\lambda}{\lambda_{+0}}\right)=C_0\varphi_1(\lambda)\left(1+\frac{T_1}{\lambda}+\frac{T_2}{\lambda^2}+\frac{T_3}{\lambda^3}\right)+\frac{\tilde{\varphi}(\lambda)}{\lambda^3}, \tag{3.15}$$

where $\tilde{\varphi}(\lambda)$ is entire function of exponential type $\leqslant a$ and

$$C_0=\prod_{-\infty}^{\infty}\left(1+\tilde{b}_n\left(\lambda_n^{(1)}\right)^{-4}\right)^{-1}.$$

Substituting (3.14) into (3.15) we obtain (3.13), where $K_1=-C_0\lambda_{+0}^{-1}\neq 0$. All the constants K_j are real due to the symmetry of the problem. The Lemma 3.2.1 is proved. ∎

LEMMA 3.2.2. *The inequality*

$$\frac{K_2}{K_1} > 0$$

holds.

Proof. Consider the function

$$\chi(\lambda)K_1^{-1} = K_1^{-1} \lim_{n\to\infty} \prod_{+0}^{n} \left(1 - \frac{\lambda}{\lambda_k}\right) \prod_{-n}^{-0} \left(1 - \frac{\lambda}{\lambda_k}\right) \tag{3.16}$$

as a perturbation of the function

$$\chi^{(0)}(\lambda) \stackrel{\text{def}}{=} -\lambda \sin \lambda a = -a\lambda^2 \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2 a^2}{\pi^2 n^2}\right).$$

Then

$$\begin{aligned}
\chi(\lambda)K_1^{-1} &= -\chi^{(0)}(\lambda)a^{-1}\lambda^{-2} \prod_{n=1}^{\infty} \left(1 - \frac{\lambda^2 a^2}{\pi^2 n^2}\right)^{-1} \\
&\quad \cdot \lim_{n\to\infty} \prod_{k=+0}^{n} \left(1 - \frac{\lambda}{\lambda_k}\right) \prod_{k=n}^{-0} \left(1 - \frac{\lambda}{\lambda_k}\right) \\
&= -a^{-1}\chi^{(0)}(\lambda) \left(\lambda^{-2} + \lambda_{+0}^{-1}\lambda_{-0}^{-1} - i\lambda^{-1}\lambda_{+0}^{-1}\lambda_{-0}^{-1}(\operatorname{Im}\lambda_{+0} + \operatorname{Im}\lambda_{-0})\right) \\
&\quad \cdot \prod_{k=1}^{\infty} \left(\lambda^{-2} + \lambda_{+k}^{-1}\lambda_{-k}^{-1} - i\lambda^{-1}\lambda_{+k}^{-1}\lambda_{-k}^{-1}(\operatorname{Im}\lambda_{+k} + \operatorname{Im}\lambda_{-k})\right) \\
&\quad \cdot (\lambda^{-2} - a^2\pi^{-2}k^{-2})^{-1}.
\end{aligned}$$

Substituting $\lambda = \frac{1}{a}\left(2\pi n + \frac{\pi}{2}\right)$, $n \in \mathbb{N}$, $n \to \infty$ we obtain

$$\begin{aligned}
\operatorname{Im}\chi(\lambda)K_1^{-1} &= -\left(\frac{\operatorname{Im}\lambda_{+0} + \operatorname{Im}\lambda_{-0}}{\lambda_{+0}\lambda_{-0}} + \sum_{k=1}^{\infty} \frac{\operatorname{Im}\lambda_{+k} + \operatorname{Im}\lambda_{-k}}{\lambda_{+k}\lambda_{-k}}\right. \\
&\quad \left. \cdot \left(\frac{1}{\lambda^2} + \frac{1}{\lambda_{+0}\lambda_{-0}}\right)\left(\frac{1}{\lambda^2} + \frac{1}{\lambda_{+k}\lambda_{-k}}\right)^{-1}\right) \\
&\quad \cdot \prod_{p=1}^{\infty} \left(\frac{1}{\lambda^2} + \frac{1}{\lambda_p\lambda_{-p}}\right)\left(\frac{1}{\lambda^2} + \frac{a^2}{\pi^2 p^2}\right)^{-1} + o(1) \\
&= -\lambda_{+0}^{-1}\lambda_{-0}^{-1} \sum_{k=0}^{\infty} (\operatorname{Im}\lambda_{+k} + \operatorname{Im}\lambda_{-k}) \prod_{p=1}^{\infty} \left(-\frac{\pi^2 p^2}{a^2 \lambda_{+p}\lambda_{-p}}\right) + o(1).
\end{aligned}$$

As $\operatorname{Im}\lambda_k > 0$ and $\lambda_k\lambda_{-k} < 0$ for all $k \in \mathbb{N}$ we compare the last equation with (3.13) and obtain $K_2/K_1 > 0$. ∎

LEMMA 3.2.3. *The inequality*

$$\frac{K_3}{K_1} + \frac{K_5}{K_2} > 0$$

holds.

Proof. Taking to account that $P_2 > 0$ and using (3.9) we obtain

$$\frac{K_2}{K_1}\left(\frac{K_3}{K_1} + \frac{K_5}{K_2}\right) > 0.$$

The result now follows from Lemma 3.2.2. ∎

LEMMA 3.2.4. *The following formulae hold:* $\operatorname{Re}\chi'(0) = 0$, $\operatorname{Im}\chi'(0) > 0$.

Proof. The first formula is a consequence of symmetry. From (3.12) we obtain

$$\operatorname{Im}\chi'(0) = \operatorname{Im}\sum_{k-0}^{\infty}(\lambda_k^{-1} + \lambda_{-k}^{-1})\chi(0) = -\sum_{k-0}^{\infty}\frac{2\operatorname{Im}\lambda_k}{\lambda_k\lambda_{-k}} > 0. \quad ∎$$

Due to Lemma 3.2.1 the following limits exist

$$K_1 = -\frac{a}{2\pi}\lim_{n\to\infty} n^{-1}\chi\left(\frac{2\pi n}{a} + \frac{\pi}{2a}\right), \tag{3.17}$$

$$K_2 = -\mathrm{i}\lim_{n\to\infty}\left(\chi\left(\frac{2\pi n}{a} + \frac{\pi}{2a}\right) + \left(2n + \frac{1}{2}\right)\frac{\pi}{a}K_1\right), \tag{3.18}$$

$$K_3 = \lim_{n\to\infty}\chi\left(\frac{2\pi n}{a}\right), \tag{3.19}$$

$$K_4 = \lim_{n\to\infty}\left(2n + \frac{1}{2}\right)\frac{\pi}{a}\left(\chi\left(\frac{2\pi n}{a} + \frac{\pi}{2a}\right) + \left(2n + \frac{1}{2}\right)\frac{\pi}{a}K_1 - \mathrm{i}K_2\right), \tag{3.20}$$

$$K_5 = -\frac{2\pi\mathrm{i}}{a}\lim_{n\to\infty} n\left(\chi\left(\frac{2\pi n}{a}\right) - K_3\right), \tag{3.21}$$

$$\begin{aligned} K_6 = -\mathrm{i}\lim_{n\to\infty}\left(\frac{2\pi n}{a} + \frac{\pi}{2a}\right)&\left(\left(\frac{2\pi n}{a} + \frac{\pi}{2a}\right)\left(\chi\frac{2\pi n}{a} + \frac{\pi}{2a}\right)\right. \\ &\left. + \left(\frac{2\pi n}{a} + \frac{\pi}{2a}\right)^2 K_1 - \mathrm{i}\left(\frac{2\pi n}{a} + \frac{\pi}{2a}\right)K_2 - K_4\right). \end{aligned} \tag{3.22}$$

Define

$$\alpha = \frac{K_2}{K_1}\left(\frac{K_3}{K_1}+\frac{K_5}{K_2}\right)^{-1}, \tag{3.23}$$

$$m = \left(\frac{K_3}{K_1}+\frac{K_5}{K_2}\right)^{-1}, \tag{3.24}$$

$$\beta = \left(\frac{K_4}{K_1}+\frac{K_6}{K_2}\right) m + \frac{K_5}{K_2}. \tag{3.25}$$

According to Lemmas 3.2.2 and 3.2.3 the inequalities $m > 0$ and $\alpha > 0$ follow from (3.24) and (3.23).

Now we construct the *Jost function*

$$\begin{aligned} e(\lambda) = (2\mathrm{i}\alpha\lambda K_1)^{-1}\mathrm{e}^{-\mathrm{i}\lambda a}\big(&\left(m\lambda^2 + \mathrm{i}(1+\alpha)\lambda - \beta\right)\chi(\lambda) \\ &+ \left(-m\lambda^2 + \mathrm{i}(\alpha-1)\lambda + \beta\right)\chi(-\lambda)\big). \end{aligned} \tag{3.26}$$

LEMMA 3.2.5. *The entire function $e(\lambda)$ has the following asymptotics*

$$e(\lambda) \underset{\substack{|\lambda|\to\infty \\ \operatorname{Im}\lambda\leqslant 0}}{=} 1 + \mathrm{O}\left(|\lambda|^{-1}\right). \tag{3.27}$$

Proof. Substituting (3.15) into (3.26) we obtain

$$e(\lambda) = 1 + \frac{\mathrm{i}K_5}{K_2\lambda} + \frac{\psi(\lambda)\mathrm{e}^{-\mathrm{i}\alpha\lambda}}{\lambda}, \tag{3.28}$$

where $\psi(\lambda)$ is an entire function of exponential type $\leqslant a$ belonging to $L_2(-\infty,\infty)$. ∎

LEMMA 3.2.6. *The function $e(\lambda)$ has no real zeroes excepting maybe $\lambda = 0$.*

Proof. If $\lambda \neq 0$ is a real zero of $e(\lambda)$, then taking to account the equality $\chi(-\lambda) = \overline{\chi(\lambda)}$ we obtain from (3.26)

$$\left(m\lambda^2 + \mathrm{i}(1+\alpha)\lambda - \beta\right)\chi(\lambda) = \left(m\lambda^2 - \mathrm{i}(\alpha-1)\lambda - \beta\right)\overline{\chi(\lambda)}.$$

The last equality contradicts the (evident) inequality

$$\left|\frac{m\lambda^2 + \mathrm{i}(1+\alpha)\lambda - \beta}{m\lambda^2 + \mathrm{i}(1-\alpha)\lambda - \beta}\right| > 1, \quad (\lambda\in\mathbb{R}, \lambda\neq 0). \quad ∎$$

LEMMA 3.2.7. *The function $e(\lambda)$ has no more than one simple zero in the closed lower half-plane. This zero, if exists, is pure imaginary.*

Proof. Let's consider the function $e(\lambda, \alpha, \beta, m)$ defined by (3.26). Its zeroes are piecewise analytic functions of each of the parameters α, β, m. Evidently,

$$e(\lambda, 1, 0, 0) = \mathrm{e}^{-\mathrm{i}\lambda a}\chi(\lambda). \tag{3.29}$$

This function has no zeroes in the closed lower half-plane. The zeroes cannot cross the real axis anywhere except the origin when we change $\alpha > 0$, $\beta \in \mathbb{R}$ and $m > 0$ (see Lemma 3.2.6). Due to Lemma 3.2.5, the zeroes do not appear at infinity in the lower half-plane. Let for some $\alpha > 0$, $\beta \in \mathbb{R}$ and $m > 0$ be valid $e(0, \alpha, \beta, m) = 0$. Then using the definition (3.26) we obtain

$$\frac{\partial}{\partial\lambda}\left(\chi(\lambda)\left(m\lambda^2 + \mathrm{i}(\alpha+1)\lambda - \beta\right) + \chi(-\lambda)\left(-m\lambda^2 + \mathrm{i}(\alpha-1)\lambda + \beta\right)\right)\Big|_{\lambda=0} = 0$$

and consequently

$$\beta = \mathrm{i}\alpha C_0^{-1}\left(\chi'(0)\right)^{-1}.$$

So for any $\alpha > 0$ there exists a unique β such that $e(0, \alpha, \beta, m) = 0$. Let us prove that the zero at the origin is simple. Suppose that

$$e(0, \alpha, \beta, m) = \frac{\partial}{\partial\lambda} e(\lambda, \alpha, \beta, m)\Big|_{\lambda=0} = 0.$$

Then using the definition (3.26) we obtain

$$\frac{\partial^2}{\partial\lambda^2}\left(\chi(\lambda)\left(m\lambda^2 + \mathrm{i}(\alpha+1)\lambda - \beta\right) + \chi(-\lambda)\left(-m\lambda^2 + \mathrm{i}(\alpha-1)\lambda + \beta\right)\right)\Big|_{\lambda=0} = 0$$

and, consequently, $\chi'(0) = 0$, what contradicts Lemma 3.2.4. Thus, there may be only one simple zero in the closed lower half plane. Due to the symmetry of the problem this zero is pure imaginary. ∎

LEMMA 3.2.8. *Let $\lambda = -\mathrm{i}\mu$ ($\mu > 0$) be the zero of $e(\lambda)$. Then*

$$m\mu^2 + \mu(\alpha - 1) + \beta > 0. \tag{3.30}$$

Proof. Using the definition (3.26) we obtain

$$\chi(\mathrm{i}\mu)(m\mu^2 + \mu(\alpha-1) + \beta) - \chi(-\mathrm{i}\mu)(m\mu^2 - \mu(\alpha+1) + \beta) = 0. \tag{3.31}$$

As

$$\left|\left(1 + \frac{\mathrm{i}\mu}{\lambda_k}\right)\left(1 + \frac{\mathrm{i}\mu}{\lambda_{-k}}\right)\right| > \left|\left(1 - \frac{\mathrm{i}\mu}{\lambda_k}\right)\left(1 - \frac{\mathrm{i}\mu}{\lambda_{-k}}\right)\right|$$

we have

$$|\chi(-\mathrm{i}\mu)| > |\chi(\mathrm{i}\mu)|. \tag{3.32}$$

Now using (3.32) we obtain (3.30) from (3.31). ∎

LEMMA 3.2.9. *If* $e(-\mathrm{i}\mu) = 0,\ \mu > 0$, *then* $e(\mathrm{i}\mu) < 0$.

Proof. Expressing $\chi(\mathrm{i}\mu)$ from (3.31) and substituting it into (3.26) at $\lambda = -\mathrm{i}\mu$ we obtain

$$e(\mathrm{i}\mu) = -\frac{2\mu\chi(-\mathrm{i}\mu)\mathrm{e}^{\mu a}}{(m\mu^2 + \mu(\alpha - 1) + \beta)\, K_1}.$$

It is easy to see that $\chi(-\mathrm{i}\mu) > 0$ and taking to account (3.30) we obtain $e(\mathrm{i}\mu) < 0$.

LEMMA 3.2.10. *If* $e(-\mathrm{i}\mu) = 0,\ \mu > 0$, *then* $\mathrm{Re}\, e'(-\mathrm{i}\mu) = 0$, $\mathrm{Im}\, e'(-\mathrm{i}\mu) > 0$.

Proof. The first assertion is a consequence of the symmetry of the problem. The second one is a consequence of the analyticity of the zero in the lower half-plane with respect to α, β, m and of the fact that $e(0) = 0$ implies $\mathrm{Im}\, e'(0) > 0$.

Now introduce the *S-matrix*:

$$S(\lambda) = \frac{e(\lambda)}{e(-\lambda)}, \tag{3.33}$$

the function

$$F_0(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - S(\lambda))\mathrm{e}^{\mathrm{i}\lambda x}\, \mathrm{d}\lambda \tag{3.34}$$

and

$$F(x) = \begin{cases} F_0(x), & \text{if } e(\lambda) \neq 0 \text{ when } \mathrm{Im}\,\lambda < 0, \\ F_0(x) + D\mathrm{e}^{-\mu x}, & \text{if } e(-\mathrm{i}\mu) = 0\ (\mu > 0). \end{cases} \tag{3.35}$$

Here

$$D = \frac{-e(\mathrm{i}\mu)}{\mathrm{Im}\, e'(-\mathrm{i}\mu)}. \tag{3.36}$$

Lemmas 3.2.9 and 3.2.10 imply $D > 0$. ∎

LEMMA 3.2.11. *The S-matrix satisfies the following conditions:*

I. (i) *$S(\lambda)$ is continuous for $\lambda \in \mathbb{R}$ and*

$$S(\lambda) = \overline{S(-\lambda)} = (S(-\lambda))^{-1},$$

(ii) *$1 - S(\lambda)$ tend to 0 as $|\lambda| \to \infty$ and $1 - S(\lambda)$ is the Fourier-transform of $F_0(x)$. The function $F_0(x)$ is bounded and belongs to $L_2(-\infty, \infty)$. There exists $F_0'(x)$ for $x > 0$ and $\int_0^{\infty} |F_0'(x)|\, \mathrm{d}x < \infty$.*

II. *The variation of the argument of* $S(\lambda)$ *satisfies the following equation*

$$\ln S(+0) - \ln S(+\infty) = 2\pi i\kappa \tag{3.37}$$

where $\kappa = 1$ *if* $e(\lambda)$ *has a zero in the lower half-plane,* $\kappa = \frac{1}{2}$ *if* $e(0) = 0$ *and* $\kappa = 0$ *if* $e(\lambda)$ *has no zeroes in the closed lower half-plane.*

Proof. Assertion (i) is a consequence of Definition (3.33). Using (3.28) we obtain

$$1 - S(\lambda) = \frac{2iK_5}{\lambda K_2} + \frac{2}{\lambda}\mathrm{Re}\,\left(\psi(\lambda)e^{i\lambda a}\right) + O\left(\frac{1}{\lambda^2}\right) \tag{3.38}$$

and consequently $1-S(\lambda) \underset{\lambda\to\pm\infty}{\longrightarrow} 0$ and $F_0(\lambda)$ is bounded and belongs to $L_2(-\infty,\infty)$. The equation (3.38) implies

$$1 - S(\lambda) = \frac{2iK_5}{\lambda K_2} + \frac{\Phi_1(\lambda)}{1+|\lambda|},$$

where $\Phi_1(\lambda) \in L_2(-\infty,\infty)$. Thus for $x > 0$

$$\begin{aligned} F_0(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2iK_5}{K_2(\lambda+i)} + \frac{\Phi_2(\lambda)}{1+|\lambda|}\right)e^{i\lambda x}\,d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\Phi_2(\lambda)e^{i\lambda x}d\lambda}{1+|\lambda|}, \quad (\Phi_2(\lambda)) \in L_2(-\infty,\infty)) \end{aligned} \tag{3.39}$$

and consequently $F_0(x)$ is absolutely continuous on $x \in (0,\infty)$ and $F_0'(x) \in L_2(-\infty,\infty)$ and $F'(x) \in L_2(-\infty,\infty)$. The function $1 - S(\lambda)$ is meromorphic and has no more than one pole in the closed upper half-plane. The pole, if exists, is simple due to Lemma 3.2.7 and Definition (3.33). In this case there exist constants $C > 0$ and $P > 0$ such that

$$|1 - S(\lambda)| < C|\lambda|^{-1}e^{2a\mathrm{Im}\,\lambda}, \quad (\mathrm{Im}\,\lambda \geqslant P).$$

So, using Jordan lemma, we obtain

$$F_0(x) = i\mathrm{Re}\,(1 - S(\lambda))\,e^{i\lambda x}\big|_{\lambda=i\mu} = \frac{ie(i\mu)e^{-\mu x}}{e'(-i\mu)} = \frac{e(i\mu)e^{-\mu x}}{\mathrm{Im}\,e'(-i\mu)}, \quad (x > 2a), \tag{3.40}$$

where $-i\mu$ is the zero of $e(\lambda)$ in the open lower half-plane. If $e(\lambda)$ has no zeroes in the open lower half-plane, then $F_0(x) \equiv 0$ for $x > 2a$. Thus $\int_0^\infty x|F_0'(x)|\,dx < \infty$. Assertion I is proved. Taking into account (3.35) we obtain

$$F(x) \equiv 0, \quad (x > 2a). \tag{3.41}$$

Using the principle of argument we have

$$\arg e(\infty) - \arg e(-\infty) = 2\pi\kappa. \tag{3.42}$$

As

$$S(0) = \begin{cases} 1, & \text{if } e(0) \neq 0, \\ -1, & \text{if } e(0) = 0, \end{cases}$$

and $\ln S(\lambda) = -2\mathrm{i}\arg e(\lambda)$ we obtain (3.37) using (3.42). ∎

We call the *scattering data* the set $\{S(\lambda)(\lambda \in \mathbb{R}), -\mathrm{i}\mu, D\}$ if $\kappa = 1$ and $\{S(\lambda)$ $(\lambda \in \mathbb{R})\}$ otherwise. According to Lemma 3.2.11 the scattering data satisfy all the conditions of V.A. Marchenko Theorem ([17], p. 218). So the integral equation

$$K(x,t) + F(x+t) + \int_x^\infty K(x,s)F(s+t)\,\mathrm{d}s = 0 \tag{3.43}$$

possesses a unique solution $K(x,t)$ and the potential

$$\tilde{q}(x) = -2\frac{\mathrm{d}K(x,x)}{\mathrm{d}x} \tag{3.44}$$

of the Sturm-Liouville problem on half-axis

$$y'' + \lambda^2 y - \tilde{q}(x)y = 0, \tag{3.45}$$

$$y(\lambda, 0) = 0 \tag{3.46}$$

satisfies the condition

$$\int_0^\infty x|\tilde{q}(x)|\,\mathrm{d}x < \infty. \tag{3.47}$$

The scatering data of the problem (3.45), (3.46) coincide with that used for the construction of $\tilde{q}(x)$. Due to (3.41) and using (3.43) and (3.44) we obtain

$$\tilde{q}(x) \equiv 0 \quad \text{for } x > a. \tag{3.48}$$

LEMMA 3.2.12. *The projection $q(x) = \tilde{q}(x)$ $(x \in [0,a])$ belongs to $L_2(0,a)$.*

Proof. We use the inequality ([17], p. 209)

$$|\tilde{q}(x)| \leqslant 4|F'(2x)| + C(x)\left(1 + C(x)\tau_1(2x)\right)\tau^2(2x),$$

where

$$\tau(x) = \int_x^\infty |F'(t)|\,\mathrm{d}t, \quad \tau_1(x) = \int_x^\infty \tau(t)\,\mathrm{d}t,$$

$C(x) \in C[0,a]$. Taking into account that $F'(x) \in L_2(0,a)$ we obtain the assertion of Lemma 3.2.12. ∎

So we have shown how to find the set $\{q(x), a, \alpha, \beta, m\}$ from the spectrum Λ. The obtained set $\{q(x), a, \alpha, \beta, m\}$ possesses the following properties: $\alpha > 0$, $a > 0$, $\beta \in \mathbb{R}$, $m > 0$, $q(x) \in L_2(0, a)$ is real and the operator A_1 defined by (A.1), (A.2) (see Appendix) is strictly positive (Corollary A.1). Denote by Q the class of sets possessing those properties. Now we prove that the problem (1.9)–(1.11) with corresponding $\{q(x), a, \alpha, \beta, m\}$ possesses the spectrum Λ which coincides with the initial one. The V.A.Marchenko Theorem mentioned above guarantees an one-to-one correspondence between scattering data and potential. It is necessary to prove the one-to-one correspondence between $q(x)$ and $e(\lambda)$.

LEMMA 3.2.13. *Let $S(\lambda)$ satisfy the conditions* I *of Lemma* 3.2.11 *and* $\ln S(+0) - \ln S(-0) = 2\pi\kappa$.

(i) *If $\kappa = 0$ or $\kappa = \frac{1}{2}$ then there exists a unique function $e(\lambda)$ holomorphic in the lower half-plane and continuous in the closed lower half-plane, admitting the asympotics*

$$e(\lambda) \underset{\substack{|\lambda| \to \infty \\ \mathrm{Im}\,\lambda \leqslant 0}}{=} 1 + o(1) \tag{3.49}$$

and such that

$$\frac{e(\lambda)}{e(-\lambda)} = S(\lambda), \quad (\lambda \in \mathbb{R}). \tag{3.50}$$

(ii) *If $\kappa = 1$ then for any positive number $\mu > 0$ there exists a unique function $e(\lambda)$ holomorphic in the lower half-plane, continuous in the closed lower half-plane, having unique simple zero at $\lambda = \mathrm{i}\mu$ and satisfying the conditions* (3.49) *and* (3.50).

Proof. To prove this lemma we have to repeat all the arguments of p. 80 in [5]. So we have proved the one-to-one correspondence between $q(x)$ and $e(\lambda)$. The inversed formula (3.26), i.e.

$$\begin{aligned}\chi(\lambda) = (2\mathrm{i}\lambda)^{-1} K_1 \left(e(\lambda)\mathrm{e}^{\mathrm{i}\lambda a}\left(-m\lambda^2 + \mathrm{i}(1+\alpha)\lambda + \beta\right)\right) \\ + e(-\lambda)\mathrm{e}^{-\mathrm{i}\lambda a}\left(m\lambda^2 - \mathrm{i}(\alpha - 1)\lambda - \beta\right)\end{aligned}$$

proves the one-to-one correspondence between the class of spectra satisfying conditions (i)–(iv) of Theorem 2.1 and the corresponding subclass of Q. We denote this subclass by Q'. The last step is to find $\{B(s), M, \nu\}$ from l (> 0) and $\{q(x), a, \alpha, \beta, m\}$. Let us rewrite (1.12), (1.15) in the form

$$\left(B^{\frac{1}{4}}(x)\right)''_{xx} - q(x)B^{\frac{1}{4}}(x) = 0 \tag{3.51}$$

(3.52) $$B^{\frac{1}{4}}{}'_x(a) + 4\beta B(a) = 0.$$

Consider the initial value problem

$$p''_{xx} - q(x)p = 0,$$
$$p'_x(a) = -\beta,$$
$$p(a) = 1.$$

This problem admits a unique solution $p(x)$ in $W_2^2(0,a)$ as $q(x) \in L_2(0,a)$. As $A_1 \gg 0$, Sturm Theorem ([9], p. 151) implies $p(x) > 0$ for all $x \in [0,a]$. The solution of (3.51), (3.52) may be expressed as follows

(3.53) $$B(x) = Cp^4(x),$$

where C is constant. The value of C may be found from the equation

(3.54) $$\int_0^a \frac{\mathrm{d}x}{B^{\frac{1}{2}}(x)} = l,$$

which is a consequence of (1.7). Substituting (3.53) into (3.54) we obtain

$$C = \left(\frac{1}{l}\int_0^a \frac{\mathrm{d}x}{p^2(x)}\right)^2$$

and

(3.55) $$B(x) = l^{-2}\left(\int_0^a p^{-2}(x)\,\mathrm{d}x\right)^2 p^4(x).$$

From (1.13) and (1.14) we obtain

$$M = mB^{\frac{1}{2}}(a),$$
$$\nu = \alpha B^{\frac{1}{2}}(a).$$

To find $B(s)$, we rewrite (1.7) in the form

$$\int_0^x B^{-\frac{1}{2}}(x')\,\mathrm{d}x' = s.$$

Finding $x(s)$ and substituting it into (3.55) we obtain $B(s) = B(x(s))$. It is clear that $B(x(s)) \in W_2^2(0,l)$ and thus $\{B(s), M, \nu\} \in \mathcal{B}_l$, so we have proved the one-to-one correspondence between the class of spectra satisfying conditions (i)–(iv) of Theorem 2.1 and corresponding subclass $\mathcal{B}'_l \subset \mathcal{B}_l$. ∎

4. COMMENTS

The implicit definition of the class $\mathcal{B}_l'$ is the following.

DEFINITION 4.1. $\mathcal{B}_l'$ is a class of sets $\{B(s), s \in [0,l], M, \nu\}$ such that the spectrum of the corresponding problem (1.4)–(1.6) satisfies the conditions (1)–(4) of Theorem 2.1.

It is difficult to describe the class $\mathcal{B}_l'$ explicitly. Let us describe the reasons for this. After finding α, m, β via formulae (3.23)–(3.25), we may find the functions

$$S(\lambda, a) \stackrel{\text{def}}{=} \frac{m\left(\chi(-\lambda) - \chi(\lambda)\right)}{2\mathrm{i}\alpha K_1 \lambda}$$
$$S'(\lambda, a) \stackrel{\text{def}}{=} -K_1^{-1} m \chi(\lambda) + (\lambda^2 m - \mathrm{i}\alpha\lambda - \beta) S(\lambda, a).$$

It is possible to prove (in the same way as in [21]) that the zeroes $\{\nu_k\}_{k=-\infty, k\neq 0}^{\infty}$ of $S(\lambda, a)$ possess the following asymptotics

$$\nu_k \underset{k\to\infty}{=} \frac{\pi k}{a} + \frac{C_1}{k} + \frac{C_2}{k^3} + \frac{b_{k1}}{k^3}, \quad (\nu_k = -\nu_{-k}), \tag{4.1}$$

where the real constants C_1 and C_2 may be expressed via K_i. The zeroes $\{\mu_k\}_{k=-\infty, k\neq 0}^{\infty}$ of the function $S'(\lambda, a)$ have the following asymptotics

$$\mu_k \underset{k\to\infty}{=} \frac{\pi\left(k - \frac{1}{2}\right)}{a} + \frac{C_1}{k} + \frac{b_{k2}}{k}, \quad (\mu_k = -\mu_{-k}), \tag{4.2}$$

where $\{b_{ki}\}_{k=-\infty, k\neq 0}^{\infty} \in l_2$, $(i = 1, 2)$. The sequences alternate, i.e. $0 < \mu_1 < \nu_1 < \mu_2 < \nu_2 < \ldots$ (see [21]). The sequence $\{\mu_k\}_{k=-\infty, k\neq 0}^{\infty}$ is the set of eigenvalues of the problem

$$y'' + \lambda^2 y - q(x) y = 0,$$
$$y(0) = y'(a) = 0,$$

with $q(x)$ obtained by formula (3.44). The sequence $\{\nu_n\}_{k=-\infty, k\neq 0}^{\infty}$ is the set of eigenvalues of the problem

$$y'' + \lambda^2 y - q(x) y = 0,$$
$$y(0) = y(a) = 0.$$

The potential $q(x)$ may be constructed by the two spectra $\{\nu_k\}_{k=-\infty, k\neq 0}^{\infty}$ and $\{\mu_k\}_{k=-\infty, k\neq 0}^{\infty}$ (see [14], [17], [15]). Now we are able to give another definition of $\mathcal{B}_l'$ (also implicit and equivalent to Definition 4.1).

DEFINITION 4.2. $\mathcal{B}'_l$ is a class of sets $\{B(s), s \in [0,l], M, \nu\}$ such that the corresponding string with both ends fixed has the spectrum with the asymptotics (4.1) and the same string with the left end fixed and the right end free and massless has the spectrum with the asymptotics (4.2).

It is clear that $\mathcal{B}'_l \subset \mathcal{B}_l$. Unfortunately, we can make the asymptotics (4.2) more precise only at the cost of making more precise the asymptotics (2.1).

APPENDIX

Introduce the following operator pencil

$$L(\lambda) = \lambda^2 F - \mathrm{i}\lambda K - A$$

where F, K, A are operators acting in $H = L_2(0,a) \oplus \mathbb{C}$ according to the formulae

$$F = \begin{pmatrix} I & 0 \\ 0 & mI \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 \\ 0 & \alpha I \end{pmatrix}, \quad D(F) = D(K) = H,$$

$$A \begin{pmatrix} y(x) \\ y(a) \end{pmatrix} = \begin{pmatrix} -y''(x) + q(x)y(x) \\ y'(a) + \beta y(a) \end{pmatrix}, \quad (q(x) \in L_2(0,a)),$$

$$D(A) = \left\{ \begin{pmatrix} y(x) \\ y(a) \end{pmatrix} : y(0) = 0, \; y(x) \in W_2^2(0,a) \right\},$$

$$D(L(\lambda)) = D(F) \cap D(K) \cap D(A) = D(A)$$

for all $\lambda \in \mathbb{C}$ by definition. It is easy to check that $A = A^*$ is bounded below, $K \geqslant 0, F \gg 0$, i.e. $F > \varepsilon I$, $(\varepsilon > 0)$. The spectrum of $L(\lambda)$ coincides with the spectrum of the problem (1.9)–(1.11).

LEMMA A.1. *The spectrum of $L(\lambda)$ is located in the open upper half-plane if and only if $A \gg 0$.*

This lemma is a consequence of the more general result of [19] (see also [20]).

Consider the operator A_1 acting in $L_2(0,a)$ according to the formulae

$$A_1 y = -y'' + q(x)y, \tag{A.1}$$

$$D(A_1) = \left\{ y \in W_2^2(0,a), \; y(0) = 0, \; y'(a) + \beta y(a) = 0 \right\}. \tag{A.2}$$

LEMMA A.2. *The operator A is strictly positive if and only if A_1 is strictly positive.*

The assertion of this lemma follows from the fact that $0 \in \rho(A) \iff 0 \in \rho(A_1)$, where $\rho(A)$ is the resolvent set of A.

COROLLARY A.1. *The spectrum of $L(\lambda)$ is located in the open upper half-plane if and only if $A_1 \gg 0$.*

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