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ON THE CONTRACTIONS IN THE CLASSES $\mathbb{A}_{n,m}$

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ABSTRACT. Let T be a contraction in the class $\mathbb{A}$ acting on a Hilbert space. Sufficient conditions in terms of the multiplicity of certain natural unitary operators associated with the $C_{0\cdot}$, $C_{\cdot 0}$, $C_{1\cdot}$ or $C_{\cdot 1}$ part of T are given to ensure that T belongs to the class $\mathbb{A}_{n,m}$, $n, m \in \mathbb{N}^*$. Along the way we obtain new relations between the boundary sets involved in arbitrary triangulations of T.

KEYWORDS: *Hilbert space, absolutely continuous contraction, dual operator algebra, minimal coisometric extension, minimal isometric dilation.*

AMS SUBJECT CLASSIFICATION: Primary 47D27; Secondary 47A20, 47A15.

1. INTRODUCTION

Let $\mathcal{H}$ be a separable, infinite-dimensional complex Hilbert space. We denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators acting on $\mathcal{H}$. Let $C^1(\mathcal{H})$ be the Banach space of trace class operators on $\mathcal{H}$ equipped with the trace norm. If $\mathcal{A}$ is a dual algebra on $\mathcal{H}$, that is, a weak*-closed unital subalgebra of $\mathcal{L}(\mathcal{H})$, then it is well-known (cf. for example [2]) that $\mathcal{A}$ can be identified with the dual space of $\mathcal{Q}_{\mathcal{A}} := C^1(\mathcal{H})/{}^{\perp}\mathcal{A}$ where ${}^{\perp}\mathcal{A}$ is the preannihilator of $\mathcal{A}$ in $C^1(\mathcal{H})$, under the pairing:

$$\langle T, [L]_{\mathcal{A}} \rangle = \mathrm{trace}(TL), \quad T \in \mathcal{A},\ [L]_{\mathcal{A}} \in \mathcal{Q}_{\mathcal{A}}.$$

The Banach space $\mathcal{Q}_{\mathcal{A}}$ is called the predual of $\mathcal{A}$. We write $[L]$ for $[L]_{\mathcal{A}}$ whenever there is no possibility of confusion. For x and y in $\mathcal{H}$, we define $x \otimes y$ by $x \otimes y(u) = (u, y)x$ for all u in $\mathcal{H}$. The cosets $[x \otimes y]_{\mathcal{A}}$ have been essential in dual algebra theory. Suppose m and n are cardinal numbers such that $1 \leqslant m$, $n \leqslant \aleph_0$.

A dual algebra $\mathcal{A}$ will be said to have property $(\mathbb{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form:

$$[x_i \otimes y_j] = [L_{i,j}], \quad 0 \leqslant i < m,\ 0 \leqslant j < n,$$

where $\{[L_{i,j}], 0 \leqslant i < m, 0 \leqslant j < n\}$ is an arbitrary array from $\mathcal{Q}_{\mathcal{A}}$, has a solution $\{x_i, 0 \leqslant i < m\}, \{y_j, 0 \leqslant j < n\}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. We write $\mathbb{D}$ for the open unit disc in the complex plane $\mathbb{C}$, and $\mathbb{T}$ for the boundary of $\mathbb{D}$. The spaces $L^p = L^p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$ are the usual Lebesgue function spaces relative to normalized Lebesgue measure m on $\mathbb{T}$. The spaces $H^p = H^p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$ are the usual Hardy spaces. It is well-known (cf. [9]) that the space H^∞ is the dual space of L^1/H_0^1 where $H_0^1 = \left\{ f \in L^1 : \int\limits_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{\mathrm{i}nt}\,\mathrm{d}t = 0,\ n = 0, 1, \dots \right\}$ and the duality is given by the pairing:

$$\langle f, [g] \rangle = \int\limits_{\mathbb{T}} fg\,\mathrm{d}m, \quad f \in H^\infty,\ [g] \in L^1/H_0^1.$$

We denote by $\mathcal{A}_T$ the dual algebra generated by $T \in \mathcal{L}(\mathcal{H})$ and by $\mathcal{Q}_T$ the predual space $\mathcal{Q}_{\mathcal{A}_T}$ of $\mathcal{A}_T$. A contraction $T \in \mathcal{L}(\mathcal{H})$ is absolutely continuous if in the canonical decomposition $T = T_1 \oplus T_2$, where T_1 is a unitary operator and T_2 is a completely non unitary contraction, T_1 is either absolutely continuous or acts on the space (0). The following is essentially Theorem 4.1 in [2]:

THEOREM 1.1. *Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then there exists a functional calculus $\Phi_T : H^\infty \to \mathcal{A}_T$ defined by $\Phi_T(f) = f(T)$ for every $f \in H^\infty$. The mapping Φ_T is a norm-decreasing weak*-continuous algebra homomorphism, and the range of Φ_T is weak*-dense in $\mathcal{A}_T$. Furthermore there exists a bounded, linear, one-to-one map φ_T of $\mathcal{Q}_T$ into L^1/H_0^1 such that $\Phi_T = \varphi_T^*$.*

In particular the coset $[x \otimes y]$ is mapped to an element of L^1/H_0^1 which we denote $x \square y$. Very often we will use the sesquilinear map "$\square$" for different absolutely continuous contractions (a.c.c.). If necessary we will write $\overset{T}{\square}$ to avoid ambiguity. Thus we write $x \square y$ either when there is only one a.c.c. for which $x \square y$ is defined or when for all a.c.c. for which $x \square y$ is defined the same value is assigned. We denote by $\mathbb{A} = \mathbb{A}(\mathcal{H})$ the class of all absolutely continuous contractions $T \in \mathcal{L}(\mathcal{H})$ for which the Sz.-Nagy–Foiaş functional calculus $\Phi_T : H^\infty \to \mathcal{A}_T$ is an isometry. Furthermore, if m and n are any cardinal numbers such that $1 \leqslant m, n \leqslant \aleph_0$, we set $\mathbb{A}_{m,n} = \mathbb{A}_{m,n}(\mathcal{H})$ to be the set of all T in $\mathbb{A}(\mathcal{H})$ such that the singly generated dual algebra $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$. We write $\mathbb{A}_n$ for $\mathbb{A}_{n,n}$.

In this paper, we continue the study of sufficient conditions for membership in the class $\mathbb{A}_{n,m}$, using improvements of techniques introduced in [8], [6], [4], [1], [12]. A lot of work has been done in this direction. For example, in [10], the authors discuss contraction operators T in the class $C_{\cdot 0} \cap \mathbb{A}$ with defect index $d_T < \infty$ ($d_T = \dim\{(\mathrm{Id} - T^*T)^{1/2}\mathcal{H}\}^-$). They show that these are particularly nice representatives of the class $\mathbb{A}_{n,\aleph_0}$. Indeed, their membership is completely determined by the multiplicity of either the shift piece of their Jordan model or the unitary piece of their minimal coisometric extension.

Our results are based upon the interplay between boundary sets, multiplicity theory and approximation techniques. In particular, we generalize the results obtained by [12] for membership in the class $\mathbb{A}_n$ by localizing the multiplicity conditions. Though this localization will not surprise the specialist, it is largely responsible for a lot of new technicalities.

In Section 2 we introduce the notation and terminology used herein. Then, in Section 3, we shall develop some functional lemmas which lead to approximation results involving multiplicity (established in Section 4). Along the way, we give some new results for some triangulation of absolutely continuous contractions (Section 4). As a sequel to this study we shall deduce some sufficient conditions for membership in the class $\mathbb{A}_{1,n}$, $\mathbb{A}_{k,1}$ and $\mathbb{A}_{k,n}$ where k and n are some positive integers.

2. PRELIMINARIES

The notation and terminology employed herein agree with those in [5] and [13]. If we suppose that T is an a.c.c. in $\mathcal{L}(\mathcal{H})$, then its minimal unitary dilation $U \in \mathcal{L}(\mathcal{U})$ ($\mathcal{H} \subset \mathcal{U}$) is also absolutely continuous.

The minimal isometric dilation U_+ of T is the restriction of U to the subspace $\mathcal{U}_+ = \operatorname{span}\{U^n\mathcal{H}, n \geqslant 0\}$, which is invariant for U. The operator U_+ has the Wold decomposition $U_+ = S_* \oplus R$ corresponding to the decomposition of $\mathcal{U}_+$ as $\mathcal{S}_* \oplus \mathcal{R}$, where S_* is a unilateral shift of some multiplicity in $\mathcal{L}(\mathcal{S}_*)$ if $\mathcal{S}_* \neq (0)$, S_* is the zero operator if $\mathcal{S}_* = (0)$, R is an absolutely continuous unitary operator in $\mathcal{L}(\mathcal{R})$ if $\mathcal{R} \neq (0)$ and R is the zero operator if $\mathcal{R} = (0)$.

The minimal coisometric extension B of T is the compression of U to the subspace $\mathcal{B} = \operatorname{span}\{U^n\mathcal{H}, n \leqslant 0\} = \operatorname{span}\{U^{*n}\mathcal{H}, n \geqslant 0\}$, invariant for U^* (hence semi-invariant for U). The operator B has the Wold decomposition $B = S^* \oplus R_*$ corresponding to the decomposition of $\mathcal{B}$ as $\mathcal{S} \oplus \mathcal{R}_*$, where S is a unilateral shift of some multiplicity in $\mathcal{L}(\mathcal{S})$ if $\mathcal{S} \neq (0)$, S is the zero operator if $\mathcal{S} = (0)$, R_* is an absolutely continuous unitary operator in $\mathcal{L}(\mathcal{R}_*)$ if $\mathcal{R}_* \neq (0)$ and R_* is the zero operator if $\mathcal{R}_* = (0)$.

Throughout the paper, expressions such as maximality, uniqueness, and equality of Borel subsets of $\mathbb{T}$ are to be interpreted as being satisfied up to Borel subsets of Lebesgue measure zero.

We write $\Sigma = \Sigma_T$ (resp. $\Sigma_* = \Sigma_{*,T}$) for the Borel subset of $\mathbb{T}$ such that $m_{|\Sigma}$ (resp. $m_{|\Sigma_*}$) is a spectral measure for R (resp. R_*). By Proposition 3.1 in [5], there exists a unique maximal Borel subset X_T of $\mathbb{T}$ such that, for any $f \in L^1(X_T), \|f\|_1 \leqslant 1$, there exist two sequences $(x_n)_n$ and $(y_n)_n$ in the unit ball of $\mathcal{H}$ such that:

$$\begin{cases} \lim_{n\to\infty} \|[f]_{L^1/H_0^1} - x_n \square y_n\| = 0; & \\ \lim_{n\to\infty} \|x_n \square w\| = 0, & w \in \mathcal{H}; \\ \lim_{n\to\infty} \|w \square y_n\| = 0, & w \in \mathcal{H}. \end{cases}$$

In fact, T belongs to the class $\mathbb{A}_{\aleph_0}$ if and only if $X_T = \mathbb{T}$ (cf. [2], with a different formulation).

We denote by E_T^{r} (resp. E_T^{l}) the Borel subset of $\mathbb{T}$ equal to $X_T \cup \Sigma_{*,T}$ (resp. $X_T \cup \Sigma_T$). It follows from Proposition 4.8 in [5] that E_T^{r} (resp. E_T^{l}) is the maximal Borel subset of $\mathbb{T}$ such that for any $f \in L^1(E_T^{\mathrm{r}})$ (resp. $f \in L^1(E_T^{\mathrm{l}})$), $\|f\|_1 \leqslant 1$, there exist two sequences $(x_n)_n$ and $(y_n)_n$ in the unit ball of $\mathcal{H}$ such that:

$$\begin{cases} \lim_{n\to\infty} \|[f]_{L^1/H_0^1} - x_n \square y_n\| = 0; \\ \lim_{n\to\infty} \|x_n \square w\| = 0 \text{ (resp. } \lim_{n\to\infty} \|w \square y_n\| = 0), \quad w \in \mathcal{H}. \end{cases}$$

The operator T belongs to the class $\mathbb{A}_{1,\aleph_0}$ (resp. $\mathbb{A}_{\aleph_0,1}$) if and only if $E_T^{\mathrm{r}} = \mathbb{T}$ (resp. $E_T^{\mathrm{l}} = \mathbb{T}$) (cf. [4]).

By Theorem 4.3 in [5], an a.c.c. $T \in \mathcal{L}(\mathcal{H})$ belongs to the class $\mathbb{A}$ if and only if $\mathbb{T} = X_T \cup \Sigma_{*,T} \cup \Sigma_T$.

If $\mathcal{M}$ is a semi-invariant subspace for T, we denote by $R^{\mathcal{M}}$ (resp. $R_*^{\mathcal{M}}$) the unitary part of the minimal isometric dilation (resp. minimal coisometric extension) of the compression $T_{\mathcal{M}}$.

We denote by Q, Q_*, A, A_* the orthogonal projections of $\mathcal{U}$ onto $\mathcal{S}, \mathcal{S}_*, \mathcal{R}, \mathcal{R}_*$ and we denote by $Q^{\mathcal{M}}, Q_*^{\mathcal{M}}, A^{\mathcal{M}}, A_*^{\mathcal{M}}$ the orthogonal projections of $\mathcal{U}^{\mathcal{M}}$, the space of the minimal unitary dilation of $T_{\mathcal{M}}$, onto $\mathcal{S}^{\mathcal{M}}, \mathcal{S}_*^{\mathcal{M}}, \mathcal{R}^{\mathcal{M}}, \mathcal{R}_*^{\mathcal{M}}$ the spaces associated in an obvious way to the minimal isometric dilation and the minimal coisometric extension of $T_{\mathcal{M}}$.

If Γ is any Borel subset of $\mathbb{T}$ (satisfying $0 \leqslant m(\Gamma) \leqslant 1$), we denote by M_Γ the absolutely continuous unitary operator on $L^2(\Gamma)$ defined by:

$$(M_\Gamma x)(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}t}x(\mathrm{e}^{\mathrm{i}t}), \quad x \in L^2(\Gamma), \quad \mathrm{e}^{\mathrm{i}t} \in \Gamma.$$

As to the multiplicity of an absolutely continuous unitary operator on a Borel subset of $\mathbb{T}$, the following (standard) formulation will be convenient for our purposes:

Definition 2.1. Let $R \in \mathcal{L}(\mathcal{R})$ be an absolutely continuous unitary operator and let σ be a Borel subset of $\mathbb{T}$. We say that the multiplicity of R is greater than or equal to $n, n \geqslant 1$ on σ if there exists a reducing subspace $\mathcal{R}_0$ for R such that $R_0 := R_{|\mathcal{R}_0}$ is unitarily equivalent to $(M_\sigma)^{(n)}$ on $(L^2(\sigma))^{(n)}$, the n-fold ampliation.

We recall that if T is an arbitrary a.c.c. in $\mathcal{L}(\mathcal{H})$ and if σ is a Borel subset of $\mathbb{T}$, then σ is said to be essential for T and we write $\sigma \subset \mathrm{ess}(T)$ (cf. Definition 3.1 in [6]) if:

$$\|f(T)\| \geqslant \|f_{|\sigma}\|_\infty, \; f \in H^\infty(\mathbb{T}).$$

We also recall that a $C_0.$ (resp. $C_{.0}$) contraction is a contraction such that $\lim_{n\to\infty} \|T^n h\| = 0$ (resp. $\lim_{n\to\infty} \|T^{*n}h\| = 0$) $h \in \mathcal{H}$. This is equivalent to $\Sigma_{*,T} = \emptyset$ (resp. $\Sigma_T = \emptyset$). On the other hand a $C_1.$ (resp. $C_{.1}$) contraction is a contraction such that $\lim_{n\to\infty} \|T^n h\| = 0 \Rightarrow h = 0$ (resp. $\lim_{n\to\infty} \|T^{*n}h\| = 0 \Rightarrow h = 0$).

We will use the very useful decomposition of a contraction $T \in \mathcal{L}(\mathcal{H})$ introduced in [13], p. 73, namely:

$$T = \begin{pmatrix} T_0 & * \\ 0 & T_1 \end{pmatrix}$$

relative to the orthogonal decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, where $\mathcal{H}_0$ is defined by $\mathcal{H}_0 := \{x \in \mathcal{H} \text{ such that } \lim_{n\to\infty} \|T^n x\| = 0\}$. By construction we have $T_0 \in C_0.$ and $T_1 \in C_1.$. We denote by E_0 the maximal essential Borel set for T_0 (unique up to Borel sets of Lebesgue measure equal zero). Since $T_0 \in C_0.$ we get that $E_0 = E^l_{T_0}$ (cf. Proposition 4.5 in [5]) and if we define E_1 by $E_1 = \mathbb{T} \setminus E_0$ we have $E_1 \subset \mathrm{ess}(T_1) \subset E^r_{T_1}$ whenever $T \in \mathbb{A}$ (cf. Proposition 1.3 in [6] and Proposition 4.5. in [5]).

We now state some elementary observations important in the sequel.

– If $\mathcal{M}$ is an invariant subspace for T and $T' = T_{|\mathcal{M}}$ then $x \overset{T}{\square} y = x \overset{T'}{\square} y'$ for all $x \in \mathcal{M}$ and $y \in \mathcal{H}$ with $y' = P_\mathcal{M} y$.

– If $\mathcal{J}$ is a semi-invariant subspace for T and Y denotes the compression of T to $\mathcal{J}$, then $x \overset{T}{\square} y = x \overset{Y}{\square} y$ for all $x, y \in \mathcal{J}$.

– With the above notation, for all $x \in \mathcal{H}$ and all $v \in \mathcal{B}$, we have the following equalities: $x \overset{U}{\square} v = x \overset{B}{\square} v = x \overset{B}{\square} P_{\mathcal{H}}v = x \overset{T}{\square} P_{\mathcal{H}}v$.

– For all $u \in \mathcal{U}_+$ and all $y \in \mathcal{H}$, $u \overset{U}{\square} y = u \overset{U_+}{\square} y = P_{\mathcal{H}}u \overset{U_+}{\square} y = P_{\mathcal{H}}u \overset{T}{\square} y$.

– For all $x, v \in \mathcal{B}, x \overset{B}{\square} v = Qx \overset{S^*}{\square} Qv + A_*x \overset{R_*}{\square} A_*v$.

– For all $u, y \in \mathcal{U}_+, u \overset{U}{\square} y = Q_*u \overset{S_*}{\square} Q_*y + Au \overset{R}{\square} Ay$.

The following technical lemmas are very useful.

LEMMA 2.2. *Let T be an a.c.c. on $\mathcal{L}(\mathcal{H})$. Then for any $w \in \mathcal{H}$ and any sequence $(u_k)_k$ (resp. $(v_k)_k$) in $\mathcal{H}$ such that $\lim_{k\to\infty} \|u_k \square w\| = 0$ (resp. $\lim_{k\to\infty} \|w \square v_k\| = 0$) we have:*

$$\lim_{k\to\infty} \|Qu_k \square w\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|A_*u_k \square w\| = 0$$

$$(\text{resp. } \lim_{k\to\infty} \|w \square Q_*v_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|w \square Av_k\| = 0).$$

LEMMA 2.3. *Let T be an a.c.c. on $\mathcal{L}(\mathcal{H})$. Then for any $w \in \mathcal{H}$ and any sequence $(u_k)_k$ in $\mathcal{H}$ which tends weakly to 0, we have:*

$$\lim_{k\to\infty} \|w \square Qu_k\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \|Q_*u_k \square w\| = 0.$$

LEMMA 2.4. *Let T be an a.c.c. on $\mathcal{H}$. Then for any $h \in \mathcal{H}$, we have:*

$$\lim_{n\to\infty} \|QT^n h\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|Q_*T^{*n} h\| = 0.$$

LEMMA 2.5. *For any x in $\mathcal{H}$ and any function f in H^∞, we have:*

$$\begin{cases} f(S^*)Qx = Q(f(T)x); \\ f(R_*)A_*x = A_*(f(T)x); \\ f(S_*^*)Q_*x = Q_*(f(T^*)x); \\ f(R^*)Ax = A(f(T^*)x). \end{cases}$$

Besides the sesquilinear map $(x, y) \in \mathcal{H} \times \mathcal{H} \to x \square y \in L^1/H_0^1$ there is also a fundamental functional sesquilinear map associated to an a.c.c. T. It is convenient to define it first for its minimal unitary dilation, U. Since U is absolutely continuous, the family $\{\mu_{x,y},\ x, y \in \mathcal{U}\}$ of elementary spectral measures attached to U (defined by $\int_{\mathbb{T}} f \,\mathrm{d}\mu_{x,y} = (f(U)x, y)$ for f continuous on $\mathbb{T}$) provides a sesquilinear map $(x, y) \to x \overset{U}{\cdot} y = \frac{\mathrm{d}\mu_{x,y}}{\mathrm{d}m}$ from $\mathcal{U} \times \mathcal{U}$ into $L^1(\mathbb{T})$. If U is M_α (the operator of multiplication by the position function) acting on some vector-valued Lebesgue Hilbert space $L^2(\mathbb{T}, E)$ then $x \cdot y(\xi) = \langle x(\xi), y(\xi)\rangle_E, \xi \in \mathbb{T}$. In particular if $E = \mathbb{C}$ then $x \cdot y = x\overline{y}$. Note also an immediate consequence of the definitions:

$$[x \cdot y] = x \square y \quad \text{for all } x, y \in \mathcal{H}.$$

If T is an arbitrary a.c.c. on $\mathcal{H}$ we define $x \overset{T}{\cdot} y$ for $x, y \in \mathcal{H}$ by $x \overset{T}{\cdot} y = x \overset{U}{\cdot} y$.

3. PRELIMINARY RESULTS

First, we give lemmas which are important steps in the proof of the next propositions.

LEMMA 3.1. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is an a.c.c. acting on $\mathcal{H}$. Then for any $h \in \mathcal{H}$ we have:*

$$\lim_{n\to\infty} \|T^n h \square w\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|w \square T^{*n} h\| = 0, \quad w \in \mathcal{H}.$$

Proof. We will just give the proof of the first assertion. The second one can be deduced from similar arguments. Indeed we have:

$$\|A_* T^n h \square w\| = (g_n(R_*) A_* T^n h, w), \quad w \in \mathcal{H}$$

for some g_n of norm 1 in $H^\infty(\mathbb{T})$; thus (recall that $A_* T^n h = R_*^n A_* h$)

$$\|A_* T^n h \square w\| = (g_n(R_*) R_*^n A_* h, w) = \langle g_n \alpha^n, A_* h \square w\rangle, \quad w \in \mathcal{H}.$$

Since the sequence $(g_n \alpha^n)_n$ converges weak* to 0 in $H^\infty(\mathbb{T})$, we obtain:

$$\lim_{n\to\infty} \|A_* T^n h \square w\| = 0, \quad w \in \mathcal{H}.$$

This combined with the fact that $\lim_{n\to\infty} \|QT^n h\| = 0$ (see Lemma 2.4) easily leads to the above lemma. ∎

Now we present two lemmas of factorization which will be important steps in the proof of the lemmas of approximation.

LEMMA 3.2. *Let σ be a Borel subset of $\mathbb{T}$, let l be in $L^1(\sigma)$ and f, g be some elements of $L^2(\sigma)$, $0 < \rho \leqslant 1/2$. Then there exist $u \in H^2$ and $(c_n)_{n\in\mathbb{N}}$ in $L^2(\sigma)$ such that:*

$$\begin{cases} l + f\cdot g = (f + \alpha^n u)\cdot c_n, & n \in \mathbb{N};\\ \|c_n\|_2 \leqslant \frac{1}{1-\rho}(\|l\|_1^{1/2} + \|g\|_2), & n \in \mathbb{N};\\ \|u\|_2 \leqslant 2\|l\|_1^{1/2}. \end{cases}$$

Similarly there exist also $v \in H^2$ and $(d_n)_{n\in\mathbb{N}}$ in $L^2(\sigma)$ such that:

$$\begin{cases} l + f\cdot g = d_n\cdot (g + \alpha^n v), & n \in \mathbb{N};\\ \|d_n\|_2 \leqslant \frac{1}{1-\rho}(\|l\|_1^{1/2} + \|f\|_2), & n \in \mathbb{N};\\ \|v\|_2 \leqslant 2\|l\|_1^{1/2}. \end{cases}$$

Proof. We will just give the proof of the first assertion. The second one can be deduced from similar arguments. Let $0 < \rho \leqslant 1/2$, $\varepsilon > 0$ such that $\varepsilon \leqslant \frac{\rho}{2-\rho}$. Since $|l| + \nu$ is log-integrable for any $\nu > 0$, there exists a function l' in H^2 such that $|l| + \nu = |l'|^2$ (cf. [11], p. 53). Moreover, we can choose $\nu > 0$ in order to have $\|l'\|_2 \leqslant (1+\varepsilon)\|l\|_1^{1/2}$. If we set $\overline{l''} := l/l'$, it is clear that $l'' \in L^2(\sigma)$ and that we have $l = l' \cdot l''$.

We set $\Omega := \{e^{it}; |f|(e^{it}) < |l'|(e^{it})\}$ and let θ be a function in H^∞ such that:

$$|\theta| = \begin{cases} 2-\rho & \text{on } \Omega; \\ \rho & \text{otherwise.} \end{cases}$$

The existence of such a function is granted by [11], p. 53. We obtain that:

$$|f + \theta\alpha^n l'| \geqslant (1-\rho)\max\{|f|, |l'|\}.$$

We set $Z := \{e^{it} \in \sigma; f(e^{it}) = -(\theta\alpha^n l')(e^{it})\}$ and we define the function c_n by:

$$\overline{c_n}(e^{it}) = \begin{cases} \dfrac{l + f\cdot g}{f + \theta\alpha^n l'}(e^{it}) & \text{if } e^{it} \in \sigma \setminus Z; \\ 0 & \text{otherwise.} \end{cases}$$

We easily get: $|c_n| \leqslant \frac{1}{1-\rho}(|l''| + |g|)$, which proves that $c_n \in L^2(\sigma)$ and moreover $\|c_n\|_2 \leqslant \frac{1}{1-\rho}(\|l\|_1^{1/2} + \|g\|_2)$. We obtain that:

$$l + f\cdot g = (f + \alpha^n u)\cdot c_n \quad \text{where } u \in H^2, u = \theta l'.$$

So we get $\|u\|_2 \leqslant \|\theta\|_\infty \|l'\|_2 \leqslant 2\|l\|_1^{1/2}$ and the proof is complete. ∎

The proof of the next lemma of factorization is left to the reader since it uses similar arguments. The starting point is the fact that any function $l \in L^1(\sigma)$ can be written $l = l' \cdot l''$ where l' and l'' are some elements of $L^2(\sigma)$.

LEMMA 3.3. *Let σ be a Borel subset of $\mathbb{T}$, let l be in $L^1(\sigma)$ and let f, g be some elements of $L^2(\sigma)$. Then there exist $u \in L^2(\sigma)$ and $(c_n)_{n\in\mathbb{N}}$ in $L^2(\sigma)$ such that:*

$$\begin{cases} l + f\cdot g = (f + \alpha^n u)\cdot c_n, & n \in \mathbb{N}; \\ \|c_n\|_2 \leqslant \|l\|_1^{1/2} + \|g\|_2, & n \in \mathbb{N}; \\ \|u\|_2 \leqslant 2\|l\|_1^{1/2}. \end{cases}$$

There exist also $v \in L^2(\sigma), (d_n)_{n\in\mathbb{N}}$ in $L^2(\sigma)$ such that:

$$\begin{cases} l + f\cdot g = d_n\cdot(g + \alpha^n v), & n \in \mathbb{N}; \\ \|d_n\|_2 \leqslant \|l\|_1^{1/2} + \|f\|_2, & n \in \mathbb{N}; \\ \|v\|_2 \leqslant 2\|l\|_1^{1/2}. \end{cases}$$

Notation. From now on, if $\mathcal{R}_*^1$ (resp. $\mathcal{R}^1$) is a reducing subspace for R_* (resp. R), we shall denote by A_{*1} (resp. A_1) the orthogonal projection onto $\mathcal{R}_*^1$ (resp. $\mathcal{R}^1$) and A_{*2} (resp. A_2) the orthogonal projection onto $\mathcal{R}_* \ominus \mathcal{R}_*^1$ (resp. $\mathcal{R} \ominus \mathcal{R}^1$).

If a unitary operator $U \in \mathcal{L}(\mathcal{U})$ is of multiplicity greater than or equal to n on a Borel set σ, we write: $\mathrm{mult}(U) \geqslant n$ on σ.

Now we give the first lemma of approximation, which can be seen as a localization (in terms of the Borel set) and generalization (in terms of the form of the vectors obtained) of Theorem 3.11 in [8].

Lemma 3.4. *Let T be an a.c.c., B its minimal coisometric extension, $B = S^* \oplus R_*$. Let σ be a Borel subset of $\mathbb{T}$. We suppose that there exists a reducing subspace $\mathcal{R}'_*$ for R_* such that:*

$$\begin{cases} \mathrm{mult}(R'_*) \geqslant 1 \text{ on } \sigma \text{ where } R'_* := R_{*|\mathcal{R}'_*}; \\ \mathrm{span}\,\{R_*{}^n(\mathcal{R}'_* \cap (A_*\mathcal{H})^-),\ n \in \mathbb{Z}\} = \mathcal{R}'_*. \end{cases}$$

Let $a \in \mathcal{H}, b \in \mathcal{R}_*, F \in L^1(\sigma), \varepsilon > 0, 0 < \rho \leqslant 1/2$. Then there exist $h \in \mathcal{H}, c_n \in \mathcal{R}_*^1$ ($\mathcal{R}_*^1$ is a reducing subspace for R_* included in $\mathcal{R}'_*$ such that $R_{*|\mathcal{R}_*^1}$ is unitarily equivalent to M_σ) such that, for any $n \geqslant 0$, we have:

$$\begin{cases} \|F + A_*a \cdot b - A_*(a + T^n h) \cdot (c_n + b_2)\| < \varepsilon \text{ where } b_2 = b - A_{*1}b; \\ \|c_n\| \leqslant \frac{1}{1-\rho}(\|A_{*1}b\| + \|F\|_1^{1/2}). \end{cases}$$

Moreover, we may assume $\|T^n h\| \leqslant 2\|F\|_1^{1/2}$ for all $n \geqslant 1$.

Proof. We consider the isometry W defined by $W := R_{*|(A_*\mathcal{H})^-}$. If $W_{|(A_*\mathcal{H})^- \cap \mathcal{R}'_*}$ is unitary, we get $\mathcal{R}'_* = \mathcal{R}'_* \cap (A_*\mathcal{H})^-$, which implies that $\mathcal{R}'_* \subset (A_*\mathcal{H})^-$. Since $\mathrm{mult}(R'_*) \geqslant 1$ on σ, there exists a reducing subspace $\mathcal{R}_*^1$ for R_* such that $\mathcal{R}_*^1 \subset (A_*\mathcal{H})^-$ and such that $R_{*|\mathcal{R}_*^1}$ is unitarily equivalent to M_σ.

If $W_{|(A_*\mathcal{H})^- \cap \mathcal{R}'_*}$ is not unitary, using the Wold decomposition, we know there exists a reducing subspace $\mathcal{W}_*^1$ for W such that $W_{|\mathcal{W}_*^1} (= R_{*|\mathcal{W}_*^1})$ is unitarily equivalent to S, the standard unilateral shift acting on H^2. Moreover we have $\mathcal{W}_*^1 \subset (A_*\mathcal{H})^-$. We recall that in this case there also exists a reducing subspace $\mathcal{R}_*^1$ for R_* such that $R_{*|\mathcal{R}_*^1}$ is unitarily equivalent to M_σ.

We write: $F + A_*a \cdot b = F + A_{*1}a \cdot A_{*1}b + A_{*2}a \cdot A_{*2}b$. First we modify $F + A_{*1}a \cdot A_{*1}b$. In the particular case where $\mathcal{R}_*^1 \subset (A_*\mathcal{H})^-$, using Lemma 3.3

and the vector-function identification $\mathcal{R}_*^1 = L^2(\sigma)$, there exist $u \in \mathcal{R}_*^1$, $(c_n)_n$ in $\mathcal{R}_*^1$ such that:

$$\begin{cases} F + A_{*1}a \cdot A_{*1}b = (A_{*1}a + R_*^n u) \cdot c_n; \\ \|c_n\| \leqslant \|F\|_1^{1/2} + \|A_{*1}b\|; \\ \|u\| \leqslant 2\|F\|_1^{1/2}. \end{cases}$$

If $\mathcal{R}_*^1 \not\subset (A_*\mathcal{H})^-$, using Lemma 3.2 and the natural vector-function identifications $\mathcal{R}_*^1 = L^2(\sigma), \mathcal{W}_*^1 = H^2$, there exist $u' \in \mathcal{W}_*^1 \subset (A_*\mathcal{H})^-, (c_n')_n$ in $\mathcal{R}_*^1$ such that:

$$\begin{cases} F + A_{*1}a \cdot A_{*1}b = (A_{*1}a + R_*^n u') \cdot c_n'; \\ \|c_n'\| \leqslant \frac{1}{1-\rho}(\|F\|_1^{1/2} + \|A_{*1}b\|); \\ \|u'\| \leqslant 2\|F\|_1^{1/2}. \end{cases}$$

Since $\mathcal{R}_*^1$ is a reducing subspace for R_*, we get:

$$F + A_*a \cdot A_*b = (A_*a + R_*^n u) \cdot (c_n + A_{*2}b) \qquad c_n, u \in \mathcal{R}_*^1 \subset (A_*\mathcal{H})^-$$

or

$$F + A_*a \cdot A_*b = (A_*a + R_*^n u') \cdot (c_n' + A_{*2}b) \qquad c_n' \in \mathcal{R}_*^1, u' \in \mathcal{W}_*^1 \subset (A_*\mathcal{H})^-.$$

We assume $\|c_n + A_{*2}b\| \neq 0$ and $\|c_n' + A_{*2}b\| \neq 0$ otherwise the proof is immediate taking $h = 0, c_n = A_{*1}b$. Now, according to the inclusion $\mathcal{R}_*^1 \subset (A_*\mathcal{H})^-$ or $\mathcal{W}_*^1 \subset (A_*\mathcal{H})^-$, we are able to find $h \in \mathcal{H}$, $h' \in \mathcal{H}$ so that:

$$\begin{cases} \|u - A_*h\| < \dfrac{\varepsilon}{2\|c_n + A_{*2}b\|}; \\ \|A_{*2}h\| < \dfrac{\varepsilon}{2\|c_n + A_{*2}b\|}; \\ \|A_*h\| < \|u\|; \end{cases} \quad \text{and} \quad \begin{cases} \|u' - A_*h'\| < \dfrac{\varepsilon}{2\|c_n' + A_{*2}b\|}; \\ \|A_{*2}h'\| < \dfrac{\varepsilon}{2\|c_n' + A_{*2}b\|}; \\ \|A_*h'\| < \|u'\|. \end{cases}$$

Since R_* is an isometry and considering the equality $R_*^n A_{*1}h = A_{*1}T^n h$, we get:

$$\|R_*^n u - A_{*1}T^n h\| < \frac{\varepsilon}{\|c_n + A_{*2}b\|}$$

and

$$\|R_*^n u' - A_{*1}T^n h'\| < \frac{\varepsilon}{\|c_n' + A_{*2}b\|},$$

which easily leads to the desired inequality. Since $\lim_{n\to\infty} \|T^n h\| = \|A_*h\|$ and since $\|A_*h\| < \|u\|$, if n is large enough, we can get $\|T^n h\| \leqslant 2\|F\|_1^{1/2}$. Thus, replacing h by $T^{n_0}h$ where n_0 is a sufficiently large integer, we may assume $\|T^n h\| \leqslant 2\|F\|_1^{1/2}$ for all $n \geqslant 1$. ∎

Now we state the dual version of the previous lemma, whose proof is left to the reader since it can be deduced from similar arguments.

LEMMA 3.5. *Let T be an a.c.c., U_+ its minimal isometric dilation, $U_+ = S_* \oplus R$. Let σ be a Borel subset of $\mathbb{T}$. We suppose that there exists a reducing subspace $\mathcal{R}'$ for R such that:*

$$\begin{cases} \text{mult}(R') \geqslant 1 \text{ on } \sigma \text{ where } R' := R_{|\mathcal{R}'}; \\ \text{span}\{R^n(\mathcal{R}' \cap (A\mathcal{H})^-), n \in \mathbb{Z}\} = \mathcal{R}'. \end{cases}$$

Let $a \in \mathcal{R}, b \in \mathcal{H}, F \in L^1(\sigma), \varepsilon > 0, 0 < \rho \leqslant 1/2$. Then there exist $h \in \mathcal{H}, d_n \in \mathcal{R}_1$ ($\mathcal{R}_1$ is a reducing subspace for R included in $\mathcal{R}'$ such that $R_{|\mathcal{R}_1}$ is unitarily equivalent to M_σ) such that, for any $n \geqslant 0$, we have:

$$\begin{cases} \|F + a \cdot Ab - (d_n + a_2) \cdot A(b + T^{*n}h)\| < \varepsilon \text{ where } a_2 = a - A_1 a; \\ \|d_n\| \leqslant \frac{1}{1-\rho}(\|A_1 a\| + \|F\|_1^{1/2}). \end{cases}$$

Moreover, we may asume $\|T^{*n}h\| \leqslant 2\|F\|_1^{1/2}$ for all $n \geqslant 1$.

For an $\mathbb{A}_{1,\aleph_0}$ version of Lemma 3.4 and an $\mathbb{A}_{\aleph_0,1}$ version of Lemma 3.5, we need the following lemma.

LEMMA 3.6. *Let σ be a Borel subset of $\mathbb{T}$. Let $(F_k)_{k\geqslant 1}$ be a norm summable sequence of functions in $L^1(\sigma)$, let $f, (g_k)_{k\geqslant 1}$ be in $L^1(\sigma), 0 < \rho \leqslant 1/2$. Then for every $k \geqslant 1$, there exist u in H^2, $(c_{k,n})_n$ in $L^2(\sigma)$ such that:*

$$\begin{cases} F_k + f \cdot g_k = (f + \alpha^n u) \cdot c_{k,n}; \\ \|u\|_2 \leqslant 2\Big(\sum\limits_{k\geqslant 1} \|F_k\|_1\Big)^{1/2}; \\ \|c_{k,n}\|_2 \leqslant \frac{1}{1-\rho}(\|F_k\|_1^{1/2} + \|g_k\|_2) \quad \text{for any } n \in \mathbb{N}. \end{cases}$$

There also exist v in H^2, $(d_{k,n})_n$ in $L^2(\sigma)$ such that:

$$\begin{cases} F_k + f_k \cdot g = d_{k,n} \cdot (g + \alpha^n v); \\ \|v\|_2 \leqslant 2\Big(\sum\limits_{k\geqslant 1} \|F_k\|_1\Big)^{1/2}; \\ \|d_{k,n}\|_2 \leqslant \frac{1}{1-\rho}(\|F_k\|_1^{1/2} + \|f_k\|_2) \quad \text{for any } n \in \mathbb{N}. \end{cases}$$

Proof. We just give the proof of the first assertion. The second one can be deduced from similar arguments. Let $F = \sum\limits_{k\geqslant 1} |F_k|$ and let $0 < \rho \leqslant 1/2$, $\varepsilon > 0$ be some positive reals such that $\varepsilon \leqslant \frac{\rho}{2-\rho}$. Since $F + \nu$ is log-integrable for any

$\nu > 0$, there exists a function $h \in H^2$ such that $F + \nu = |h|^2$. Moreover we can choose $\nu > 0$ in order to have $\|h\|_2 \leqslant (1+\varepsilon)\|F\|_1^{1/2}$. We consider the Borel set $\Omega = \{e^{it}; |f|(e^{it}) < |h|(e^{it})\}$ and we define a function $\theta \in H^\infty$ such that:

$$|\theta| = \begin{cases} 2-\rho & \text{on } \Omega; \\ \rho & \text{otherwise.} \end{cases}$$

The existence of the functions h and θ are granted by [11], p. 53. Then, for a given n, define the measurable function $c_{k,n}$ by:

$$\overline{c_{k,n}} = \begin{cases} \dfrac{F_k + f \cdot g_k}{f + \theta\alpha^n h} & \text{on } \sigma \setminus \{f + \theta\alpha^n h = 0\}; \\ 0 & \text{elsewhere.} \end{cases}$$

As in Lemma 3.2, we easily get $c_{k,n} \in L^2(\sigma)$ and $\|c_{k,n}\|_2 \leqslant \frac{1}{1-\rho}(\|F_k\|_1^{1/2} + \|g_k\|_2)$. Now we set $u = \theta h \in L^2(\sigma)$. So we get $\|u\|_2 \leqslant 2\Big(\sum\limits_{k\geqslant 1} \|F_k\|_1\Big)^{1/2}$ and $F_k + f \cdot g_k = (f + \alpha^n u) \cdot c_{k,n}$, which ends the proof of the lemma. ∎

Using the previous lemma, we can state the following propositions:

PROPOSITION 3.7. *Let T be an a.c.c., $B = S^* \oplus R_*$ its minimal coisometric extension. Let σ be a Borel subset of $\mathbb{T}$. We suppose that there exists a reducing subspace $\mathcal{R}'_*$ for R_* such that:*

$$\begin{cases} \operatorname{mult}(R'_*) \geqslant 1 \text{ on } \sigma \text{ where } R'_* := R_{*|\mathcal{R}'_*}; \\ \operatorname{span}\{R_*^n(\mathcal{R}'_* \cap (A_*\mathcal{H})^-), n \in \mathbb{Z}\} = \mathcal{R}'_*. \end{cases}$$

Let $a \in \mathcal{H}, (b_k)_k \in \mathcal{R}_*, (F_k)_k$ a norm summable sequence of functions in $L^1(\sigma), \varepsilon > 0, 0 < \rho \leqslant 1/2$. Then there exist $h \in \mathcal{H}$ and $(c_{n,k})_n \in \mathcal{R}_*^1$ ($\mathcal{R}_*^1$ is some reducing subspace for R_* included in $\mathcal{R}'_*$ such that $R_{*|\mathcal{R}_*^1}$ is unitarily equivalent to M_σ) such that, for any $n, k \geqslant 0$, we have:

$$\begin{cases} \|F_k + A_*a \cdot b_k - A_*(a + T^n h) \cdot (c_{n,k} + b_{2,k})\| < \varepsilon \text{ where } b_{2,k} = b_k - A_{*1}b_k; \\ \|c_{n,k}\| \leqslant \frac{1}{1-\rho}(\|A_{*1}b_k\| + \|F_k\|_1^{1/2}). \end{cases}$$

Moreover, we may assume $\|T^n h\| \leqslant 2\Big(\sum\limits_{k\geqslant 1} \|F_k\|_1\Big)^{1/2}$.

As usual this lemma has a dual version whose statement is left to the reader.

PROPOSITION 3.8. *Let T be an a.c.c., U_+ its minimal isometric dilation, $U_+ = S_* \oplus R$. Let σ be a Borel subset of $\mathbb{T}$. We suppose that there exists a reducing subspace $\mathcal{R}'$ for R such that:*

$$\begin{cases} \operatorname{mult}(R') \geqslant 1 \text{ on } \sigma \text{ where } R' := R_{|\mathcal{R}'}; \\ \operatorname{span}\{R^n(\mathcal{R}' \cap (A\mathcal{H})^-), n \in \mathbb{Z}\} = \mathcal{R}'. \end{cases}$$

Let $(a_k)_k \in \mathcal{R}, b \in \mathcal{H}, (F_k)_k$ a norm summable sequence of functions in $L^1(\sigma)$, $\varepsilon > 0, 0 < \rho \leqslant 1/2$. Then there exist $h \in \mathcal{H}, (d_{n,k})_n \in \mathcal{R}_1$ ($\mathcal{R}_1$ is some reducing subspace for R included in $\mathcal{R}'$ such that $R_{|\mathcal{R}_1}$ is unitarily equivalent to M_σ) such that, for any $n \geqslant 0$, we have:

$$\begin{cases} \|F_k + a_k \cdot Ab - (d_{n,k} + a_{2,k}) \cdot A(b + T^{*n}h)\| < \varepsilon \text{ where } a_{2,k} = a_k - A_1 a_k; \\ \|d_{n,k}\| \leqslant \frac{1}{1-\rho}(\|A_1 a_k\| + \|F_k\|_1^{1/2}). \end{cases}$$

Moreover, we may assume $\|T^{*n}h\| \leqslant 2\Big(\sum_{k\geqslant 1} \|F_k\|_1\Big)^{1/2}$.

Using matricial tools (see for example Part I, Section 1.4 in [7]) which are helpful in making certain arguments more transparent, we easily state the following proposition:

PROPOSITION 3.9. *Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c.c. and let $\{f^{i,j},\ 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant n\}$ (where k and n are some positive integers) be a collection of functions in the unit ball of $L^1(X_T)$. Then there exist sequences of $\mathcal{H}$, $(x^i_m)_m$ and $(y^j_m)_m$ such that:*

$$\begin{cases} \lim\limits_{m\to\infty} \|[f^{i,j}] - x^i_m \Box y^j_m\| = 0; & \\ \lim\limits_{m\to\infty} \|w \Box y^j_m\| = 0, & w \in \mathcal{H}, 1 \leqslant j \leqslant n; \\ \lim\limits_{m\to\infty} \|x^i_m \Box w\| = 0, & w \in \mathcal{H}, 1 \leqslant i \leqslant k; \\ \|x^i_m\| \leqslant n, & m \geqslant 1, 1 \leqslant i \leqslant k; \\ \|y^j_m\| \leqslant k, & m \geqslant 1, 1 \leqslant j \leqslant n. \end{cases}$$

The next propositions (Propositions 3.10, 3.11, and 3.12) are very important tools in the proof of the main results. They show we can get some results of approximation where we can choose the sequences of approximation in the space $\mathcal{H}_1$ or $\mathcal{H}_0$ (it depends on the choice of the elements of the predual of H^∞) with a "vanishing condition" extended to the whole space (recall that the subspaces $\mathcal{H}_1$ and $\mathcal{H}_0$ are those involved in the canonical $C_0. - C_1.$ triangulation introduced in Section 2). The following proposition slightly generalizes Proposition 2.2, Part V in [7]. The main improvement is the fact we achieve the approximation for a collection of functions.

PROPOSITION 3.10. *Suppose T is an a.c.c.. Let $(f^j)_{1\leqslant j\leqslant n}$ be a finite sequence of elements of $L^1(\Sigma_{*,1})$ such that $\|f^j\|_1 \leqslant 1$ (the integer n is arbitrarily large) and let a, b^j be some elements in $\mathcal{H}_1$, $1 \leqslant j \leqslant n$. Let ρ be a positive real such that $\rho < 1/2$. Then there exist sequences $(x_m)_m$ and $(y^j_m)_m$ in $\mathcal{H}_1$ $(1 \leqslant j \leqslant n)$ such that:*

$$\begin{cases} \lim\limits_{m\to\infty} \|[f^j] + a\Box b^j - x_m\Box y^j_m\| = 0, & 1 \leqslant j \leqslant n; \\ \lim\limits_{m\to\infty} \|(x_m - a)\Box w\| = 0, & w \in \mathcal{H}; \\ \|a - x_m\| \leqslant 2\Big(\sum\limits_{j=1}^{n} \|f^j\|_1\Big)^{1/2}, & m \geqslant 1; \\ \|y^j_m\| \leqslant \frac{1}{1-\rho}(\|b^j\| + \|f^j\|_1{}^{1/2}), & 1 \leqslant j \leqslant n, m \geqslant 1. \end{cases}$$

The following proposition is the dual version of Proposition 3.10.

PROPOSITION 3.11. *Suppose T is an a.c.c. and let $(f^i)_{1\leqslant i\leqslant k}$ be a finite sequence of elements of $L^1(\Sigma_1)$ (the integer k is arbitrarily large) and $a^i, b, 1 \leqslant i \leqslant k$ in $\mathcal{H}_1$. Then there exist sequences $(x^i_n)_n$ and $(y_n)_n$ in $\mathcal{H}_1$ $(1 \leqslant i \leqslant k)$ such that:*

$$\begin{cases} \lim\limits_{n\to\infty} \|[f^i] + a^i\Box b - x^i_n\Box y_n\| = 0, & 1 \leqslant i \leqslant k; \\ \lim\limits_{n\to\infty} \|w\Box(y_n - b)\| = 0, & w \in \mathcal{H}; \\ \|y_n - b\| \leqslant 2\Big(\sum\limits_{i=1}^{k} \|f^i\|_1\Big)^{1/2}, & n \geqslant 1; \\ \|x^i_n\| \leqslant \frac{1}{1-\rho}(\|a^i\| + \|f^i\|_1{}^{1/2}), & 1 \leqslant i \leqslant k, n \geqslant 1. \end{cases}$$

The next proposition states precisely the approximation for a collection of functions in $L^1(E_0)$.

PROPOSITION 3.12. *Suppose T is an a.c.c. and let $(f^i)_{1\leqslant i\leqslant k}$ be a finite sequence of elements of $L^1(E_0)$ (the integer k is arbitrarily large) and $a^i, b, 1 \leqslant i \leqslant k$ in $\mathcal{H}_0$. Then there exist sequences $(x^i_n)_n$ and $(y_n)_n$ in $\mathcal{H}_0$ $(1 \leqslant i \leqslant k)$ such that:*

$$\begin{cases} \lim\limits_{n\to\infty} \|[f^i] + a^i\Box b - x^i_n\Box y_n\| = 0, & 1 \leqslant i \leqslant k; \\ \lim\limits_{n\to\infty} \|w\Box(y_n - b)\| = 0, & w \in \mathcal{H}; \\ \|y_n - b\| \leqslant 2\Big(\sum\limits_{i=1}^{k} \|f^i\|_1\Big)^{1/2}, & n \geqslant 1; \\ \|x^i_n\| \leqslant \frac{1}{1-\rho}(\|a^i\| + \|f^i\|_1{}^{1/2}), & 1 \leqslant i \leqslant k, n \geqslant 1. \end{cases}$$

4. TECHNIQUES OF APPROXIMATION INVOLVING MULTIPLICITY

Let $W \in \mathcal{L}(\mathcal{W})$ be an absolutely continuous isometry and let $\widetilde{W} \in \mathcal{L}(\widetilde{\mathcal{W}})$ be its minimal unitary extension. We define the multiplicity of the isometry W as the multiplicity of $\widetilde{W}$.

LEMMA 4.1. *Let $W \in \mathcal{L}(\mathcal{W})$ be an absolutely continuous isometry. Let $\mathcal{M}$ be a subset of $\mathcal{W}$ and let $\mathcal{M}_r$ be the reducing subspace for W generated by $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ the reducing subspace for $\widetilde{W}$ generated by $\mathcal{M}$. Then we have:*

(i) *$\widetilde{\mathcal{M}_r} = \widetilde{\mathcal{M}}$, that is to say, the reducing subspace for $\widetilde{W}$ generated by $\mathcal{M}_r$ is equal to $\widetilde{\mathcal{M}}$, and*

(ii) *$\widetilde{W}^n(\mathcal{W} \ominus \mathcal{M}_r) \perp \widetilde{\mathcal{M}}$, $n \in \mathbb{Z}$ and consequently, the minimal unitary extension of $W_{|\mathcal{W}\ominus\mathcal{M}_r}$ is $\widetilde{W}_{|\widetilde{\mathcal{M}}^\perp}$.*

Proof. By definition we have $\mathcal{M} \subset \widetilde{\mathcal{M}_r}$ where $\widetilde{\mathcal{M}_r}$ is a reducing subspace for $\widetilde{W}$. Hence we get $\widetilde{\mathcal{M}} \subset \widetilde{\mathcal{M}_r}$. If we suppose that the inclusion is a strict one we get a contradiction by minimality of the unitary extension $\widetilde{W}$.

Now, for any $y \in \mathcal{W} \ominus \mathcal{M}_r$ we get:

$$(\widetilde{W}^k y, \widetilde{W}^n \mathcal{M}) = (W^k y, W^n \mathcal{M}) \quad \text{for any } k, n \geqslant 0.$$

Thus we have $\widetilde{W}^n(\mathcal{W} \ominus \mathcal{M}_r) \perp \widetilde{\mathcal{M}}$, $n \in \mathbb{N}$, which implies that the minimal unitary extension of $W_{|\mathcal{W}\ominus\mathcal{M}_r}$ lives on $\widetilde{\mathcal{M}}^\perp$. In fact we have:

$$W = W_{|\mathcal{M}_r} \oplus W_{|\mathcal{W}\ominus\mathcal{M}_r} \quad \text{and} \quad \widetilde{W} = \widetilde{W}_{|\widetilde{\mathcal{M}}} \oplus \widetilde{W}_{|\widetilde{\mathcal{M}}^\perp}$$

where $\widetilde{W}_{|\widetilde{\mathcal{M}}}$ is the minimal unitary extension of $W_{|\mathcal{M}_r}$. By minimality of $\widetilde{W}$ and since the minimal unitary extension of $W_{|\mathcal{W}\ominus\mathcal{M}_r}$ is defined on $\widetilde{\mathcal{M}}^\perp$, we get that the minimal unitary extension of $W_{|\mathcal{W}\ominus\mathcal{M}_r}$ is $\widetilde{W}_{|\widetilde{\mathcal{M}}^\perp}$, which ends the proof of the lemma. ∎

NOTATION. If $U \in \mathcal{L}(\mathcal{U})$ is a unitary operator and if $a_1, \ldots, a_k$ are some elements of $\mathcal{U}$, we denote by $\mathrm{red}_U(a_1, \ldots, a_k)$ the reducing subspace for U generated by $a_1, \ldots, a_k$.

Now, we are able to prove the following lemma which is essential to our understanding of multiplicity.

LEMMA 4.2. *Let σ be a Borel subset of $\mathbb{T}$ and let $T \in \mathcal{L}(\mathcal{H})$ be an a.c.c. Suppose that* $\mathrm{mult}(R_*) \geqslant n$ *(resp.* $\mathrm{mult}(R) \geqslant n$*) on σ and let $x_1, \ldots, x_{n-1}$ be some elements of $\mathcal{H}$. Then there exists a reducing subspace $\mathcal{R}'_*$ (resp. $\mathcal{R}'$) for R_* (resp. R) such that:*

(i) $\mathcal{R}'_* \perp \mathrm{red}_{R_*}(A_*x_1, \ldots, A_*x_{n-1})$*(resp.* $\mathcal{R}' \perp \mathrm{red}_R(Ax_1, \ldots, Ax_{n-1})$*);*

(ii) $\mathrm{mult}(R_{*|\mathcal{R}'_*}) \geqslant 1$*(resp.* $\mathrm{mult}(R_{|\mathcal{R}'}) \geqslant 1$*) on σ;*

(iii) $\mathrm{span}\,\{R_*^n(\mathcal{R}'_* \cap (A_*\mathcal{H})^-), n \in \mathbb{Z}\} = \mathcal{R}'_*$ *(resp.* $\mathrm{span}\,\{R^n(\mathcal{R}' \cap (A\mathcal{H})^-), n \in \mathbb{Z}\} = \mathcal{R}'$*).*

In particular we have $\mathcal{R}'_* \cap (A_*\mathcal{H})^- \neq (0)$ (resp. $\mathcal{R}' \cap (A\mathcal{H})^- \neq (0)$).

Proof. We establish this lemma in the case $\mathrm{mult}(R_*) \geqslant n$ on σ. The assertion of this lemma in the case $\mathrm{mult}(R) \geqslant n$ on σ can be proved using similar arguments. We set $\mathcal{R}''_* := \mathrm{red}_{R_*}(A_*x_1, \ldots, A_*x_{n-1})$. Remark that $\mathcal{R}''_*$ is a closed subspace of $(A_*\mathcal{H})^-$ satisfying $\mathrm{mult}(R_{*|\mathcal{R}''_*}) \leqslant n-1$ on σ. Let us set $W := R_{*|(A_*\mathcal{H})^-}$ and consider $\mathcal{M}_r := \mathrm{red}_W\{A_*x_1, \ldots, A_*x_{n-1}\}$. Using Lemma 4.1, we have:

$$\mathrm{red}_{R_*}\{\mathcal{M}_r\} = \mathrm{red}_{R_*}\{A_*x_1, \ldots, A_*x_{n-1}\} = \mathcal{R}''_*.$$

If $\mathcal{R}''_* = (A_*\mathcal{H})^-$, since R_* is the minimal unitary extension of $W = R_{*|(A_*\mathcal{H})^-}$, we get that $\mathcal{R}''_* = \mathcal{R}_*$, which contradicts the hypothesis $\mathrm{mult}(R_*) \geqslant n$ on σ. Hence we consider the non-trivial invariant subspace $\mathcal{W}'_*$ defined by $\mathcal{W}'_* := (A_*\mathcal{H})^- \ominus \mathcal{R}''_*$. Using Lemma 4.1, if we set $\mathcal{R}'_* := \mathrm{span}\,\{R_*^n\mathcal{W}'_*, n \in \mathbb{Z}\}$, we get:

$$\mathcal{R}'_* \perp \mathcal{R}'_* \quad \text{and} \quad R_* = R_{*|\mathcal{R}'_*} \oplus R_{*|\mathcal{R}''_*}.$$

In particular we have $\mathrm{mult}(R_{*|\mathcal{R}_*'}) \geqslant 1$ on σ and since $\mathcal{W}'_* \subset \mathcal{R}'_* \cap (A_*\mathcal{H})^-$, we get $\mathrm{span}\,\{R_*^n(\mathcal{R}'_* \cap (A_*\mathcal{H})^-), n \in \mathbb{Z}\} = \mathcal{R}'_*$. ∎

Now we can state the following propositions which are essential steps in the proof of the main results, that is to say, sufficient conditions for being in the class $\mathbb{A}_{n,m}$.

PROPOSITION 4.3. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is an a.c.c. and let σ be a Borel set of Σ_*. We also suppose that mult$(R_*) \geqslant k$ on σ. Let $\{f^{i,j},\ 1 \leqslant i \leqslant k,\ j \geqslant 1\}$ be a collection of fuctions in $L^1(\sigma)$ be such that:*

$$\sum_{j \geqslant 1} \|f^{i,j}\|_1 < \infty \quad \textit{for any } i \in \{1, \ldots, k\}$$

and let $a^i \in \mathcal{H}, b^j \in \mathcal{R}_*, 1 \leqslant i \leqslant k, j \geqslant 1$. *Then there exist sequences* $(a^i_n)_n$ *in* $\mathcal{H}$, $(b^j_n)_n$ *in* $\mathcal{R}_*$ *such that:*

$$\begin{cases} \lim_{n\to\infty} \|f^{i,j} + A_*a^i \cdot b^j - A_*a^i_n \cdot b^j_n\| = 0, & 1 \leqslant i \leqslant k,\ j \geqslant 1; \\ \lim_{n\to\infty} \|Q(a^i_n - a^i)\| = 0, & 1 \leqslant i \leqslant k; \\ \lim_{n\to\infty} \|(a^i_n - a^i)\square w\| = 0, & w \in \mathcal{H}\ ,\ 1 \leqslant i \leqslant k. \end{cases}$$

Proof. First we show how to construct a^i_1, b^j_1 for $1 \leqslant i \leqslant k$ and $j \geqslant 1$. Let $t_1, \ldots, t_q$ a finite sequence of elements of $\mathcal{H}, \varepsilon_1 > 0, \nu_1 > 0$ be given. We will proceed via k steps in order to construct those elements. If ${\mathcal{R}^1_*}^{(i)}$ is a reducing subspace for R_*, $i \in \mathbb{N}$, we shall denote by $A^{(i)}_{*1}$ the orthogonal projection onto ${\mathcal{R}^1_*}^{(i)}$ and by $A^{(i)}_{*2}$ the orthogonal projection onto $\mathcal{R}_* \ominus {\mathcal{R}^1_*}^{(i)}$.

First, we approximate $(f^{1,j})_{j\geqslant 1}$ using Proposition 3.7 and Lemma 4.2. Let ${\mathcal{R}^1_*}^{(1)}$ be a reducing subspace for R_* orthogonal to the reducing subspace generated by $\{A_*a_i, 2 \leqslant i \leqslant k\}$, such that $\operatorname{mult}(R_{*|{\mathcal{R}^1_*}^{(1)}}) \geqslant 1$ on σ and such that $\operatorname{span}\{R^n_*({\mathcal{R}^1_*}^{(1)} \cap (A_*\mathcal{H})^-), n \in \mathbb{Z}\} = {\mathcal{R}^1_*}^{(1)}$. Let ρ be a real number satisfying $0 < \rho \leqslant 1/2$. By Proposition 3.7, there exist $(c^{(1)}_{n,j})_{n\geqslant 1}$ in ${\mathcal{R}^1_*}^{(1)}$ and h^1 in $\mathcal{H}$ such that:

$$\begin{cases} \|f^{1,j} + A_*a^1 \cdot A_*b^j - A_*(a^1 + T^n h^1) \cdot (c^1_{n,j} + A^{(1)}_{*2} b^j)\| < \varepsilon_1, & n \geqslant 1; \\ \|c^1_{n,j}\| \leqslant \frac{1}{1-\rho}(\|A^{(1)}_{*1} b^j\| + \|f^{1,j}\|_1^{1/2}), & n \geqslant 1. \end{cases}$$

By Lemma 2.4 and Lemma 3.1, there exists $n_1 \in \mathbb{N}$ such that, for any $n \geqslant n_1$ we have:

$$\begin{cases} \|QT^n h^1\| < \nu_1; \\ \|T^n h^1 \square t_p\| < \nu_1, \quad 1 \leqslant p \leqslant q. \end{cases}$$

Now we set $\underline{a}^1 = a^1 + T^{n_1} h^1$, $\underline{b}^j_1 = c^1_{n_1,j} + A^{(1)}_{*2} b^j$. For any $i \in \{2, \ldots, k\}$, by the choice of ${\mathcal{R}^1_*}^{(1)}$, we get:

$$A_*a^i \cdot A_*b^j = A_*a^i \cdot \underline{b}^j_1, j \geqslant 1.$$

Next, we approximate $(f^{2,j})_{j\geqslant 1}$. Let ${\mathcal{R}^1_*}^{(2)}$ be a reducing subspace for R_* orthogonal to the reducing subspace generated by $\{A_*\underline{a}^1, A_*a_i, 3 \leqslant i \leqslant k\}$ such that $\operatorname{mult}(R_{*|{\mathcal{R}^1_*}^{(2)}}) \geqslant 1$ on σ and such that $\operatorname{span}\{R^n_*({\mathcal{R}^1_*}^{(2)} \cap (A_*\mathcal{H})^-), n \in \mathbb{Z}\} = {\mathcal{R}^1_*}^{(2)}$. By Proposition 3.7, there exist $(c^2_{n,j})_{n\geqslant 1}$ in ${\mathcal{R}^1_*}^{(2)}$ and h^2 in $\mathcal{H}$ such that:

$$\begin{cases} \|f^{2,j} + A_*a^2 \cdot \underline{b}^j_1 - A_*(a^2 + T^n h^2) \cdot (c^2_{n,j} + A^{(2)}_{*2} \underline{b}^j_1)\| < \varepsilon_1, & n \geqslant 1; \\ \|c^1_{n,j}\| \leqslant \frac{1}{1-\rho}(\|A^{(2)}_{*1} \underline{b}^j_1)\| + \|f^{2,j}\|_1^{1/2}), & n \geqslant 1. \end{cases}$$

By Lemma 2.4 and Lemma 3.1, there exists $n_2 \in \mathbb{N}$ such that, for any $n \geqslant n_2$ we have:

$$\begin{cases} \|QT^n h^2\| < \nu_1; \\ \|T^n h^2 \Box t_p\| < \nu_1, \quad 1 \leqslant p \leqslant q. \end{cases}$$

Now we set $\underline{a}^2 = a^2 + T^{n_2} h^2$, $\underline{b}_2^j = c_{n_2,j}^2 + A_{*2}^{(2)} \underline{b}_1^j$. By the choice of $\mathcal{R}_*^{1^{(2)}}$, we get:

$$\begin{cases} A_* a^i \cdot A_* \underline{b}_1^j = A_* a^i \cdot \underline{b}_2^j, & j \geqslant 1,\ 3 \leqslant i \leqslant k; \\ A_* \underline{a}^1 \cdot A_* \underline{b}_1^j = A_* \underline{a}^1 \cdot A_* \underline{b}_2^j, & j \geqslant 1. \end{cases}$$

Then we proceed by a finite induction. The last step consists in the approximation of $(f^{k,j})_{j \geqslant 1}$.

We suppose that $\underline{a}^1, \dots, \underline{a}^{k-1}$ and $\underline{b}_1^j, \dots, \underline{b}_{k-1}^j$ are constructed. Let $\mathcal{R}_*^{1^{(k)}}$ be a reducing subspace for R_* orthogonal to the reducing subspace generated by $\{A_* \underline{a}^i, 1 \leqslant i \leqslant k-1\}$, such that $\operatorname{mult}(R_{*|\mathcal{R}_*^{1(k)}}) \geqslant 1$ on σ and such that we have $\operatorname{span}\{R_*^n(\mathcal{R}_*^{1^{(k)}} \cap (A_*\mathcal{H})^-), n \in \mathbb{Z}\} = \mathcal{R}_*^{1^{(k)}}$. By Proposition 3.7, there exist $(c_{n,j}^k)_{n \geqslant 1}$ in $\mathcal{R}_*^{1^{(k)}}$ and h^k in $\mathcal{H}$ such that:

$$\begin{cases} \|f^{k,j} + A_* a^k \cdot \underline{b}_{k-1}^j - A_*(a^k + T^n h^k) \cdot (c_{n,j}^k + A_{*2}^{(k)} \underline{b}_{k-1}^j)\| < \varepsilon_1, & n \geqslant 1; \\ \|c_{n,j}^k\| \leqslant \frac{1}{1-\rho}(\|A_{*1}^{(k)} \underline{b}_{k-1}^j\| + \|f^{k,j}\|_1^{1/2}), & n \geqslant 1. \end{cases}$$

By Lemma 2.4 and Lemma 3.1, there exists $n_k \in \mathbb{N}$ such that, for any $n \geqslant n_k$ we have:

$$\begin{cases} \|QT^n h^k\| < \nu_1; \\ \|T^n h^k \Box t_p\| < \nu_1, \quad 1 \leqslant p \leqslant q. \end{cases}$$

We set $\underline{a}^k = a^k + T^{n_k} h^k$ and $\underline{b}_k^j = c_{n_k,j}^k + A_{*2}^k \underline{b}_{k-1}^j, j \geqslant 1$. By the choice of $\mathcal{R}_*^{1^{(k)}}$ we get:

$$A_* \underline{a}^i \cdot A_* \underline{b}_{k-1}^j = A_* \underline{a}^i \cdot A_* \underline{b}_k^j, \quad 1 \leqslant i \leqslant k-1.$$

Then we can easily verify that, setting $a_1^i := \underline{a}^i, 1 \leqslant i \leqslant k$ and $b_1^j := \underline{b}_k^j, j \geqslant 1$ we obtain:

$$f^{i,j} + A_* a^i \cdot b^j - A_* a_1^i \cdot b_1^j = f^{i,j} + A_* a^i \cdot \underline{b}_{i-1}^j - A_* \underline{a}^i \cdot \underline{b}_i^j,$$

which implies that: $\|f^{i,j} + A_* a^i \cdot b^j - A_* a_1^i \cdot b_1^j\| \leqslant \varepsilon_1$.

Moreover, by construction, we have:

$$\|Q(a_1^i - a^i)\| < \nu_1, \|(a_1^i - a^i)\Box t_p\| < \nu_1, \quad 1 \leqslant p \leqslant q.$$

The proof of this proposition results from iterations of the previous process, taking $(t_i)_i$ a sequence dense in $\mathcal{H}$ and $(\varepsilon_n)_n, (\nu_n)_n$ some sequences of positive reals decreasing to 0. ∎

REMARK 4.4. By construction, and by Proposition 3.7, we have:

$$\begin{cases} \|a_n^i - a^i\| \leqslant 2\Big(\sum\limits_{j\geqslant 1} \|f^{i,j}\|_1\Big)^{1/2}, & 1 \leqslant i \leqslant k; \\ \|b_n^j\| \leqslant \frac{1}{1-\rho}\Big(\|b^j\| + \sum\limits_{i=1}^{k} \|f^{i,j}\|_1^{1/2}\Big), & j \geqslant 1. \end{cases}$$

PROPOSITION 4.5. *Let T be an a.c.c., $\sigma \subset \Sigma_{*;T}$. Suppose that* $\operatorname{mult}(R_*) \geqslant k$ *on σ. If $\{f^{i,j},\ 1 \leqslant i \leqslant k,\ j \geqslant 1\}$ is a collection of functions in $L^1(\sigma)$ such that $\sum\limits_{j\geqslant 1} \|f^{i,j}\|_1 < \infty$ $(1 \leqslant i \leqslant k)$, and $a^1, \dots, a^k, (b_n^j)_n$ are in $\mathcal{H}$, then there exist sequences $(a_m^i)_m$, $(b_m^j)_m$ $(1 \leqslant i \leqslant k,\ j \geqslant 1)$ in $\mathcal{H}$ such that:*

$$\begin{cases} \lim\limits_{m\to\infty} \|[f^{i,j}] + a^i \Box b^j - a_m^i \Box b_m^j\| = 0, & 1 \leqslant i \leqslant k,\ j \geqslant 1; \\ \lim\limits_{m\to\infty} \|(a_m^i - a^i)\Box w\| = 0, & w \in \mathcal{H},\ 1 \leqslant i \leqslant k; \\ \|a_m^i - a^i\| \leqslant 2\Big(\sum\limits_{j\geqslant 1} \|f^{i,j}\|_1\Big)^{1/2}, & 1 \leqslant i \leqslant k; \\ \|b_m^j\| \leqslant 3\Big(\|b^j\| + \sum\limits_{i=1}^{k} \|f^{i,j}\|_1^{1/2}\Big), & j \geqslant 1; \\ a_m^i - a^i \rightharpoonup 0, & 1 \leqslant i \leqslant k. \end{cases}$$

Proof. We can write:

$$[f^{i,j}] + a^i \Box b^j = [f^{i,j}] + A_* a^i \Box A_* b^j + Q a^i \Box Q b^j.$$

Using multiplicity on σ and Proposition 4.3, we can find sequences $(a_m^i)_m$ $(1 \leqslant i \leqslant k)$ in $\mathcal{H}$ and $(b'^j_m)_m$ $(j \geqslant 1)$ in $\mathcal{R}_*$ such that:

$$\begin{cases} (1) \quad \lim\limits_{m\to\infty} \|[f^{i,j}] + A_* a^i \Box A_* b^j - (A_* a_m^i \Box A_* b'^j_m)\| = 0, & 1 \leqslant i \leqslant k,\ j \geqslant 1; \\ (2) \quad \lim\limits_{m\to\infty} \|(a_m^i - a^i)\Box w\| = 0, & w \in \mathcal{H},\ 1 \leqslant i \leqslant k; \\ (3) \quad \lim\limits_{m\to\infty} \|Q(a_m^i - a^i)\| = 0, & 1 \leqslant i \leqslant k; \\ (4) \quad \|a_m^i - a^i\| \leqslant 2\Big(\sum\limits_{j\geqslant 1} \|f^{i,j}\|\Big)^{1/2}, & 1 \leqslant i \leqslant k; \\ (5) \quad \|A_* b'^j_m\| \leqslant 2\Big(\|b^j\| + \sum\limits_{i=1}^{k} \|f^{i,j}\|_1^{1/2}\Big), & j \geqslant 1. \end{cases}$$

Moreover, for $1 \leqslant i \leqslant k$, $j \geqslant 1$, we have:

$$\lim_{m\to\infty} (A_* a_m^i \Box A_* b'^j_m) + Q a^i \Box Q b^j = \lim_{m\to\infty} a_m^i \Box (Q b^j + A_* b'^j_m) - Q a_m^i \Box Q b^j + Q a^i \Box Q b^j.$$

But, we can work on the two last items as follows ($1 \leqslant i \leqslant k, j \geqslant 1$):

$$Qa^i \Box Qb^j - Qa^i_m \Box Qb^j = Q(a^i - a^i_m) \Box Qb^j = (a^i - a^i_m) \Box Qb^j$$

which tends to 0 when m becomes large by (2). Finally, the approximation is established with $b^j_m = Qb^j + A_* b'^j_m$, which gives, via (5),

$$\|b^j_m\| \leqslant 2\|A_* b^j\| + \|Qb^j\| + 2\sum_{i=1}^{k} \|f^{i,j}\|_1^{1/2},$$

that is:

$$\|b^j_m\| \leqslant 3\Big(\|b^j\| + \sum_{i=1}^{k} \|f^{i,j}\|_1^{1/2}\Big), \quad j \geqslant 1, m \geqslant 1. \quad \blacksquare$$

Now we give (without proof) a dual version of the Proposition 4.3.

PROPOSITION 4.6. *Suppose $T \in \mathcal{L}(\mathcal{H})$ is an a.c.c. and let σ be a Borel set of Σ. We also suppose that* $\mathrm{mult}(R) \geqslant n$ *on σ. Let $\{f^{i,j},\ i \geqslant 1,\ 1 \leqslant j \leqslant n\}$ be a collection of functions in $L^1(\sigma)$ be such that:*

$$\sum_{i \geqslant 1} \|f^{i,j}\|_1 < \infty \qquad \textit{for any } j \in \{1, \dots, n\}$$

and let $a^i \in \mathcal{R}, b^j \in \mathcal{H}, 1 \leqslant j \leqslant n, i \geqslant 1$. Then there exist sequences $(a^i_m)_m$ in $\mathcal{R}$, $(b^j_m)_m$ in $\mathcal{H}$ such that:

$$\begin{cases} \lim_{m\to\infty} \|f^{i,j} + a^i \cdot Ab^j - a^i_m \cdot Ab^j_m\| = 0, & 1 \leqslant j \leqslant n,\ i \geqslant 1; \\ \lim_{m\to\infty} \|Q_*(b^j_m - b^j)\| = 0, & 1 \leqslant j \leqslant n; \\ \lim_{m\to\infty} \|w \Box (b^j_m - b^j)\| = 0, & w \in \mathcal{H},\ 1 \leqslant j \leqslant n. \end{cases}$$

REMARK 4.7. Moreover, if ρ is a real satisfying $0 < \rho \leqslant 1/2$. we have:

$$\begin{cases} \|b^j_m - b^j\| \leqslant 2\Big(\sum_{i\geqslant 1} \|f^{i,j}\|_1\Big)^{1/2}, & 1 \leqslant j \leqslant n; \\ \|a^i_m\| \leqslant \frac{1}{1-\rho}\Big(\|a^i\| + \sum_{j=1}^{n} \|f^{i,j}\|_1^{1/2}\Big), & i \geqslant 1. \end{cases}$$

We now give a dual version of the Proposition 4.5:

PROPOSITION 4.8. *Let T be an a.c.c., $\sigma \subset \Sigma_T$. Suppose that* $\operatorname{mult}(R) \geqslant n$ *on σ. If $\{f^{i,j},\ 1 \leqslant j \leqslant n,\ i \geqslant 1\}$ is a collection of functions in $L^1(\sigma)$ such that $\sum_{i\geqslant 1} \|f^{i,j}\|_1 < \infty$, and $b^1, \dots, b^n, a^i,\ i \geqslant 1$ are in $\mathcal{H}$, then there exist sequences $(b^j_m)_m$ $(1 \leqslant j \leqslant n)$, $(a^i_m)_m$ $(j \geqslant 1)$ in $\mathcal{H}$ such that:*

$$\begin{cases} \lim\limits_{m\to\infty} \|[f^{i,j}] + a^i \square b^j - a^i_m \square b^j_m\| = 0, & i \geqslant 1,\ 1 \leqslant j \leqslant n; \\ \lim\limits_{m\to\infty} \|w \square (b^j_m - b^j)\| = 0, & w \in \mathcal{H},\ 1 \leqslant j \leqslant n; \\ \|(b^j_m - b^j)\| \leqslant 2\Big(\sum\limits_{i\geqslant 1} \|f^{i,j}\|_1\Big)^{1/2}, & 1 \leqslant j \leqslant n; \\ \|a^i_m\| \leqslant 3\Big(\|a^i\| + \sum\limits_{j=1}^{n} \|f^{i,j}\|_1^{1/2}\Big), & i \geqslant 1; \\ b^j_m - b^j \rightharpoonup 0, & 1 \leqslant j \leqslant n. \end{cases}$$

Now we give a new result for a certain triangulation of an absolutely continuous contraction.

THEOREM 4.9. *Let $T \in \mathcal{L}(\mathcal{H})$ be an a.c.c. such that: $T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}$ relative to some orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Then we have:*

$$E^r_T = E^r_{T_1} \cup E^r_{T_2} \quad \text{and} \quad E^l_T = E^l_{T_1} \cup E^l_{T_2}.$$

Proof. What we have to do is to show that for any $F \in L^1(\mathbb{T} \backslash E^r_{T_2})$, $\|F\|_1 \leqslant 1$, the class $[F]$ is such that there exist two sequences $(u_n)_n, (v_n)_n$ in $\mathcal{H}_1$ with

$$\begin{cases} \lim\limits_{n\to\infty} \|[F] - u_n \square v_n\| = 0; \\ \lim\limits_{n\to\infty} \|u_n \square w\| = 0, & w \in \mathcal{H}_1. \end{cases}$$

For this, it is sufficient to prove that given $\varepsilon > 0$, $w_1, \dots, w_p \in \mathcal{H}_1$, there exist $u, v \in \mathcal{H}_1$ such that:

$$\begin{cases} \|[F] - u \square v\| < \varepsilon; \\ \|u \square w_q\| < \varepsilon, & q = 1, \dots, p. \end{cases}$$

Suppose $\mathbb{T} = E^r_T$ up to a Borel set of Lebesgue measure 0. For any $\lambda \in \mathbb{D}$, there exists a sequence $(x_{n,\lambda})_n$ in $\mathcal{H}$ such that:

$$\begin{cases} \lim\limits_{n\to\infty} \|E_\lambda - x_{n,\lambda} \square x_{n,\lambda}\| = 0; \\ \lim\limits_{n\to\infty} \|x_{n,\lambda} \square w\| = 0, & w \in \mathcal{H}. \end{cases}$$

We set $x_{n,\lambda}^1 = P_{\mathcal{H}_1}x_{n,\lambda}$ and $x_{n,\lambda}^2 = P_{\mathcal{H}_2}x_{n,\lambda}$. Since $\|E_\lambda\| = 1$, without loss of generality we may assume that $\|x_{n,\lambda}\| = 1$ (all n, all λ) and also (removing for each λ a suitable subsequence) that the sequence $(\|x_{n,\lambda}^2\|)_n$ is convergent. Let $\gamma_\lambda = \lim_{n\to\infty} \|x_{n,\lambda}^2\|$. Relative to the choice of the sequences $(x_{n,\lambda})_n$ we define, for $0 < \gamma < 1$, the sets:

$$\mathcal{D}_\gamma := \{\lambda \in \mathbb{D};\ \gamma_\lambda < \gamma\}.$$

Note that if $\lambda \notin \mathcal{D}_\gamma$ then:

$$\limsup_{n\to\infty} \|E_\lambda - x_{n,\lambda}^2 \Box x_{n,\lambda}^2\| \leqslant \limsup_{n\to\infty} \|x_{n,\lambda} \Box x_{n,\lambda}^1\| \leqslant (1-\gamma)^{1/2}.$$

Moreover, we have:

$$\lim_{n\to\infty} \|x_{n,\lambda}^2 \Box w\| = \lim_{n\to\infty} \|x_{n,\lambda} \Box w\|, \quad w \in \mathcal{H}_2.$$

Thus we obtain that $\mathrm{NTL}(\mathbb{D} \setminus \mathcal{D}_\gamma) \subset E_{T_2}^{\mathrm{r}}$ (where $\mathrm{NTL}(E)$ denotes the set of all nontangential limits of points in E), that is, $\mathbb{T} \setminus E_{T_2}^{\mathrm{r}} \subset \mathrm{NTL}(\mathcal{D}_\gamma)$ for any $\gamma \in]0,1[$. Let γ be a positive real satisfying $\gamma < \min\left\{\frac{\varepsilon}{10}, \frac{\varepsilon}{4\max\{\|w_q\|, 1\leqslant q\leqslant p\}}\right\}$. Since $\mathcal{D}_\gamma$ is dominating for $\mathbb{T} \setminus E_{T_2}^{\mathrm{r}}$, there exist $\lambda_1, \dots \lambda_n$ in $\mathbb{D}$ such that $\gamma_{\lambda_j} < \gamma, j = 1, \dots, n$, and $\alpha_1, \dots, \alpha_n \in \mathbb{D}$ such that $\sum_{j=1}^n |\alpha_j| < 1$ satisfying $\left\|[F] - \sum_{j=1}^n \alpha_j E_{\lambda_j}\right\| < \frac{\varepsilon}{10}$. For each $j = 1, \dots, n$ we may assume (throwing away if necessary a finite number of terms in each of the n sequences $(x_{i,\lambda_j})_i$) that $\|x_{i,\lambda_j}^2\| < \gamma'$ (where γ' has been chosen such that $\gamma_j < \gamma' < \gamma, 1 \leqslant j \leqslant n$) and

$$\begin{cases} \|E_{\lambda_j} - x_{i,\lambda_j} \Box x_{i,\lambda_j}\| < \dfrac{\varepsilon}{10}, & i \geqslant 1,\ 1 \leqslant j \leqslant n; \\ \|x_{i,\lambda_j} \Box w_q\| < \dfrac{\varepsilon}{4n}, & 1 \leqslant q \leqslant p,\ 1 \leqslant j \leqslant n,\ i \geqslant 1. \end{cases}$$

We set $x_{\nu_j} := x_{\nu_j,\lambda_j}, x_{\nu_j}^1 := P_{\mathcal{H}_1}x_{\nu_j}, x_{\nu_j}^2 := P_{\mathcal{H}_2}x_{\nu_j}$, and if $\nu := (\nu_1, \dots, \nu_n)$, we set $x_\nu := \sum_{j=1}^n \sqrt{\alpha_j} x_{\nu_j}, \overline{x}_\nu = \sum_{j=1}^n \overline{\sqrt{\alpha_j}} x_{\nu_j}, x_\nu^1 = P_{\mathcal{H}_1}x_\nu, x_\nu^2 = P_{\mathcal{H}_2}x_\nu, \overline{x}_\nu^1 = P_{\mathcal{H}_1}\overline{x}_\nu, \overline{x}_\nu^2 = P_{\mathcal{H}_2}\overline{x}_\nu$.

We get:

$$\begin{cases} \left\|[F] - \sum_{j=1}^n \alpha_j x_{\nu_j} \Box x_{\nu_j}\right\| < \dfrac{3\varepsilon}{10}; \\ \|x_\nu \Box w_q\| < \dfrac{\varepsilon}{4}, & q = 1, \dots, p. \end{cases}$$

Since $\|x_\nu^2\| < \gamma$ by construction, we get:

$$\|x_\nu^1 \Box w_q\| < \frac{\varepsilon}{4} + \gamma \max\{\|w_q\|, q = 1, \dots, p\} < \frac{\varepsilon}{2}, q = 1, \dots, p.$$

Since $A_*^{\mathcal{H}_1}h = A_*h$ for any $h \in \mathcal{H}_1$, and since $\|x_{\nu_j} - x_{\nu_j}^1\| = \|x_{\nu_j}^2\| < \gamma < \frac{\varepsilon}{10}$ we can easily conclude that:

$$\left\| \sum_{j=1}^{n} \alpha_j A_* x_{\nu_j} \Box A_* x_{\nu_j} - \sum_{j=1}^{n} \alpha_j A_*^{\mathcal{H}_1} x_{\nu_j}^1 \Box A_*^{\mathcal{H}_1} x_{\nu_j}^1 \right\| < \frac{\varepsilon}{5}.$$

Using the vanishing condition satisfied by the sequence $(x_{i,\lambda_j})_i$ we can find ν such that:

$$\left\| \sum_{j=1}^{n} \alpha_j Q x_{\nu_j} \Box Q x_{\nu_j} - Q x_\nu \Box Q \overline{x}_\nu \right\| < \frac{\varepsilon}{10}$$

(see for example Proposition 1.3, Part III in [6]). For such a ν, we transform $\sum_{j=1}^{n} \alpha_j A_*^{\mathcal{H}_1} x_{\nu_j}^1 \Box A_*^{\mathcal{H}_1} x_{\nu_j}^1$ in $A^{\mathcal{H}_1}(x_\nu^1 + \tau) \cdot b$ where $\tau \in \mathcal{H}_1, b \in \mathcal{R}_*^{\mathcal{H}_1}$, with the following inequalities: $\|\tau \Box w_q\| < \frac{\varepsilon}{2}, 1 \leqslant q \leqslant p$ and $\|Q^{\mathcal{H}_1} \tau \Box \overline{x}_\nu^1\| < \frac{\varepsilon}{10}$. In fact we get:

$$\|[F] - (x_\nu^1 + \tau) \Box P_{\mathcal{H}_1}(Q^{\mathcal{H}_1} \overline{x}_\nu^1 + b)\| < \varepsilon$$
$$\|(x_\nu^1 + \tau) \Box w_q\| < \varepsilon \qquad 1 \leqslant q \leqslant p.$$

If $\mathbb{T} \neq E_T^{\mathrm{r}}$ (up to a Borel set of Lebesgue measure 0) we set $\widetilde{T} = M_\sigma \oplus T$, where $\sigma = \mathbb{T} \setminus E_T^{\mathrm{r}}$. Relative to the decomposition $\widetilde{\mathcal{H}} = L^2(\sigma) \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$, we have the following representation of the operator $\widetilde{T}$:

$$\begin{pmatrix} M_\sigma & 0 & 0 \\ 0 & T_1 & * \\ 0 & 0 & T_2 \end{pmatrix}.$$

By construction $E_{\widetilde{T}}^{\mathrm{r}} = \mathbb{T}$ since we have $E_{\widetilde{T}}^{\mathrm{r}} = E_{M_\sigma}^{\mathrm{r}} \cup E_T^{\mathrm{r}} = \sigma \cup E_T^{\mathrm{r}}$. If we set $\widetilde{T}_1 = M_\sigma \oplus T_1$, by what precedes, we get:

$$E_{\widetilde{T}}^{\mathrm{r}} = E_{\widetilde{T}_1}^{\mathrm{r}} \cup E_{T_2}^{\mathrm{r}} = \sigma \cup E_{T_1}^{\mathrm{r}} \cup E_{T_2}^{\mathrm{r}} = \sigma \cup E_T^{\mathrm{r}}.$$

Thus we get $E_T^{\mathrm{r}} = E_{T_1}^{\mathrm{r}} \cup E_{T_2}^{\mathrm{r}}$. Using the equality $X_{T^*} = \overline{X_T}$ and $\Sigma_{*,T^*} = \overline{\Sigma_T}$, (see Proposition 3.5 in [5]) and the equality $E_{T^*}^{\mathrm{r}} = E_{T_1^*}^{\mathrm{r}} \cup E_{T_2^*}^{\mathrm{r}}$, we easily get:

$$E_T^{\mathrm{l}} = E_{T_1}^{\mathrm{l}} \cup E_{T_2}^{\mathrm{l}} \text{ for any triangulation of } T. \quad \blacksquare$$

We can add that for any triangulation of an a.c.c. T such that $T = \begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}$ relative to some orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, we have:

$$\Sigma_{*,T} = \Sigma_{*,T_1} \cup \Sigma_{*,T_2} \quad \text{and} \quad \Sigma_T = \Sigma_{T_1} \cup \Sigma_{T_2}.$$

The above equalities are trivial consequences of Lemma 1.4 in [3]. The flavour of these results is that the boundary sets E_T^{r} and E_T^{l} behave well with respect to (arbitrary) triangulations. With regard to the sets X_T, this behaviour is not completely settled. The inclusion $X_{T_1} \cup X_{T_2} \subset X_T$ is always valid, with equality if T_1 or T_2 is C_0. (cf. Proposition 3.5 and Corollary 6.4 in [5]), but the question wether the equality holds in general is still open.

5. MAIN RESULTS

Recall that T is in the class $\mathbb{A}_{1,\aleph_0}$ if and only if $\mathbb{T} = E_T^{\mathrm{r}}$ (see Theorem 4.6 in [5]). The following result shows how much the multiplicity of $R^{\mathcal{H}_0}$ on $\mathbb{T} \setminus E_T^{\mathrm{r}}$ (equal to the multiplicity of R on $\mathbb{T} \setminus E_T^{\mathrm{r}}$) "pushes" the operator into the class $\mathbb{A}_{1,n}$.

THEOREM 5.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be in the class $\mathbb{A}$ such that* $\mathrm{mult}(R^{\mathcal{H}_0}) \geqslant n$ *on $\mathbb{T} \setminus E_T^{\mathrm{r}}$. Then T belongs to the class $\mathbb{A}_{1,n}$.*

Proof. Of course if $\mathbb{T} = E_T^{\mathrm{r}}$ the conclusion holds since T is in the class $\mathbb{A}_{1,\aleph_0}$. So we consider the case where the Borel set $\mathbb{T} \setminus E_T^{\mathrm{r}}$ has positive Lebesgue measure. We first show how to, approximately and simultaneously, transform elements of the type $[f^j] + a\Box b^j$ in the form $\widetilde{a}\Box\widetilde{b^j}$. Let $f^1, \ldots, f^n$ be in $L^1(\mathbb{T})$, $\varepsilon > 0$, and $b^1, \ldots, b^n, a$ in $\mathcal{H}$. We split the $f^j, 1 \leqslant j \leqslant n$ into pieces: $f^j = f^j_\sigma + f^j_{X_T} + f^j_{\sigma'}$ where $\sigma = \mathbb{T} \setminus E_T^{\mathrm{r}}$, and $\sigma' = E_T^{\mathrm{r}} \setminus (X_T)$ $(\subset \Sigma_{*,T_1})$. Then we get for $1 \leqslant j \leqslant n$:

$$[f^j] + a\Box b^j = ([f^j_\sigma] + a_0\Box b^j_0) + ([f^j_{\sigma'}] + a_1\Box b^j_1) + [f^j_{X_T}] + a_1\Box b^j_0$$

where we refer to notation in preliminaries. We first deal with $[f^j_\sigma] + a_0\Box b^j_0 = [f^j_\sigma] + A^{\mathcal{H}_0}a_0\Box A^{\mathcal{H}_0}b^j_0 + Q_*^{\mathcal{H}_0}a_0\Box Q_*^{\mathcal{H}_0}b^j_0$. Using multiplicity on σ and Proposition 4.8, we can find two sequences $(a_{0,m})_m$ and $(b^j_{0,m})_m$ in $\mathcal{H}_0$ such that:

$$\begin{cases} (1) \quad \lim_{m\to\infty} \|[f^j_\sigma] + a_0\Box b^j_0 - a_{0,m}\Box b^j_{0,m}\| = 0, & 1 \leqslant j \leqslant n; \\ (2) \quad \lim_{m\to\infty} \|w\Box(b^j_{0,m} - b^j_0)\| = 0, & w \in \mathcal{H},\ 1 \leqslant j \leqslant n; \\ (3) \quad \|b^j_{0,m} - b^j_0\| \leqslant 2\|f^j_\sigma\|_1^{1/2}, & m \geqslant 1,\ 1 \leqslant j \leqslant n; \\ (4) \quad \|a_{0,m}\| \leqslant 3\Big(\|a_0\| + \sum_{j=1}^{n} \|f^j_\sigma\|_1^{1/2}\Big), & m \geqslant 1. \end{cases}$$

In fact, using Proposition 4.8, we get that the vanishing condition (2) is obtained for any $w \in \mathcal{H}_0$, but since $w\Box(b^j_{0,m} - b^j_0) = Qw\Box Q(b^j_{0,m} - b^j_0)$ where $(b^j_{0,m} - b^j_0)_j$ tends weakly to 0 and using Lemma 2.2, we get the vanishing condition (2) for any $w \in \mathcal{H}$. Next we use the fact that $\sigma' \subset \Sigma_{*,T_1}$ and Proposition 3.10 to find sequences $(a_{1,p})_p$ and $(b^j_{1,p})_p$ in $\mathcal{H}_1$, $(1 \leqslant j \leqslant n)$ such that:

$$\begin{cases} (5) \quad \lim_{p\to\infty} \|[f^j_{\sigma'}] + a_1\Box b^j_1 - (a_{1,p}\Box b^j_{1,p})\| = 0, & 1 \leqslant j \leqslant n; \\ (6) \quad \lim_{p\to\infty} \|(a_{1,p} - a_1)\Box w\| = 0, & w \in \mathcal{H}; \\ (7) \quad \|b^j_{1,p}\| \leqslant 2(\|b^j_1\| + \|f^j_{\sigma'}\|_1^{1/2}), & 1 \leqslant j \leqslant n; \\ (8) \quad \|a_1 - a_{1,p}\| \leqslant 3\sum_{j=1}^{n} \|f^j_{\sigma'}\|_1^{1/2}, & p \geqslant 1. \end{cases}$$

Finally, we use the property of the set X_T. Using Proposition 3.9, we can find sequences $(x_q)_q$ and $(y_q^j)_q$ in $\mathcal{H}$, $1 \leqslant j \leqslant n$, such that:

$$\begin{cases} (9) \quad \lim_{q\to\infty} \|[f_{X_T}^j] - x_q \square y_q^j\| = 0, & 1 \leqslant j \leqslant n; \\ (10) \quad \lim_{q\to\infty} \|x_q \square w\| = 0, & w \in \mathcal{H}; \\ (11) \quad \lim_{q\to\infty} \|w \square y_q^j\| = 0, & w \in \mathcal{H},\ 1 \leqslant j \leqslant n; \\ (12) \quad \|y_q^j\| \leqslant \|f_{X_T}^j\|_1^{1/2}, & q \geqslant 1,\ 1 \leqslant j \leqslant n; \\ (13) \quad \|x_q\| \leqslant \sum_{j=1}^{n} \|f_{X_T}^j\|_1^{1/2}, & q \geqslant 1. \end{cases}$$

We now put the pieces together; from (1), (5), (9) and the initial decomposition of the elements $[f^j] + a\square b^j$ $(1 \leqslant j \leqslant n)$ we easily deduce the existence of integers M, P, Q such that for any $m > M$, $p > P$, $q > Q$ we have:

$$\|[f^j] + a\square b^j - (a_{0,m}\square b_{0,m}^j + a_{1,p}\square b_{1,p}^j + x_q\square y_q^j + a_1\square b_0^j)\| \leqslant \frac{\varepsilon}{4}, \quad 1 \leqslant j \leqslant n.$$

We can write, for any $1 \leqslant j \leqslant n$:

$$\begin{aligned} a_{0,m}\square b_{0,m}^j + a_{1,p}\square b_{1,p}^j + x_q\square y_q^j + a_1\square b_0^j &= (a_{0,m} + a_{1,p} + x_q)\square(b_{0,m}^j + b_{1,p}^j + y_q^j) \\ &+ (-a_{1,p}\square b_{0,m}^j + a_1\square b_0^j) - (a_{0,m} + a_{1,p})\square y_q^j - x_q\square(b_{0,m}^j + b_{1,p}^j). \end{aligned}$$

Moreover we have:

$$a_1\square b_0^j - a_{1,p}\square b_{0,m}^j = (a_1 - a_{1,p})\square b_0^j + a_{1,p}\square(b_0^j - b_{0,m}^j).$$

Now we use relations (2), (6), (10), (11), and we choose successively,

$p > P$ so that $\|(a_1 - a_{1,p})\square b_0^j\| < \frac{\varepsilon}{4}$, $1 \leqslant j \leqslant n$,

$m > M$ so that $\|(a_{1,p}\square(b_0^j - b_{0,m}^j)\| \leqslant \frac{\varepsilon}{4}$, $1 \leqslant j \leqslant n$ and

$q > Q$ so that $\|(a_{0,m} + a_{1,p})\square y_q^j\| + \|x_q\square(b_{0,m}^j + b_{1,p}^j)\| < \frac{\varepsilon}{4}$, $1 \leqslant j \leqslant n$.

Thus, upon setting $\widetilde{y}^j = y_q^j$, $\widetilde{x} = x_q$ in $\mathcal{H}$, $\widetilde{b}_1^j = b_{1,p}^j$, $\widetilde{a}_1 = a_{1,p}$ in $\mathcal{H}_1$, $\widetilde{b}_0^j = b_{0,m}^j$,

$\widetilde{a}_0 = a_{0,m}$ in $\mathcal{H}_0$ $(1 \leqslant j \leqslant n)$ we have:

$$\begin{cases} \|[f^j] + a\Box b^j - (\widetilde{a}_0 + \widetilde{a}_1 + \widetilde{x})\Box(\widetilde{b}_0^j + \widetilde{b}_1^j + \widetilde{y}^j)\| < \varepsilon, & 1 \leqslant j \leqslant n; \\ \|b_0^j - \widetilde{b}_0^j\| \leqslant 2\|f^j\|_1^{1/2}, & 1 \leqslant j \leqslant n; \\ \|\widetilde{a}_1 - a_1\| \leqslant 3 \sum_{j=1}^{n} \|f^j\|_1^{1/2}; & \\ \|\widetilde{a}_0\| \leqslant 3\Big(\|a_0\| + \sum_{j=1}^{n} \|f^j\|_1^{1/2}\Big); & \\ \|\widetilde{b}_1^j\| \leqslant 2(\|b_1^j\| + \|f^j\|_1^{1/2}), & 1 \leqslant j \leqslant n; \\ \|\widetilde{y}^j\| \leqslant \|f^j\|_1^{1/2}, & 1 \leqslant j \leqslant n; \\ \|\widetilde{x}\| \leqslant \sum_{j=1}^{n} \|f^j\|_1^{1/2}. & \end{cases}$$

Now we use this result to start the standard self improving process, which leads to the property $\mathbb{A}_{1,n}$. Let us take $(\varepsilon_m)_m$ a sequence of positive reals decreasing to 0 such that $\varepsilon_m < 1/2^m, m \geqslant 1$. Suppose we have found vectors $\widetilde{a}_{0,m}$, $\widetilde{a}_{1,m}$, $\widetilde{x}_m$, $\widetilde{b}_{0,m}^j$, $\widetilde{b}_{1,m}^j$, $\widetilde{y}_m^j$ such that: $\|[f^j] - (\widetilde{a}_{1,m} + \widetilde{a}_{0,m} + \widetilde{x}_m)\Box(\widetilde{b}_{1,m}^j + \widetilde{b}_{0,m}^j + \widetilde{y}_m^j)\| < \varepsilon_m$, $1 \leqslant j \leqslant n$; then, by the above, we can find vectors $\widetilde{a}_{0,m+1}$, $\widetilde{a}_{1,m+1}$, $\widetilde{x}_{m+1}$, $\widetilde{b}_{0,m+1}^j$, $\widetilde{b}_{1,m+1}^j$, $\widetilde{y}_{m+1}^j$ such that:

$$\begin{aligned} \|[f^j] &- (\widetilde{a}_{1,m} + \widetilde{a}_{0,m} + \widetilde{x}_m)\Box(\widetilde{b}_{1,m}^j + \widetilde{b}_{0,m}^j + \widetilde{y}_m^j) \\ &+ (\widetilde{a}_{1,m} + \widetilde{a}_{0,m} + \widetilde{x}_m)\Box(\widetilde{b}_{1,m}^j + \widetilde{b}_{0,m}^j + \widetilde{y}_m^j) \\ &- (\widetilde{a}_{1,m+1} + \widetilde{a}_{0,m+1} + \widetilde{x}_{m+1})\Box(\widetilde{b}_{1,m+1}^j + \widetilde{b}_{0,m+1}^j + \widetilde{y}_{m+1}^j)\| < \varepsilon_{m+1} \end{aligned}$$

with the following control of the norms:

$$\begin{cases} \|\widetilde{b}_{0,m+1}^j - \widetilde{b}_{0,m}^j\| < 2\varepsilon_m^{1/2}, & 1 \leqslant j \leqslant n; \\ \|\widetilde{a}_{1,m+1} - \widetilde{a}_{1,m}\| < 3n\varepsilon_m^{1/2}; & \\ \|\widetilde{a}_{0,m+1}\| < 3(\|\widetilde{a}_{0,m} + \widetilde{x}_{0,m}\| + n\varepsilon_m^{1/2}); & \\ \|\widetilde{b}_{1,m+1}^j\| < 2(\|\widetilde{b}_{1,m}^j + \widetilde{y}_{1,m}^j\| + \varepsilon_m^{1/2}), & 1 \leqslant j \leqslant n; \\ \|\widetilde{x}_{m+1}\| < n\varepsilon_m^{1/2}; & \\ \|\widetilde{y}_{m+1}^j\| < \varepsilon_m^{1/2}, & 1 \leqslant j \leqslant n. \end{cases}$$

Since $\|\widetilde{x}_{0,m}\| \leqslant \|\widetilde{x}_m\| < n\varepsilon_{m-1}^{1/2}$ and since $\|\widetilde{y}_{1,m}^j\| \leqslant \|\widetilde{y}_m^j\| < \varepsilon_{m-1}^{1/2}, 1 \leqslant j \leqslant n$, we obtain:

$$\begin{cases} \|\widetilde{a}_{0,m+1}\| < 3(\|\widetilde{a}_{0,m}\| + 2n\varepsilon_{m-1}^{1/2}); & \\ \|\widetilde{b}_{1,m+1}^j\| < 2(\|\widetilde{b}_{1,m}^j\| + 2\varepsilon_{m-1}^{1/2}), & 1 \leqslant j \leqslant n; \end{cases}$$

We then get Cauchy sequences $(\widetilde{b}^j_{0,m})_m$ and $(\widetilde{a}_{1,m})_m$ $(1 \leqslant j \leqslant n)$, which converge to $\overline{b}^j_0$ and $\overline{a}_1$ respectively. The sequences $(\widetilde{x}_m)_m$ and $(\widetilde{y}^j_m)_m$ converge to 0. Moreover $(\widetilde{b}^j_{1,m})_m$, $(\widetilde{a}_{0,m})_m$, $(1 \leqslant j \leqslant n)$, have weak cluster point $\overline{b}^j_1$, $\overline{a}_0$ respectively. Using mixed continuity of $(x, y) \to x\Box y$ we can write:

$$[f^j] = (\overline{a}_0 + \overline{a}_1)\Box(\overline{b}^j_0 + \overline{b}^j_1), \quad 1 \leqslant j \leqslant n$$

which ends the proof. ∎

Since $T \in \mathbb{A}_{1,n}$ is equivalent to $T^* \in \mathbb{A}_{n,1}$, the following theorem is easily deduced:

THEOREM 5.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be in the class $\mathbb{A}$. If $\operatorname{mult}(R^{\mathcal{H}_0^*}) \geqslant n$ on $\mathbb{T}\backslash E^{\mathrm{r}}_{T^*}$, then T belongs to the class $\mathbb{A}_{n,1}$, where $R^{\mathcal{H}_0^*}$ is the unitary part of the minimal isometric dilation of the C_0. part ot T^*.*

Remark that the condition T does not belong to the class $\mathbb{A}_{\aleph_0,1}$ means that the Borel set $\mathbb{T} \setminus E^{\mathrm{r}}_{T^*}$ has a positive Lebesgue measure. Indeed, T in the class $\mathbb{A}$ belongs to the $\mathbb{A}_{\aleph_0,1}$ if and only if $\mathbb{T} = E^{\mathrm{l}}_T$ and $E^{\mathrm{l}}_T = \{\overline{\zeta}, \zeta \in E^{\mathrm{r}}_{T^*}\}$.

Now we give another sufficient condition for an operator T in the class $\mathbb{A}$ to be in the class $\mathbb{A}_{k,1}$. Recall that if T is C_0. then $T \in \mathbb{A}_{\aleph_0,1}$ (see Proposition 4.5 in [5]).

THEOREM 5.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be in the class $\mathbb{A}$. If $\operatorname{mult}(R^{\mathcal{H}_1}_*) \geqslant k$ on $E_1 \setminus X_T$, then $T \in \mathbb{A}_{k,1}$.*

Proof. Of course, if $E_1 \setminus X_T = \emptyset$ then $T \in \mathbb{A}_{\aleph_0,1}$ and the conclusion holds. So we consider the case where $E_1 \setminus X_T$ has positive Lebesgue measure. Let $f^1, \ldots, f^k$ be in $L^1(\mathbb{T})$, $\varepsilon > 0$, and $a^1, \ldots, a^k, b$ in $\mathcal{H}$. We split the functions f^i $(1 \leqslant i \leqslant k)$ into pieces: $f^i = f^i_\sigma + f^i_{X_T} + f^i_{\sigma'}$ where $\sigma = E_1 \setminus X_T$, and $\sigma' = \mathbb{T} \setminus (\sigma \cup X_T)$. Then we get for $1 \leqslant i \leqslant k$,

$$[f^i] + a^i\Box b = ([f^i_\sigma] + a^i_1\Box b_1) + ([f^i_{\sigma'}] + a^i_0\Box b_0) + [f^i_{X_T}] + a^i_1\Box b_0,$$

where we refer to notation from the preliminaries. By Proposition 4.5, we know there exist sequences $(a^i_{1,n})_n$ and $(b_{1,n})_n$ in $\mathcal{H}_1$, $(1 \leqslant i \leqslant k)$ such that:

$$\begin{cases} (1) \quad \lim_{n\to\infty} \|[f^i_\sigma] + a^i_1\Box b_1 - a^i_{1,n}\Box b_{1,n}\| = 0, & 1 \leqslant i \leqslant k; \\ (2) \quad \lim_{n\to\infty} \|(a^i_{1,n} - a^i_1)\Box w\| = 0, & w \in \mathcal{H},\ 1 \leqslant i \leqslant k; \\ (3) \quad \|a^i_{1,n} - a^i_1\| \leqslant 2\|f^i_\sigma\|_1^{1/2}, & 1 \leqslant i \leqslant k,\ n \geqslant 1; \\ (4) \quad \|b_{1,n}\| \leqslant 3\Big(\|b_1\| + \sum_{i=1}^{k} \|f^i_\sigma\|_1^{1/2}\Big), & n \geqslant 1. \end{cases}$$

Next we use the fact that $\sigma' \subset \Sigma_{T_0}$ and Proposition 3.12 to find sequences $(a^i_{0,p})_p$ and $(b_{0,p})_p$ in $\mathcal{H}_0$ $(1 \leqslant i \leqslant k)$ such that:

$$\begin{cases} (5) \quad \lim_{p\to\infty} \|[f^i_{\sigma'}] + a^i_0 \Box b_0 - a^i_{0,p}\Box b_{0,p}\| = 0, & 1 \leqslant i \leqslant k; \\ (6) \quad \lim_{p\to\infty} \|w\Box(b_{0,p} - b_0)\| = 0, & w \in \mathcal{H}; \\ (7) \quad \|a^i_{0,p}\| \leqslant 2(\|a^i_0\| + \|f^i_{\sigma'}\|_1^{1/2}), & 1 \leqslant i \leqslant k,\ p \geqslant 1; \\ (8) \quad \|b_0 - b_{0,p}\| \leqslant 3 \sum_{i=1}^{k} \|f^i_{\sigma'}\|_1^{1/2}. \end{cases}$$

Finally, we use the property of the set X_T and Proposition 3.9 to find sequences $(x^i_m)_m$ and $(y_m)_m$ in $\mathcal{H}$ such that:

$$\begin{cases} (9) \quad \lim_{m\to\infty} \|[f^i_{X_T}] - x^i_m\Box y_m\| = 0, & 1 \leqslant i \leqslant k; \\ (10) \quad \lim_{m\to\infty} \|x^i_m\Box w\| = 0, & 1 \leqslant i \leqslant k,\ w \in \mathcal{H}; \\ (11) \quad \lim_{m\to\infty} \|w\Box y_m\| = 0, & w \in \mathcal{H}; \\ (12) \quad \|x^i_m\| \leqslant \|f^i_{X_T}\|_1^{1/2}, & m \geqslant 1,\ 1 \leqslant i \leqslant k; \\ (13) \quad \|y_m\| \leqslant \sum_{i=1}^{k} \|f^i_{X_T}\|_1^{1/2}, & m \geqslant 1. \end{cases}$$

We now put the pieces together. Let us take N, M, P such that, for any $n > N$, $m > M$, $p > P$, we get:

$$\|[f^i] + a^i\Box b - (a^i_{1,n}\Box b_{1,n} + a^i_{0,p}\Box b_{0,p} + x^i_m\Box y_m + a^i_1\Box b_1)\| < \frac{\varepsilon}{4}.$$

We can write for any $1 \leqslant i \leqslant k$ that $a^i_{1,n}\Box b_{1,n} + a^i_{0,p}\Box b_{0,p} + x^i_m\Box y_m + a^i_1\Box b_0$ is equal to the following expression:

$$(-a^i_{1,n}\Box b_{0,p} + a^i_1\Box b_0) - (a^i_{1,n} + a^i_{0,p})\Box y_m - x^i_m\Box(b_{1,n} + b_{0,p}) \\ + (a^i_{1,n} + a^i_{0,p} + x^i_m)\Box(b_{1,n} + b_{0,p} + y_m).$$

Moreover, we have:

$$a^i_1\Box b_0 - a^i_{1,n}\Box b_{0,p} = (a^i_1 - a^i_{1,n})\Box b_0 + a^i_{1,n}\Box(b_0 - b_{0,p}).$$

Now we use relations (2), (6), (10), (11), and we choose successively,

$$\begin{aligned} &n > N \quad \text{so that} \quad \|(a^i_1 - a^i_{1,n})\Box b_0\| < \frac{\varepsilon}{4}, \\ &p > P \quad \text{so that} \quad \|(a^i_{1,n}\Box(b_0 - b_{0,p})\| < \frac{\varepsilon}{4} \text{ and} \\ &m > M \quad \text{so that} \quad \|(a^i_{1,n} + a^i_{0,p})\Box y_m\| + \|x^i_m\Box(b_{1,n} + b_{0,p})\| < \frac{\varepsilon}{4}. \end{aligned}$$

Thus we have found vectors $\widetilde{a}_1^i, \widetilde{b}_1$ in $\mathcal{H}_1$, $\widetilde{b}_0, \widetilde{a}_0^i$ in $\mathcal{H}_0$, $\widetilde{x}^i$, $\widetilde{y}$ in $\mathcal{H}$, $(1 \leqslant i \leqslant k)$ such that:

$$\begin{cases} \|[f^i] + a^i \square b - (\widetilde{a}_0^i + \widetilde{a}_1^i + \widetilde{x}^i)\square(\widetilde{b}_0 + \widetilde{b}_1 + \widetilde{y})\| < \varepsilon, & 1 \leqslant i \leqslant k; \\ \|\widetilde{a}_1^i - a_1^i\| \leqslant 2\|f^i\|_1^{1/2}, & 1 \leqslant i \leqslant k, \\ \|b_0 - \widetilde{b}_0\| \leqslant 3 \sum\limits_{i=1}^{k} \|f^i\|_1^{1/2}; & \\ \|\widetilde{a}_0^i\| \leqslant 2(\|a_0^i\| + \|f^i\|_1^{1/2}), & 1 \leqslant i \leqslant k; \\ \|\widetilde{b}_1\| \leqslant 3(\|b_1\| + \sum\limits_{i=1}^{k} \|f^i\|_1^{1/2}); & \\ \|\widetilde{x}^i\| \leqslant \|f^i\|_1^{1/2}, & 1 \leqslant i \leqslant k; \\ \|\widetilde{y}\| \leqslant \sum\limits_{i=1}^{k} \|f^i\|_1^{1/2}. & \end{cases}$$

By a standard self improving process, the proof can now be completed as in the proof of Theorem 5.1. ∎

The following result generalizes Theorem 5.1 and Theorem 5.3. We give sufficient conditions for an operator T in the class $\mathbb{A}$ for being in the class $\mathbb{A}_{k,n}$.

THEOREM 5.4. *Suppose T is in the class $\mathbb{A}\setminus(\mathbb{A}_{\aleph_0,1}\cup\mathbb{A}_{1,\aleph_0})$. If* $\operatorname{mult}(R^{\mathcal{H}_0}) \geqslant n$ *on* $\sigma_0 \subset \Sigma_0 \setminus X_T$ *and if* $\operatorname{mult}(R_*^{\mathcal{H}_1}) \geqslant k$ *on* $\sigma_1 \subset \Sigma_{*1} \setminus X_T$ *where σ_0 and σ_1 are some Borel subsets of $\mathbb{T}$ such that $\sigma_0 \cup \sigma_1 \cup X_T = \mathbb{T}$, then T belongs to the class $\mathbb{A}_{k,n}$.*

REMARK 5.5. (i) If $\operatorname{mult}(R^{\mathcal{H}_0}) \geqslant n$ on $\mathbb{T} \setminus E_T^r$ and if $\operatorname{mult}(R_*^{\mathcal{H}_1}) \geqslant k$ on $\Sigma_{*1} \setminus X_T$, then $T \in \mathbb{A}_{k,n}$.

(ii) If $\operatorname{mult}(R^{\mathcal{H}_0}) \geqslant n$ on $E_0 \setminus X_T$ and if $\operatorname{mult}(R_*^{\mathcal{H}_1}) \geqslant k$ on $E_1 \setminus X_T$, then $T \in \mathbb{A}_{k,n}$.

Proof of Theorem 5.4. Let us consider $\{f^{i,j},\ 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant n\}$ a finite sequence of functions in $L^1(\mathbb{T})$, $\varepsilon > 0$, and $a^1, \ldots, a^k, b^1, \ldots, b^n$ in $\mathcal{H}$. We write $f^{i,j} = f_{\sigma_0}^{i,j} + f_{\sigma_1}^{i,j} + f_{X_T}^{i,j}$. Once again we have to deal with terms such as

$$[f^{i,j}] + a^i \square b^j = ([f_{\sigma_0}^{i,j}] + a_0^i \square b_0^j) + ([f_{\sigma_1}^{i,j}] + a_1^i \square b_1^j) + [f_{X_T}^{i,j}] + a_1^i \square b_0^j,$$

$1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n$. Using Proposition 4.8, we can find sequences $(a_{0,m}^i)_m$ and

$(b^j_{0,m})_m$ in $\mathcal{H}_0$ such that, if $1 \leqslant i \leqslant k$ and if $1 \leqslant j \leqslant n$ we have:

$$\begin{cases} (1) & \lim_{m\to\infty} \|[f^{i,j}_{\sigma_0}] + a^i_0 \Box b^j_0 - (a^i_{0,m} \Box b^j_{0,m})\| = 0, & 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant n; \\ (2) & \lim_{m\to\infty} \|w \Box (b^j_{0,m} - b^j_0)\| = 0, & w \in \mathcal{H},\ 1 \leqslant j \leqslant n; \\ (3) & \|b^j_{0,m} - b^j_0\| \leqslant 2\Big(\sum_{i=1}^k \|f^{i,j}_{\sigma_0}\|_1\Big)^{1/2}, & m \geqslant 1,\ 1 \leqslant j \leqslant n; \\ (4) & \|a^i_{0,m}\| \leqslant 3(\|a^i_0\| + \sum_{j=1}^n \|f^{i,j}_{\sigma_0}\|_1^{1/2}), & m \geqslant 1,\ 1 \leqslant i \leqslant k. \end{cases}$$

Using Proposition 4.5, we can find sequences $(a^i_{1,p})_p$ and $(b^j_{1,p})_p$ in $\mathcal{H}_1$ such that:

$$\begin{cases} (5) & \lim_{p\to\infty} \|[f^{i,j}_{\sigma_1}] + a^i_1 \Box b^j_1 - a^i_{1,p} \Box b^j_{1,p}\| = 0, & 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant n; \\ (6) & \lim_{p\to\infty} \|(a^i_{1,p} - a^i_1) \Box w\| = 0, & w \in \mathcal{H},\ 1 \leqslant i \leqslant k; \\ (7) & \|a^i_{1,p} - a^i_1\| \leqslant 2 \sum_{j=1}^n \|f^{i,j}_{\sigma_1}\|_1^{1/2}, & 1 \leqslant i \leqslant k,\ p \geqslant 1; \\ (8) & \|b^j_{1,p}\| \leqslant 3(\|b^j_1\| + \sum_{i=1}^k \|f^{i,j}_{\sigma_1}\|_1^{1/2}), & 1 \leqslant j \leqslant n,\ p \geqslant 1. \end{cases}$$

We now use Proposition 3.9 to find two sequences $(x^i_q)_q$ and $(y^j_q)_q$ in $\mathcal{H}$ such that:

$$\begin{cases} (9) & \lim_{q\to\infty} \|[f^{i,j}_{X_T}] - x^i_q \Box y^j_q\| = 0, & 1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant n; \\ (10) & \lim_{q\to\infty} \|x^i_q \Box w\| = 0, & w \in \mathcal{H},\ 1 \leqslant i \leqslant k; \\ (11) & \lim_{q\to\infty} \|w \Box y^j_q\| = 0, & w \in \mathcal{H},\ 1 \leqslant j \leqslant n; \\ (12) & \|x^i_q\| \leqslant \sum_{j=1}^n \|f^{i,j}_{X_T}\|_1^{1/2}, & 1 \leqslant i \leqslant k,\ q \geqslant 1; \\ (13) & \|y^j_q\| \leqslant \sum_{i=1}^k \|f^{i,j}_{X_T}\|_1^{1/2}, & 1 \leqslant j \leqslant n,\ q \geqslant 1. \end{cases}$$

We now put the pieces together. Let us take M, P, Q such that if $m > M$, $p > P$, $q > Q$, for any $1 \leqslant i \leqslant k$ and any $1 \leqslant j \leqslant n$ we have:

$$\|[f^{i,j}] + a^i \Box b^j - (a^i_{1,p} \Box b^j_{1,p} + a^i_{0,m} \Box b^j_{0,m} + x^i_q \Box y^j_q + a^i_1 \Box b^j_0)\| < \frac{\varepsilon}{4}.$$

We can write, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n$:

$$\begin{aligned} a^i_{1,p} \Box b^j_{1,p} + a^i_{0,m} \Box b^j_{0,m} + x^i_q \Box y^j_q + a^i_1 \Box b^j_0 &= (a^i_{1,p} + a^i_{0,m} + x^i_q) \Box (b^j_{1,p} + b^j_{0,m} + y^j_q) \\ &+ (-a^i_{1,p} \Box b^j_{0,m} + a^i_1 \Box b^j_0) - (a^i_{1,p} + a^0_{0,m}) \Box y^j_q - x^i_m \Box (b^j_{1,p} + b^j_{0,m}). \end{aligned}$$

Moreover we have:

$$a_1^i \square b_0^j - a_{1,p}^i \square b_{0,m}^j = (a_1^i - a_{1,p}^i) \square b_0^j + a_{1,p}^i \square (b_0^j - b_{0,m}^j).$$

Now we use relations (2), (6), (10), (11), and we choose $(1 \leqslant i \leqslant k, 1 \leqslant j \leqslant n)$:

$$\begin{aligned} &p > P \quad \text{so that} \quad \|(a_1^i - a_{1,p}^i \square b_0^j\| < \frac{\varepsilon}{4} \\ &m > M \quad \text{so that} \quad \|(a_{1,p}^i \square (b_0^j - b_{0,m}^j)\| < \frac{\varepsilon}{4} \\ &q > Q \quad \text{so that} \quad \|(a_{1,p}^i + a_{0,m}^i)\square y_q^j\| + \|x_q^i \square (b_{1,p}^j + b_{0,m}^j)\| < \frac{\varepsilon}{4}. \end{aligned}$$

Finally we have found vectors $\widetilde{a}_0^i, \widetilde{b}_0^j$ in $\mathcal{H}_0$, $\widetilde{a}_1^i, \widetilde{b}_1^j$ in $\mathcal{H}_1$ and $\widetilde{x}^i, \widetilde{y}^j$ in $\mathcal{H}$, such that:

$$\begin{cases} \|[f^{i,j}] + a^i \square b^j - (\widetilde{a}_0^i + \widetilde{a}_1^i + \widetilde{x}^i)\square(\widetilde{b}_0^j + \widetilde{b}_1^j + \widetilde{y}^j)\| < \varepsilon, & 1 \leqslant i \leqslant k; \\ \|\widetilde{a}_1^i - a_1^i\| \leqslant 2 \sum_{j=1}^{n} \|f^{i,j}\|_1^{1/2}, & 1 \leqslant i \leqslant k; \\ \|b_0^j - \widetilde{b}_0^j\| \leqslant 2 \sum_{i=1}^{k} \|f^i\|_1^{1/2}, & 1 \leqslant j \leqslant n; \\ \|\widetilde{a}_0^i\| \leqslant 3\Big(\|a_0^i\| + \sum_{j=1}^{n} \|f^{i,j}\|_1^{1/2}\Big), & 1 \leqslant i \leqslant k; \\ \|\widetilde{b}_1^j\| \leqslant 3\Big(\|b_1^j\| + \sum_{i=1}^{k} \|f^{i,j}\|_1^{1/2}\Big), & 1 \leqslant j \leqslant n; \\ \|\widetilde{x}^i\| \leqslant \sum_{j=1}^{n} \|f^{i,j}\|_1^{1/2}, & 1 \leqslant i \leqslant k; \\ \|\widetilde{y}^j\| \leqslant \sum_{i=1}^{k} \|f^{i,j}\|_1^{1/2}, & 1 \leqslant j \leqslant n. \end{cases}$$

This result is the core of the standard self improving process, which leads to property $\mathbb{A}_{k,n}$ in the same way as in the proofs of the previous theorems. ∎

In view of the above results, the following conjecture seems a reasonable first step towards obtaining necessary conditions for membership in the classes $\mathbb{A}_{k,n}$.

CONJECTURE. If $T \in \mathcal{L}(\mathcal{H})$ belongs to the class $\mathbb{A}$, then $T \in \mathbb{A}_{1,n}$ if and only if $\operatorname{mult}(R^{\mathcal{H}_0}) \geqslant n$ on $\mathbb{T} \setminus E_T^r$.

If we were able to prove this conjecture, we could easily prove that:

$$\bigcap_{n \geqslant 1} \mathbb{A}_{1,n} = \mathbb{A}_{1,\aleph_0}.$$

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