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THE CONJUGATE OPERATOR METHOD FOR LOCALLY REGULAR HAMILTONIANS

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ABSTRACT. We develop a version of the conjugate operator method for an arbitrary pair of self-adjoint operators: the *hamiltonian* H and the *conjugate operator* A. We obtain optimal results concerning the regularity properties of the boundary values $(H - \lambda \mp \mathrm{i}0)^{-1}$ of the resolvent of H as functions of λ. Our approach allows one to eliminate the spectral gap hypothesis on H without asking the invariance of the domain or of the form domain of the hamiltonian under the unitary group generated by A (previous versions of the theory assume at least one of theses conditions). In particular one may treat singular hamiltonians with spectrum equal $\mathbb{R}$, e.g. strongly singular perturbations of Stark hamiltonians or simply characteristic operators.

KEYWORDS: *Mourre estimate, conjugate operator, boundary values of the resolvent.*

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INTRODUCTION

The conjugate operator method is a very efficient method for studying the spectral and scattering properties of a hamiltonian H. This method has been initiated by E. Mourre (see [15], [16]) and developed by many authors e.g. [17], [13], [14], [11], [12], [2], [4], [5] and more recently [7], [8] (see [9] and [2] for other references). All the versions of the theory developed in the preceding references assume either that H has a spectral gap (see [2] or [5]) or that the domain or the form domain of H is invariant under the action of the unitary group generated by conjugate operator A (as in [15], [16], [17], [4]). But the spectral gap hypothesis is quite restrictive in some applications e.g. it excludes the Stark effect hamiltonians (see

[9] or [12]) or the simply characteristic operators (see [1]). On the other hand, in several important cases, it seems difficult to verify the condition of invariance of the domain or of the form domain of H. This is the case, for example, if H is simply characteristic with "natural" short range perturbations as in Chapter 14 from [10] (see, however, [3] and references therein for the kind of results that can be obtained by using the Mourre theory) or if H contains very singular interactions (see [5]).

Our purpose in this work is to develop a version of the conjugate operator method for locally regular hamiltonians. More precisely, we prove that for each pair of self-adjoint operators H, A in a Hilbert space $\mathcal{H}$ there is a natural real open set (which, however, could be empty) such that for each λ in this set the boundary values $R(\lambda \pm \mathrm{i}0)$ of the resolvent $R(z) = (H-z)^{-1}$ of H exist in a certain sense, and we describe the continuity properties of these boundary values as functions of λ. As a consequence, we eliminate the spectral gap hypothesis without asking the invariance of the domain or of the form domain of the hamiltonian under the action of the unitary group of A.

We begin by observing that there is a largest real open set, denoted by $\Omega_1^A(H)$, such that for each $\varphi \in C_0^\infty(\Omega_1^A(H))$ the operator $\varphi(H)$ is of class $C^1(A)$ (we use the terminology of [2] which, however, is recalled in Section 1); we say that H is locally of class $C^1(A)$ on $\Omega_1^A(H)$. Similarly, there are largest open real sets $\Omega_{1+0}^A(H)$, $\Omega_s^A(H)$ (for $1 < s < 2$) on which H is locally of class $C^{1+0}(A)$ or $C^s(A)$ respectively, and $\Omega_s^A(H) \subset \Omega_t^A(H) \subset \Omega_{1+0}^A(H) \subset \Omega_1^A(H)$ if $1 < t \leqslant s$. H is locally of class $C^s(A)$ if and only if $\Omega_s^A(H)$ equal to $\mathbb{R}$.

The next step is to give a meaning to the symbol $[H, \mathrm{i}A]$ in this context: this is the purpose of Proposition 2.1. We show that the symmetric form defined by $2\Re\langle Hf, \mathrm{i}Af\rangle$ on the domain $D(A) \cap \mathcal{H}_c^A(H)$ (where $\mathcal{H}_c^A(H)$ is the space consisting of all the vectors that have a compact support included in $\Omega_1^A(H)$ in the spectrum representation of H) has a unique extension to a continuous symmetric form denoted $[H, \mathrm{i}A]$ on $\mathcal{H}_c^A(H)$. The equality $[H, \mathrm{i}A] = H\mathrm{i}A - \mathrm{i}AH$ holds (in the form sense) only on the subspace $D(A) \cap \mathcal{H}_c^A(H)$. If $\varphi \in C_0^\infty(\Omega_1^A(H))$ then $\varphi(H)[H, \mathrm{i}A]\varphi(H)$ is a continuous symmetric operator in $\mathcal{H}$. So, we can define the set $\mu^A(H)$ of the real numbers $\lambda \in \Omega_1^A(H)$ for which there are a function $\varphi \in C_0^\infty(\Omega_1^A(H))$ with $\varphi(\lambda) \neq 0$ and a number $a > 0$ such that

$$\varphi(H)[H, \mathrm{i}A]\varphi(H) \geqslant a\varphi^2(H). \tag{0.1}$$

We say that A is locally strictly conjugate to H on $\mu^A(H)$. Let us set $\mu_{1+0}^A(H) = \mu^A(H) \cap \Omega_{1+0}^A(H)$ and $\mu_s^A(H) = \mu^A(H) \cap \Omega_s^A(H)$ for $1 < s < 2$. So, the main ingredient for the study of the spectral and propagation properties of H, namely the Mourre estimate (0.1), has a meaning in our context.

Let us denote by $\{\mathcal{H}_s\}_{s\in\mathbb{R}}$ the Sobolev scale associated to A (that is described in details in the next section). By interpolation we obtain the Besov spaces $\mathcal{K} = (\mathcal{H}_1, \mathcal{H})_{1/2,1}$.

Our first aim is to prove that H has nice spectral properties in $\mu^A_{1+0}(H)$. By using the Virial Theorem one sees that H has no eigenvalues in $\mu^A_{1+0}(H)$. Moreover, if we denote $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \Im z > 0\}$ we get the following theorem usually called the *Limiting Absorption Principle.*

THEOREM 0.1. *Let H, A be self-adjoint operators in the Hilbert space $\mathcal{H}$. Then the map $z \mapsto R(z) \in B(\mathcal{K}, \mathcal{K}^*)$, which is holomorphic on the half-plane $\mathbb{C}_\pm$, extends to a weak* continuous function on $\mathbb{C}_\pm \cup \mu^A_{1+0}(H)$. In particular, H has no singularly continuous spectrum in $\mu^A_{1+0}(H)$.*

The most important particular case is that when H is of class $C^{1+0}(A)$. Then we clearly obtain a result which extends those referred to above. Note that we assume neither that H has a spectral gap (i.e. the spectrum of H can be equal to $\mathbb{R}$) nor that the domain or the form domain of H is invariant under the unitary group generated by A. We have explained before why this fact is interesting and we mention that our result allows the treatment of simply characteristic operators and of the Stark effect hamiltonians under quite general conditions (these applications are treated in [18]; see also [19]). Unfortunately our proof of Theorem 0.1 does not extend to hamiltonians H of class $C^{1,1}(A)$ (this class would be optimal on the Besov scale $C^{s,p}(A)$, cf. [2]).

The preceding theorem can be stated as follows: the map $\lambda \mapsto R(\lambda \pm \mathrm{i}0) \in B(\mathcal{K}, \mathcal{K}^*)$ is locally weakly* continuous in $\mu^A_{1+0}(H)$. Let us remark that $B(\mathcal{K}, \mathcal{K}^*) \subset B(\mathcal{H}_s, \mathcal{H}_{-s})$ for each $s > 1/2$. Then it is natural to study the continuity properties of the $B(\mathcal{H}_s, \mathcal{H}_{-s})$-valued function $\lambda \mapsto R(\lambda \pm \mathrm{i}0)$; this is the aim of the next theorem.

THEOREM 0.2. *Let H, A be self-adjoint operators in the Hilbert space $\mathcal{H}$, let $1/2 < s < 1$ and set $\alpha = s - 1/2$. Then the map $\lambda \mapsto R(\lambda \pm \mathrm{i}0) \in B(\mathcal{H}_s \mathcal{H}_{-s})$ is locally of class Λ^α on $\mu^A_{\alpha+1}(H)$.*

This result is optimal on the Sobolev scale $\{\mathcal{H}_s\}$, on the Lipschitz-Zygmund scale $\{\Lambda^\alpha\}$ and on the scale $C^\alpha(A)$ (represented here by $\mu^A_{\alpha+1}(H)$). This optimality is proved in [6] (see also [18]).

In order to illustrate these considerations we reconsider the preceding particular case. If we assume that H is of class $C^{s+1/2}(A)$ then we obtain a result similar to that of [6] (see also [16], [13]), but without any condition on the spectrum or on the domain or the form domain of H. Finally, we note that by making a Fourier transform (as in [7] and [18]) we may deduce local decay estimate.

Let us denote by $\Pi_\pm$ the spectral projectors of A associated to $\mathbb{R}_\pm = \{x \in \mathbb{R} \mid \pm x \geqslant 0\}$. Now we intend to describe the propagation properties of H.

THEOREM 0.3. (i) *Let s be a real number such that $1/2 < s < 1$ and set $\alpha = s - 1/2$. Then for $\lambda \in \mu^A_{1+\alpha}(H)$ we have $\Pi_\mp(H - \lambda \mp \mathrm{i}0)^{-1}\mathcal{H}_s \subset \mathcal{H}_{s-1,\infty}$ and the map $\lambda \mapsto \Pi_\mp(H - \lambda \mp \mathrm{i}0)^{-1} \in B(\mathcal{H}_s, \mathcal{H}_{s-1,\infty})$ is locally weakly* continous on $\mu^A_{1+\alpha}(H)$.*

(ii) *Let β be a real number such that $0 < \beta < \alpha$. Then the map $\lambda \mapsto \Pi_\mp(H - \lambda \mp \mathrm{i}0)^{-1} \in B(\mathcal{H}_s, \mathcal{H}_{s-1-\beta})$ is locally of class Λ^β on $\mu^A_{1+\alpha}(H)$.*

This result is optimal on the same scales as in the case of Theorem 0.2. If we consider the particular case when H is of class $C^{s+1/2}(A)$ we see that our result is better than the results of [16], [13], [7] in the sense explained before. We note also that we can deduce certain propagation estimates as in [7] (see also [18]).

The paper is organized as follows. In Section 1 we give a detailed description of all the spaces which appear in our results. In Sections 2 and 3 we introduce the local regularity classes, we make some elementary remarks concerning these classes, and then we show that the Mourre estimate has a natural meaning in this context. In Section 4 we recall the regularization technique introduced in [4] and we prove some preliminary estimates. The Sections 5 and 6 are devoted to the proof of the Theorems 0.1, 0.2 and 0.3.

1. NOTATIONS

1.1. Let $(E, \|\cdot\|)$ be a Banach space, $0 < \alpha < 1$ and let $f : \mathbb{R} \to E$ be a bounded continuous function. We say that f is of class Λ^α if there is a finite constant C such that:

$$\sup_{x\in\mathbb{R}} \|f(x+\varepsilon) - f(x)\| \leqslant C\varepsilon^\alpha, \quad \forall \varepsilon \in (0,1).$$

So, Λ^α is just the space of Hölder continuous functions of order α. f is of class $\Lambda^{1+\alpha}$ $(0 < \alpha < 1)$ if and only if it is continuously differentiable and its derivative f' is of class Λ^α. We recall also that a function f is Dini-continuous if

$$\int_0^1 \sup_{x\in\mathbb{R}} \|f(x+\varepsilon) - f(x)\| \frac{\mathrm{d}\varepsilon}{\varepsilon} < \infty.$$

Finally, if f is of class C^1 and f' is Dini-continuous we say that f is of class C^{1+0}.

1.2. Let A be an unbounded self-adjoint operator in the Hilbert space $\mathcal{H}$. For real $s \geqslant 0$ we denote $\mathcal{H}_s$ the Hilbert space $D(|A|^s)$ endowed with the graph norm $\|u\|_s = \|\langle A\rangle^s u\|$, where $\langle A\rangle = (1+A^2)^{1/2}$. If $s < 0$ then $\mathcal{H}_s$ is defined as the completion of $\mathcal{H}$ for the norm $\|\cdot\|_s$. For each $0 < t < s$ we have strict dense and continuous embedding: $\mathcal{H}_s \subset \mathcal{H}_t \subset \mathcal{H} = \mathcal{H}_0 \subset \mathcal{H}_{-t} \subset \mathcal{H}_{-s}$. We always identify $\mathcal{H}$ with its adjoint space $\mathcal{H}^*$ (by Riesz lemma) and so we obtain a canonical identification $\mathcal{H}_s^* = \mathcal{H}_{-s}$.

We also introduce the Besov space $\mathcal{K}$ defined in terms of the behaviour at infinity of its elements in the spectral representation of A. We shall denote by $E_A(r)$ the spectral projector of A associated to the set $r < |x| < 2r$. Then $f \in \mathcal{K}$ if and only if

$$\|f\|_{\mathcal{K}} := \|f\| + \int_1^\infty \|E_A(r)f\| \frac{\mathrm{d}r}{\sqrt{r}} < \infty.$$

$\mathcal{K}$ equipped with this norm is a Banach space such that $\mathcal{H}_s \subset \mathcal{K} \subset \mathcal{H}$ continuously and densely for each $s > 1/2$ hence $\mathcal{H} \subset \mathcal{K}^* \subset \mathcal{H}_{-s}$ continuously.

1.3. We shall introduce certain regularity classes of operators with respect to the self-adjoint operator A in $\mathcal{H}$. Let $W_\tau = \mathrm{e}^{\mathrm{i}A\tau}$ and denote by $\mathcal{W}_\tau$ the automorphism group on $B(\mathcal{H})$ induced by A and defined by $\mathcal{W}_\tau T = \mathrm{e}^{-iA\tau} T \mathrm{e}^{\mathrm{i}A\tau}$ for $T \in B(\mathcal{H})$. Clearly the function $\tau \mapsto \mathcal{W}_\tau T \in B(\mathcal{H})$ is strongly continuous. If this function is strongly (resp. in norm) of class C^k for some $k \in \mathbb{N} \cup \{\infty\}$, then we say that T if of class $C^k(A)$ (resp. $C^k_{\mathrm{u}}(A)$). If T is of class $C^1(A)$ then the sesquilinear form $[T, A] = TA - AT$ (with domain $D(A)$) extends to a bounded operator in $\mathcal{H}$ denoted $\mathcal{A}[T]$ and given by $\mathrm{i}\mathcal{A}T = \frac{\mathrm{d}}{\mathrm{d}\tau}|_{\tau=0} \mathcal{W}_\tau T$, where the derivative exists in the strong topology. If no confusion can arise we identify $\mathcal{A}[T] = [T, A]$.

Let $0 < \alpha < 1$. We say that $T \in B(\mathcal{H})$ is of class $\mathcal{C}^{1+\alpha}(A)$ if the function $\tau \mapsto \mathcal{W}_\tau T \in B(\mathcal{H})$ is of class $\Lambda^{1+\alpha}$ or, equivalently, if T is of class $C^1(A)$ and $\sup_{0<\varepsilon<1} \|\varepsilon^{-\alpha}(\mathcal{W}_\varepsilon - 1)\mathcal{A}[T]\| < \infty$. Now if $T \in C^1(A)$ and the function $\tau \mapsto \mathcal{W}_\tau \mathcal{A}T \in B(\mathcal{H})$ is Dini-continuous, which is equivalent to $\int_0^1 \|(\mathcal{W}_\varepsilon - 1)\mathcal{A}T\| \frac{\mathrm{d}\varepsilon}{\varepsilon} < \infty$, then we say that T is of class $C^{1+0}(A)$. Note that in this case one has:

$$\int_0^1 \left\| \frac{(\mathcal{W}_\varepsilon - 1)T}{\varepsilon} - \mathrm{i}\mathcal{A}T \right\| \frac{\mathrm{d}\varepsilon}{\varepsilon} < \infty.$$

Finally, let H be a self-adjoint operator in the Hilbert space $\mathcal{H}$ and let $0 < \alpha < 1$. We say that H is of class $C^1(A)$ (resp. $C^1_{\mathrm{u}}(A)$, $C^{1+0}(A)$ or $\mathcal{C}^{1+\alpha}(A)$) if there exists $z \in \mathbb{C} \setminus \sigma(H)$ such that the resolvent $(H-z)^{-1}$ is of class $C^1(A)$

(resp. $C^1_{\mathrm{u}}(A)$, $C^{1+0}(A)$ or $C^{1+\alpha}(A)$). This property is independent of z and if H is bounded then it has the same regularity class as its resolvent.

2. LOCAL REGULARITY CLASSES

2.1. It is easy to show that for each $\varphi \in C_0^\infty(\mathbb{R})$ the operator $\varphi(H)$ has the same regularity class as the self-adjoint operator H. This makes natural the introduction of the following local versions of the preceding regularity classes (cf. Chapter 8 in [2]). Let Ω be an open real set. We say that the self-adjoint operator H is *locally of class* $C^1(A)$ in Ω, if for each $\varphi \in C_0^\infty(\Omega)$ the operator $\varphi(H)$ is of class $C^1(A)$. If $\Omega \in \mathbb{R}$, we say that H is *locally of class* $C^1(A)$. One similarly defines the local classes $C^{1+\alpha}(A)$ for $0 < \alpha < 1$ or $C^{1+0}(A)$.

The difference between the local regularity classes and the global classes is illustrated in the following example. Let $\mathcal{H} = L^2(\mathbb{R})$, $H = h(Q)$ the operator of multiplication by the homeomorphism $h : \mathbb{R} \to \mathbb{R}$, and let $A = P = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$. If h is a locally Lipschitz function then the operator H is locally of class $C^1(A)$. But since $[(H-z)^{-1}, \mathrm{i}P] = [h'(h-z)^{-2}](Q)$, H is of class $C^1(A)$ if and only if the derivative h' satisfies an estimate of the form $|h'(x)| \leqslant c(1 + h^2(x))$.

Unfortunately, if H is more complicated than the operator considered in the previous example (a Schrödinger hamiltonian for example) it seems difficult to show that $\varphi(H) \in C^1(A)$ without knowing that $(H-z)^{-1} \in C^1(A)$ (i.e. that H is of class $C^1(A)$). However the local regularity classes are important for technical reasons, and this is due to the fact that these classes are preserved by the C^∞ functional calculus (see Proposition 3.3).

2.2. Let us fix two arbitrary self-adjoint operators A, H in the Hilbert space $\mathcal{H}$. Then one may easily show that

$$\Omega_1^A(H) = \{\lambda \in \mathbb{R} \mid H \text{ is locally of class } C^1(A) \text{ in some neighbourhood of } \lambda\}$$

is the largest open real set on which H is locally of class $C^1(A)$. One may similarly define the largest real open sets $\Omega_{1+0}^A(H)$ and $\Omega_s^A(H)$ on which H is locally of class $C^{1+0}(A)$ or $C^s(A)$ respectively. Obviously we have for each $1 < s < t < 2$:

$$\Omega_t^A(H) \subset \Omega_s^A(H) \subset \Omega_{1+0}^A(H) \subset \Omega_1^A(H).$$

Note that we always have $\mathbb{R} \setminus \sigma(H) \subset \Omega_t^A(H)$.

Let E be the spectral measure of H. For each $f \in \mathcal{H}$ let $\operatorname{supp}_H f$ be the smallest closed real set Λ such that $E(\Lambda)f = f$ (i.e. the support of the measure $\|E(\cdot)f\|^2$). Then

$$\mathcal{H}_c^A(H) = \left\{ f \in \mathcal{H} \mid \operatorname{supp}_H f \text{ is a compact subset of } \Omega_1^A(H) \right\}$$

is a linear subspace of $\mathcal{H}$, densely embedded in the closed subspace $E(\Omega_1^A(H))\mathcal{H}$. Clearly $\mathcal{H}_c^A(H) = \bigcup_K E(K)\mathcal{H}$, union over all compact subsets of $\Omega_1^A(H)$. We shall equip $\mathcal{H}_c^A(H)$ with the natural inductive limit topology associated to this representation. For example, a sequence $\{f_n\}$ converges in $\mathcal{H}_c^A(H)$ to some f if and only if there is a compact set $K \subset \Omega_1^A(H)$ such that $\operatorname{supp}_H f_n \subset K$ for each n and $f_n \to f$ in $\mathcal{H}$.

PROPOSITION 2.1. (i) *$D(A) \cap \mathcal{H}_c^A(H)$ is a dense subspace of $\mathcal{H}_c^A(H)$.*

(ii) *The symmetric sesquilinear form defined by $2\Re\langle Hf, \mathrm{i}Af\rangle$ on $D(A) \cap \mathcal{H}_c^A(H)$ has a unique extension to a continuous symmetric form on $\mathcal{H}_c^A(H)$; denote by $[H, \mathrm{i}A]$ this extension.*

(iii) *If φ_1, φ_2 are bounded Borel functions on $\mathbb{R}$ such that* $\operatorname{supp}\varphi_1$, $\operatorname{supp}\varphi_2$ *are compact subsets of $\Omega_1^A(H)$, then $\varphi_1(H)[H, \mathrm{i}A]\varphi_2(H)$ is a continuous sesquilinear form on $\mathcal{H}$, and so it is identified with a bounded operator on $\mathcal{H}$. If $\psi \in C_0^\infty(\Omega_1^A(H))$ and $\psi(x) = x$ on a neighborhood of* $\operatorname{supp}\varphi_1 \cup \operatorname{supp}\varphi_2$ *then*

$$\varphi_1(H)[H, \mathrm{i}A]\varphi_2(H) = \varphi_1(H)[\psi(H), \mathrm{i}A]\varphi_2(H). \tag{2.1}$$

In particular, if $\varphi \in C_0^\infty(\Omega_1^A(H))$ is a real function then the continuous symmetric operator $\varphi(H)[H, \mathrm{i}A]\varphi(H)$ is given by

$$\varphi(H)[H, \mathrm{i}A]\varphi(H) = [H\varphi^2(H), \mathrm{i}A] - 2\Re(H\varphi(H)[\varphi(H), \mathrm{i}A]). \tag{2.2}$$

Proof. **Step 1.** Assertion (i) follows from the following more precise fact: if U is an arbitrary open subset of $\Omega_1^A(H)$ and $f \in E(U)\mathcal{H}$, then there is a sequence $\{f_n\}$ of vectors in $\mathcal{H}$ such that $\operatorname{supp}_H f_n$ is a compact subset of U and $f_n \in D(A)$ for each n, and $f_n \longrightarrow f$ in $\mathcal{H}$. Choose $\theta \in C_0^\infty(\mathbb{R})$ with $\theta(0) = 1$ and set $\theta_n(x) = \theta(x/n)$; then $\theta_n(A) \longrightarrow I$ strongly on $\mathcal{H}$. Now let $\varphi_1 \leqslant \varphi_2 \leqslant \cdots \leqslant \varphi_n \leqslant \cdots \leqslant 1$ be an increasing sequence of functions of class $C_0^\infty(U)$ such that $\varphi_n(x) \longrightarrow 1\ \forall x \in U$. Then $\varphi_n(H) \longrightarrow E(U)$ strongly on $\mathcal{H}$ and so $\varphi_n(H)\theta_n(A) \longrightarrow E(U)$ strongly on $\mathcal{H}$. Since $\varphi_n(H)$ is of class $C^1(A)$, we have $\varphi_n(H)D(A) \subset D(A)$. So it is sufficient to take $f_n = \varphi_n(H)\theta_n(A)f$.

Step 2. For each $\psi \in C_0^\infty(\Omega_1^A(H))$ we define a continuous quadratic form Q_ψ on $\mathcal{H}$ by setting $Q_\psi(f) = \langle f, [\psi(H), \mathrm{i}A]f\rangle$. Let U be an open real set whose closure is a compact subset of $\Omega_1^A(H)$. Assume that $f \in E(U)\mathcal{H}$ and $\psi(x) = x$ on U. Then the number $Q_\psi(f)$ does not depend on ψ: indeed, if $\{f_n\}$ is a sequence as in Step 1 then $f_n \in D(A)$ and $\psi(H)^* f_n = \psi(H) f_n = H f_n$, hence

$$\begin{aligned} Q_\psi(f) &= \lim_{n\to\infty} Q_\psi(f_n) = \lim_{n\to\infty} \{\langle \psi(H)^* f_n, \mathrm{i}Af_n\rangle + \langle \mathrm{i}Af_n, \psi(H)f_n\rangle\} \\ &= \lim_{n\to\infty} 2\Re\langle Hf_n, \mathrm{i}Af_n\rangle. \end{aligned}$$

So, if we consider the restriction of Q_ψ to $E(U)\mathcal{H}$, we obtain a continuous quadratic form on $E(U)\mathcal{H}$ independent of ψ. We denote by Q_U this restriction. Moreover, if $f \in D(A) \cap E(U)\mathcal{H}$ then $Q_U(f) = 2\Re\langle Hf, \mathrm{i}Af\rangle$ because:

$$Q_U(f) = \langle f, [\psi(H), \mathrm{i}A]f\rangle = \langle \psi(H)^* f, \mathrm{i}Af\rangle + \langle \mathrm{i}Af, \psi(H)f\rangle = 2\Re\langle Hf, \mathrm{i}Af\rangle.$$

Finally, observe that if $U_1 \subset U_2$ are open sets with the same properties as U, then Q_{U_1} is equal to the restriction of Q_{U_2} to $E(U_1)\mathcal{H}$ (which is a subspace of $E(U_2)\mathcal{H}$).

Step 3. Now the existence of a continuous extension of the quadratic form $2\Re\langle Hf, \mathrm{i}Af\rangle$ to $\mathcal{H}_c^A(H)$ is obvious: if $f \in E(U)\mathcal{H}$ with U as above, we set $\langle Hf, \mathrm{i}Af\rangle = Q_U(f)$. The uniqueness follows from (i). Formula (2.1) is clear by the preceding construction. The continuity of $\varphi_1(H)[H, \mathrm{i}A]\varphi_2(H)$ on $\mathcal{H}$ follows, for example, from (2.1). ∎

The most important particular case of the preceding considerations is when H is locally of class $C^1(A)$, i.e. $\Omega_1^A(H) = \mathbb{R}$. Then $\mathcal{H}_c^A(H) = \mathcal{H}_c(H)$ is the dense subspace of $\mathcal{H}$ consisting of the vectors f such that $\operatorname{supp}_H f$ is compact. So, if H is locally of class $C^1(A)$ then $[H, \mathrm{i}A]$ is a densely defined symmetric sesquilinear form in $\mathcal{H}$ with domain $\mathcal{H}_c(H)$. If H is of class $C^1(A)$ then this form has a unique continuous extension to $D(H)$ ($\mathcal{H}_c(H)$ is a dense subspace of $D(H)$, the later space being equipped with the graph topology).

3. MOURRE ESTIMATE

For each $\lambda \in \mathbb{R}$ we set $E(\lambda,\varepsilon) = E((\lambda-\varepsilon,\lambda+\varepsilon))$. Then by Proposition 2.1 for each $\lambda \in \Omega_1^A(H)$ there is $\varepsilon_0 > 0$ such that $\forall\varepsilon \in (0,\varepsilon_0)$ the operator $E(\lambda,\varepsilon)[H,\mathrm{i}A]E(\lambda,\varepsilon)$ is symmetric and bounded in $\mathcal{H}$ and is zero on the orthogonal complement of the subspace $E(\lambda,\varepsilon)\mathcal{H}$. Hence there is $a > -\infty$ such that $E(\lambda,\varepsilon)[H,\mathrm{i}A]E(\lambda,\varepsilon) \geqslant aE(\lambda,\varepsilon)$. The supremum of the numbers a such that the preceding inequality holds for some $\varepsilon > 0$ is denoted by $\rho_H^A(\lambda)$. So one may consider the set $\mu^A(H)$ of the numbers $\lambda \in \Omega_1^A(H)$ such that $\rho_H^A(\lambda) > 0$ or, equivalently, the set of the numbers λ for which there are $\varepsilon, a > 0$ such that $[\lambda-\varepsilon,\lambda+\varepsilon] \subset \Omega_1^A(H)$ and $E(\lambda,\varepsilon)[H,\mathrm{i}A]E(\lambda,\varepsilon) \geqslant aE(\lambda,\varepsilon)$. We say that A is locally strictly conjugate to H on the real open set $\mu^A(H)$. We set $\mu_{1+0}^A(H) = \mu^A(H) \cap \Omega_{1+0}^A(H)$ and $\mu_s^A(H) = \mu^A(H) \cap \Omega_s^A(H)$ for each $1 < s < 2$. We may similarly define the open real set $\tilde{\mu}^A(H)$ of $\lambda \in \Omega_1^A(H)$ for which there are $\varepsilon, a > 0$ with $[\lambda-\varepsilon,\lambda+\varepsilon] \subset \Omega_1^A(H)$ and a compact operator K in $\mathcal{H}$ such that $E(\lambda,\varepsilon)[H,\mathrm{i}A]E(\lambda,\varepsilon) \geqslant aE(\lambda,\varepsilon) + K$. The difference between the sets $\mu^A(H)$ and $\tilde{\mu}^A(H)$ is described in the next proposition. Note that $\mu^A(H), \tilde{\mu}^A(H)$ are open real sets such that $\mu^A(H) \subset \tilde{\mu}^A(H)$.

PROPOSITION 3.1. *The set $\tilde{\mu}^A(H) \setminus \mu^A(H)$ is discrete and closed in $\tilde{\mu}^A(H)$, and it consists of eigenvalues of H of finite multiplicity.*

One may prove this result by the method of Section 7.2.2. in [2]. We also mention the following *Virial Theorem*:

PROPOSITION 3.2. *If f is an eigenvector of H associated to an eigenvalue $\lambda \in \Omega_1^A(H)$, then $\langle f, [H,\mathrm{i}A]f\rangle = 0$.*

Proof. Let $\psi \in C_0^\infty(\Omega_1^A(H))$ real with $\psi(x) = x$ on some neighborhood of λ. If $f \in D(H)$ and $Hf = \lambda f$, then $\operatorname{supp}_H f \subset \{\lambda\}$. By Proposition 2.1, with the notation $A_\tau = (\mathrm{e}^{\mathrm{i}A\tau} - 1)/\tau$, we get:

$$\langle f, [H,\mathrm{i}A]f\rangle = \langle f, [\psi(H),\mathrm{i}A]f = \lim_{\tau\to 0}\langle f, [\psi(H), A_\tau]f\rangle = 0. \quad \blacksquare$$

PROPOSITION 3.3. *Let V be an open real set such that $\sigma(H) \subset V$, and let u be a C^∞ diffeomorphism of V onto an open real set U. Then H is locally of class $C^1(A)$ (resp $C^{1+0}(A)$ or $C^s(A)$) on V if and only if $u(H)$ is locally of class $C^1(A)$ (resp $C^{1+0}(A)$ or $C^s(A)$) on U. Moreover, if $\lambda \in \Omega_1^A(H) \cap V$ then $\rho_{u(H)}^A(u(\lambda)) = u'(\lambda)\rho_H^A(\lambda)$. In particular, if u is increasing then $\mu^A(u(H)) \cap U = u(\mu^A(H) \cap V)$ and this property is valid also for μ_{1+0}^A or μ_s^A.*

Proof. **Step 1.** The first assertion of the proposition is obvious. We shall give only the proof of the formula $\rho_{u(H)}^A(u(\lambda)) = u'(\lambda)\rho_H^A(\lambda)$. Note first that this is easily shown if u is linear: indeed, we clearly have $\rho^A_{H-d}(\lambda - d) = \rho^A_H(\lambda)$ and $\rho^A_{cH}(c\lambda) = c\rho^A_H(\lambda)$. So, if $S = H - \lambda$ and $v(x) = (u'(\lambda))^{-1}(u(x+\lambda) - u(\lambda))$, it is sufficient to prove that $\rho^A_{v(S)}(0) = \rho^A_S(0)$; note that $v(0) = 0$ and $v'(0) = 1$. By changing the notations, we are reduced to the case $\lambda = 0 \in \Omega^A_1(H) \cap V$ and the function u has the properties $u(0) = 0, u'(0) = 1$. Under these conditions we have to show that $\rho^A_{u(H)}(0) = \rho^A_H(0)$.

Step 2. Let $V_0 \subset \Omega^A_1(H) \cap V$ be a neighborhood of 0 and set $U_0 = u(V_0)$. Fix a real function $\psi \in C^\infty_0(U_0)$ such that $\psi(x) = 1$ near zero and set $\varphi = \psi \circ u$. So $\varphi \in C^\infty_0(V_0)$ and $\varphi(x) = 1$ near zero. If $v \in C^\infty_0(V_0)$ and $v(x) = u(x)$ on $\operatorname{supp}\varphi$, then

$$\psi(u(H))[u(H), \mathrm{i}A]\psi(u(H)) = \varphi(H)[v(H), \mathrm{i}A]\varphi(H).$$

Set $\xi(x) = v(x)x^{-1}$, then $\xi \in C^\infty_0(V_0)$ and $\xi(0) = 1$. Then if $\eta \in C^\infty_0(V_0)$ and $\eta(x) = x$ on $\operatorname{supp} v$, we have $v = \xi\eta$. Hence:

$$\begin{aligned}&\psi(u(H))[u(H), \mathrm{i}A]\psi(u(H))\\ &\quad= \varphi(H)[\xi(H), \mathrm{i}A]\eta(H)\varphi(H) + \varphi(H)\xi(H)[\eta(H), \mathrm{i}A]\varphi(H).\end{aligned}$$

Now, by using the fact that $\eta(x) = x$ on $\operatorname{supp}\varphi$, we obtain:

$$\begin{aligned}&\psi(u(H))[u(H), \mathrm{i}A]\psi(u(H))\\ &\quad= \varphi(H)[\xi(H), \mathrm{i}A]H\varphi(H) + \varphi(H)(\xi(H) - 1)[\eta(H), \mathrm{i}A]\varphi(H) + \varphi(H)[H, \mathrm{i}A]\varphi(H).\end{aligned}$$

Let E be the spectral measure of H, set $E(\varepsilon) = E(0, \varepsilon)$ and let E_u be the spectral measure of $u(H)$. Then we have $E_u(J) = E(u^{-1}(J))$ if J is a Borel subset of U, so for $\varepsilon > 0$ small enough we have $E(\varepsilon) = E_u(J_\varepsilon)$ where $J_\varepsilon = u((-\varepsilon, \varepsilon))$ is an open interval containing zero and which shrinks to $\{0\}$ as $\varepsilon \to 0$. Now assume that ε is so small that $\varphi(x) = 1$ on $(-\varepsilon, \varepsilon)$. We get:

$$\begin{aligned}E(\varepsilon)[u(H), \mathrm{i}A]E(\varepsilon) = E(\varepsilon)[H, \mathrm{i}A]E(\varepsilon) + E(\varepsilon)(\xi(H) - 1)[\eta(H), \mathrm{i}A]E(\varepsilon)\\ + E(\varepsilon)[\xi(H), \mathrm{i}A]HE(\varepsilon).\end{aligned}$$

We have $\|E(\varepsilon)(\xi(H) - 1)\| \leqslant \sup\{|\xi(x) - 1|/|x| < \varepsilon\} \leqslant C\varepsilon$ for some constant C and $\|HE(\varepsilon)\| \leqslant \varepsilon$. So the last two terms in the preceding equality have norms bounded by $C\varepsilon$ for a constant C. Now we get $\rho^A_{u(H)}(0) = \rho^A_H(0)$ by a straightforward argument. ∎

COROLLARY 3.4. *Assume that H is locally of class $C^1(A)$ and that $u : \mathbb{R} \to U$ is an increasing C^∞ diffeomorphism of $\mathbb{R}$ onto the open real interval U. Then the operator $u(H)$ is locally of class $C^1(A)$ on U and we have $\rho^A_{u(H)} \circ u = u'\rho^A_H$. In particular, $\mu^A(u(H)) \cap U = u(\mu^A(H))$.*

4. REGULARIZATION WITH RESPECT TO A

4.1. We keep the notations introduced in Section 1.3. In [4] (see also [6] and [8]) a functional calculus was associated to the operator $\mathcal{A}$ (acting in the Banach space $B(\mathcal{H})$) which allows one to approximate an operator $S \in B(\mathcal{H})$ by operators of class $C^\infty(A)$ in a way which is convenient for our further developments. We shall recall here only the facts that we shall need later on (for details see the preceding references). For $\psi \in \mathcal{S} \equiv \mathcal{S}(\mathbb{R})$ (Schwartz's test function space) define a linear continuous operator $\psi(\mathcal{A}) : B(\mathcal{H}) \to B(\mathcal{H})$ by

$$\psi(\mathcal{A})[S] = \int_{-\infty}^{+\infty} \mathcal{W}(\tau)[S]\widehat{\psi}(\tau)\underline{\mathrm{d}}\tau = \int_{-\infty}^{+\infty} W(-\tau)SW(\tau)\widehat{\psi}(\tau)\underline{\mathrm{d}}\tau.$$

Then $\mathcal{S} \ni \psi \mapsto \psi(\mathcal{A}) \in B(B(\mathcal{H}))$ is a homomorphism and $(\psi(\mathcal{A})[S])^* = \psi^+[S^*]$ with $\psi^+(x) = \overline{\psi}(-x)$.

For each $\varepsilon \in \mathbb{R} \setminus \{0\}$ we set $\psi(\varepsilon\mathcal{A}) = \psi^\varepsilon(\mathcal{A})$ where $\psi^\varepsilon(x) = \psi(\varepsilon x)$. Then $\psi(\varepsilon\mathcal{A})[S] = \int_{-\infty}^{+\infty} \mathcal{W}(\varepsilon\tau)[T]\widehat{\psi}(\tau)\underline{\mathrm{d}}\tau$, so the mapping $\varepsilon \mapsto \psi(\varepsilon\mathcal{A})[S] \in B(B(\mathcal{H}))$ is strongly continuous and $\operatorname{s-lim}_{\varepsilon\to 0} \psi(\varepsilon\mathcal{A})[S] = \psi(0)S$ for each $S \in B(\mathcal{H})$. Clearly $\psi(\mathcal{A})[S]$ is of class $C^\infty(A)$ and $\mathcal{A}^k\psi(\mathcal{A})[S] = \psi_{(k)}(\mathcal{A})[S]$ if $k \in \mathbb{N}$ and $\psi_{(k)}(x) = x^k\psi(x)$. So if $\psi(0) = 1$ then for each $\varepsilon \neq 0$ the operator $\psi(\varepsilon\mathcal{A})[S]$ is of class $C^\infty(A)$ and tends strongly to S as $\varepsilon \to 0$. Moreover, this convergence holds in norm if and only if S is of class $C^0_{\mathrm{u}}(A)$. The map $\varepsilon \mapsto \psi(\varepsilon\mathcal{A})[S]$ is of class C^∞ in norm on $\mathbb{R} \setminus \{0\}$ and $\frac{\mathrm{d}^k}{\mathrm{d}\varepsilon^k}\psi(\varepsilon\mathcal{A})[S] = \mathcal{A}^k\psi^{(k)}(\varepsilon\mathcal{A})[S] = \varepsilon^{-k}\psi_k(\varepsilon\mathcal{A})[S]$ where $\psi_k(x) = x^k\psi^{(k)}(x)$.

4.2. Let us fix a symmetric operator $S \in B(\mathcal{H})$. We choose now a real function $\xi \in C_0^\infty(\mathbb{R})$ such that $\xi(0) = \xi'(0) = 1$ and set $S(\varepsilon) = \xi(\varepsilon\mathcal{A})[S]$ for each $\varepsilon \in \mathbb{R}\setminus\{0\}$, $S(0) = S$. Then $S(\varepsilon)^* = S(-\varepsilon)$ and $\operatorname{s-lim}_{\varepsilon\to 0} S(\varepsilon) = S(0) = S$ (but not in norm in general, since S is not supposed of class $C^0_{\mathrm{u}}(A)$). The operator $S(\varepsilon)$ is of class $C^\infty(A)$ for each $\varepsilon \neq 0$, the map $\varepsilon \mapsto S(\varepsilon) \in B(\mathcal{H})$ is of class C^∞ in norm on $\mathbb{R} \setminus \{0\}$ and for each $\varepsilon \neq 0$ one has:

$$S'(\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}S(\varepsilon) = \varepsilon^{-1}\xi_1(\varepsilon\mathcal{A})[S] = \mathcal{A}\xi'(\varepsilon\mathcal{A})[S]. \tag{4.1}$$

Moreover, if we set $\eta(x) = (\xi(x) - \xi(-x))/2x$ if $x \neq 0$ and $\eta(0) = 1$ which is of class $C_0^\infty(\mathbb{R})$ and $\eta = \eta^+$, we get easily for each $\varepsilon \neq 0$

$$\Im S(\varepsilon) = -\mathrm{i}\varepsilon\mathcal{A}\eta(\varepsilon\mathcal{A})[S]. \tag{4.2}$$

We noticed before that the operators $S(\varepsilon)$ do not converge to S in norm (unless S is of class $C^0_{\mathrm{u}}(A)$). By using (4.2) one may easily show that $\lim_{\varepsilon\to 0} \mathrm{i}\varepsilon^{-1}\Im S(\varepsilon) = \mathcal{A}[S]$ as sesquilinear forms on $D(A)$; but $\mathcal{A}[S]$ is not a bounded operator in general and the family $\mathrm{i}\varepsilon^{-1}\Im S(\varepsilon)$ is norm convergent in $B(\mathcal{H})$ if and only if S is of class $C^1_{\mathrm{u}}(A)$. We shall describe these convergence properties in the context of the local regularity classes. We recall for this that the operator $\theta(S)[S,\mathrm{i}A]\theta(S)$ is well defined and belongs to $B(\mathcal{H})$ for each bounded function θ such that $\operatorname{supp}\theta$ is a compact subset of $\Omega^A_1(H)$ (see Proposition 2.1).

LEMMA 4.1. *Let θ be a real function in $C_0^\infty(\Omega^A_{1+0}(S))$ and set $\Phi = \theta(S)$.*

(i) *For each real ε the operator $\Phi S(\varepsilon)\Phi$ is of class $C^1(A)$. The map $\varepsilon \mapsto [A, \Phi S(\varepsilon)\Phi] \in B(\mathcal{H})$ is norm continuous on $\mathbb{R}$.*

(ii) *The map $\varepsilon \mapsto \Phi S(\varepsilon)\Phi \in B(\mathcal{H})$ is norm C^1 on $\mathbb{R}$ and one has*

$$\lim_{\varepsilon\to 0} \|\Phi S'(\varepsilon)\Phi - \Phi[S,A]\Phi\| = 0.$$

In particular:

$$\lim_{\varepsilon\to 0} \|\varepsilon^{-1}\Im\Phi S(\varepsilon)\Phi + \Phi[S,\mathrm{i}A]\Phi\| = 0. \tag{4.3}$$

Moreover, $N_\varepsilon := \|[\Phi(S(\varepsilon)-S)\Phi, A]\| + \|\Phi S'(\varepsilon)\Phi - \Phi[S,A]\Phi\|$ is integrable on $(0,1)$ with respect to the measure $\varepsilon^{-1}\mathrm{d}\varepsilon$.

(iii) *If $0 < \alpha < 1$ and $\theta \in C_0^\infty(\Omega^A_{1+\alpha}(S))$ then $N_\varepsilon = O(\varepsilon^\alpha)$.*

Proof. (i) If $\varepsilon = 0$ then $\Phi S(\varepsilon)\Phi = S\Phi^2$ is of class $C^1(A)$ (by hypothesis), while for $\varepsilon \neq 0, \Phi S(\varepsilon)\Phi$ is the product of three operators of class $C^1(A)$, so it is of class $C^1(A)$. Then for each $\varepsilon \neq 0$ we have:

$$[\Phi S(\varepsilon)\Phi, A] = \mathcal{A}[\Phi S(\varepsilon)\Phi] = [\Phi, A]S(\varepsilon)\Phi + \Phi S(\varepsilon)[\Phi, A] + \varepsilon^{-1}\Phi\xi_{(1)}(\varepsilon\mathcal{A})[S]\Phi.$$

The three terms in the right hand side are C^∞ functions of ε on $\mathbb{R}\setminus\{0\}$. So, in order to prove the norm continuity property on $\mathbb{R}$ it suffices to show that $\|\mathcal{A}[\Phi(S(\varepsilon) - S)\Phi]\| \to 0$ as $\varepsilon \to 0$. If we set $\Phi_\sigma = \mathcal{W}(-\sigma)\Phi$, we get:

$$\mathcal{A}[\Phi S(\varepsilon)\Phi] = \int_{-\infty}^{+\infty} \mathcal{W}(\tau\varepsilon)\mathcal{A}[\Phi_{\tau\varepsilon} S \Phi_{\tau\varepsilon}]\widehat{\xi}(\tau)\underline{\mathrm{d}}\tau.$$

By writing $\Phi_{\tau\varepsilon}S\Phi_{\tau\varepsilon} = (\Phi_{\tau\varepsilon} - \Phi)S\Phi_{\tau\varepsilon} + \Phi S(\Phi_{\tau\varepsilon} - \Phi) + \Phi S\Phi$ and since $\Phi S\Phi = \int_{-\infty}^{+\infty} \Phi S\Phi\widehat{\xi}(\tau)\underline{d}\tau$ (because $\xi(0) = 1$), we deduce that:

$$\begin{aligned}\mathcal{A}[\Phi(S(\varepsilon) - S)\Phi] = &\int_{-\infty}^{+\infty} (\mathcal{W}(\tau\varepsilon) - 1)\mathcal{A}[\Phi S\Phi]\widehat{\xi}(\tau)\underline{d}\tau \\ &+ \int_{-\infty}^{+\infty} \mathcal{W}(\tau\varepsilon)\mathcal{A}[(\Phi_{\tau\varepsilon} - \Phi)S\Phi_{\tau\varepsilon}]\widehat{\xi}(\tau)\underline{d}\tau \\ &+ \int_{-\infty}^{+\infty} \mathcal{W}(\tau\varepsilon)\mathcal{A}[\Phi S(\Phi_{\tau\varepsilon} - \Phi)]\widehat{\xi}(\tau)\underline{d}\tau.\end{aligned} \tag{4.4}$$

The norm-convergence to zero of the right hand side of (4.4) as $\varepsilon \to 0$ is assured by the dominated convergence theorem.

(ii) The first part of assertion (ii) of the lemma is shown by using similar arguments. Indeed, we know that the function $\varepsilon \mapsto \Phi S(\varepsilon)\Phi$ is strongly continuous on $\mathbb{R}$ and of class C^∞ on $\mathbb{R}\setminus\{0\}$. So, the norm-C^1 property of this function follows if we show that $\|\Phi S'(\varepsilon)\Phi - \Phi\mathcal{A}[S]\Phi\| \to 0$ as $\varepsilon \to 0$. But we have from (4.1):

$$\begin{aligned}\Phi S'(\varepsilon)\Phi = \mathcal{A}\xi'(\varepsilon\mathcal{A})[\Phi S\Phi] &+ \int_{-\infty}^{\infty} \mathcal{W}(\tau\varepsilon)\frac{\Phi(\tau\varepsilon) - \Phi}{\tau\varepsilon}S\Phi\tau\widehat{\xi_1}(\tau)\underline{d}\tau \\ &+ \int_{-\infty}^{\infty} \mathcal{W}(\tau\varepsilon)\Phi S\frac{\Phi(\tau\varepsilon) - \Phi}{\tau\varepsilon}\tau\widehat{\xi_1}(\tau)\underline{d}\tau \\ &+ \varepsilon\int_{-\infty}^{\infty} \mathcal{W}(\tau\varepsilon)\frac{\Phi(\tau\varepsilon) - \Phi}{\tau\varepsilon}S\frac{\Phi(\tau\varepsilon) - \Phi}{\tau\varepsilon}\tau^2\widehat{\xi_1}(\tau)\underline{d}\tau.\end{aligned}$$

The rest of the argument is similar to that in part (i).

Let us now prove the integrability of N_ε over $(0,1))$ with respect to the measure $\varepsilon^{-1}d\varepsilon$. As above it suffices to show that $\|\mathcal{A}[\Phi(S(\varepsilon) - S)\Phi]\|$ is integrable over $(0,1)$ with respect to the measure $\varepsilon^{-1}d\varepsilon$. Concerning the first terms in the right hand side of (4.4) we have

$$\left\| \int_{-\infty}^{+\infty} (\mathcal{W}(\tau\varepsilon) - 1)\mathcal{A}[\Phi S\Phi]\widehat{\xi}(\tau)\underline{d}\tau \right\| \leqslant \int_{-\infty}^{+\infty} \|(\mathcal{W}(\tau\varepsilon) - 1)\mathcal{A}[\Phi S\Phi]\| \cdot |\widehat{\xi}(\tau)|\underline{d}\tau. \tag{4.5}$$

Hence

$$\int_0^1 \varepsilon^{-1}\mathrm{d}\varepsilon \int_{-\infty}^{+\infty} \|(\mathcal{W}(\tau\varepsilon)-1)\mathcal{A}[\Phi S\Phi]\| \cdot |\widehat{\xi}(\tau)|\underline{\mathrm{d}}\tau$$
$$= \int_{-\infty}^{+\infty} \underline{\mathrm{d}}\tau|\widehat{\xi}(\tau)| \int_0^{\tau} \sigma^{-1}\,\mathrm{d}\sigma\|(\mathcal{W}(\sigma)-1)\mathcal{A}[\Phi S\Phi]\|.$$

Since $\widehat{\xi}(\tau)$ is rapidly decreasing at infinity, it suffices to estimate

$$\int_0^1 \sigma^{-1}\mathrm{d}\sigma\|(\mathcal{W}(\sigma)-1)\mathcal{A}[\Phi S\Phi]\|.$$

But the finiteness of this term is equivalent to the fact that $\mathcal{A}[\Phi S\Phi]$ is of class $C^{+0}(A)$ which holds because the function $\varphi(x)=x\theta^2(x)$ belongs to $C_0^\infty(\Omega_{1+0}^A(H))$.

We must prove the same integrability property for the sum of the two last terms in the right hand side of (4.4), which is equal to the following sum:

$$\begin{aligned}(4.6)\quad &\int_{-\infty}^{+\infty} \mathcal{W}(\tau\varepsilon)\mathcal{A}[(\Phi_{\tau\varepsilon}-\Phi)]S\Phi_{\tau\varepsilon}\widehat{\xi}(\tau)\underline{\mathrm{d}}\tau + \int_{-\infty}^{+\infty} \mathcal{W}(\tau\varepsilon)(\Phi_{\tau\varepsilon}-\Phi)\mathcal{A}[S\Phi_{\tau\varepsilon}]\widehat{\xi}(\tau)\underline{\mathrm{d}}\tau\\ &+\int_{-\infty}^{+\infty} \mathcal{W}(\tau\varepsilon)\mathcal{A}[\Phi S](\Phi_{\tau\varepsilon}-\Phi)\widehat{\xi}(\tau)\underline{\mathrm{d}}\tau + \int_{-\infty}^{+\infty} \mathcal{W}(\tau\varepsilon)\Phi S\mathcal{A}[(\Phi_{\tau\varepsilon}-\Phi)]\widehat{\xi}(\tau)\underline{\mathrm{d}}\tau.\end{aligned}$$

The norm of the first term in the right hand side is obviously estimated by

$$\int_{-\infty}^{+\infty} \|(\mathcal{W}(\tau\varepsilon)-1)\mathcal{A}\Phi\| \cdot \|S\Phi_{\tau\varepsilon}\| \cdot |\widehat{\xi}(\tau)|\underline{\mathrm{d}}\tau \leqslant C\int_{-\infty}^{+\infty} \|(\mathcal{W}(\tau\varepsilon)-1)\mathcal{A}\Phi\| \cdot |\widehat{\xi}(\tau)|\underline{\mathrm{d}}\tau.$$

So, the desired integrability property of this term follows as above (because $\mathcal{A}\Phi \in C^{+0}(A)$). The last two terms in the right hand side of (4.6) are similar to the first term. It remains to show that the second terms in the right hand side of (4.6) has the same property . For this we rewrite it as follows:

$$\int_{-\infty}^{+\infty} (\Phi-\Phi_{-\tau\varepsilon}) \cdot (\mathcal{A}\mathcal{W}(\tau\varepsilon)[S\Phi_{\tau\varepsilon}])\widehat{\xi}(\tau)\underline{\mathrm{d}}\tau.$$

By writing $\mathcal{W}(\tau\varepsilon)\mathcal{A} = \frac{1}{\mathrm{i}\varepsilon}\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{W}(\tau\varepsilon)$, we obtain after an integration by parts that the above expression is equal to:

$$\begin{aligned}(4.7)\qquad &\int_{-\infty}^{+\infty}(\mathcal{W}(\tau\varepsilon)\mathcal{A}\Phi)(\mathcal{W}(\tau\varepsilon)S\Phi_{\tau\varepsilon})\widehat{\xi}(\tau)\underline{\mathrm{d}}\tau + \int_{-\infty}^{+\infty}\frac{\Phi-\Phi_{-\tau\varepsilon}}{\mathrm{i}\tau\varepsilon}\mathcal{W}(\tau\varepsilon)(S\Phi_{\tau\varepsilon})\widehat{\zeta}(\tau)\underline{\mathrm{d}}\tau \\ &-\int_{-\infty}^{+\infty}(\Phi-\Phi_{-\tau\varepsilon})(S\mathcal{W}(\tau\varepsilon)\mathcal{A}\Phi)\widehat{\xi}(\tau)\underline{\mathrm{d}}\tau \equiv \mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3,\end{aligned}$$

with $\zeta(x) = (x\xi)'(x)$. $\mathcal{L}_3$ is obviously similar to the first term in right hand side of (4.6). So, it suffices to control the contribution of $\mathcal{L}_1 + \mathcal{L}_2$ which is obviously equal to

$$\begin{aligned}\mathcal{L}_1+\mathcal{L}_2 = &\int_{-\infty}^{+\infty}[\mathcal{W}(\tau\varepsilon)\mathcal{A}\Phi - \mathcal{A}\Phi]\mathcal{W}(\tau\varepsilon)(S\Phi_{\tau\varepsilon})\widehat{\xi}(\tau)\underline{\mathrm{d}}\tau \\ (4.8)\qquad &-\int_{-\infty}^{+\infty}\mathcal{A}\Phi\mathcal{W}(\tau\varepsilon)(S\Phi_{\tau\varepsilon})(\widehat{\zeta}(\tau)-\widehat{\xi}(\tau))\underline{\mathrm{d}}\tau \\ &+\int_{-\infty}^{+\infty}\left[\frac{\Phi-\Phi_{-\tau\varepsilon}}{\mathrm{i}\tau\varepsilon}+\mathcal{A}\Phi\right]\mathcal{W}(\tau\varepsilon)(S\Phi_{\tau\varepsilon})\widehat{\zeta}(\tau)\underline{\mathrm{d}}\tau.\end{aligned}$$

The first term in the right hand side of (4.8) is similar to the first term in the right hand side of (4.6) and the second term in the right hand side of (4.8) is equal to:

$$\mathcal{A}[\Phi]\left(\int_{-\infty}^{+\infty}\mathcal{W}(\tau\varepsilon)S\widehat{\xi_1}(\tau)\underline{\mathrm{d}}\tau\right)\Phi = \mathcal{A}[\Phi]\xi_1(\varepsilon\mathcal{A})[S]\Phi.$$

Hence

$$\begin{aligned}\int_0^1 \|\mathcal{A}[\Phi]\xi_1(\varepsilon\mathcal{A})[S]\Phi\|\frac{\mathrm{d}\varepsilon}{\varepsilon} &= \int_0^1 \|\mathcal{A}[\Phi]\varepsilon\mathcal{A}\xi'(\varepsilon\mathcal{A})[S]\Phi\frac{\mathrm{d}\varepsilon}{\varepsilon} \\ &= \int_0^1 \|\mathcal{A}[\Phi]\mathcal{A}\xi'(\varepsilon\mathcal{A})[S]\Phi d\varepsilon < \infty.\end{aligned}$$

Finally the integrability of the last term in the right hand side of (4.8) follows from the fact that Φ is of class $C^{1+0}(A)$ (see Section 1.3).

(iii) We shall now prove that if the operator S is locally of class $\mathcal{C}^{1+\alpha}(A)$ with $0 < \alpha < 1$ on a neighborhood of $\operatorname{supp}\theta$ then $\|[\Phi(S(\varepsilon) - S)\Phi, A]\| = O(\varepsilon^\alpha)$. This follows easily from the preceding arguments. For example the first term in the right hand side of (4.4) is treated as follows. By using the fact that Φ is of class $\mathcal{C}^{1+\alpha}(A)$ (i.e. $\|(\mathcal{W}(\sigma) - 1)\mathcal{A}[\Phi S \Phi]\| \leqslant C\sigma^\alpha$) and the fast decay of $\widehat{\xi}$ at infinity we may bound the right hand side of (4.5) by const$\cdot\varepsilon^\alpha$. Similarly we prove the same property for all other terms and also for $\|\Phi S'(\varepsilon)\Phi - \Phi[S, A]\Phi\|$. ∎

5. LIMITING ABSORPTION PRINCIPLE

This section is devoted to the proof of Theorem 0.1. More explicitly, we shall prove that if H, A are self-adjoint operators in the Hilbert space $\mathcal{H}$ then the boundary values $R(\lambda + \mathrm{i}0) := \lim_{\mu \to +0} R(\lambda + \mathrm{i}\mu)$ exist weakly* in $B(\mathcal{K}, \mathcal{K}^*)$ locally uniformly in $\lambda \in \mu^A_{1+0}(H)$.

5.1. Let $u : \mathbb{R} \to U$ be an increasing C^∞-diffeomorphism of $\mathbb{R}$ onto some open real bounded interval U. Then $u(H)$ is a symmetric bounded operator in $\mathcal{H}$. According to Corollary 3.4, $\lambda \in \mu^A_{1+0}(H)$ if and only if $u(\lambda) \in \mu^A_{1+0}(u(H))$. Now let $f \in \mathcal{H}$ and denote by σ the measure $\|E(\cdot)f\|^2$. Then

$$F(\lambda, \mu) \equiv \langle f, (H - \lambda - \mathrm{i}\mu)^{-1} f\rangle = \int_{\mathbb{R}} \frac{1}{x - \lambda - \mathrm{i}\mu}\sigma(\mathrm{d}x)$$

$$F_u(\lambda, \mu) \equiv \langle f, (u(H) - u(\lambda) - \mathrm{i}\mu)^{-1} f\rangle = \int_{\mathbb{R}} \frac{1}{u(x) - u(\lambda) - \mathrm{i}\mu}\sigma(\mathrm{d}x).$$

Let K be a compact real set. We prove that $\lim_{\mu \to +0} F(\lambda, \mu)$ *exists uniformly in* $\lambda \in K$ *if and only if* $\lim_{\mu \to +0} F_u(\lambda, \mu)$ *exists uniformly in* $\lambda \in K$. Indeed, we have:

$$F_u(\lambda, \mu) = \frac{1}{u'(\lambda)} F\left(\lambda, \frac{\mu}{u'(\lambda)}\right) - \int_{\mathbb{R}} \frac{u(x) - u(\lambda) - u'(\lambda)(x - \lambda)}{[u(x) - u(\lambda) - \mathrm{i}\mu] \cdot [u'(\lambda)(x - \lambda) - \mathrm{i}\mu]}\sigma(\mathrm{d}x).$$

But

$$\left|\frac{u(x) - u(\lambda) - u'(\lambda)(x - \lambda)}{[u(x) - u(\lambda) - \mathrm{i}\mu][u'(\lambda)(x - \lambda) - \mathrm{i}\mu]}\right| \leqslant \frac{|u(x) - u(\lambda) - u'(\lambda)(x - \lambda)|}{|u(x) - u(\lambda)| \cdot u'(\lambda)|x - \lambda|} \leqslant C$$

with a constant C independent of $\lambda \in K$ and $\mu > 0$. So by the dominated convergence theorem the limit as $\mu \to +0$ of the second term in the last member of the preceding equality exists uniformly in $\lambda \in K$, which finishes the proof.

The conclusion of the preceding remarks is that we can assume without loss of generality that H is a bounded self-adjoint operator in $\mathcal{H}$ (otherwise we work with $u(H)$ instead of H).

5.2. Let E be the spectral measure of the operator H and let $\lambda_0 \in \mu^A_{1+0}(H)$. We know that one may find strictly positive numbers $a_0, \delta_0 > 0$ such that $[\lambda_0 - \delta_0, \lambda_0 + \delta_0] \subset \mu^A_{1+0}(H)$ and $E(\lambda_0, \delta_0)[H, \mathrm{i}A]E(\lambda_0, \delta_0) \geqslant a_0 E(\lambda_0, \delta_0)$. In particular for each δ, a such that $0 < \delta < \delta_0$ and $0 < a < a_0$, and each real function $\theta \in C_0^\infty([\lambda_0 - \delta, \lambda_0 + \delta])$ we have $\theta(H)[H, \mathrm{i}A]\theta(H) \geqslant a_0\theta^2(H)$.

Let $H(\varepsilon) = \xi(\varepsilon A)H$ where the function ξ is as in Subsection 4.2. Then by using (4.3) we see that for each number $\nu > 0$ there is $\varepsilon_1 > 0$ such that $\varepsilon^{-1}\Im TH^*(\varepsilon)T \geqslant a_0T^2 - \nu$ for $|\varepsilon| \leqslant \varepsilon_1$, where $T = \theta(H)$. If $\varphi \in C_0^\infty((\lambda_0 - \delta_0, \lambda_0 + \delta_0))$ then the preceding function θ may be chosen such that $\theta(x) = 1$ on $\operatorname{supp}\varphi$ so $\theta\varphi = \varphi$. By pre and post-multiplying the preceding inequality by $\varphi(H)$ we then get $\varepsilon^{-1}\Im\varphi(H)H^*(\varepsilon)\varphi(H) \geqslant a_0\varphi^2(H) - \nu\varphi^2(H)$ for $0 < |\varepsilon| < \varepsilon_1$. Finally, by choosing $\nu = a_0 - a$, we see that there is $\varepsilon_1 > 0$ such that for $0 < \varepsilon \leqslant \varepsilon_1$:

$$\Im[\varphi(H)H^*(\varepsilon)\varphi(H)] \geqslant a\varepsilon\varphi^2(H). \tag{5.1}$$

5.3. Let us fix a point $\lambda_0 \in \mu^A_{1+0}(H)$, strictly positive numbers $\delta_0 > \delta > 0$ and $a_0 > a > 0$ and a function $\varphi \in C_0^\infty(\mathbb{R})$ such that: $[\lambda_0 - \delta_0, \lambda_0 + \delta_0] \subset \mu^A_{1+0}(H)$, $\operatorname{supp}\varphi \subset (\lambda_0 - \delta_0, \lambda_0 + \delta_0)$, $\varphi(x) = 1$ on the neighborhood of $[\lambda_0 - \delta, \lambda_0 + \delta]$ and $0 \leqslant \varphi(x) \leqslant 1$. We set $\Phi = \varphi(H)$ and $\Phi^\perp = 1 - \Phi^2$, and we define H_ε for any real ε by:

$$H_\varepsilon = H(1 - \Phi^2) + \Phi H(\varepsilon)\Phi. \tag{5.2}$$

We list now some properties of these operators which are immediate consequences of the results of Subsection 4.2 and of the fact that Φ is a self-adjoint operator that commutes with H:

(a) $H_0 = H$, $H_\varepsilon^* = H_{-\varepsilon}$, $\forall\varepsilon \in \mathbb{R}$;

(b) $\exists\varepsilon_1 > 0$ such that $\Im H_\varepsilon{}^* \geqslant a\varepsilon\Phi^2$ for all $0 \leqslant \varepsilon \leqslant \varepsilon_1$;

(c) the map $\varepsilon \mapsto H_\varepsilon \in B(\mathcal{H})$ is of class C^1 in norm on $\mathbb{R}$ and its derivative at 0 is given by $H_0' = \Phi[H, A]\Phi$;

(d) there is a finite constant C such that $\|H_\varepsilon - H\| \leqslant C|\varepsilon|$ for real ε.

We stress the fact that, due to the term $H(1 - \Phi^2)$ the operator H_ε is not of class $C^1(A)$ in general. The term $\Phi H(\varepsilon)\Phi$ is of class $C^1_{\mathrm{u}}(A)$, in fact, it is even of class $C^{1+0}(A)$, but it is not of higher order even if $\varepsilon \neq 0$ (because of the factor Φ).

5.4. From now on we shall always denote by λ an element of the interval $[\lambda_0 - \delta, \lambda_0 + \delta]$ and by μ a positive real number; then we set $z = \lambda + i\mu$. We recall that our aim is to prove that for each $f, g \in \mathcal{K}$, $\lim_{\mu \to +0} \langle g, (H-z)^{-1} f\rangle$ exists. But $\langle g, (H-z)^{-1}f\rangle = \langle g, (H-z)^{-1}\Phi^2 f\rangle + \langle g, (H-z)^{-1}\Phi^{\perp} f\rangle$ and the second term in the right hand side is clearly an analytic function of z in a complex neighborhood of the interval $|\lambda - \lambda_0| \leqslant \delta$. So, it suffices to show that $\lim_{\mu \to +0} \langle g, (H-z)^{-1}\Phi^2 f\rangle$ exists for each $f, g \in \mathcal{K}$, uniformly in $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$.

5.5. The aim of this step of the proof is to show that for each z as above and for $\varepsilon > 0$ sufficiently small, the operator $H_\varepsilon - z$ (with H_ε the operator defined by (5.2)) is invertible in $B(\mathcal{H})$ and its inverse $G_\varepsilon(z)$ satisfies certain estimates uniformly in μ.

We state the next lemma in a slightly more general setting. We consider an arbitrary bounded symmetric operator H and a function $\varphi \in C_0^\infty(\mathbb{R})$ with $0 \leqslant \varphi(x) \leqslant 1$ and $\varphi(x) = 1$ on a neighborhood of a compact interval $[\lambda_0 - \delta, \lambda_0 + \delta]$. The notations $\Phi, \Phi^\perp$ and $z = \lambda + i\mu$ have the same meaning as above.

LEMMA 5.1. *Let $\{H_\varepsilon\}_{0 \leqslant \varepsilon \leqslant \varepsilon_1}$ be a family of bounded operators such that: $\lim_{\varepsilon \to 0} H_\varepsilon = H_0 = H$ in norm in $B(\mathcal{H})$ and $\Im H_\varepsilon{}^* \geqslant a\varepsilon\Phi^2$ for some fixed number $a > 0$ and all $0 < \varepsilon < \varepsilon_1$. Then there are constants $C < \infty$ and $\varepsilon_0 > 0$ such that for all $\lambda, \mu, \varepsilon$ with $|\lambda - \lambda_0| < \delta$, $\mu > 0$, $0 < \varepsilon < \varepsilon_0$ and $\mu + \varepsilon > 0$ the operator $H_\varepsilon - z$ is invertible in $B(\mathcal{H})$ and its inverse $G_\varepsilon = G_\varepsilon(z) = (H_\varepsilon - z)^{-1}$ has the property $\Im G_\varepsilon \geqslant 0$ and satisfies the estimates:*

$$\|G_\varepsilon\| \leqslant C(\mu + \varepsilon)^{-1} + C, \tag{5.3}$$

$$\|G_\varepsilon f\|^2 \leqslant C(\mu + \varepsilon)^{-1}\Im\langle f, G_\varepsilon f\rangle + C\|f\|^2, \tag{5.4}$$

$$\|G_\varepsilon^* f\|^2 \leqslant C(\mu + \varepsilon)^{-1}\Im\langle f, G_\varepsilon f\rangle + C\|f\|^2. \tag{5.5}$$

For the proof see [15] or Lemmas 7.3.2 and 7.3.3 from [2].

The special form (5.2) of the operators H_ε allows us to get new properties of G_ε. From now on the constant ε_0 is that found in Lemma 5.1 and we assume $0 \leqslant \varepsilon \leqslant \varepsilon_0$, $|\lambda - \lambda_0| \leqslant \delta$, $\mu > 0$. The constant C is different from place to place but is always independent of $\lambda, \varepsilon, \mu$ (subject to the preceding conditions).

LEMMA 5.2. (i) *There is a constant C, independent of $\lambda, \varepsilon, \mu$, such that $\|G_\varepsilon \Phi^\perp\| \leqslant C$ and $\|\Phi^\perp G_\varepsilon\| \leqslant C$.*

(ii) *For each $\lambda, \varepsilon, \mu$ one may find operators $K_\varepsilon = K_\varepsilon(z)$ and $L_\varepsilon = L_\varepsilon(z)$ such that $\Phi G_\varepsilon(1-\Phi) = K_\varepsilon \Phi$ and $(1-\Phi)G_\varepsilon \Phi = \Phi L_\varepsilon$, and such that $\|K_\varepsilon\| + \|L_\varepsilon\| \leqslant C < \infty$ for some constant independent of $\lambda, \varepsilon, \mu$.*

Proof. We have

$$\begin{aligned} G_\varepsilon(1-\Phi) &= G_\varepsilon(H - H_\varepsilon + H_\varepsilon - z)(H-z)^{-1}(1-\Phi) \\ &= G_\varepsilon \Phi(H - H(\varepsilon))\Phi(H-z)^{-1}(1-\Phi) + (H-z)^{-1}(1-\Phi). \end{aligned}$$

Let $\psi \in C_0^\infty(\mu_{1+0}^A(H))$ such that $\psi\varphi = \varphi$. Then $\Phi\psi(H) = \psi(H)\Phi = \Phi$ and if we set

$$K_{1\varepsilon} = G_\varepsilon \Phi \psi(H)(H - H(\varepsilon))\psi(H)(H-z)^{-1}(1-\Phi)$$

then we obtain $G_\varepsilon(1-\Phi) = K_{1\varepsilon}\Phi + (H-z)^{-1}(1-\Phi)$. From (5.3) and Lemma 4.1 we clearly get $\|K_{1\varepsilon}\| \leqslant C < \infty$, for some constant C independent of $\lambda, \varepsilon, \mu$. This proves the assertions concerning ΦG_ε with $K_\varepsilon = \Phi K_{1\varepsilon} + (H-z)^{-1}(1-\Phi)$. For $G_\varepsilon \Phi$ the proof is similar and one may take

$$L_\varepsilon = (1-\Phi)(H-z)^{-1}[\psi(H)(H - H(\varepsilon))\psi(H)\Phi G_\varepsilon \Phi + 1]. \quad \blacksquare$$

5.6. Since the operator H_ε is not of class $C^1(A)$, the operator G_ε is not of class $C^1(A)$. We shall see below that $\Phi G_\varepsilon \Phi$ behaves better from this point of view. Observe first that we have, in the sense of sesquilinear forms on $D(A)$,

$$[A, \Phi G_\varepsilon \Phi] = [A, \Phi]G_\varepsilon \Phi + \Phi G_\varepsilon [A, \Phi] + \Phi[A, G_\varepsilon]\Phi. \tag{5.6}$$

The last term in the right hand side is a well defined continuous sesquilinear form on $D(A)$ because Φ is of class $C^1(A)$ and so $\Phi D(A) \subset D(A)$ and the operator induced by Φ in $D(A)$ is bounded. The first two terms in the right hand side in (5.6) are bounded operators in $\mathcal{H}$. Hence $\Phi G_\varepsilon \Phi$ is of class $C^1(A)$ if and only if $\Phi[A, G_\varepsilon]\Phi$ is a bounded operator in $\mathcal{H}$. In order to prove this fact we recall a method of computing the sesquilinear form $[A, T]$, with domain $D(A)$, for an arbitrary $T \in B(\mathcal{H})$. Set $A_\tau = (\mathrm{i}\tau)^{-1}(1 - \mathrm{e}^{\mathrm{i}A\tau})$ for $\tau \neq 0$ and note that $A_\tau^* = A_{-\tau}$. Then for all $f, g \in D(A)$ we have

$$\begin{aligned} \langle f, [A, T]g\rangle &= \langle Af, Tg\rangle - \langle Tf, Ag\rangle \\ &= \lim_{\tau\to 0}\langle A_{-\tau}f, Tg\rangle - \langle f, TA_\tau g\rangle = \lim_{\tau\to 0}\langle f, [A_\tau, T]g\rangle. \end{aligned}$$

Here $[A_\tau, T]$ are bounded operators. So, the boundedness of the form $[A, T]$ is a consequence of the existence of the weak limit $\text{w-}\lim_{\tau\to 0}[A_\tau, T]$ in $B(\mathcal{H})$. On the other hand, even if $[A, T]$ is not a bounded operator, the expression $\Phi[A, T]\Phi$ makes sense as sesquilinear form on $D(A)$ (because $\Phi D(A) \subset D(A)$) and we have $\langle f, \Phi[A, T]\Phi g\rangle = \lim_{\tau\to 0}\langle f, \Phi[A_\tau, T]\Phi g\rangle$ for each $f, g \in D(A)$. So $\Phi[A, T]\Phi$ is a bounded operator in $\mathcal{H}$ if $\text{w-}\lim_{\tau\to 0}\Phi[A_\tau, T]\Phi$ exists in $B(\mathcal{H})$.

LEMMA 5.3. (i) *The sesquilinear form $\Phi[A, H_\varepsilon]\Phi$ (with domain $D(A)$) is a bounded operator in $\mathcal{H}$ and we have*

$$\Phi[A, H_\varepsilon]\Phi = \text{s-}\lim_{\tau\to 0}\Phi[A_\tau, H_\varepsilon]\Phi = [A, H\Phi^2(1-\Phi^2)] \\ - [A, \Phi]H\Phi(1-\Phi^2) - H\Phi(1-\Phi^2)[A, \Phi] + \Phi[A, \Phi H(\varepsilon)\Phi]\Phi. \tag{5.7}$$

(ii) *The sesquilinear form $\Phi[A, G_\varepsilon]\Phi$ (with domain $D(A)$) is a bounded operator in $\mathcal{H}$ and we have*

$$\Phi[A, G_\varepsilon]\Phi = (\Phi G_\varepsilon + K_\varepsilon)\Phi[H_\varepsilon, A]\Phi(L_\varepsilon + G_\varepsilon\Phi). \tag{5.8}$$

Proof. (i) According to the preceding discussion it is sufficient to show that $\text{s-}\lim_{\tau\to 0}\Phi[A_\tau, H_\varepsilon]\Phi$ exists and equals the last member of (5.7) (note that each term in the last member of (5.7) is a bounded operator). Since Φ and $\Phi H(\varepsilon)\Phi$ are of class $C^1(A)$, it is sufficient to treat the term $H(1-\Phi^2) \equiv H\Phi^\perp$ from (5.2). But we clearly have (A_τ being bounded):

$$\Phi[A_\tau, H\Phi^\perp]\Phi = [A_\tau, H\Phi^2\Phi^\perp] - [A_\tau, \Phi]H\Phi\Phi^\perp - H\Phi\Phi^\perp[A_\tau, \Phi].$$

Since $H\Phi^2\Phi^\perp = H\Phi^2(1-\Phi^2)$ and Φ are of class $C^1(A)$, the right hand side is strongly convergent and this proves (i).

(ii) As before it suffices to show that $\text{s-}\lim_{\tau\to 0}\Phi[A_\tau, G_\varepsilon]\Phi$ exists and is equal to the right hand side of (5.8). Since A_τ is bounded and $\Phi G_\varepsilon = (\Phi G_\varepsilon + K_\varepsilon)\Phi, G_\varepsilon\Phi = \Phi(L_\varepsilon + G_\varepsilon\Phi)$, by Lemma 5.2, we have:

$$\Phi[A_\tau, G_\varepsilon]\Phi = \Phi G_\varepsilon[H_\varepsilon, A_\tau]G_\varepsilon\Phi = (\Phi G_\varepsilon + K_\varepsilon)\Phi[H_\varepsilon, A_\tau]\Phi(L_\varepsilon + G_\varepsilon\Phi).$$

Now the result follows from (i). ∎

5.7. We need one more identity that we shall establish in the next lemma.

LEMMA 5.4. *The map $\varepsilon \mapsto G_\varepsilon$ is norm C^1 on $[0, \varepsilon_0]$ and the operator $\Phi G_\varepsilon \Phi$ is of class $C^1(A)$ for each $\varepsilon \in [0, \varepsilon_0]$. If we set $G'_\varepsilon = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} G_\varepsilon$ and*

$$M_\varepsilon = \Phi[H, A]\Phi - \Phi H'(\varepsilon)\Phi + \Phi[\Phi(H(\varepsilon) - H)\Phi, A]\Phi,$$

then we have:

$$\begin{aligned}\Phi G'_\varepsilon \Phi + [A, \Phi G_\varepsilon \Phi] = K_\varepsilon \Phi[H_\varepsilon, A]\Phi L_\varepsilon + ([A, \Phi] + K_\varepsilon \Phi[H_\varepsilon, A]\Phi) G_\varepsilon \Phi \\ + \Phi G_\varepsilon([A, \Phi] + \Phi[H_\varepsilon, A]\Phi L_\varepsilon) + \Phi G_\varepsilon M_\varepsilon G_\varepsilon \Phi.\end{aligned} \tag{5.9}$$

Proof. The differentiability of G_ε is easily obtained from the identity $G_\varepsilon - G_{\varepsilon'} = -G_\varepsilon(H_\varepsilon - H_{\varepsilon'})G_{\varepsilon'}$ and the estimate $\|G_\varepsilon\| \leqslant C\mu^{-1} + C$. We observe that $G'_\varepsilon = -G_\varepsilon H_\varepsilon{}' G_\varepsilon$. The property $\Phi G_\varepsilon \Phi \in C^1(A)$ follows from (5.6) and Lemma 5.3 Then (5.9) follows from (5.6) and (5.8) by observing that $M_\varepsilon = \Phi[H_\varepsilon, A]\Phi - H'_\varepsilon$. ∎

LEMMA 5.5. *Let $\{f_\varepsilon\}$ be a family of vectors in $\mathcal{H}$ such that $\varepsilon \mapsto f_\varepsilon$ is strongly C^1 and $f_\varepsilon \in D(A)$ for each ε. Set $F_\varepsilon = \langle \Phi f_\varepsilon, G_\varepsilon \Phi f_\varepsilon \rangle$. Then $\varepsilon \mapsto F_\varepsilon$ is of class C^1 and its derivative F'_ε satisfies:*

$$\begin{aligned}F'_\varepsilon = \langle f_\varepsilon, K_\varepsilon \Phi[H_\varepsilon, A]\Phi L_\varepsilon f_\varepsilon \rangle + \langle \Phi f'_\varepsilon - A\Phi f_\varepsilon + \Phi[A, H_\varepsilon]\Phi K^*_\varepsilon f_\varepsilon, G_\varepsilon \Phi f_\varepsilon \rangle \\ + \langle G^*_\varepsilon \Phi f_\varepsilon, \Phi f'_\varepsilon + A\Phi f_\varepsilon + \Phi[H_\varepsilon, A]\Phi L_\varepsilon f_\varepsilon \rangle + \langle G^*_\varepsilon \Phi f_\varepsilon, M_\varepsilon G_\varepsilon \Phi f_\varepsilon \rangle.\end{aligned} \tag{5.10}$$

Let $l(\varepsilon) = \|f'_\varepsilon\| + \|Af_\varepsilon\| + \|f_\varepsilon\|$. Then there is a constant C independent of $\lambda, \varepsilon, \mu$ such that F_ε satisfies the following differential inequality:

$$| F'_\varepsilon | \leqslant C\|f_\varepsilon\|(l(\varepsilon) + \|M_\varepsilon\| \cdot \|f_\varepsilon\|) + C\frac{l(\varepsilon)}{\sqrt{\varepsilon}} | F_\varepsilon |^{\frac{1}{2}} + \frac{C}{\varepsilon}\|M_\varepsilon\| \cdot | F_\varepsilon |. \tag{5.11}$$

Proof. (5.10) is a straightforward consequence of (5.8). Note that $\Phi A f_\varepsilon + [A, \Phi]f_\varepsilon = A\Phi f_\varepsilon$, because $f_\varepsilon \in D(A)$ and $\Phi D(A) \subset D(A)$. From (5.7) and 4.2 it follows that $\|\Phi[H_\varepsilon, A]\Phi\|$ is bounded by a constant independent of $\varepsilon \in (0, \varepsilon_0)$, λ and μ. Since we have a similar bound for $\|K_\varepsilon\|$ and $\|L_\varepsilon\|$ and $\|A\Phi f_\varepsilon\| \leqslant \|Af_\varepsilon\| + \|[A, \Phi]\| \, \|f_\varepsilon\|$, we see that there is a constant C_1 (independent of $\lambda, \varepsilon, \mu$) such that

$$| F'_\varepsilon | \leqslant C_1\|f_\varepsilon\|^2 + C_1 l(\varepsilon)(\|G_\varepsilon \Phi f_\varepsilon\| + \|G^*_\varepsilon \Phi f_\varepsilon\|) + \|M_\varepsilon\| \cdot \|G_\varepsilon \Phi f_\varepsilon\| \cdot \|G^*_\varepsilon \Phi f_\varepsilon\|.$$

On the other hand, (5.4) and (5.5) imply:

$$\|G^{(*)}_\varepsilon \Phi f_\varepsilon\| \leqslant \frac{C}{\sqrt{\varepsilon}} | F_\varepsilon |^{\frac{1}{2}} + C\|\Phi f_\varepsilon\|$$

where $G^{(*)}_\varepsilon$ is either G_ε and G^*_ε. The last two inequalities clearly imply (5.11) (for a new constant C). ∎

5.8. We are now in a position to deduce the main estimates of the theory.

PROPOSITION 5.6. *There is a constant C, independent of $\lambda, \varepsilon, \mu$, such that $\|G_\varepsilon\|^{(1/2,1)} := \|G_\varepsilon\|_{\mathcal{K}\to\mathcal{K}^*} \leqslant C$. In particular for each $s > 1/2$ there exists $C_s < \infty$ such that $\|G_\varepsilon\|^{(s)} := \|G_\varepsilon\|_{\mathcal{H}_s\to\mathcal{H}_s^*} \leqslant C_s$. Moreover, one may choose C such that:*

$$\|G_\varepsilon\|_{\mathcal{K}\to\mathcal{H}} + \|G_\varepsilon\|_{\mathcal{H}\to\mathcal{K}^*} \leqslant C\varepsilon^{-\frac{1}{2}}. \tag{5.12}$$

$$\|G_\varepsilon\|_{\mathcal{H}_s\to\mathcal{H}} + \|G_\varepsilon\|_{\mathcal{H}\to\mathcal{H}_{-s}} \leqslant C\varepsilon^{-\frac{1}{2}}. \tag{5.13}$$

Proof. By following the proof from Section 4.6 of [6] we get $|\langle\Phi f, G_\varepsilon\Phi f\rangle| \leqslant C\|f\|_{\mathcal{K}}^2$ for all $f \in \mathcal{K}$ (note that $\int_0^{\varepsilon_0} \|M_\varepsilon\|\varepsilon^{-1}\,\mathrm{d}\varepsilon < \infty$, as has been shown in Lemma 4.1 (b)). This gives $\|\Phi G_\varepsilon \Phi\|^{(1/2,1)} \leqslant C$ for (another) constant C. Then $\|G_\varepsilon\|^{(1/2,1)} \leqslant C$ is consequence of Lemma 5.2 (i). The other estimates are proved as in Section 4.6 of [6]. ∎

Theorem 0.1 is an easy consequence of the Proposition 5.6, see Section 4.7 in [6].

6. REGULARITY OF THE BOUNDARY VALUES OF THE RESOLVENT FAMILY

In the previous section we have shown the existence of the boundary values $R(\lambda \pm \mathrm{i}0)$ in $B(\mathcal{K},\mathcal{K}^*)$ if $\lambda \in \mu^A_{1+0}(H)$. But for each $s > 1/2$ we have $B(\mathcal{K},\mathcal{K}^*) \subset B(\mathcal{H}_s,\mathcal{H}_{-s})$. We shall now prove that for $1/2 < s < 1$ the map $\lambda \mapsto R(\lambda \pm \mathrm{i}0) \in B(\mathcal{H}_s,\mathcal{H}_{-s})$ is locally of class Λ^α on $\mu^A_{1+\alpha}(H)$, with $\alpha = s - 1/2$.

This section is devoted to the proof of this assertion. We keep the notations of the preceding section and we assume that $[\lambda_0 - \delta_0, \lambda_0 + \delta_0] \subset \mu^A_{1+\alpha}(H)$. Clearly it suffices to show that the map $\lambda \mapsto \Phi R(\lambda + \mathrm{i}0)\Phi \in B(\mathcal{H}_s, \mathcal{H}_s^*)$ is of class Λ^α on $[\lambda_0 - \delta, \lambda_0 + \delta]$. Let

$$F_\varepsilon = F(\lambda,\varepsilon) = \theta(\varepsilon A)\Phi G_\varepsilon \Phi\theta(\varepsilon A) = \theta(\varepsilon A)\Phi(H_\varepsilon - \lambda - \mathrm{i}\mu)^{-1}\Phi\theta(\varepsilon A)$$

where $\theta \in C_0^\infty(\mathbb{R})$ with $\theta(0) = 1$, $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta] \subset \mu^A_{1+\alpha}(H)$, $\mu > 0$, $\varepsilon \in (0, \varepsilon_0)$. The assertion that we want to prove is a simple consequence of the following estimate

$$\left\|\frac{\mathrm{d}}{\mathrm{d}\lambda}F_\varepsilon\right\|^{(s)} + \left\|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}F_\varepsilon\right\|^{(s)} \leqslant C\varepsilon^{-1+\alpha}. \tag{6.1}$$

Indeed, by applying the Proposition A.1 of [6] to the function $[\lambda_0 - \delta, \lambda_0 + \delta] \times (0, \varepsilon_0) \ni (\lambda, \varepsilon) \mapsto \langle f, F(\lambda,\varepsilon) f\rangle$ for $f \in \mathcal{H}_s$, one sees that the map $\lambda \mapsto \langle f, F(\lambda, 0) f\rangle = \langle f, \Phi R(\lambda + \mathrm{i}\mu)\Phi f\rangle$ is of class Λ^α on $[\lambda_0 - \delta, \lambda_0 + \delta]$, uniformly in μ.

So, if we make $\mu \to +0$ we obtain that the map $\lambda \mapsto \langle f, \Phi R(\lambda + \mathrm{i}0)\Phi f\rangle$ is of class Λ^α on $[\lambda_0 - \delta, \lambda_0 + \delta]$, and the desired assertion follows from the polarization formula and the Banach-Steinhauss theorem.

Now we shall prove the estimate (6.1). For this we calculate the derivative with respect to ε:

$$F'_\varepsilon = A\theta'(\varepsilon A)\Phi G_\varepsilon \Phi\theta(\varepsilon A) + \theta(\varepsilon A)\Phi G'_\varepsilon \Phi\theta(\varepsilon A) + \theta(\varepsilon A)\Phi G_\varepsilon \Phi A\theta'(\varepsilon A).$$

By Lemma 5.4, $\Phi G'_\varepsilon \Phi = \mathcal{A}[\Phi G_\varepsilon \Phi] + R_\varepsilon$, with R_ε is the right hand side of the identity (5.9); and since $\mathcal{A}[\Phi G_\varepsilon \Phi] + A\Phi G_\varepsilon \Phi = \Phi G_\varepsilon \Phi A$ is valid in the sense of sesquilinear forms on $D(A)$, we have

$$\varepsilon F'_\varepsilon = \varphi_1(\varepsilon A)\Phi G_\varepsilon \Phi\theta(\varepsilon A) + \theta(\varepsilon A)\Phi G_\varepsilon \Phi\varphi_2(\varepsilon A) + \varepsilon\theta(\varepsilon A)R_\varepsilon\theta(\varepsilon A) \tag{6.2}$$

with the notations $\varphi_1(x) = x(\theta'(x) - \theta(x))$ and $\varphi_2(x) = x(\theta'(x) + \theta(x))$. The first two terms in the right hand side of (6.2) are similar, so it suffices to estimate only the first term (for example):

$$\|\langle A\rangle^{-s}\varphi_1(\varepsilon A)\Phi G_\varepsilon \Phi\theta(\varepsilon A)\langle A\rangle^{-s}\| \leqslant C\|\langle A\rangle^{-s}\varphi_1(\varepsilon A)\| \cdot \|\Phi G_\varepsilon \Phi\langle A\rangle^{-s}\| \tag{6.3}$$

where C is a finite constant independent of $\lambda, \varepsilon, \mu$ (as above). Since for each $0 < s \leqslant 1$ we have $\|\varphi_1(\varepsilon A)\langle A\rangle^{-s}\| \leqslant C\varepsilon^s$ as $\varepsilon \to 0$. So by taking into account the estimate (5.13), the right hand side of (6.3) is dominated by a constant independent of $\lambda, \varepsilon, \mu$ times $\varepsilon^{s-1/2} = \varepsilon^\alpha$ for each $\varepsilon \in (0, \varepsilon_0)$. It remains to show the same estimate for $\varepsilon R_\varepsilon$. For this we recall the explicit form of R_ε:

$$\begin{aligned} R_\varepsilon = {} & K_\varepsilon \Phi\mathcal{A}[H_\varepsilon]\Phi L_\varepsilon + (-\mathcal{A}[\Phi] + K_\varepsilon \Phi\mathcal{A}[H_\varepsilon]\Phi)G_\varepsilon \Phi \\ & + \Phi G_\varepsilon(-\mathcal{A}[\Phi] + \Phi\mathcal{A}[H_\varepsilon]\Phi)L_\varepsilon + \Phi G_\varepsilon M_\varepsilon G_\varepsilon \Phi. \end{aligned} \tag{6.4}$$

Then by Lemma 5.2 there is a constant C (independent of $\lambda, \varepsilon, \mu$) such that:

$$\|\varepsilon K_\varepsilon \Phi\mathcal{A}[H_\varepsilon]\Phi L_\varepsilon\|^{(s)} \leqslant C\varepsilon\|\Phi\mathcal{A}[H_\varepsilon]\Phi\|.$$

By Lemma 5.2 and relation (5.7) the right hand side of the preceding estimate is dominated by a constant (similar to C) times ε and this is better than we need because $\alpha < 1/2 < 1$. The second and the third terms in the right hand side of (6.4) are similar, so we have to estimate the second term for example:

$$\begin{aligned} & \|\varepsilon\langle A\rangle^{-s}(-\mathcal{A}[\Phi] + K_\varepsilon \Phi\mathcal{A}[H_\varepsilon]\Phi)\Phi G_\varepsilon\langle A\rangle^{-s}\| \\ & \quad \leqslant \varepsilon\|\langle A\rangle^{-s}(-\mathcal{A}[\Phi] + K_\varepsilon \Phi\mathcal{A}[H_\varepsilon]\Phi)\| \cdot \|\Phi G_\varepsilon\langle A\rangle^{-s}\|. \end{aligned}$$

By the same argument as above, the right hand side is bounded by $C\varepsilon^{1/2}$ (with a constant C which is independent of $\lambda, \varepsilon, \mu$) and this is better than we need because $\alpha < 1/2 < 1$. Finally by the same argument there is a constant C independent of $\lambda, \varepsilon, \mu$ such that

$$\begin{aligned}\|\varepsilon\langle A\rangle^{-s}\Phi G_\varepsilon M_\varepsilon G_\varepsilon \Phi\langle A\rangle^{-s}\| &\leqslant \varepsilon\|\langle A\rangle^{-s}\Phi G_\varepsilon\| \cdot \|M_\varepsilon\| \cdot \|G_\varepsilon\Phi\langle A\rangle^{-s}\| \\ &\leqslant C\varepsilon\varepsilon^{-1}\|M_\varepsilon\| \equiv C\|M_\varepsilon\|.\end{aligned}$$

Now we apply Lemma 4.1 (iii): the fact that $[\lambda_0 - \delta_0, \lambda_0 + \delta_0] \subset \mu^A_{1+\alpha}(H)$ implies that $\|M_\varepsilon\| = \mathrm{O}(\varepsilon^\alpha)$, so the right hand side of the preceding estimate satisfies the desired estimate. In conclusion, we have $\|\varepsilon\langle A\rangle^{-s}F'_\varepsilon\langle A\rangle^{-s}\| = \mathrm{O}(\varepsilon^\alpha)$.

It remains to prove a similar estimate for $(\mathrm{d}/\mathrm{d}\lambda)F_\varepsilon$.For this we derive (6.2) with respect to λ and, by taking into account the relation $\frac{\mathrm{d}}{\mathrm{d}\lambda}G_\varepsilon = G_\varepsilon^2$ we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\varepsilon F'_\varepsilon = \varphi_1(\varepsilon A)\Phi G_\varepsilon^2\Phi\theta(\varepsilon A) + \theta(\varepsilon A)\Phi G_\varepsilon^2\Phi\varphi_2(\varepsilon A) + \varepsilon\theta(\varepsilon A)\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}R_\varepsilon\right)\theta(\varepsilon A). \tag{6.5}$$

As above we get

$$\left\|\langle A\rangle^{-s}\frac{\mathrm{d}}{\mathrm{d}\lambda}\varepsilon F'_\varepsilon\langle A\rangle^{-s}\right\| = \mathrm{O}(\varepsilon^{-1+\alpha}),$$

in other terms

$$\left\|\langle A\rangle^{-s}\frac{\mathrm{d}}{\mathrm{d}\lambda}F'_\varepsilon\langle A\rangle^{-s}\right\| = \mathrm{O}(\varepsilon^{-2+\alpha}).$$

By integrating with respect to ε over an interval $(\varepsilon, \varepsilon_0)$ and by taking into account that $\alpha - 1 < 0$ we obtain

$$\left\|\langle A\rangle^{-s}\frac{\mathrm{d}}{\mathrm{d}\lambda}F_\varepsilon\langle A\rangle^{-s}\right\| = \mathrm{O}(\varepsilon^{-1+\alpha}).$$

Thus (6.1) is proved and so the proof of Theorem 0.2 is finished. ∎

The proof of Theorem 0.3 is rather similar: one uses the estimates that we have established in Section 5 and the techniques of the proof of Theorem 6.7 from [8]. Details can be found in [18].

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REFERENCES

1. S. AGMON, L. HÖRMANDER, Asymptotic properties of solutions of differential equations with simple characteristics, *J. Anal. Math.* **30**(1976), 1–38.
2. W. AMREIN, A. BOUTET DE MONVEL, V. GEORGESCU, *C_0-Groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians*, Birkhäuser, Progr. Math., vol. 135, 1996.
3. G. ARSU, Spectral analysis for simply characteristic operators by Mourre's Method. III, *J. Operator Theory* **34**(1995), 177–187.
4. A. BOUTET DE MONVEL, V. GEORGESCU, Locally conjugate operators, boundary values of the resolvent and wave operators, *C.R. Acad. Sci. Paris Sér. I* **313**(1991), 13–18.
5. A. BOUTET DE MONVEL, V. GEORGESCU, Some developments and applications of the abstract Mourre theory, *Astérisque* **210**(1992), 27–48.
6. A. BOUTET DE MONVEL, V. GEORGESCU, Boundary values of the resolvent of a self-adjoint operator: higher order estimates, in *Algebraic and Geometrics Methods in Mathematical Physics*, Kaciveli Workshop, Crimé 1993, A. Boutet de Monvel, V. Marchenko eds., Kluwer, 1996, pp. 9–52.
7. A. BOUTET DE MONVEL, V. GEORGESCU, J. SAHBANI, Boundary values of resolvent families and propagation properties, *C.R. Acad. Sci. Paris Sér. I* **322**(1996), 289–294.
8. A. BOUTET DE MONVEL, V. GEORGESCU, J. SAHBANI, Higher order estimates in the conjugate operator theory, preprint de l'Institut de Mathématiques de Jussieu, no. 59, 1996.
9. H.L. CYCON, R.G. FROESE, W. KIRSCH, B. SIMON, *Schrödinger Operators with Applications to Quantum Mechanics and Global Geometry*, Springer, Berlin 1987.
10. L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators*, vol. 2, Springer, Berlin, 1985.
11. A. JENSEN, Propagation estimates for Schrödinger-type operators, *Trans. Amer. Math. Soc.* **291**(1985), 129–144.
12. A. JENSEN, Scattering theory for Stark hamiltonians, *Proc. Indian Acad. Sci. Math.*, vol. 104, 1994, pp. 207–225.
13. A. JENSEN, E. MOURRE, P. PERRY, Multiple commutator estimates and resolvent smoothness in quantum scattering theory, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **41**(1984), 207–225.
14. A. JENSEN, P. PERRY, Commutator methods and Besov space estimates for Schrödinger operators, *J. Operator Theory* **14**(1985), 181–189.
15. E. MOURRE, Absence of singular continuous spectrum for certain self-adjoint operators, *Comm. Math. Phys.* **78**(1981), 391–408.
16. E. MOURRE, Opérateurs conjugués et propriétés de propagation, *Comm. Math. Phys.* **91**(1983), 279–300.
17. P. PERRY, I.M. SIGAL, B. SIMON, Spectral analysis of N-body Schrödinger operators, *Ann. Math. (2)* **114**(1981), 519–567.

18. J. SAHBANI, Théorèmes de propagation, hamiltoniens localement réguliers et applications, Ph.D. Dissertation, Université Paris 7, 1996.
19. J. SAHBANI, Propagation theorems for some classes of pseudo-differential operators, *J. Math. Anal. Appl.* **211**(1997), 481–497.

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