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WICK ALGEBRA OF GENERALIZED OPERATORS INVOLVING QUANTUM WHITE NOISE

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ABSTRACT. By virtue of the chaos decomposition and symbol calculus of generalized operators generated by quantum white noise, Wick product of generalized operators is introduced in an physically intuitive way and characterized analytically and algebraically. The set of generalized operators is a commutative unital involutive associative algebra under Wick product. Some fundamental properties of the Wick algebra are investigated.

KEYWORDS: *Free Bose fields, quantum white noise, Wick algebra, generalized operators.*

AMS SUBJECT CLASSIFICATION: Primary 47D30; Secondary 47D45.

0. INTRODUCTION

In the mathematical descriptions of quantum field theory, the field operator at a point of space-time is a highly singular object ([1], [2], [10]), only exists as a generalized operator (operator valued distribution). The product (i.e. composition) of such operators is nonsense, let alone nonlinear functions of them. However, physics always necessitates treatment of functions of fields. To this end renormalizations are inevitable, one method of such procedures — the so called Wick ordering — has been utilized by physicists for a long time, though their manipulations are formal ([2]).

We shall try to define nonlinear functions of Bose fields, or more precisely algebra structures over fields, by virtue of the notion of Wick product within the framework of white noise analysis.

In white noise approach to the calculus over free Bose fields ([3]), it is possible to introduce generalized (including pointwisely defined) annihilation and creation operators as rigorous mathematical objects within the framework of Hilbert space plus distribution theory (Gelfand triplet). The generalized annihilation (resp. creation) operators are continuous on Hida's testing (resp. generalized) function space, thus the field operators as linear combinations of annihilation and creation operators are generalized operators, i.e., continuous from testing functional space to generalized functional space. Moreover, any generalized operator admits a unique chaos decomposition ([5]), each chaos is a formal integral kernel operator generated by quantum white noise (the family of pointwisely defined annihilation and creation operators). If the formal product of generalized operators is renormalized by taking Wick ordering ([2], [10]), i.e., moving every creation operator to the left of annihilation operators, then the formal product of generalized operators makes sense as a generalized operator. This is the physical origin of Wick product of generalized operators. Due to Obata's symbol characterization of generalized operators ([9]), such Wick product can be characterized analytically.

The paper is organized as follows. In Section 1 we assemble general notions of white noise calculus and quantum white noise. In Section 2, following ([6], [8]), we introduce the symbol calculus of generalized operators and prepare some lemmas including operator's chaos decomposition and symbol characterization. In Section 3 we introduce Wick product of generalized operators following physicist's intuitive and heuristic procedure. In Section 4 we characterize Wick product by appealing to the symbol characterization of operators, and study some fundamental properties of Wick algebra involving quantum white noise. Some subalgebras such as the annihilation and creation operator algebras are observed.

We should mention that Wick product of generalized white noise or Wiener functionals is extensively studied by many author (see, e.g. [3], [7], [8]) and Wick product of generalized operators is essentially along the same line, but at the operator level and possesses its own special features. Also, the present approach should be compared with Segal's work ([10]).

1. QUANTUM WHITE NOISE

All spaces in this paper are assumed to be real, this is just for notational simplicity and comlexification is straightforward.

To recall the general idea and notion of white noise initiated by Hida ([3]), we follow the presentation of [4], [5].

Let $H = L^2(\mathbb{R})$ be the Hilbert space of square integrable (with respect to Lebesgue measure) functions on $\mathbb{R}$ with the standard norm $|\cdot|_0, E = S(\mathbb{R})$ the Schwartz space of rapidly decreasing C^∞ functions and $E^* = S^*(\mathbb{R})$, the space of tempered distributions.

Let $A = -\frac{d^2}{dt^2} + t^2 + 1$ be the simple harmonic oscillator on H (note that A extends to a positive self-adjoint operator on H). For $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$, put $E_k = \operatorname{Dom} A^k$, then E_k is a Hilbert space with norm $|\cdot|_k = |A^k \cdot|_0$. Denote by E_{-k} its dual space with dual norm $|\cdot|_{-k}$, $E = \bigcap\limits_k E_k$ is the projective limit of E_k and $E^* = \bigcup\limits_k E_{-k}$ is the inductive limit of E_{-k}. By Minlos Theorem, there exists a unique Gaussian measure associated with the Gelfand triplet

$$E \subset H \subset E^*,$$

satisfying

$$\int\limits_{E^*} e^{i\langle x, \xi\rangle} \mu(dx) = e^{-\frac{1}{2}|\xi|_0^2}, \quad \xi \in E,$$

where $\langle\cdot, \cdot\rangle$ is the dual pairing between E^* and E.

(E^*, μ) is called the *white noise space* and serves as the basic probability space in white noise calculus.

Let $(L^2) = L^2(E^*, \mu)$ with norm $\|\cdot\|_0$ be the space of square integrable white noise functionals and

$$\Phi(H) = \bigoplus_{n=0}^{\infty} H^{\widehat{\otimes} n}$$

be the Bose-Fock space over H, then via the well known Wiener-Itô-Segal chaos decomposition, (L^2) is isometrically isomorphic to $\Phi(H)$. Therefore, the second quantized operator ([1])

$$\Gamma(A) = \bigoplus_{n=0}^{\infty} A^{\otimes n}$$

acting in $\Phi(H)$ can be lifted to an operator acting in (L^2) naturally.

For $k \in \mathbb{N}$, put $(E_k) = \operatorname{Dom} \Gamma(A^k)$, then (E_k) is a Hilbert space with norm $\|\cdot\|_k = \|\Gamma(A^k) \cdot\|_0$. Denote by (E_{-k}) the dual space of (E_k) with dual norm

$\|\cdot\|_{-k}$. The projective limit $(E) = \bigcap_k (E_k)$ is a nuclear Fréchet reflexive space and the inductive limit $(E)^* = \bigcup_k (E_{-k})$ is its topological dual. Thus, we obtain a second Gelfand triplet

$$(E) \subset (L^2) \subset (E)^*$$

with dual pairing $\langle\langle \cdot, \cdot \rangle\rangle$.

(E) (resp. $(E)^*$) is called *Hida's testing* (resp. *generalized*) *functional space.*

For any $h \in H$, the exponential vector (coherent state)

$$\varepsilon(h) = \mathrm{e}^{\langle \cdot, h\rangle - \frac{1}{2}|h|_0^2}$$

belongs to (L^2). It is well known that $\{\varepsilon(h) \mid h \in H\}$ is total in (L^2) and $h \to \varepsilon(h)$ is continuous from H to (L^2).

REMARK 1.1. $\langle \cdot, h\rangle$ should be stochastically approximated in the L^2-sense by $\langle \cdot, \xi\rangle$, $\xi \in E$, or be interpreted as a Wiener integral of h with respect to Brownian motion over white noise space (cf. [3]).

When developing calculus over white noise space, it is more convenient to use

$$\mathbf{E}_0 = \{\varepsilon(\xi) \mid \xi \in E\}$$

which is total in (E) as testing vector space.

For any $y \in E^*$, the annihilation operator D_y is defined by

$$[D_y\varphi](x) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(x + ty)|_{t=0}, \quad \varphi \in (E).$$

It is known that D_y is a continuous derivation on (E), and $\{D_y \mid y \in E^*\}$ is a commuting family. For $y \in E$, D_y can be extended to a continuous linear operator from $(E)^*$ to $(E)^*$. In a dual fashion, for any $y \in E^*$, the adjoint operator D_y^* is a continuous linear operator from $(E)^*$ to $(E)^*$ and $\{D_y^* \mid y \in E^*\}$ is a commuting family. For $y \in E$, D_y^* restricts to a continuous linear operator from (E) to (E).

When $y = \delta(t)$, $t \in \mathbb{R}$, the Dirac delta function, we denote $D_{\delta(t)}$ and $D_{\delta(t)}^*$ by ∂_t and ∂_t^* respectively. $\{\partial_t, \partial_t^* \mid t \in \mathbb{R}\}$ is referred to as *quantum white noise* and serves as the basis for operator calculus over white noise. Note that when ∂_t^* is considered as an operator on (L^2), its domain is just $\{0\}$ (cf. [1], [10]). This is the reason for introducing generalized operators.

Quantum white noise constitutes a representation of the CCR:

$$[\partial_s, \partial_t] = 0 = [\partial_s^*, \partial_t^*], \quad [\partial_s, \partial_t^*] = \delta(s - t)I, \quad s, t \in \mathbb{R}.$$

2. SYMBOL CALCULUS OF GENERALIZED OPERATORS

Let $\mathcal{L} = L((E),(E)^*)$ be the space of continuous linear operators from (E) to $(E)^*$, elements of $\mathcal{L}$ are called *generalized operators*. In particular, both ∂_t and ∂_t^* can be regarded as elements of $\mathcal{L}$. For any $T \in \mathcal{L}$, its adjoint T^* also belongs to $\mathcal{L}$ since (E) is reflexive. Since $\mathbf{E}_0 = \{\varepsilon(\xi) \mid \xi \in E\}$ is total in (E), any generalized operator T is uniquely determined by its behavior on $\mathbf{E}_0$, i.e., by the infinite matrix

$$\{\langle\!\langle T\varepsilon(\xi), \varepsilon(\eta)\rangle\!\rangle \mid \xi, \eta \in E\}.$$

This leads to

DEFINITION 2.1. For any $T \in \mathcal{L}$, its *symbol* $\widehat{T}$ is defined by

$$\widehat{T}(\xi,\eta) = \mathrm{e}^{-\langle \xi,\eta\rangle} \langle\!\langle T\varepsilon(\xi), \varepsilon(\eta)\rangle\!\rangle, \quad \xi, \eta \in E.$$

REMARK 2.2. This definition due to Meyer ([8]) differs slightly from the original definition of Krée ([6]) and Obata ([9]) by the factor $\mathrm{e}^{-\langle \xi,\eta\rangle}$ which is adapted to ensure $\widehat{I}(\xi,\eta) = 1$, and is well adapted to the formulation of Wick algebra.

By direct calculations (cf. [5]), we have

LEMMA 2.3. *For any* $T \in \mathcal{L}$, $\widehat{T^*}(\xi,\eta) = \widehat{T}(\eta,\xi)$, $\xi, \eta \in E$. *Moreover:*
(i) $\widehat{\partial_t}(\xi,\eta) = \xi(t)$, $\widehat{\partial_t^*}(\xi,\eta) = \eta(t)$;
(ii) *if* $K \in L((E),(E)^*)$, *then*

$$\widehat{K\partial_t}(\xi,\eta) = \xi(t)\widehat{K}(\xi,\eta), \quad \widehat{\partial_t^* K}(\xi,\eta) = \eta(t)\widehat{K}(\xi,\eta);$$

(iii) *if* $K \in L((E),(E))$, *then*

$$\widehat{\partial_t K}(\xi,\eta) = \mathrm{e}^{-\langle \xi,\eta\rangle} \frac{\delta}{\delta\eta(t)} [\widehat{K}(\xi,\eta)\mathrm{e}^{\langle \xi,\eta\rangle}],$$

if $K \in L((E)^*,(E)^*)$, *then*

$$\widehat{K\partial_t^*}(\xi,\eta) = \mathrm{e}^{-\langle \xi,\eta\rangle} \frac{\delta}{\delta\xi(t)} [\widehat{K}(\xi,\eta)\mathrm{e}^{\langle \xi,\eta\rangle}];$$

where $\frac{\delta}{\delta\xi(t)}, \frac{\delta}{\delta\eta(t)}$ *stand for Fréchet functional derivatives and* $L((E),(E))$ *(resp.* $L((E)^*,(E)^*)$*) is the space of continuous linear operators on* (E) *(resp.* $(E)^*$*).*

The following important characterization result is due to Obata ([9]), we restate it as

LEMMA 2.4. *If $T \in \mathcal{L}$, then*

(i) *$\widehat{T}$ is analytic in the sense that for any $\xi, \xi_1, \eta, \eta_1 \in E$, the function*

$$(t, s) \to \widehat{T}(t\xi + \xi_1, s\eta + \eta_1), \quad (t, s) \in \mathbb{R}^2$$

is real analytic;

(ii) *there exist constant numbers $c_1, c_2 > 0, p \in \mathbb{N}$ such that*

$$|\widehat{T}(\xi, \eta)| \leqslant c_1 \exp\{c_2(|\xi|_p^2 + |\eta|_p^2)\}, \quad \xi, \eta \in E.$$

Conversely, assume that an $\mathbb{R}$-valued function θ on $E \times E$ satisfies the above conditions (i) *and* (ii), *then there exists a unique $T \in \mathcal{L}$ such that $\widehat{T} = \theta$.*

REMARK 2.5. In view of the fact that $|\xi|_0^2 \leqslant |\xi|_p^2$, and thus

$$|\mathrm{e}^{-\langle \xi, \eta \rangle}| \leqslant \exp\Big\{\frac{1}{2}(|\xi|_p^2 + |\eta|_p^2)\Big\}, \quad p \in \mathbb{N},$$

the above result is essentially the same as that in [9].

The chaos decomposition of square-integrable white noise functionals plays an important role in white noise calculus. Similarly, the chaos decomposition of generalized operators plays a crucial role in calculus over quantum white noise. We shall utilize it to construct Wick product of generalized operators.

Note that for any $\varphi, \psi \in (E)$, $l, m \in \mathbb{N}$, the function $\eta_{\varphi,\psi}$ on $\mathbb{R}^l \times \mathbb{R}^m$ defined by

$$(s_1, \ldots, s_l; t_1, \ldots, t_m) \to \langle\!\langle \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} \varphi, \psi \rangle\!\rangle$$

belongs to $E^{\widehat{\otimes} l} \otimes E^{\widehat{\otimes} m}$, therefore for $\kappa \in (E^{\widehat{\otimes} l} \otimes E^{\widehat{\otimes} m})^*$, there exists a unique continuous linear operator

$$\Xi_{l,m}(\kappa): \ (E) \to (E)^*$$

such that

$$\langle\!\langle \Xi_{l,m}(\kappa)\varphi, \psi \rangle\!\rangle = \langle \kappa, \eta_{\varphi,\psi} \rangle,$$

here $\langle \cdot, \cdot \rangle$ is the dual pairing between $(E^{\widehat{\otimes} l} \otimes E^{\widehat{\otimes} m})^*$ and $E^{\widehat{\otimes} l} \otimes E^{\widehat{\otimes} m}$.

The symbol of $\Xi_{l,m}(\kappa)$ is

$$\widehat{\Xi_{l,m}(\kappa)}(\xi, \eta) = \langle \kappa, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle.$$

It is heuristic to write $\Xi_{l,m}(\kappa)$ in the formal integral expression ([4]).

$$\begin{aligned}
\Xi_{l,m}(\kappa) &= \int\limits_{\mathbb{R}^l \times \mathbb{R}^m} \kappa(s_1, \ldots, s_l; t_1, \ldots, t_m) \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} \mathrm{d}s_1 \cdots \mathrm{d}s_l \, \mathrm{d}t_1 \cdots \mathrm{d}t_m \\
&= \int\limits_{\mathbb{R}^l \times \mathbb{R}^m} \kappa(\bar{s}; \bar{t}) \partial_{\bar{s}}^* \partial_{\bar{t}} \mathrm{d}\bar{s} \mathrm{d}\bar{t}.
\end{aligned}$$

This kind of expression already appeared in [1].

The following result is due to Huang ([5]).

LEMMA 2.6. *Every $K \in \mathcal{L}$ (resp. $K \in L((E),(E))$) has a unique decomposition*

$$K = \sum_{l,m} \Xi_{l,m}(K_{l,m}),$$

where $K_{l,m} \in (E^{\widehat{\otimes} l} \otimes E^{\widehat{\otimes} m})^*$ (resp. $E^{\widehat{\otimes} l} \otimes (E^{\widehat{\otimes} m})^*$), $l, m \in \mathbb{N}$.

The series converges in the sense that

$$K\varphi = \sum_{l,m} \Xi_{l,m}(K_{l,m})\varphi, \quad \varphi \in E$$

converges in $(E)^*$ (resp. (E)). Moreover

$$\widehat{K}(\xi,\eta) = \sum_{l,m} \langle K_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m}\rangle, \quad \xi,\eta \in E.$$

3. WICK PRODUCT

In terms of quantum white noise, the free Bose field operator $p(x)$ may be represented as

$$p(x) = D_x^* + D_x, \quad x \in E^*.$$

Since for any $x \in E^*, D_x \in L((E),(E))$ and $D_x^* \in L((E)^*,(E)^*)$, $p(x)$ can be regarded as an element of $\mathcal{L}$, i.e., a generalized operator.

It is important to construct functions of field operators ([2], [10]) through certain renormalization procedure. To this end, we shall construct Wick product of generalized operators.

Firstly, observe that for general $x, y \in E^*$, the composition $D_x^* D_y$ makes sense as a generalized operator, but it is not the case for $D_y D_x^*$. For $x_1, \dots, x_n \in E^*$, consider the formal product of field operators (the notation $:$ indicates Wick product which will be explained below)

$$p(x_1) : \cdots : p(x_n) = (D_{x_1}^* + D_{x_1}) : \cdots : (D_{x_n}^* + D_{x_n}).$$

This expression is interpreted in the following way:

(1) multiply out formally and express the result as a sum of monomial terms;

(2) for each monomial term, move all creation operators to the left of annihilation operators.

The final result is

$$\sum_{\Lambda \subset \{1,2,\dots,n\}} \Big(\prod_{i\in\Lambda} D_{x_i}^*\Big)\Big(\prod_{j\in\Lambda^c} D_{x_j}\Big)$$

which belongs to $\mathcal{L}$. This procedure is the so called Wick ordering.

REMARK 3.1. The expression is well defined due to the CCR.

Secondly, observe that for $x, y \in E^*$,

$$D_x^* = \Xi_{1,0}(x) = \int_{\mathbb{R}} x(s)\partial_s^* \,\mathrm{d}s, \quad D_y = \Xi_{0,1}(y) = \int_{\mathbb{R}} y(t)\partial_t \,\mathrm{d}t,$$

hence formally

$$D_x^* : D_y = D_x^* D_y = \Xi_{1,0}(x)\Xi_{0,1}(y) = \int_{\mathbb{R}} x(s)\partial_s^* \,\mathrm{d}s \int_{\mathbb{R}} y(t)\partial_t \,\mathrm{d}t$$
$$= \int_{\mathbb{R}\times\mathbb{R}} x(s)y(t)\partial_s^*\partial_t \,\mathrm{d}s\mathrm{d}t = \Xi_{1,1}(x \otimes y).$$

More generally, for $\kappa \in (E^{\widehat{\otimes} p} \otimes E^{\widehat{\otimes} q})^*, \zeta \in (E^{\widehat{\otimes} i} \otimes E^{\widehat{\otimes} j})^*$, and

$$\Xi_{p,q}(\kappa) = \int_{\mathbb{R}^p\times\mathbb{R}^q} \kappa(\bar{s},\bar{t})\partial_{\bar{s}}^*\partial_{\bar{t}}\mathrm{d}\bar{s}\mathrm{d}\bar{t}, \quad \Xi_{i,j}(\zeta) = \int_{\mathbb{R}^i\times\mathbb{R}^j} \zeta(\bar{u},\bar{v})\partial_{\bar{u}}^*\partial_{\bar{v}}\mathrm{d}\bar{u}\mathrm{d}\bar{v},$$

it is reasonable to define

$$\Xi_{p,q}(\kappa) : \Xi_{i,j}(\zeta) = \int_{\mathbb{R}^{p+i}\times\mathbb{R}^{q+j}} \kappa\overline{\otimes}\zeta(\bar{s},\bar{u};\bar{t},\bar{v})\partial_{\bar{s}}^*\partial_{\bar{u}}^*\partial_{\bar{t}}\partial_{\bar{v}}\mathrm{d}\bar{s}\mathrm{d}\bar{u}\mathrm{d}\bar{t}\mathrm{d}\bar{v};$$

here $\kappa\overline{\otimes}\zeta$ is the tensor product of κ and ζ, symmetrized with the first $p+i$ and last $q+j$ variables independently. It is determined by

$$\langle \kappa\overline{\otimes}\zeta, \xi^{\otimes p+i} \otimes \eta^{\otimes q+j}\rangle = \langle \kappa, \xi^{\otimes p} \otimes \eta^{\otimes q}\rangle\langle \zeta, \xi^{\otimes i} \otimes \eta^{\otimes j}\rangle.$$

Obviously, $\Xi_{p,q}(\kappa) : \Xi_{i,j}(\zeta) \in \mathcal{L}$.

Finally, by linear combinations and Lemma 2.6, we propose

DEFINITION 3.2. For any $K, J \in L((E),(E)^*)$ with chaos decompositions

$$K = \sum_{p,q} \Xi_{p,q}(K_{p,q}), \quad J = \sum_{i,j} \Xi_{i,j}(J_{i,j}),$$

the *Wick product* of K and J is defined as

$$K : J = \sum_{l,m} \Xi_{l,m}\Big(\sum_{\substack{p+i=l\\ q+j=m}} K_{p,q}\overline{\otimes}J_{i,j}\Big)$$

provided the series converges in the sense of Lemma 2.6.

REMARK 3.3. This definition is intuitive from the Wick ordering procedure and physical point of view. In the next section, we shall characterize this product in terms of symbols and show that the product $:$ is always well defined and well adapted to the Wick ordering.

4. WICK ALGEBRA

Topologize $L((E),(E))$ (resp. $L((E)^*,(E)^*)$) with uniform convergence on bounded subsets of (E) (resp. $(E)^*$). Then, under composition of operators, $L((E),(E))$ (resp. $L((E)^*,(E)^*)$) is a non-commutative associative algebra with unit I and the operation $\cdot$ is separatively continuous. $\{D_x|x \in E^*\}$ (resp. $\{D_x^*|x \in E^*\}$) is a commutative subalgebra of $L((E),(E))$ (resp. $L((E)^*,(E)^*)$), so is $\{D_y^* \mid y \in E\}$ (resp. $\{D_y \mid y \in E\}$). Moreover, $L((E),(E))$ and $L((E)^*,(E)^*)$ can be regarded as each other's dual.

Equip $\mathcal{L}$ with the topology of uniform convergence on bounded subsets of (E); then both $L((E),(E))$ and $L((E)^*,(E)^*)$ are closed subspaces of $\mathcal{L}$. Note $\mathcal{L}$ is not an algebra in the usual sense, since in general, composition of generalized operators is meaningless.

In view of Lemma 2.6, $\mathcal{L}$ is generated by quantum white noise $\{\partial_t, \partial_t^* \mid t \in \mathbb{R}\}$ in certain sense.

Let $\Theta = \{\widehat{T} \mid T \in \mathcal{L}\}$ with the topology inherited from $\mathcal{L}$, then Θ is a locally convex topological vector space.

PROPOSITION 4.1. *Θ is an algebra under pointwisely multiplication, and the multiplication is separatively continuous.*

Proof. Suppose that $\widehat{T}, \widehat{S} \in \Theta$; it is obvious that $\widehat{T} \cdot \widehat{S}$ is analytic and by Lemma 2.4, there exist constant numbers $a, b, c, d > 0, p, q \in \mathbb{N}$ such that

$$|\widehat{T}(\xi,\eta)| \leqslant a \exp\{b(|\xi|_p^2 + |\eta|_p^2)\}, \quad |\widehat{S}(\xi,\eta)| \leqslant c \exp\{d(|\xi|_q^2 + |\eta|_q^2)\}, \quad \xi, \eta \in E.$$

Therefore

$$|\widehat{T}(\xi,\eta)\widehat{S}(\xi,\eta)| \leqslant ac \exp\{(b+d)(|\xi|_{p\vee q}^2 + |\eta|_{p\vee q}^2)\}.$$

Consequently, $\widehat{T} \cdot \widehat{S} \in \Theta$. The separative continuity of multiplication is obvious. ∎

Now, the Wick product can be characterized as

PROPOSITION 4.2. *For $K, J \in \mathcal{L}$, we have*

$$\widehat{K : J} = \widehat{K} \cdot \widehat{J}.$$

Proof. Suppose that K, J have chaos decompositions

$$K = \sum_{p,q} \Xi_{p,q}(K_{p,q}), \quad J = \sum_{i,j} \Xi_{i,j}(J_{i,j}).$$

In view of Lemma 2.6, for $\xi, \eta \in E$,

$$\widehat{K}(\xi,\eta) = \sum_{p,q} \langle K_{p,q}, \eta^{\otimes p} \otimes \xi^{\otimes q}\rangle, \quad \widehat{J}(\xi,\eta) = \sum_{i,j} \langle J_{i,j}, \eta^{\otimes i} \otimes \xi^{\otimes j}\rangle,$$

therefore

$$\begin{aligned}\widehat{K}(\xi,\eta)\cdot\widehat{J}(\xi,\eta) &= \sum_{l,m} \sum_{\substack{p+i=l\\ q+j=m}} \langle K_{p,q}, \eta^{\otimes p} \otimes \xi^{\otimes q}\rangle \cdot \langle J_{i,j}, \eta^{\otimes i} \otimes \xi^{\otimes j}\rangle \\ &= \sum_{l,m} \sum_{\substack{p+i=l\\ q+j=m}} \langle K_{p,q}\overline{\otimes} J_{i,j}, \eta^{\otimes p+i} \otimes \xi^{\otimes q+j}\rangle.\end{aligned}$$

On the other hand, by Definition 3.2 and Lemma 2.6,

$$\widehat{K:J}(\xi,\eta) = \sum_{l,m} \sum_{\substack{p+i=l\\ q+j=m}} \langle K_{p,q}\overline{\otimes} J_{i,j}, \eta^{\otimes p+i} \otimes \xi^{\otimes q+j}\rangle.$$

Consequently

$$\widehat{K:J} = \widehat{K}\cdot\widehat{J}. \quad \blacksquare$$

According to the above two propositions, the Wick product of any two generalized operator is always well defined.

THEOREM 4.3. *$(\mathcal{L},\ :\ )$ is a nuclear Wick algebra in the sense that:*
(i) *$\mathcal{L}$ is a nuclear space;*
(ii) *$:$ is a separatively continuous bilinear mapping from $\mathcal{L} \times \mathcal{L}$ to $\mathcal{L}$;*
(iii) *$:$ is associative and commutative;*
(iv) *$I : T = T$, $\forall T \in \mathcal{L}$, where I is the identity operator;*
(v) *$\partial_t^* : T = \partial_t^* T$, $\partial_t : T = T\partial_t$, $\forall t \in \mathbb{R}$, $T \in \mathcal{L}$;*
(vi) *$(K : J)^* = K^* : J^*$, $\forall K, J \in \mathcal{L}$.*

Proof. (i), (ii), (iii) and (iv) are obvious, and (v) follows from Lemma 2.3 and Proposition 4.2. It remains to prove (vi). In fact

$$(\widehat{K:J})^*(\xi,\eta) = \widehat{K:J}(\eta,\xi) = \widehat{K}(\eta,\xi)\cdot\widehat{J}(\eta,\xi) = \widehat{K^*}(\xi,\eta)\widehat{J^*}(\xi,\eta) = \widehat{K^*:J^*}(\xi,\eta),$$

therefore $(K : J)^* = K^* : J^*$. $\blacksquare$

Note that $(L((E),(E)),\ \cdot)$ is itself a non-commutative algebra, while the Wick algebra $(\mathcal{L},\ :\ )$ is commutative, therefore $(L((E),(E)),\ \cdot)$ cannot be embedded into $(\mathcal{L},\ :\ )$ even though $L((E),(E))$ is a subspace of $\mathcal{L}$. However, there exist non-trivial subalgebras of $(L((E),(E)),\ \cdot)$ which can be embedded in $(\mathcal{L},\ :\ )$. In order to characterize them, we shall study operator algebra generated by annihilation (resp. creation) operators more extensively.

PROPOSITION 4.4. *The following are equivalent:*

(i) $T \in L((E),(E))$ *and* $T\partial_t = \partial_t T$, $\forall t \in \mathbb{R}$;

(ii) $T \in \mathcal{L}$ *admits the chaos decomposition*

$$T = \sum_m \Xi_{0,m}(T_{0,m}), \quad T_{0,m} \in E^{*\widehat{\otimes} m}, \ m \in \mathbb{N}.$$

Proof. (i) $\Rightarrow$ (ii) In view of Lemma 2.3,

$$\begin{aligned}
\widehat{T\partial_t}(\xi,\eta) &= \xi(t)\widehat{T}(\xi,\eta) \\
\widehat{\partial_t T}(\xi,\eta) &= e^{-\langle\xi,\eta\rangle}\frac{\delta}{\delta\eta(t)}[\widehat{T}(\xi,\eta)e^{\langle\xi,\eta\rangle}] \\
&= e^{-\langle\xi,\eta\rangle}\left[e^{\langle\xi,\eta\rangle}\frac{\delta}{\delta\eta(t)}\widehat{T}(\xi,\eta) + \widehat{T}(\xi,\eta)\xi(t)e^{\langle\xi,\eta\rangle}\right] \\
&= \frac{\delta}{\delta\eta(t)}\widehat{T}(\xi,\eta) + \xi(t)\widehat{T}(\xi,\eta).
\end{aligned}$$

Since $T\partial_t = \partial_t T$ implies $\widehat{T\partial_t} = \widehat{\partial_t T}$, we have

$$\frac{\delta}{\delta\eta(t)}\widehat{T}(\xi,\eta) = 0, \quad \forall t \in \mathbb{R}.$$

Thus $\widehat{T}(\xi,\eta)$ is independent of η, and the conclusion follows from Lemma 2.6.

(ii) $\Rightarrow$ (i) Just reversing the above argument, we obtain $T\partial_t = \partial_t T$, $\forall t \in \mathbb{R}$, and by Lemma 2.6, we have $T \in L((E),(E))$. ∎

REMARK 4.5. This proposition is intuitive in view of Lemma 2.6 and the CCR of quantum white noise.

In a dual fashion, we have

PROPOSITION 4.6. *The following are equivalent:*

(i) $T \in L((E)^*,(E)^*)$ *and* $T\partial_t^* = \partial_t^* T$, $\forall t \in \mathbb{R}$;

(ii) $T \in \mathcal{L}$ *admits the chaos decomposition*

$$T = \sum_l \Xi_{l,0}(T_{l,0}), \quad T_{l,0} \in E^{*\widehat{\otimes} l}, \ l \in \mathbb{N}.$$

Combining the above two propositions, we obtain the following well known fact about the irreducibility of CCR in terms of quantum white noise.

COROLLARY 4.7. *If* $T \in L((E),(E)) \cap L((E)^*,(E)^*)$, *and*

$$T\partial_t = \partial_t T, \quad T\partial_t^* = \partial_t^*, \quad \forall t \in \mathbb{R},$$

then there exists a constant c *such that* $T = cI$.

Let $\mathcal{A} = \{A \in \mathcal{L} \mid A\partial_t = \partial_t A,\ \forall t \in \mathbb{R}\}$, then $\mathcal{A}$ is a common subalgebra of $(L(E),(E)),\cdot)$ and $(\mathcal{L},\ :\ )$. In a dual fashion, $\mathcal{A}^* = \{A \in \mathcal{L} \mid A\partial_t^* = \partial_t^* A,\ \forall t \in \mathbb{R}\}$ is a common subalgebra of $(L(E)^*,(E)^*),\cdot)$ and $(\mathcal{L},\ :\ )$. They are referred to as annihilation algebra and creation algebra respectively, and are dual to each other.

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