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ESSENTIALLY QUASINILPOTENT ELEMENTS WITH RESPECT TO ARBITRARY NORM CLOSED TWO-SIDED IDEALS IN VON NEUMANN ALGEBRAS

ANTON STRÖH and LÁSZÓ ZSIDÓ

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ABSTRACT. In this paper we prove that a part of the Riesz decomposition theory for compact operators holds in maximal generality in the realm of von Neumann algebras. More precisely, if an element x of a von Neumann algebra M is essentially quasinilpotent with respect to an arbitrary norm closed two-sided ideal of M, then the supremum (in the projection lattice of M) of the kernel projections of all positive integer powers of $1-x$ belongs to the ideal. It seems to be an interesting question, whether the above statement holds in arbitrary AW^*-algebras.

KEYWORDS: *Essentially quasinilpotent elements, norm closed ideals and von Neumann algebras.*

AMS SUBJECT CLASSIFICATION: 46L10, 47B06, 47D25.

In order to extend the classical Riesz theory of compact operators to the elements of the norm closed (two-sided) ideal $\mathcal{I}_{\text{fin}}$ generated by the finite projections of a von Neumann algebra M, M. Breuer ([2]) made the following conjecture:

If $x \in \mathcal{I}_{\text{fin}}$ and if $e_n \in M$ with $n \geqslant 1$ is the orthogonal projection of x onto the kernel of $(1-x)^n$ and if e_∞ is the supremum of the sequence $e_1 \leqslant e_2 \leqslant \cdots$ in the complete lattice of all projections of M, then e_∞ belongs to $\mathcal{I}_{\text{fin}}$, that is, e_∞ is a finite projection.

Assuming the validity of the above conjecture, a substantial Riesz decomposition theorem was proved in [2] for the elements of $\mathcal{I}_{\text{fin}}$.

Breuer's conjecture was proved in [6] in the case of a von Neumann algebra M acting on a separable Hilbert space. Subsequently, it was proved in full generality, independently, in [3] and [7]. We notice that, according to [8], Breuer's conjecture holds even for every $x \in M$, whose canonical image in $M/\mathcal{I}_{\mathrm{fin}}$ is quasinilpotent.

The aim of this paper is to prove that for every norm closed two-sided ideal $\mathcal{I}$ in a von Neumann algebra M and for every $x \in M$, whose canonical image in $M/\mathcal{I}$ is quasinilpotent, we have $e_\infty \in \mathcal{I}$, where e_∞ is as defined above.

1. LEMMAS ON ESSENTIALLY QUASINILPOTENT ELEMENTS

In this section M will denote an AW^*-algebra ([1], Definition 4.2), $\mathcal{I}$ a norm closed two-sided ideal in M, and x an element of M, such that its canonial image $x/\mathcal{I}$ in $M/\mathcal{I}$ is quasinilpotent. Let us further denote, for every integer $n \geqslant 1$, by e_n the unique projection in M satisfying

$$\{y \in M;\ (1-x)^n y = 0\} = e_n M.$$

Of course, e_n is the greatest projection $e \in M$ with $(1-x)^n e = 0$. Finally, let $g_1 = e_1$ and $g_n = e_n - e_{n-1}$, $n \geqslant 2$. Since $e_1 \leqslant e_2 \leqslant \cdots$, it follows that $g_1, g_2, \ldots$ are mutually orthogonal projections in M. It will follow from Lemma 1.3 that $g_n \in \mathcal{I}$, when $n \geqslant 1$.

LEMMA 1.1. *For all integers $k, n \geqslant 1$ we have:*
(i) $x^k e_{n+1} = e_n x^k e_{n+1} + g_{n+1}$;
(ii) $g_n x^k g_n = g_n$.

Proof. For every $n \geqslant 1$,

$$(1-x)^n(1-x)e_{n+1} = (1-x)^{n+1}e_{n+1} = 0,$$

which implies successively that

$$(1-x)e_{n+1} = e_n(1-x)e_{n+1};$$

$$xe_{n+1} = e_{n+1} - e_n(1-x)e_{n+1} = e_n x e_{n+1} + g_{n+1}.$$

Now, since for every $k, n \geqslant 1$

$$\begin{aligned} x^{k+1}e_{n+1} - e_n x^{k+1} e_{n+1} &= (x^k - e_n x^k) x e_{n+1} = (x^k - e_n x^k)(e_n x e_{n+1} + g_{n+1}) \\ &= (x^k e_{n+1} - e_n x^k e_{n+1})(e_n x e_{n+1} + g_{n+1}), \end{aligned}$$

part (i) follows by induction on k.

To prove (ii), since $xe_1 = e_1$, we have for every $k \geqslant 1$ that $x^k e_1 = e_1$ and $g_1 x^k g_1 = e_1 x^k e_1 = e_1 = g_1$. If $n \geqslant 2$, using (i) with $n-1$ instead of n, it follows for all $k \geqslant 1$ that

$$x^k e_n = e_{n-1} x^k e_n + g_n;$$
$$g_n x^k e_n = g_n;$$
$$g_n x^k g_n = g_n. \quad \blacksquare$$

LEMMA 1.2. *For every $\varepsilon > 0$ there are an integer $k \geqslant 1$ and a projection $e \in \mathcal{I}$, such that*

$$\|(1-e)x^k\| \leqslant \varepsilon^k.$$

Proof. Let $0 < \delta < \varepsilon$. By the Beurling formula for the spectral radius there is an integer $k \geqslant 1$ with

$$\|(x^k)/\mathcal{I}\| \leqslant \delta^k.$$

Now from the spectral theorem (see the proof of Proposition 7.3 in [1]), there exists a projection $e \in M$, commuting with $x^k(x^k)^*$, such that

$$ex^k(x^k)^* \geqslant \varepsilon^{2k} e$$

and

$$(1-e)x^k(x^k)^* \leqslant \varepsilon^{2k}(1-e).$$

Then

$$\varepsilon^{2k}\|e/\mathcal{I}\| \leqslant \|(ex^k(x^k)^*)/\mathcal{I}\| \leqslant \|(x^k)/\mathcal{I}\|^2 \leqslant \delta^{2k}.$$

Hence from

$$\|e/\mathcal{I}\| \leqslant \left(\frac{\delta}{\varepsilon}\right)^{2k} < 1,$$

it follows that $e \in \mathcal{I}$.

On the other hand,

$$\|(1-e)x^k\|^2 \leqslant \|(1-e)x^k(x^k)^*\| \leqslant \varepsilon^{2k}. \quad \blacksquare$$

Note that the property proven in the above lemma characterizes the essential quasinilpotentness of x with respect to $\mathcal{I}$ (compare with Theorem 3.2 in [8]).

LEMMA 1.3. *There are an integer $\ell_0 \geqslant 1$ and a projection $e \in \mathcal{I}$, such that*

$$g_n e g_n \geqslant \frac{1}{\ell_0} g_n, \quad n \geqslant 1.$$

Proof. Let $0 < \varepsilon < (4\|x\|)^{-1}$. By Lemma 1.2 there are an integer $k \geqslant 1$ and a projection $e \in \mathcal{I}$, such that

$$\|(1-e)x^k\| \leqslant \varepsilon^k.$$

For every $n \geqslant 1$, using Lemma 1.1, we obtain

$$\begin{aligned}\|g_n - g_n e x^k g_n (x^k)^* e g_n\| &= \|g_n x^k g_n (x^k)^* g_n - g_n e x^k g_n (x^k)^* e g_n\| \\ &\leqslant \|g_n(1-e)x^k g_n (x^k)^* g_n\| + \|g_n e x^k g_n (x^k)^* (1-e) g_n\| \\ &\leqslant \|(1-e)x^k\| \cdot \|x^k\| + \|x^k\| \cdot \|(x^k)^*(1-e)\| \\ &= 2\|(1-e)x^k\| \cdot \|x^k\| \leqslant 2\varepsilon^k \|x\|^k < 2\frac{1}{4^k} \leqslant \frac{1}{2}.\end{aligned}$$

Hence

$$\frac{1}{2} g_n \leqslant g_n e x^k g_n (x^k)^* e g_n \leqslant \|x^k\|^2 g_n e g_n.$$

Choosing now an integer $\ell_0 \geqslant 2\|x^k\|^2$, it follows for all $n \geqslant 1$ that

$$g_n e g_n \geqslant \frac{1}{\ell_0} g_n. \quad ∎$$

2. LEMMAS ON PROJECTIONS

Let us agree to denote

$$e = e_1 + e_2 + \cdots$$

for $e, e_1, e_2, \ldots$ projections in an AW^*-algebra M whenever $e_1, e_2, \ldots$ are mutually orthogonal and with supremum e in the lattice of all projections of M. In this case e is equally the supremum of the partial sum sequence

$$e_1,\ e_1 + e_2,\ e_1 + e_2 + e_3, \ldots,$$

which justifies the notation.

LEMMA 2.1. *Let M be an AW^*-algebra, $e \in M$ a projection and $\ell_0 \geqslant 1$ an integer. Then there are central projections $p_1, \ldots, p_{\ell_0}, p_\infty$ in M with*

$$p_1 + \cdots + p_{\ell_0} + p_\infty = 1$$

such that:

(i) *for every $1 \leqslant \ell < \ell_0$ there are projections $e_{\ell,1}, \ldots, e_{\ell,\ell+1} \in M$ with*

$$\begin{aligned} p_\ell &= \sum_{k=1}^{\ell+1} e_{\ell,k}, \\ e_{\ell,1} &= e p_\ell \quad \textit{and is finite,} \\ e_{\ell,k} &\sim e p_\ell \quad \textit{for } 1 < k \leqslant \ell, \\ e_{\ell,\ell+1} &\prec e p_\ell; \end{aligned}$$

(ii) *there are projections* $e_{\ell_0,1}, \dots, e_{\ell_0,\ell_0} \in M$ *with*

$$p_{\ell_0} \geqslant \sum_{k=1}^{\ell_0} e_{\ell_0,k},$$
$$e_{\ell_0,1} = ep_{\ell_0} \quad \text{and is finite,}$$
$$e_{\ell_0,k} \sim ep_{\ell_0} \quad \text{for } 1 < k \leqslant \ell_0;$$

(iii) *there is an infinite sequence of projections* $e_{\infty,1}, e_{\infty,2}, \dots \in M$ *with*

$$ep_\infty = e_{\infty,1} + e_{\infty,2} + \cdots,$$
$$e_{\infty,k} \sim ep_\infty \quad \text{for } k \geqslant 1.$$

Proof. By Proposition 4.8 (iii) in [1], eMe is an AW^*-subalgebra of M. According to Theorem 15.1 and Proposition 6.4 in [1], there exists a central projection p_∞ in M such that ep_∞ is properly infinite and $e(1 - p_\infty)$ is finite.

By Theorem 17.1 in [1], our statement (iii) holds. Set $q = 1 - p_\infty$, so that the projection eq is finite. We prove inductively that there are mutually orthogonal central projections $q_1, q_2, \dots \leqslant q$, and there are mutually orthogonal projections $f_1, f_2, \dots$ with $e = f_1$ such that, for every ℓ,

$$(1 - f_1 - \cdots - f_\ell)q_\ell \prec eq_\ell; \quad f_{\ell+1} \sim e(q - q_1 - \cdots - q_\ell).$$

As required, first let $f_1 = e$. Since M has generalized comparability (see Corollary 1 of Proposition 14.7 in [1]), there exists a central projection $q_1 \leqslant q$ in M with

$$(1 - f_1)q_1 \prec eq_1; \quad (1 - f_1)(q - q_1) \succ e(q - q_1).$$

By the last relation there exists a projection $f_2 \in M$ with

$$(1 - f_1)(q - q_1) \geqslant f_2 \sim e(q - q_1).$$

Let us now assume that, for some $m \geqslant 1$, $q_1, \dots, q_m$, $f_1, \dots, f_m, f_{m+1}$ have been constructed. Since M has generalized comparability, there exists a central projection $q_{m+1} \leqslant q - q_1 - \cdots - q_m$ in M with

$$(1 - f_1 - \cdots - f_{m+1})q_{m+1} \prec eq_{m+1};$$

$$(1 - f_1 - \cdots - f_{m+1})(q - q_1 - \cdots - q_{m+1}) \succ e(q - q_1 - \cdots - q_{m+1}).$$

By the last relation there exists a projection $f_{m+2} \in M$ with

$$(1 - f_1 - \cdots - f_{m+1})(q - q_1 - \cdots - q_{m+1}) \geqslant f_{m+2} \sim e(q - q_1 - \cdots - q_{m+1}),$$

which completes the proof of the induction step.

Let $1 \leqslant \ell < \ell_0$. If we let $p_\ell = q_\ell$, $e_{\ell,k} = f_k q_\ell$ for $1 \leqslant k \leqslant \ell$, $e_{\ell,\ell+1} = (1 - f_1 - \cdots - f_\ell)q_\ell$, it follows that

$$\sum_{k=1}^{\ell+1} e_{\ell,k} = q_\ell = p_\ell;$$
$$e_{\ell,1} = f_1 q_\ell = ep_\ell \leqslant eq \quad \text{is finite;}$$
$$e_{\ell,k} = f_k q_\ell \sim e(q - q_1 - \cdots - q_{k-1})q_\ell = eq_\ell = ep_\ell \quad \text{for } 1 < k \leqslant \ell;$$
$$e_{\ell,\ell+1} = (1 - f_1 - \cdots - f_\ell)q_\ell \prec eq_\ell = ep_\ell.$$

Also, if we let $p_{\ell_0} = q - q_1 - \cdots - q_{\ell_0 - 1}$, $e_{l_0,k} = f_k p_{\ell_0}$ for $1 \leqslant k \leqslant \ell_0$, it follows that

$$\sum_{k=1}^{\ell_0} e_{\ell_0,k} \leqslant p_{\ell_0};$$
$$e_{\ell_0,1} = f_1 p_{\ell_0} = ep_{\ell_0} \leqslant eq \quad \text{is finite;}$$
$$e_{\ell_0,k} = f_k p_{\ell_0} \sim e(q - q_1 - \cdots - q_{k-1})p_{\ell_0} = ep_{\ell_0} \quad \text{for } 1 < k \leqslant \ell_0. \quad \blacksquare$$

The following known result is included in order to have a convenient reference in the sequel.

LEMMA 2.2. *Let e, g be projections in a C^*-algebra A such that, for some $\varepsilon > 0$, $geg \geqslant \varepsilon g$. Then $g \prec e$.*

Proof. Since $0 \leqslant g - geg \leqslant (1 - \varepsilon)g$, it follows from $\|g - geg\| \leqslant 1 - \varepsilon < 1$ that geg is invertible in gAg. Let $a \geqslant 0$ denote the inverse of $geg \geqslant 0$ in gAg and set

$$v = ega^{\frac{1}{2}}.$$

Then

$$v^*v = a^{\frac{1}{2}} gega^{\frac{1}{2}} = g\ ,$$

and from

$$(1 - e)vv^* = (1 - e)egage = 0$$

it is clear that

$$vv^* = evv^* \leqslant e. \quad \blacksquare$$

LEMMA 2.3. *Let M be a von Neumann algebra, $e \in M$ a finite projection and $g_1, g_2, \ldots \in M$ mutually orthogonal projections such that, for some integer $\ell_0 \geqslant 1$,*

$$g_n e g_n \geqslant \frac{1}{\ell_0} g_n, \quad n \geqslant 1.$$

If $e'_1 = e, e'_2, \ldots, e'_{\ell_0} \in M$ are mutually orthogonal equivalent projections then

$$\sum_{n=1}^{\infty} g_n \prec \sum_{k=1}^{\ell_0} e'_k.$$

Proof. First of all we notice that, according to Lemma 2.2, $g_n \prec e$, $n \geqslant 1$.

In particular, all projections g_n are finite. By Lemma 7.3 in [9] or Chapter V, Lemma 2.2 in [10] it is enough to prove that, for every $m \geqslant 1$,

$$\sum_{n=1}^{m} g_n \prec \sum_{k=1}^{\ell_0} e'_k.$$

Passing from M to the von Neumann algebra generated by $e'_1, \ldots, e'_{\ell_0}, g_1, \ldots, g_m$, we can assume without loss of generality that M is finite.

Let $x \mapsto x^\natural$ be the center valued trace on M (see Theorem 7.11 in [9] or Chapter V, Theorem 2.6 in [10]). Then for any $m \geqslant 1$,

$$\begin{aligned}\left(\sum_{n=1}^{m} g_n\right)^\natural &= \sum_{n=1}^{m} g_n^\natural \leqslant \ell_0 \sum_{n=1}^{m} (g_n e g_n)^\natural = \ell_0 \sum_{n=1}^{m} (e g_n e)^\natural \\ &= \ell_0 \left(e \sum_{n=1}^{m} g_n e\right)^\natural \leqslant \ell_0 e^\natural = \sum_{k=1}^{\ell_0} (e'_k)^\natural = \left(\sum_{k=1}^{\ell_0} e'_k\right)^\natural,\end{aligned}$$

thus

$$\sum_{n=1}^{m} g_n \prec \sum_{k=1}^{\ell_0} e'_k. \qquad \blacksquare$$

LEMMA 2.4. *Let M be a von Neumann algebra, $e \in M$ a projection and $g_1, g_2, \ldots \in M$ mutually orthogonal projections such that, for some integer $\ell_0 \geqslant 1$,*

$$g_n e g_n \geqslant \frac{1}{\ell_0} g_n, \quad n \geqslant 1.$$

Then there are mutually orthogonal projections $e'_1 = e, e'_2, \ldots, e'_{\ell_0} \in M$ such that

$$e'_1 \succ e'_2 \succ \ldots \succ e'_{\ell_0}, \qquad \sum_{n=1}^{\infty} g_n \prec \sum_{k=1}^{\ell_0} e'_k.$$

In particular, $\sum\limits_{n=1}^{\infty} g_n$ belongs to every two-sided ideal of M containing e.

Proof. Applying Lemma 2.1 to e and ℓ_0 and using the same notations, let us set

$$e'_1 = e = ep_\infty + \sum_{\ell=1}^{\ell_0} e_{\ell,1};$$

$$e'_k = \sum_{\ell=k-1}^{\ell_0} e_{\ell,k} \quad \text{for } 1 < k \leqslant \ell_0.$$

The projections $e'_1 = e, e'_2, \dots, e'_{\ell_0} \in M$ are clearly mutually orthogonal. Since $e_{1,2} \prec ep_1$ and $e_{\ell,2} \sim ep_\ell$ for $2 \leqslant \ell \leqslant \ell_0$, we have

$$e'_2 = e_{1,2} + \sum_{\ell=2}^{\ell_0} e_{\ell,2} \prec ep_1 + \sum_{\ell=2}^{\ell_0} ep_\ell \leqslant e = e'_1.$$

For $1 < k < \ell_0$, since $e_{k,k+1} \prec ep_k \sim e_{k,k}$ and $e_{\ell,k+1} \sim ep_\ell \sim e_{\ell,k}$ for $k+1 \leqslant \ell \leqslant \ell_0$, we also have

$$e'_{k+1} = e_{k,k+1} + \sum_{\ell=k+1}^{\ell_0} e_{\ell,k+1} \prec e_{k,k} + \sum_{\ell=k+1}^{\ell_0} e_{\ell,k} \leqslant e'_k.$$

Hence $e'_1 \succ e'_2 \succ \cdots \succ e'_{\ell_0}$. Note that

$$\begin{aligned}\sum_{k=1}^{\ell_0} e'_k &= ep_\infty + \sum_{\ell=1}^{\ell_0} e_{\ell,1} + \sum_{k=2}^{\ell_0} \sum_{\ell=k-1}^{\ell_0} e_{\ell,k} \\ &= ep_\infty + \sum_{\ell=1}^{\ell_0-1} \sum_{k=1}^{\ell+1} e_{\ell,k} + \sum_{k=1}^{\ell_0} e_{\ell_0,k} \\ &= ep_\infty + \sum_{\ell=1}^{\ell_0-1} p_\ell + \sum_{k=1}^{\ell_0} e_{\ell_0,k}.\end{aligned}$$

Hence, in order to complete the proof we have to show that

$$\sum_{n=1}^{\infty} g_n p_\infty \prec ep_\infty, \tag{2.1}$$

$$\sum_{n=1}^{\infty} g_n \sum_{\ell=1}^{\ell_0-1} p_\ell \prec \sum_{\ell=1}^{\ell_0-1} p_\ell \tag{2.2}$$

and

$$\sum_{n=1}^{\infty} g_n p_{\ell_0} \prec \sum_{\ell=1}^{\ell_0} e_{\ell_0,k}. \tag{2.3}$$

To show (2.1), we notice that according to Lemma 2.2 and property (iii) from Lemma 2.1, we have $g_n p_\infty \prec e p_\infty \sim e_{\infty,n}$ for all $n \geqslant 1$, and

$$\sum_{n=1}^{\infty} g_n p_\infty \prec \sum_{n=1}^{\infty} e_{\infty,n} = e p_\infty.$$

Relation (2.2) holds trivially, because

$$\sum_{n=1}^{\infty} g_n \sum_{\ell=1}^{\ell_0-1} p_\ell \leqslant \sum_{\ell=1}^{l_0-1} p_\ell.$$

Since $e_{\ell_0,1} = e p_{\ell_0}, e_{\ell_0,2}, \ldots, e_{\ell_0,\ell_0}$ are mutually orthogonal equivalent finite projections, (2.3) follows from Lemma 2.3. ∎

3. THE MAIN RESULT

THEOREM 3.1. *Let M be a von Neumann algebra, $\mathcal{I}$ a norm closed two-sided ideal in M and $x \in M$, such that the canonical image of x in $M/\mathcal{I}$ is quasinilpotent. Let us denote, for every integer $n \geqslant 1$, by $e_n(x)$ the null projection of $(1-x)^n$, that is the greatest projection $e \in M$ with $(1-x)^n e = 0$, and by $f_n(x)$ the range projection of $(1-x)^n$, that is the smallest projection $f \in M$ with $f(1-x)^n = (1-x)^n$. Let $e_\infty(x)$ be the supremum of $e_1(x) \leqslant e_2(x) \leqslant \cdots$ in the complete lattice of all projections of M, and $f_\infty(x)$ the infimum of $f_1(x) \geqslant f_2(x) \geqslant \cdots$ in the same lattice.*

Then $e_\infty(x)$ and $1 - f_\infty(x)$ belong to $\mathcal{I}$.

Proof. Setting $g_1 = e_1(x)$, $g_n = e_n(x) - e_{n-1}(x)$, $n \geqslant 2$ and applying Lemmas 1.3 and 2.4, yield

$$e_\infty(x) = \sum_{n=1}^{\infty} g_n \in \mathcal{I}.$$

The statement concerning $f_\infty(x)$ follows by observing that the canonical image of x^* in $M/\mathcal{I}$ is quasinilpotent and that $1 - f_\infty(x) = e_\infty(x^*)$. ∎

REMARK 3.2. In spite of the fact that both $e_\infty(x)$ and $1 - f_\infty(x)$ are in $\mathcal{I}$ under the most general conditions, Riesz type decomposition theorems like Theorem 3 in [3], Theorem 3 in [7] and Theorem 4.2 in [8] do not hold generally. A simple example is the following one:

Let M be the von Neumann algebra of all bounded linear operators on ℓ^2, $\mathcal{I} = M$, s the forward shift $s(\xi_1, \xi_2, \ldots) = (0, \xi_1, \xi_2, \ldots)$, and $x = 1 - s$.

Obviously, the canonical image of x in $M/\mathcal{I}$ is quasinilpotent. It is easy to see that both $e_\infty(x)$ and $f_\infty(x)$ are zero, therefore neither do we have that $e_\infty(x) \vee f_\infty(x) = 1$, nor that $e_\infty(x)$ and $1 - f_\infty(x)$ are equivalent.

REMARK 3.3. The proof of Theorem 3.1 had a strong AW^*-algebra flavour, so it is quite natural to ask: does it hold for every AW^*-algebra M?

The only point where we essentially used the fact that M is a von Neumann algebra, was in the proof of Lemma 2.3. If Kaplansky's conjecture on the linearity of the canonical quasitrace of a finite AW^*-factor would hold, then Lemma 2.3 would follow for arbitrary AW^*-algebras.

However, we need slightly less than the validity of the general Kaplansky conjecture.

Namely, let us assume the following

$$(*)\qquad \begin{cases} \text{if } M \text{ is a finite } AW^*\text{-factor, generated by the projections} \\ e, g_1, \ldots, g_m, \text{ where } g_1, \ldots, g_m \text{ are mutually orthogonal} \\ \text{and if, for some } \varepsilon > 0, \text{ we have that} \\ \qquad\qquad g_n e g_n \geqslant \varepsilon g_n,\ 1 \leqslant n \leqslant m, \\ \text{then the canonical quasitrace of } M \text{ is linear.} \end{cases}$$

Then Proposition 4.6 in [4] would imply that every quasitrace on a C^*-algebra generated by a finite number of projections $e, g_1, \ldots, g_m$, where $g_1, \ldots, g_m$ are mutually orthogonal and, for some $\varepsilon > 0$, $g_n e g_n \geqslant \varepsilon g_n$, $1 \leqslant n \leqslant m$, is linear. This would suffice to prove Lemma 2.3 for arbitrary AW^*-algebras. So, let us ask, does $(*)$ hold?

We notice that, if under the assumptions from $(*)$ the C^*-algebra generated by $e, g_1, \ldots, g_m$ would be exact (see, for example, [5] and [11]), then Corollary 5.13 in [4] would imply the conclusion from $(*)$.

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ANTON STRÖH
Department of Mathematics
and Applied Mathematics
University of Pretoria
0002 Pretoria
REPUBLIC OF SOUTH AFRICA

LÁSZLÓ ZSIDÓ
Dipartimento di Matematica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica
00133 Roma
ITALY

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