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A CHARACTERIZATION OF POSITIVE ELEMENTARY OPERATORS

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Abstract. A global structure theorem is obtained for positive elementary operators from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$, where $\mathcal{B}(\mathcal{H})$ is the C*-algebra of all (bounded linear) operators on complex Hilbert space $\mathcal{H}$. Several simple criteria are given which ensure a positive elementary operator to be completely positive.

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1. INTRODUCTION

Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ are complex Hilbert spaces and $\langle\cdot,\cdot\rangle$ stands for the inner product in both of them. $\mathcal{B}(\mathcal{H},\mathcal{K})$ ($\mathcal{B}(\mathcal{H})$ when $\mathcal{K}=\mathcal{H}$) is the Banach space of all (bounded linear) operators from $\mathcal{H}$ into $\mathcal{K}$. $A\in\mathcal{B}(\mathcal{H})$ is self-adjoint if $A=A^*$ (A^* is the conjugate operator of A); and A is positive, denoted by $A\geqslant 0$, if A is self-adjoint with spectrum falling in the interval $[0,\infty)$ (or equivalently, $\langle Ax,x\rangle\geqslant 0$ for all $x\in\mathcal{H}$). For any positive integer n, $\mathcal{H}^{(n)}$ denotes the direct sum of n copies of $\mathcal{H}$. It is clear that every operator $A\in\mathcal{B}(\mathcal{H}^{(n)},\ \mathcal{K}^{(m)})$ can be written in an $n\times m$ operator matrix $\mathbf{A}=(A_{ij})_{i,j}$ with $A_{ij}\in\mathcal{B}(\mathcal{H},\mathcal{K})$, $i=1,2,\ldots,m$; $j=1,2,\ldots,n$; we will write $\mathbf{A}^{\mathrm{T}}=(A_{ij})^{\mathrm{T}}$ for the transpose matrix of $(A_{ji})_{i,j}$ and $A^{(n)}$ for $(A_{ij})\in\mathcal{B}(\mathcal{H}^{(n)},\ \mathcal{K}^{(n)})$ with $A_{ii}=A$ and $A_{ij}=0$ if $i\neq j$. If Φ is a linear map from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$, we can define a linear map Φ_n: $\mathcal{B}(\mathcal{H}^{(n)})\to\mathcal{B}(\mathcal{K}^{(n)})$ by $\Phi_n((A_{ij}))=(\Phi(A_{ij}))$. Recall that Φ is said to be positive (resp. hermitian-preserving) if $A\in\mathcal{B}(\mathcal{H})$ is positive (resp. self-adjoint) implies that $\Phi(A)$ is positive (resp. self-adjoint). If Φ_n is positive we say Φ is n-positive; if Φ_n is positive for every integer $n > 0$, we say that Φ is completely positive. Obviously, Φ is completely positive $\Rightarrow$ Φ is positive $\Rightarrow$ Φ is hermitian-preserving. Φ is called an elementary operator if there are two finite sequences $\{A_i\}_{i=1}^n \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $\{B_i\}_{i=1}^n \subset \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\Phi(X) = \sum_{i=1}^n A_i X B_i$ for all $X \in \mathcal{B}(\mathcal{H})$.

Unlike completely positive linear maps, the structure of positive linear maps on C*-algebras is drastically nontrivial even for the finite dimensional case ([1]–[3], [14]–[15]). Since every linear map from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$ is an elementary operator in the finite dimensional case, it might be appropriate to centralize our attention firstly on elementary operators when we attempt to give some thorough description of the structure of positive linear maps. The global structures of hermitian-preserving and completely positive elementary operators are quite clear. In fact we have the following result.

THEOREM 1.1. *Let $\Phi = \sum_{i=1}^n A_i(\cdot)B_i$ be an elementary operator from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$. Then*

(i) *Φ is hermitian-preserving iff there are $C_1, \ldots, C_r \in [A_1, \ldots, A_n]$ such that*

$$\Phi = \sum_{i=1}^r \varepsilon_i C_i(\cdot)C_i^*,$$

where $\varepsilon_i \in \{-1, 1\}$;

(ii) *Φ is completely positive iff there are $C_1, \ldots, C_r \in [A_1, \ldots, A_n]$ such that*

$$\Phi = \sum_{i=1}^r C_i(\cdot)C_i^*.$$

Here we use notation $[\mathcal{S}]$ for the linear span of the set $\mathcal{S}$. Theorem 1.1 (i) and (ii) were established by dePillis ([15]) and Choi ([1]), respectively, for the finite dimensional case. For the general case, see Mathieu ([12]) and Hou ([6]). It is useful to notice that, by a result in [5], the operator set $\{C_1, \ldots, C_r\}$ can be chosen so that it is linearly independent.

The main purpose of this paper is to establish a global structure theorem for positive elementary operators in terms of local linear combination. We find that this characterization is much helpful in some sense for us to understand the differences of these three kinds of elementary operators. Our results and method allow us to give several simple criteria to ensure a positive elementary operator to be completely positive, one of them strengthens a result announced in [13].

2. A CHARACTERIZATION OF POSITIVE ELEMENTARY OPERATORS

Before stating the main result in this section, we need some definitions.

DEFINITION 2.1. Let $k, l \in \mathbb{N}$ (the set of all natural numbers), and let $A_1, \ldots, A_k$, and $C_1, \ldots, C_l \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. If, for each $x \in \mathcal{H}^{(n)}$, there exists an $l \times k$ complex matrix $(\alpha_{ij}(x))$ (depending on x) such that

$$C_i^{(n)} x = \sum_{j=1}^{k} \alpha_{ij}(x) A_j^{(n)} x, \quad i = 1, 2, \ldots, l,$$

we say that $(C_1, \ldots, C_l)$ is an n-locally linear combination of $(A_1, \ldots, A_k)$, $(\alpha_{ij}(x))$ is called a *local coefficient matrix* at x. Furthermore, if there exists a constant $M > 0$, and a local coefficient matrix $(\alpha_{ij}(x))$ at every $x \in \mathcal{H}^{(n)}$ can be chosen so that the norm $\|(\alpha_{ij}(x))\|$ of it is bounded by M, we say that $(C_1, \ldots, C_l)$ is an *n-regular locally linear combination* of $(A_1, \ldots, A_k)$; if $M \leqslant 1$, $(C_1, \ldots, C_l)$ is an *n-contractive locally linear combination* of $(A_1, \ldots, A_k)$; if there is a matrix (α_{ij}) such that $C_i = \sum\limits_{j=1}^{k} \alpha_{ij} A_j$ for all i, we say that $(C_1, \ldots, C_l)$ is a *linear combination* of $(A_1, \ldots, A_k)$ with coefficient matrix (α_{ij}).

We will omit "n" in the case $n = 1$. Sometimes we also write $\{A_i\}_{i=1}^k$ for $(A_1, \ldots, A_k)$.

Let $\mathcal{B}_M(\mathcal{H}, \mathcal{K}) = \{X \in \mathcal{B}(\mathcal{H}, \mathcal{K}) : \|X\| \leqslant M\}$. It is clear that $(C_1, \ldots, C_l)$ is an n-locally linear combination (resp. contractive locally linear combination) of $(A_1, \ldots, A_k)$ iff there exists a map $\Omega : \mathcal{H}^{(n)} \to \mathcal{B}(\mathbb{C}^{(k)}, \mathbb{C}^{(l)})$ (resp. $\mathcal{H}^{(n)} \to \mathcal{B}_1(\mathbb{C}^{(k)}, \mathbb{C}^{(l)})$) such that

$$(\, C_1^{(n)} x \quad \cdots \quad C_l^{(n)} x \,)^{\mathrm{T}} = \Omega(x)(\, A_1^{(n)} x \quad \cdots \quad A_k^{(n)} x \,)^{\mathrm{T}}.$$

Here we agree to use $(\lambda_{ij})(T_{jk})$ for the product of two operator matrices $((\lambda_{ij}) \otimes I)(T_{jk}) = (\lambda_{ij} I)(T_{jk})$ when (λ_{ij}) is a numerical matrix.

Indeed, we will discuss in a more general setting.

For an infinite sequence $\mathbf{A} = (A_1, \ldots, A_n, \ldots)$, we will denote by $\mathbf{A}^{\mathrm{T}}$ also the formal transpose of $\mathbf{A}$, that is,

$$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} A_1 \\ \vdots \\ A_n \\ \vdots \end{pmatrix}.$$

If $A_i \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and if both $\Big\| \sum\limits_{i=1}^{\infty} A_i^* A_i \Big\|$ and $\Big\| \sum\limits_{i=1}^{\infty} A_i A_i^* \Big\|$ are finite, then both $\mathbf{A}$ and $\mathbf{A}^{\mathrm{T}}$ are bounded operators from $\mathcal{H}^{(\infty)}$ into $\mathcal{K}$ and from $\mathcal{H}$ into $\mathcal{K}^{(\infty)}$, respectively. The following lemma is critical for our purpose.

LEMMA 2.2. *Let* $\{A_i\}_{i=1}^{\infty}$ *and* $\{C_j\}_{j=1}^{\infty} \subset \mathcal{B}(\mathcal{H},\mathcal{K})$ *such that* $\left\| \sum_{i=1}^{\infty} A_i^* A_i \right\| < \infty$, $\left\| \sum_{i=1}^{\infty} A_i A_i^* \right\| < \infty$, $\left\| \sum_{j=1}^{\infty} C_j^* C_j \right\| < \infty$ *and* $\left\| \sum_{j=1}^{\infty} C_j C_j^* \right\| < \infty$. *Then, the following statements are equivalent:*

(i) $\sum_{i=1}^{\infty} A_i P A_i^* \geqslant \sum_{j=1}^{\infty} C_j P C_j^*$ *for all positive operators* $P \in \mathcal{B}(\mathcal{H})$*;*

(ii) $\sum_{i=1}^{\infty} A_i P A_i^* \geqslant \sum_{j=1}^{\infty} C_j P C_j^*$ *for all rank-one projections* $P \in \mathcal{B}(\mathcal{H})$*;*

(iii) *there exists a map* $\Omega : \mathcal{H} \to \mathcal{B}_1(l_2)$ *such that*

$$\mathbf{C}^{\mathrm{T}} x = \Omega(x)\mathbf{A}^{\mathrm{T}} x \quad \textit{for every} \;\; x \in \mathcal{H};$$

(iv) $\sum_{i=1}^{\infty} A_i^* P A_i \geqslant \sum_{j=1}^{\infty} C_j^* P C_j$ *for all positive operators* $P \in \mathcal{B}(\mathcal{K})$*;*

(v) *there exists a map* $\Gamma : \mathcal{K} \to \mathcal{B}_1(l_2)$ *such that*

$$\mathbf{C}^* y = \Gamma(y)\mathbf{A}^* y \quad \textit{for every } y \in \mathcal{K}.$$

Proof. It is obvious that (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii) Given any $x \in \mathcal{H}$, without loss of generality, we may assume that $\|x\| = 1$, then $P_x = x \otimes x$ is a rank-one projection. From (ii) we have

$$\sum_{i=1}^{\infty} A_i P_x A_i^* \geqslant \sum_{j=1}^{\infty} C_j P_x C_j^*. \tag{2.1}$$

Let

$$\mathbf{T} = \begin{pmatrix} A_1 P_x & \cdots & A_i P_x & \cdots \\ 0 & \cdots & 0 & \cdots \\ 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \mathbf{S} = \begin{pmatrix} C_1 P_x & \cdots & C_j P_x & \cdots \\ 0 & \cdots & 0 & \cdots \\ 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

be operators from $\mathcal{H}^{(\infty)}$ into $\mathcal{K}^{(\infty)}$. The inequality (2.1) implies that $\mathbf{TT}^* \geqslant \mathbf{SS}^*$. So by [4] there exists a unique contraction $\mathbf{X} = (X_{ij}) \in \mathcal{B}(\mathcal{H}^{(\infty)})$ such that $\ker \mathbf{X}^* \supseteq \ker \mathbf{T}$ and $\mathbf{S} = \mathbf{TX}$. Since $(\,0 \;\; \cdots \;\; 0 \;\; x_i \;\; 0 \;\; \cdots)^{\mathrm{T}} \in \ker \mathbf{T}$ for each $x_i \in \ker A_i P_x$, we have $X_{ij}^* x_i = 0$ for all $j = 1, 2, \ldots$. Hence, $\ker X_{ij}^* \supseteq \ker A_i P_x$ for all i and j. It follows that X_{ij} are operators of rank at most one and there exist vectors $y_{ij} \in \mathcal{H}$ such that

$$X_{ij} = x \otimes y_{ij}, \quad i, j = 1, 2, \ldots.$$

Now $\mathbf{S} = \mathbf{T}\mathbf{X}$ leads to

$$\begin{aligned}(\,C_1x \quad \cdots \quad C_jx \quad \cdots)^{\mathrm{T}} &= (\,C_1P_xx \quad \cdots \quad C_jP_xx \quad \cdots)^{\mathrm{T}} \\ &= \Big(\sum_{i=1}^{\infty} A_iP_xX_{i1}x \quad \cdots \quad \sum_{i=1}^{\infty} A_iP_xX_{ij}x \quad \cdots\Big)^{\mathrm{T}} \\ &= \Big(\sum_{i=1}^{\infty} \langle x, y_{i1}\rangle A_ix \quad \cdots \quad \sum_{i=1}^{\infty} \langle x, y_{ij}\rangle A_ix \quad \cdots\Big)^{\mathrm{T}}.\end{aligned}$$

Let $\omega_{ji}(x) = \langle x, y_{ij}\rangle$ for $i, j = 1, 2, \ldots$, and let $\Omega(x) = (\omega_{ji}(x))_{j,i}$. Then we have

$$\mathbf{C}^{\mathrm{T}}x = (\,C_1x \quad \cdots \quad C_jx \quad \cdots)^{\mathrm{T}} = \Omega(x)(\,A_1x \quad \cdots \quad A_ix \quad \cdots)^{\mathrm{T}} = \Omega(x)\mathbf{A}^{\mathrm{T}}x.$$

Moreover, since $X_{ij}P_x = \langle x, y_{ij}\rangle P_x = \omega_{ij}(x)P_x$, by regarding $\Omega(x)$ as an operator from l_2 into itself, we get

$$\|\Omega(x)\| = \|\Omega(x) \otimes P_x\| = \|\mathbf{X}P_x^{(\infty)}\| \leqslant \|\mathbf{X}\| \leqslant \mathbf{1}.$$

Therefore, (ii) holds implies that (iii) holds.

(iii) $\Rightarrow$ (iv) Assume that (iii) holds. Then for any $x \in \mathcal{H}$, there is a contractive matrix $\Omega(x) = (\omega_{ji}(x))_{j,i} \in \mathcal{B}_1(l_2)$ such that

$$(\,C_1x \quad \cdots \quad C_jx \quad \cdots)^{\mathrm{T}} = \Big(\sum_{i=1}^{\infty} \omega_{1i}(x)A_ix \quad \cdots \quad \sum_{i=1}^{\infty} \omega_{ji}(x)A_ix \quad \cdots\Big)^{\mathrm{T}}.$$

If $P \in \mathcal{B}(\mathcal{K})$ is a positive operator, then

$$\begin{aligned}\Big\langle\Big(\sum_{j=1}^{\infty} C_j^*PC_j\Big)x, x\Big\rangle &= \sum_{j=1}^{\infty} \|P^{\frac12}C_jx\|^2 = \sum_{j=1}^{\infty} \Big\|\sum_{i=1}^{\infty} \omega_{ji}(x)P^{\frac12}A_ix\Big\|^2 \\ &= \|\Omega(x)(\,P^{\frac12}A_1 \quad \cdots \quad P^{\frac12}A_i \quad \cdots)^{\mathrm{T}}x\|^2 \\ &\leqslant \|(\,P^{\frac12}A_1 \quad \cdots \quad P^{\frac12}A_i \quad \cdots)^{\mathrm{T}}x\|^2 \\ &= \sum_{j=1}^{\infty} \|P^{\frac12}A_ix\|^2 = \Big\langle\Big(\sum_{i=1}^{\infty} A_i^*PA_i\Big)x, x\Big\rangle,\end{aligned}$$

that is, (iv) is true.

It is clear from above arguments that (iv) $\Rightarrow$ (v) $\Rightarrow$ (i). ∎

To see further what happens when a hermitian-preserving elementary operator

$$\Phi = \sum_{i=1}^{k} C_i(\cdot)C_i^* - \sum_{j=1}^{l} D_j(\cdot)D_j^*$$

is completely positive, we need the following lemma.

LEMMA 2.3. *Let* $\{A_i\}_{i=1}^{\infty}$ *and* $\{C_j\}_{j=1}^{\infty} \subset \mathcal{B}(\mathcal{H},\mathcal{K})$ *with* $\left\|\sum_{i=1}^{\infty} A_i^* A_i\right\| < \infty$, $\left\|\sum_{i=1}^{\infty} A_i A_i^*\right\| < \infty$, $\left\|\sum_{j=1}^{\infty} C_j^* C_j\right\| < \infty$ *and* $\left\|\sum_{j=1}^{\infty} C_j C_j^*\right\| < \infty$. *The following statements are equivalent:*

(i) $\sum_{i=1}^{\infty} A_i^{(n)} P {A_i^{(n)}}^* \geqslant \sum_{j=1}^{\infty} C_j^{(n)} P {C_j^{(n)}}^*$ *for all positive operators* $P \in \mathcal{B}(\mathcal{H}^{(n)})$ *and every* $n = 1, 2, \ldots$;

(ii) *there exists a contractive matrix* $\Omega = (\omega_{ji})_{j,i} \in \mathcal{B}_1(l_2)$ *such that*

$$(\, C_1 \quad \cdots \quad C_j \quad \cdots)^{\mathrm{T}} = (\omega_{ji})_{j,i} (\, A_1 \quad \cdots \quad A_i \quad \cdots)^{\mathrm{T}}.$$

Proof. (ii) $\Rightarrow$ (i) Obvious by Lemma 2.2.

(i) $\Rightarrow$ (ii) Let $\mathbf{A} = (\, A_1 \quad \cdots \quad A_i \cdots)$ and $\mathbf{C} = (\, C_1 \quad \cdots \quad C_j \quad \cdots)$. Let

$$\mathcal{B} = \{\Gamma = (\gamma_{ji})_{j,i} \mathbf{A}^{\mathrm{T}} : \Gamma \in \mathcal{B}_1(l_2)\ \}.$$

It is clear that $\mathcal{B}$ is closed in the strong operator topology in $\mathcal{B}(\mathcal{H}, \mathcal{K}^{(\infty)})$. Take $\varepsilon > 0$. For any $x_1, \ldots, x_n \in \mathcal{H}$, let $\mathbf{x} = (\, x_1 \quad \cdots \quad x_n\,) \in \mathcal{H}^{(n)}$. It follows from (ii) and Lemma 2.2, there exists an $\Omega(\mathbf{x}) = (\omega_{ji}(x)) \in \mathcal{B}_1(l_2)$ such that $\Omega(\mathbf{x}) {\mathbf{A}^{(n)}}^{\mathrm{T}} \mathbf{x} = {\mathbf{C}^{(n)}}^{\mathrm{T}} \mathbf{x}$. Therefore,

$$\Omega(\mathbf{x}) \mathbf{A}^{\mathrm{T}} x_k = \mathbf{C}^{\mathrm{T}} x_k \quad \text{for every } k = 1, \ldots, n.$$

Thus

$$\Omega(\mathbf{x}) \mathbf{A}^{\mathrm{T}} \in \{\mathbf{X} \in \mathcal{B}(\mathcal{H}, \mathcal{K}^{(\infty)}) : \|\mathbf{X} x_k - \mathbf{C}^{\mathrm{T}} x_k\| < \varepsilon \text{ for } k = 1, \ldots, n\ \}.$$

But this means that every strong neighborhood of $\mathbf{C}^{\mathrm{T}}$ has a nonempty intersection with $\mathcal{B}$ and hence, $\mathbf{C}^{\mathrm{T}} \in \mathcal{B}$. So, there exists an $\Omega \in \mathcal{B}(l_2)$ such that $\mathbf{C}^{\mathrm{T}} = \Omega \mathbf{A}^{\mathrm{T}}$. ∎

THEOREM 2.4. *Let* $\{A_i\}_{i=1}^{\infty}$ *and* $\{C_j\}_{j=1}^{\infty} \subset \mathcal{B}(\mathcal{H},\mathcal{K})$ *with* $\left\|\sum_{i=1}^{\infty} A_i^* A_i\right\| < \infty$, $\left\|\sum_{i=1}^{\infty} A_i A_i^*\right\| < \infty$, $\left\|\sum_{j=1}^{\infty} C_j^* C_j\right\| < \infty$ *and* $\left\|\sum_{j=1}^{\infty} C_j C_j^*\right\| < \infty$. *Let* $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ *be a linear map defined by*

$$\Phi(X) = \sum_{i=1}^{\infty} A_i X A_i^* - \sum_{j=1}^{\infty} C_j X C_j^*$$

for every $X \in \mathcal{B}(\mathcal{H})$. *Then*

(i) Φ *is positive iff there exists a map* $\Omega : x \in \mathcal{H} \mapsto \Omega(x) = (\omega_{ji}(x))_{j,i} \in \mathcal{B}_1(l_2)$ *such that*

$$\mathbf{C}^{\mathrm{T}} x = \Omega(x) \mathbf{A}^{\mathrm{T}} x$$

for every $x \in \mathcal{H}$.

(ii) Φ *is completely positive iff there exists a contractive matrix* $\Omega = (\omega_{ji})_{j,i} \in \mathcal{B}(l_2)$ *such that*

$$\mathbf{C}^{\mathrm{T}} = \Omega \mathbf{A}^{\mathrm{T}}.$$

Moreover, if $I - \Omega^*\Omega$ *is diagonalizable (in particular, if it is compact), then there exists a sequence* $\{D_i\}_{i=1}^{\infty} \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$ *such that*

$$\Phi(X) = \sum_{i=1}^{\infty} D_i X D_i^*$$

for every $X \in \mathcal{B}(\mathcal{H})$.

Proof. Since

$$\Phi_n(\mathbf{X}) = \sum_{i=1}^{\infty} A_i^{(n)} \mathbf{X} A_i^{(n)^*} - \sum_{j=1}^{\infty} C_j^{(n)} \mathbf{X} C_j^{(n)^*}$$

for every $\mathbf{X} \in \mathcal{B}(\mathcal{H}^{(n)}, \mathcal{K}^{(n)})$, by Lemma 2.2 and 2.3, we need only to prove the second part of (ii). Now, we have, for any $X \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned}\Phi(X) &= \mathbf{A} X^{(\infty)} \mathbf{A}^* - \mathbf{C} X^{(\infty)} \mathbf{C}^* = \mathbf{A} X^{(\infty)} \mathbf{A}^* - \mathbf{A} \Omega^{\mathrm{T}} X^{(\infty)} \Omega^{*\mathrm{T}} \mathbf{A}^* \\ &= \mathbf{A} X^{(\infty)} \mathbf{A}^* - \mathbf{A} X^{(\infty)} \Omega^{\mathrm{T}} \Omega^{*\mathrm{T}} \mathbf{A}^* = \mathbf{A} X^{(\infty)} (I - \Omega^*\Omega)^{\mathrm{T}} \mathbf{A}^*.\end{aligned}$$

Since $I - \Omega^*\Omega \geqslant 0$ as $\|\Omega\| \leqslant 1$, the diagonalization of $I - \Omega^*\Omega$ implies that there is a sequence $\{d_i\}$ with $0 \leqslant d_i \leqslant 1$ and a unitary matrix $\Gamma = (\gamma_{ij})_{i,j} \in \mathcal{B}(l_2)$ such that $\Gamma^*(I - \Omega^*\Omega)^{\mathrm{T}}\Gamma = \mathrm{diag}\{d_i^2\}$. Let $\mathbf{E} = (E_1 \quad \cdots \quad E_j \quad \cdots) = \mathbf{A}\Gamma$; then

$$\begin{aligned}\Phi(X) &= \mathbf{A} X^{(\infty)} (I - \Omega^*\Omega)^{\mathrm{T}} \mathbf{A}^* = \mathbf{E}\Gamma^* X^{(\infty)} (I - \Omega^*\Omega)^{\mathrm{T}} \Gamma \mathbf{E}^* \\ &= \mathbf{E} X^{(\infty)} \Gamma^* (I - \Omega^*\Omega)^{\mathrm{T}} \Gamma \mathbf{E}^* = \sum_{i=1}^{\infty} d_i^2 E_i X E_i^*.\end{aligned}$$

Let $D_i = d_i E_i$, we get

$$\Phi(X) = \sum_{i=1}^{\infty} D_i X D_i^*$$

for every X. ∎

The following theorem gives a global structure characterization for positive elementary operators.

THEOREM 2.5. *Let* $\Phi = \sum_{i=1}^{n} A_i(\cdot)B_i$ *be an elementary operator from* $\mathcal{B}(\mathcal{H})$ *into* $\mathcal{B}(\mathcal{K})$. Φ *is* m*-positive iff there exist* $C_1, \ldots, C_k$ *and* $D_1, \ldots, D_l$ *in* $[A_1, \ldots, A_n]$ *with* $k+l \leqslant n$ *such that* $(D_1, \ldots, D_l)$ *is an* m*-contractive locally linear combination of* $(C_1, \ldots, C_k)$ *and*

$$\Phi = \sum_{i=1}^{k} C_i(\cdot)C_i^* - \sum_{j=1}^{l} D_j(\cdot)D_j^*. \tag{2.2}$$

Furthermore, Φ *in* (2.2) *is completely positive iff* $(D_1, \ldots, D_l)$ *is a linear combination of* $(C_1, \ldots, C_k)$ *with a contractive coefficient matrix.*

NOTE: C_i and D_j in (2.2) can be chosen so that $\{C_1, \ldots, C_k, D_1, \ldots, D_l\}$ is linearly independent.

Proof. If Φ is m-positive, then it is hermitian-preserving. Therefore, by Theorem 1.1 (i), there exist operators C_i and $D_j \in [A_1, \ldots, A_n]$, $i = 1, \ldots, k$, $j = 1, \ldots, l$ with $k + l \leqslant n$ such that $\Phi = \sum_{i=1}^{k} C_i(\cdot)C_i^* - \sum_{j=1}^{l} D_j(\cdot)D_j^*$. Now it is easy to see that the theorem is an immediate consequence of Theorem 2.4 by considering finite sequences. ∎

Denote by $\mathcal{M}_n$ the C*-algebra of all $n \times n$ complex matrices. Since all linear maps from $\mathcal{M}_n$ into $\mathcal{M}_m$ are elementary operators, the next corollary is trivial from Theorem 2.5.

COROLLARY 2.6. *A linear map* $\Phi : \mathcal{M}_n \to \mathcal{M}_m$ *is positive iff there exist* $m \times n$ *matrices* $C_1, \ldots, C_k$ *and* $D_1, \ldots, D_l$ *such that* $(D_1, \ldots, D_l)$ *is a contractive locally linear combination of* $(C_1, \ldots, C_k)$ *and*

$$\Phi(X) = \sum_{i=1}^{k} C_i X C_i^* - \sum_{j=1}^{l} D_j X D_j^*$$

for all $X \in \mathcal{M}_n$.

REMARK 2.7. For the special case when $\{C_1, C_2, \ldots, C_j, \ldots\} = \{C, 0, \ldots, 0, \ldots\}$, Lemma 2.2 and 2.3 were obtained in [7].

3. WHEN POSITIVITY IMPLIES COMPLETE POSITIVITY

It is interesting to observe from the discussion in Section 2 that, for elementary operators, the question when positivity ensures complete positivity may be reduced to the question when regular locally linear combination implies linear combination. This connection allows us to look more deeply into the relationship and the difference between positivity and complete positivity, and obtain some simple criteria to check whether a positive elementary operator is completely positive. In fact, as we will see, the length of an elementary operator turns out to play a decisive role. One of the first results in this direction was proved by M. Mathieu ([11]): if the length is one, i.e., if $\Phi(X) = AXB$, then positivity already implies complete positivity. In this section, we will show that if the length n is greater than one, then $[\frac{n}{2}]$-positivity implies complete positivity. Recall that the notation $[t]$ stands for the integer part of a real number t.

If $\mathcal{S} \subset \mathcal{B}(\mathcal{H}, \mathcal{K})$, we will denote by $\mathcal{S}_F$ the subset of all finite-rank operators in $\mathcal{S}$. The following lemma is taken from [8]. Though only the case of regular locally linear combinations is needed for our purpose, we list both cases of locally linear combinations and regular locally linear combinations for comparison.

LEMMA 3.1. *Let $A_1, \dots, A_k$ and $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. If any one of the following conditions holds, then $C \in [A_1, \dots, A_k]$.*

(i) *C is a locally linear combination of A_1;*

(ii) *C is a locally linear combination of $(A_1, \dots, A_k)$ and $[A_1, \dots, A_k]_F = \{0\}$;*

(iii) *C is a regular locally linear combination of (A_1, A_2);*

(iv) *C is a regular locally linear combination of $(A_1, \dots, A_k)$ and there is a vector $x \in \mathcal{H}$ such that $\{A_i x\}_{i=1}^k$ is linearly independent;*

(v) *C is a regular locally linear combination of $(A_1, \dots, A_k)$ and $\dim[A_1, \dots, A_k]_F \leqslant 2$.*

NOTE: As pointed out in [7], there exist C and A_1, A_2 such that C is a locally linear combination of (A_1, A_2) but C is not a linear combination of A_1 and A_2; there exist C and (A_1, A_2, A_3) such that C is a regular locally linear combination of (A_1, A_2, A_3) but C is not a linear combination of them.

THEOREM 3.2. *Assume that $\Phi = \sum\limits_{i=1}^{k} A_i(\cdot)B_i : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a positive elementary operator. If any one of the following conditions hold, then Φ is completely positive:*

(i) *$k \leqslant 3$;*

(ii) *$\dim[A_1, \dots, A_k]_F \leqslant 2$ (or $\dim[B_1, \dots, B_k]_F \leqslant 2$);*

(iii) *there exists a vector* $x \in \mathcal{H}$ *(or* $y \in \mathcal{K}$*) such that* $\{A_i x\}_{i=1}^k$ *(or* $\{B_i y\}_{i=1}^k$*) is linearly independent.*

Proof. (i) Without loss of generality we may assume that both $\{A_1, A_2, A_3\}$ and $\{B_1, B_2, B_3\}$ are linearly independent. Since Φ is positive, by Theorem 2.5, there is a linearly independent set $\{D_1, D_2, D_3\} \subset [A_1, A_2, A_3]$ and an integer l with $1 < l \leqslant 3$ such that

$$\Phi = \sum_{i=1}^{l} D_i(\cdot)D_i^* - \sum_{i=l+1}^{3} D_i(\cdot)D_i^*.$$

If $l \neq 3$, then D_3 is a regular locally linear combination of D_1 or (D_1, D_2). By applying Lemma 3.1 (i) or (iii), we get a contradiction that D_3 is a linear combination of D_1 and D_2. Hence, we must have $l = 3$, i.e.

$$\Phi = \sum_{i=1}^{3} D_i(\cdot)D_i^*$$

is completely positive.

(ii) and (iii) can be treated similarly. ∎

COROLLARY 3.3. *If* $\Phi = \sum_{i=1}^{k} C_i(\cdot)C_i^* - \sum_{j=1}^{l} D_j(\cdot)D_j^*$ *is positive but not completely positive, then*

(i) $k \geqslant 3$*;*

(ii) *there is at least one* D_j *which is not in* $[C_1, \ldots, C_k]$*;*

(iii) *every* D_j*,* $j = 1, \ldots, l$*, is a finite-rank perturbation of some linear combination of* $\{C_i\}_{i=1}^k$*.*

Proof. (i) and (ii) are obvious. (iii) is a consequence of a result in [10] because D_j is a locally linear combination of $\{C_i\}$. ∎

COROLLARY 3.4. *A non-completely positive linear map* $\Phi : \mathcal{M}_2 \to \mathcal{M}_2$ *is positive iff it has the form*

$$\Phi = \sum_{i=1}^{3} C_i(\cdot)C_i^* - D(\cdot)D^*,$$

where $\{C_1, C_2, C_3, D\}$ *is linearly independent and* D *is a contractive locally linear combination of* (C_1, C_2, C_3)*, i.e., for every* $x \in \mathbb{C}^{(2)}$*, there exist scalars* $\lambda_1(x)$*,* $\lambda_2(x)$ *and* $\lambda_3(x)$ *with* $|\lambda_1(x)|^2 + |\lambda_2(x)|^2 + |\lambda_3(x)|^2 \leqslant 1$ *such that* $Dx = \lambda_1(x)C_1x + \lambda_2(x)C_2x + \lambda_3(x)C_3x$*.*

Using the above results one can construct with little difficulty examples of elementary operators which are positive but not completely positive. For example, let

$$C_1 = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C_3 = C_2^* \quad \text{and} \quad D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},$$

where c_i and $d_i \in \mathbb{C}$ satisfy the condition that $|c_i| \geqslant |d_i|$ for $i = 1, 2$, $0 < |c_1\overline{c}_2 - d_1\overline{d}_2| < 1$ and $c_1 d_2 - c_2 d_1 \neq 0$. Then

$$\Phi = \sum_{i=1}^{3} C_i(\cdot)C_i^* \; - D(\cdot)D^*$$

is positive on $\mathcal{M}_2$ but not completely positive.

Mathieu announced a result in [13] that an elementary operator $\Phi = \sum\limits_{i=1}^{n} A_i(\cdot)B_i$ is completely positive iff Φ is n-positive. Using our method, this result can be strengthened as follows.

Theorem 3.5. *Let $\Phi = \sum\limits_{i=1}^{n} A_i(\cdot)B_i$ be an elementary operator from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{K})$ with $n > 1$, then Φ is completely positive iff it is $\left[\frac{n}{2}\right]$-positive.*

To prove this theorem, we first need to investigate further the relation between local linear combinations and linear combinations of operators. Again, only the case of regular locally linear combination is needed in the proof of Theorem 3.5, but we give a thorough discussion in both cases of regular locally linear combination and locally linear combination for comparison and completeness.

It is clear from Definition 2.1 that if C is an n-locally linear combination (resp. n-regular locally linear combination) of $\{A_i\}$, then C is also an m-locally linear combination (resp. m-regular locally linear combination) of $\{A_i\}$ for all $1 \leqslant m \leqslant n$; and if $C \in [A_1, \ldots, A_k]$, then C is an n-locally linear combination (resp. n-regular locally linear combination) of $\{A_i\}$ for all $n = 1, 2, \ldots$.

Lemma 3.6. *Let $C, A_1, \ldots, A_n \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then $C \in [A_1, \ldots, A_n]$ iff C is an $\left[\frac{n+2}{2}\right]$-locally linear combination of $\{A_1, \ldots, A_n\}$.*

Proof. We need only to prove the "if" part. Supposing that C is an $\left[\frac{n+2}{2}\right]$-locally linear combination of $\{A_i\}_{i=1}^n$, we have to show that C is a linear combination of $\{A_i\}_{i=1}^n$. We will do this by induction. Without loss of generality, assume that $\{A_i\}_{i=1}^n$ is linearly independent in the sequel.

It is clear that from Lemma 2.2 in [6] that the lemma holds for $n = 1$ and 2.

Suppose that the assertion holds for all m with $1 \leqslant m < n$ and C is an $\left[\frac{n+2}{2}\right]$-locally linear combination of $\{A_i\}_{i=1}^n$. Let

$$k = \max\{\dim[A_1x, \ldots, A_nx] : x \in \mathcal{H}\}.$$

Obviously, $1 \leqslant k \leqslant n$. We may assume that $n \geqslant 3$.

If $k = 1$, then $\{A_ix, A_jx\}$ is linearly dependent for every $x \in \mathcal{H}$ and every pair (i, j) with $i, j = 1, \ldots, n$. Hence, there must be a vector $x_\circ \in \mathcal{H}$ and n vectors $y_1, \ldots, y_n \in \mathcal{K}$ such that $\{y_1, \ldots, y_n\}$ is linearly independent and $A_i = x_\circ \otimes y_i$, $i = 1, \ldots, n$. (Here we have used the fact that if $\{Ax, Bx\}$ is linearly dependent for every vector x, then either $\{A, B\}$ is linearly dependent or $A = x_\circ \otimes f_1$ and $B = x_\circ \otimes f_2$. A proof of this fact may be found in [8]). Since C is a locally linear combination of $\{A_i\}$, there exists a vector $f \in \mathcal{K}$ such that $C = x_\circ \otimes f$. It is clear that $\{f\}^\perp \supseteq \bigcap\limits_{i=1}^{n} \{y_i\}^\perp = [y_1, \ldots, y_n]^\perp$. Therefore, $f \in [y_1, \ldots, y_n]$ and hence $C \in [A_1, \ldots, A_n]$.

If $k = n$, then there is a vector $x_\circ \in \mathcal{H}$ such that $\{A_1x_\circ, \ldots, A_nx_\circ\}$ is linearly independent and hence there are uniquely determined scalars α_i such that

$$Cx_\circ = \alpha_1A_1x_\circ + \cdots + \alpha_nA_nx_\circ.$$

Now, for any $x \in \mathcal{H}$, C is a 2-locally linear combination of $\{A_i\}$ since $\left[\frac{n+2}{2}\right] \geqslant 2$, therefore there exist scalars λ_i depending on x such that

$$\begin{pmatrix} Cx_\circ \\ Cx \end{pmatrix} = \begin{pmatrix} \sum\limits_{i=1}^{n} \lambda_iA_ix_\circ \\ \sum\limits_{i=1}^{\infty} \lambda_iA_ix \end{pmatrix}.$$

It follows that $\lambda_i = \alpha_i$, $i = 1, \ldots, n$, and $Cx = \sum\limits_{i=1}^{n} \alpha_iA_ix$ for all $x \in \mathcal{H}$. Hence, $C \in [A_1, \ldots, A_n]$.

If $k = n - 1$, without loss of generality we may assume that there is a vector $x_\circ \in \mathcal{H}$ such that $\{A_ix_\circ\}_{i=1}^{n-1}$ is linearly independent. Thus, there are uniquely determined numbers $\lambda_i, i = 1, \ldots, n-1$, so that $A_nx_\circ = \sum\limits_{i=1}^{n-1} \lambda_iA_ix_\circ$. Taking any α_i's such that $Cx_\circ = \sum\limits_{i=1}^{n} \alpha_iA_ix_\circ$, we have

$$Cx_\circ = \sum_{i=1}^{n-1}(\alpha_i + \lambda_i\alpha_n)A_ix_\circ = \sum_{i=1}^{n-1} \gamma_iA_ix_\circ.$$

Because $\{A_1x_\circ,\dots,A_{n-1}x_\circ\}$ is linearly independent, γ_i are constants. Thus $Cx_\circ$ is of the form

$$Cx_\circ=\sum_{i=1}^{n-1}(\gamma_i-\lambda_i t)A_ix_\circ+tA_nx_\circ \tag{3.1}$$

with $t\in\mathbb{C}$ arbitrary. Since C is certainly a 2-locally linear combination of $\{A_i\}$, for any $x\in\mathcal{H}$, by virtue of (3.1) and by taking $\mathbf{x}=(\,x_\circ\quad x\,)\in\mathcal{H}^{(2)}$, it is easy to see that there is a $t(x)\in\mathbb{C}$

such that

$$\begin{aligned}Cx&=\sum_{i=1}^{n-1}(\gamma_i-\lambda_i t(x))A_ix+t(x)A_nx\\&=\Big(\sum_{i=1}^{n-1}\gamma_1A_i\Big)x+t(x)\Big(A_n-\sum_{i=1}^{n-1}\lambda_iA_i\Big)x.\end{aligned}\tag{3.2}$$

Let $S=\sum\limits_{i=1}^{n-1}\gamma_iA_i$ and $T=A_n-\sum\limits_{i=1}^{n-1}\lambda_iA_i$; then (3.2) means that $C-S$ is a locally linear combination of T. So, we have $C-S\in[T]$ and hence $C\in[S,T]\subseteq[A_1,\dots,A_n]$.

If $1<k<n-1$, again we may assume that there exists a vector $x_\circ$ such that $\{A_ix_\circ\}_{i=1}^k$ is linearly independent. Then, there exists a uniquely determined $k\times(n-k)$ matrix (λ_{ij}) such that

$$A_{k+j}x_\circ=\sum_{i=1}^{k}\lambda_{ij}A_ix_\circ,\quad j=1,\dots,n-k.$$

Since $Cx_\circ\in[A_1x_\circ,\dots,A_nx_\circ]$, it is clear that there exist scalars $\gamma_1,\dots,\gamma_k$ such that $Cx_\circ$ is of the form

$$Cx_\circ=\sum_{i=1}^{k}\Big(\gamma_i-\sum_{j=1}^{n-k}\lambda_{ij}t_j\Big)A_ix_\circ+\sum_{j=1}^{n-k}t_jA_{k+j}x_\circ \tag{3.3}$$

with $(\,t_1\quad\cdots\quad t_{n-k}\,)\in\mathbb{C}^{(n-k)}$ arbitrary. Recall that C is a $\big[\frac{n+2}{2}\big]$-locally linear combination of $\{A_i\}_{i=1}^n$. By taking $\mathbf{x}=(\,x_\circ\quad x_1\quad\cdots\quad x_{[\frac{n}{2}]}\,)$ and using (3.3), we observe that, for every $(\,x_1\quad\cdots\quad x_{[\frac{n}{2}]}\,)\in\mathcal{H}^{([\frac{n}{2}])}$, there is a corresponding $(\,t_1\quad\cdots\quad t_{n-k}\,)\in\mathbb{C}^{(n-k)}$ such that

$$Cx_r=\sum_{i=1}^{k}\Big(\gamma_i-\sum_{j=1}^{n-k}\lambda_{ij}t_j\Big)A_ix_r+\sum_{j=1}^{n-k}t_jA_{k+j}x_r \tag{3.4}$$

for all $r = 1, \dots, \left[\frac{n}{2}\right]$. Let

$$S = \sum_{i=1}^{k} \gamma_i A_i \quad \text{and} \quad T_j = A_{k+j} - \sum_{i=1}^{k} \lambda_{ij} A_i, \quad j = 1, \dots, n-k. \tag{3.5}$$

It follows from (3.4) that $C - S$ is a $\left[\frac{n}{2}\right]$-locally linear combination of $\{T_1, \dots, T_{n-k}\}$. Now, since $k \geqslant 2$ and $\left[\frac{n-k+2}{2}\right] \leqslant \left[\frac{n}{2}\right]$, by the induction assumption, we conclude that

$$C - S \in [T_1, \dots, T_{n-k}]$$

and hence

$$C \in [S, T_1, \dots, T_{n-k}] \subset [A_1, \dots, A_n]. \quad \blacksquare$$

COROLLARY 3.7. *C is a linear combination of $\{A_i\}_{i=1}^n$ iff C is an n-locally linear combination of $\{A_i\}_{i=1}^n$.*

As to the case of regular locally linear combination, we have

LEMMA 3.8. *Let $A_1, \dots, A_n$ and $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then $C \in [A_1, \dots, A_n]$ iff C is a $\left[\frac{n+1}{2}\right]$-regular locally linear combination of $\{A_i\}_{i=1}^n$.*

Proof. Similar to the proof of Lemma 3.6, we need only to show the sufficiency by induction on n. Lemma 3.1 says this is true for cases $n = 1$ or 2.

Assuming that the lemma holds for all m with $1 \leqslant m < n$ and C is a $\left[\frac{n+1}{2}\right]$-regular locally linear combination of $\{A_i\}_{i=1}^n$, we have to show that $C \in [A_1, \dots, A_n]$. We may regard n as an integer greater than 2, this assures that $\left[\frac{n+1}{2}\right] \geqslant 2$.

Again, let $k = \max\{\dim[A_1 x, \dots, A_n x] : x \in \mathcal{H}\}$. It is clear from the proof of Lemma 3.6 that if $k = 1, n-1$, or n, then $C \in [A_1, \dots, A_n]$. So we may assume that $2 \leqslant k \leqslant n-2$ and without loss of generality, there exists a vector $x_\circ$ such that $\{A_i x_\circ\}_{i=1}^k$ is linearly independent. Thus, there are constants γ_i and λ_{ij}, $i = 1, \dots, k, j = 1, \dots, n-k$, such that $Cx_\circ$ is of the form as in (3.3) but with $|t_j| \leqslant M$ for all j. By taking $\mathbf{x} = (\, x_\circ \quad x_1 \quad \cdots \quad x_{\left[\frac{n-1}{2}\right]} \,)$, one can get that, for any $(\, x_1 \quad \cdots \quad x_{\left[\frac{n-1}{2}\right]} \,) \in \mathcal{H}^{(\left[\frac{n-1}{2}\right])}$, there is a corresponding $(\, t_1 \quad \cdots \quad t_{n-k} \,) \in \mathbb{C}^{(n-k)}$ with $|t_j| \leqslant M$ such that (3.4) holds for every $r = 1, \dots, \left[\frac{n-1}{2}\right]$. Let S and T_j, $j = 1, \dots, n-k$, as that in (3.5); then $C - S$ is a $\left[\frac{n-1}{2}\right]$-regular locally linear combination of $\{T_j\}_{j=1}^{n-k}$. Note that $\left[\frac{n-k+1}{2}\right] \leqslant \left[\frac{n-1}{2}\right]$. Using the induction assumption for $n-k$, we obtain that $C - S$ is a linear combination of $\{T_j\}_{j=1}^{n-k}$, and hence, C is a linear combination of $\{A_i\}_{i=1}^n$ as well. $\blacksquare$

COROLLARY 3.9. *C is a linear combination of $\{A_i\}_{i=1}^n$ with $n \geqslant 2$ iff C is a $(n-1)$-regular locally linear combination of $\{A_i\}_{i=1}^n$.*

REMARK 3.10. It is worthwhile to point out that Lemma 3.8 and Corollary 3.9 hold true for linear maps on complex vector spaces, and Lemma 3.6 and Corollary 3.7 hold true for both cases of real and complex vector spaces.

It is the time to prove Theorem 3.5.

Proof of Theorem 3.5. We may assume that both $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ are linearly independent.

By Theorem 3.2 we may assume that $n \geqslant 4$. If Φ is $\left[\frac{n}{2}\right]$-positive, then by Theorem 2.2, there exists a linearly independent set $\{C_i\}_{i=1}^n$ in $[A_1, \ldots, A_n]$ and a $k \geqslant 3$ such that

$$\Phi = \sum_{i=1}^{k} C_i(\cdot)C_i^* \; - \sum_{i=k+1}^{n} C_i(\cdot)C_i^*.$$

We need only to prove $k = n$. If, on the contrary, $k \leqslant n-1$, since Φ is $\left[\frac{n}{2}\right]$-positive, we must have from Theorem 2.5 that C_n is a $\left[\frac{n}{2}\right]$-regular locally linear combination of $\{C_i\}$. Because $k \leqslant n-1$ and $\left[\frac{k+1}{2}\right] \leqslant \left[\frac{n}{2}\right]$, by Lemma 3.8, we obtain a contradiction that C_n is a linear combination of $\{C_i\}_{i=1}^k$. Therefore, $k = n$ and Φ is completely positive by virtue of Theorem 1.1 (ii). ∎

REMARK 3.11. By a similar argument as that in [9], one can see that Theorem 3.2 (suitably stated) and Theorem 3.5 still hold for the elementary operators on a prime C^*-algebra $\mathcal{A}$ with Soc($\mathcal{A}$)$\neq \{0\}$, where Soc($\mathcal{A}$) is the sum of all minimal left ideals in $\mathcal{A}$.

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