THE CENTRAL HAAGERUP TENSOR PRODUCT
OF A C*-ALGEBRA

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Abstract. Let $A$ be a $C^*$-algebra with an identity and let $\theta_Z$ be the canonical map from $A \otimes_Z A$, the central Haagerup tensor product of $A$, to $CB(A)$, the algebra of completely bounded operators on $A$. It is shown that if every Glimm ideal of $A$ is primal then $\theta_Z$ is an isometry. This covers unital quasi-standard $C^*$-algebras and quotients of $AW^*$-algebras.

Keywords: $C^*$-algebra, Haagerup tensor product, primal ideal.

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1. INTRODUCTION

If $A$ is a $C^*$-algebra the Haagerup norm $\| \cdot \|_h$ is defined on an element $x$ in the algebraic tensor product $A \otimes A$ by

$$\|x\|_h = \inf \left\{ \left\| \sum_{i=1}^n a_i a_i^* \right\|^{1/2} \left\| \sum_{i=1}^n b_i^* b_i \right\|^{1/2} \right\},$$

where the infimum is taken over all possible representations of $x$ as a finite sum $x = \sum_{i=1}^n a_i \otimes b_i$, $a_i, b_i \in A$. The completion of $A \otimes A$ in this norm is called the Haagerup tensor product of $A$ with itself. There is a natural contraction $\theta : A \otimes_h A \to CB(A)$ (where $CB(A)$ is the algebra of completely bounded operators on $A$ with the completely bounded norm $\| \cdot \|_{cb}$) given by $\theta \left( \sum_{i=1}^n a_i \otimes b_i \right)(c) = \sum_{i=1}^n a_i c b_i$, $c \in A$. It is clear that $\theta$ is not injective if $A$ is not a prime $C^*$-algebra, but if $A$ is prime then $\theta$ is an isometry ([3], 3.9).
Suppose that $A$ is unital and that $z \in Z(A)$, the centre of $A$. Then it is easy to see that the element $az \otimes b - a \otimes zb$, $a, b \in A$, belongs to $\ker \theta$. Thus if $J_A$ is the closed ideal of $A \otimes_h A$ generated by such elements, one can consider the induced map $\theta_Z : A \otimes_h A/J_A \to CB(A)$, and ask whether it is injective or isometric. The Banach algebra $A \otimes_h A/J_A$, with the quotient norm $\|\cdot\|_Z$, is called the central Haagerup tensor product of $A$, and denoted $A \otimes_Z A$. It is known that $\theta_Z$ is isometric if $A$ is a von Neumann algebra or if $A$ has Hausdorff primitive ideal space ([10]), or if $A$ is boundedly centrally closed ([3]). On the other hand, if $Z(A) \cong \mathbf{C}$ then $\theta_Z$ is $\theta$, so $\theta_Z$ is not injective in this case, unless $A$ is prime. One could try factoring by $\ker \theta$, but an example in [10] shows that even this can fail to produce an isometry.

Von Neumann algebras and $C^*$-algebras with Hausdorff primitive ideal space and boundedly centrally closed $C^*$-algebras are all prominent examples of quasi-standard $C^*$-algebras, that is, $C^*$-algebras $A$ for which $\text{Glimm}(A)$ and $\text{MinPrimal}(A)$ (defined below) coincide as topological spaces. This makes it natural to wonder if $\theta_Z$ is isometric whenever $A$ is a unital quasi-standard $C^*$-algebra. The main result of this paper is that this is indeed so, and in fact we only require that $\text{Glimm}(A)$ and $\text{MinPrimal}(A)$ should coincide as sets. This weaker condition is always satisfied by quotients of von Neumann algebras, which need not necessarily be quasi-standard.

We also characterize the injectivity of $\theta_Z$ (every Glimm ideal of $A$ must be 2-primal), and show that a necessary condition for $\theta_Z$ to be an isometry is that every Glimm ideal of $A$ should be 3-primal. Thus the exact characterization of $\theta_Z$ being an isometry lies somewhere between the conditions that every Glimm ideal be 3-primal, and that every Glimm ideal be primal.

2. PRELIMINARIES

Let $A$ be a $C^*$-algebra and let $\text{Id}(A)$ denote the set of all ideals of $A$ (ideal means closed, two-sided ideal in this paper). Then $\text{Id}(A)$ has a natural topology $\tau_w$ obtained by taking as a sub-base all sets of the form $\{I \in \text{Id}(A) : I \not\supset J\}$, where $J$ is allowed to vary through $\text{Id}(A)$. When restricted to $\text{Prim}(A)$, the set of primitive ideals of $A$, $\tau_w$ is simply the hull-kernel topology. A second topology $\tau_s$ is defined on $\text{Id}(A)$ as the weakest topology making the functions $I \to \|a + I\|$, $I \in \text{Id}(A)$, continuous for all $a \in A$. This topology is stronger than $\tau_w$, and $(\text{Id}(A), \tau_s)$ is a compact, Hausdorff space (see [4] for a discussion of the history and properties of $\tau_w$ and $\tau_s$).
Recall from [8], p. 351 that if $A$ is a unital $C^*$-algebra then the Glimm ideals are the closed ideals of $A$ generated by the maximal ideals of the centre of $A$. The set of Glimm ideals of $A$ is denoted $\text{Glimm}(A)$, and is equipped with the topology from the maximal ideal space of the centre of $A$, so that $\text{Glimm}(A)$ is a compact, Hausdorff space, homeomorphic to the maximal ideal space of the centre of $A$. Thus we can identify the centre of $A$ with the algebra of continuous complex-valued functions on $\text{Glimm}(A)$. Furthermore, for each $a \in A$ the map $G \to \|a + G\|$ ($G \in \text{Glimm}(A)$) is upper semi-continuous on $\text{Glimm}(A)$ ([15], Theorem 1; [12], Lemma 9).

Let us say that an ideal $I$ of $A$ is $n$-primal ($n \geq 2$) if whenever $J_1, \ldots, J_n$ are $n$ ideals of $A$ with $J_1 \cdots J_n = 0$ then $J_i \subseteq I$ for at least one value of $i$. If $I$ is $n$-primal for all $n$ then $I$ is primal. Note that prime (and hence primitive) ideals are primal. Let $n$-Primal($A$), respectively Primal($A$), denote the set of $n$-primal, respectively primal ideals of $A$. It is not difficult to see, using [5], 3.2, that a 2-primal ideal must contain a unique Glimm ideal. An ideal is $n$-primal if and only if the intersection of any $n$ primitive ideals containing it is primal ([7], 1.3). It is shown in [5], p. 59 that for any $n$ there is a $C^*$-algebra with an $n$-primal ideal which is not primal. An argument involving Zorn’s Lemma shows that every primal ideal contains a minimal primal ideal. Let MinPrimal($A$) denote the set of minimal closed primal ideals. Primal($A$) is a $\tau_w$-closed subset of $\text{Id}(A)$, hence a compact Hausdorff space in the $\tau_s$-topology, and the topologies $\tau_s$ and $\tau_w$ coincide on MinPrimal($A$) ([4]).

A $C^*$-algebra $A$ is said to be quasi-standard if MinPrimal($A$) and Glimm($A$) coincide, both as sets and as topological spaces. This is equivalent, for separable $C^*$-algebras, to $A$ being isomorphic to a continuous field of $C^*$-algebras in which the set of primitive fibres is dense ([8], 3.5). Examples include AW* -algebras and $C^*$-algebras with Hausdorff primitive ideal space ([8]), boundedly centrally closed $C^*$-algebras ([19]), and the $C^*$-algebras of various groups, such as discrete amenable groups, see [13]. If $A$ is a quotient of an AW*-algebra then MinPrimal($A$) and Glimm($A$) coincide as sets, but not necessarily as topological spaces ([18], 2.8).
3. RESULTS

We begin with a description of the central Haagerup norm, along the lines of [18], 2.3. For an ideal \( I \) in a \( C^* \)-algebra \( A \), and for \( u \in A \otimes_h A \), we shall use \( u^I \) to denote the image of \( u \) in the quotient algebra \( A \otimes_h A / (I \otimes_h A + A \otimes_h I) \) (which is isometrically isomorphic to \( A/I \otimes_h A/I \) by [2], 2.6).

**Theorem 1.** Let \( A \) be a \( C^* \)-algebra with an identity and let \( u \in A \otimes_h A \). Then

\[
\|u\|_Z = \sup \{\|u^G\|_h : G \in \text{Glimm}(A)\}.
\]

Hence \( J_A = \bigcap \{G \otimes_h A + A \otimes_h G : G \in \text{Glimm}(A)\} \).

**Proof.** It is enough to prove equality when \( u \) has the form \( u = \sum_{i=1}^{n} a_i \otimes b_i \), with \( a_i, b_i \in A \). Set \( \alpha = \sup \{\|u^G\|_h : G \in \text{Glimm}(A)\} \). Since \( J_A \subseteq G \otimes_h A + A \otimes_h G \) for all \( G \in \text{Glimm}(A) \) it is clear that \( \|u\|_Z \geq \alpha \). Suppose that \( \varepsilon > 0 \) is given. For each \( G \in \text{Glimm}(A) \) there exists, by [10], Lemma 2.3, an invertible \( n \times n \) matrix \( S \) such that if \( (a_i') = (a_i)S^{-1} \) and \( (b_i') = S(b_i) \) then

\[
\left\| \sum_{i=1}^{n} (a_i' a_i'^* + G) \right\|, \left\| \sum_{i=1}^{n} (b_i'^* b_i' + G) \right\| < \alpha + \varepsilon.
\]

By the upper semi-continuity of the norm functions on \( \text{Glimm}(A) \) there is a neighbourhood \( N \) of \( G \) such that

\[
\left\| \sum_{i=1}^{n} (a_i' a_i'^* + G') \right\|, \left\| \sum_{i=1}^{n} (b_i'^* b_i' + G') \right\| < \alpha + \varepsilon
\]

for all \( G' \in N \). Thus by the compactness of \( \text{Glimm}(A) \) there exist open subsets \( \{N_j\}_{j=1}^{m} \) of \( \text{Glimm}(A) \) and invertible \( n \times n \) matrices \( \{S_j\}_{j=1}^{m} \) such that the \( N_j \)'s cover \( \text{Glimm}(A) \) and such that if \( G \in N_j \) then

\[
\left\| \sum_{i=1}^{n} (a_i' a_i'^* + G) \right\|, \left\| \sum_{i=1}^{n} (b_i'^* b_i' + G) \right\| < \alpha + \varepsilon,
\]

where \( (a_i') = (a_i)S_j^{-1} \) and \( (b_i') = S_j(b_i) \). Let \( \{z_j\}_{j=1}^{m} \) be a partition of the identity on \( \text{Glimm}(A) \) subordinate to the cover \( \{N_j\}_{j=1}^{m} \), and set

\[
v = \sum_{j=1}^{m} \sum_{i=1}^{n} a_i' z_j^{1/2} \otimes z_j^{1/2} b_i'.
\]
The central Haagerup tensor product of a C*-algebra

Then
\[ v = \sum_{j=1}^{m} \left( \sum_{i=1}^{n} a_{ji}^l \otimes b_{ji}^l \right) (z_j^{1/2} \otimes z_j^{1/2}) = \sum_{j=1}^{m} u(z_j^{1/2} \otimes z_j^{1/2}), \]
so
\[ u - v = u \left( 1 - \sum_{j=1}^{m} (z_j^{1/2} \otimes z_j^{1/2}) \right) = u \left( \sum_{j=1}^{m} z_j \otimes 1 - z_j^{1/2} \otimes z_j^{1/2} \right) \]
\[ = u \left( \sum_{j=1}^{m} (z_j^{1/2} \otimes 1)(1 - 1 \otimes z_j^{1/2}) \right). \]

Hence \( u - v \in J_A \). But for \( G \in \text{Glimm}(A) \)
\[ \left\| \sum_{j=1}^{m} \sum_{i=1}^{n} z_j a_{ji}^l a_{ji}^{*l} + G \right\| = \left\| \sum_{j=1}^{m} (z_j + G) \left( \sum_{i=1}^{n} a_{ji}^l a_{ji}^{*l} + G \right) \right\| < \alpha + \varepsilon, \]
and similarly for \( G' \in \text{Glimm}(A) \)
\[ \left\| \sum_{j=1}^{m} \sum_{i=1}^{n} z_j b_{ji}^{*l} b_{ji}^l + G' \right\| < \alpha + \varepsilon. \]

Since \( \bigcap \{ G : G \in \text{Glimm}(A) \} = \{0\} \) it follows that
\[ \|u\|_Z \leq \|v\|_h \leq \left\| \sum_{j=1}^{m} \sum_{i=1}^{n} z_j a_{ji}^l a_{ji}^{*l} \right\|^{1/2} \left\| \sum_{j=1}^{m} \sum_{i=1}^{n} z_j b_{ji}^{*l} b_{ji}^l \right\|^{1/2} < \alpha + \varepsilon, \]
as required.

Remarks. (i) A subspace \( X \) of a Banach space \( Y \) is said to be proximinal if every element of \( Y \) attains its distance to \( X \). Ideals in C*-algebras are proximinal ([1], 4.3), and so too is the centre of a unital C*-algebra ([20]). This makes it natural to wonder if \( J_A \) is proximinal in \( A \otimes h A \).

(ii) An ideal in \( A \otimes h A \) is said to be upper, see [2], 6.7 (ii), if it is the intersection of the primitive ideals containing it. If \( J \) is a proper ideal of \( A \) then \( J \otimes h A + A \otimes h J \) is upper; in fact \( J \otimes h A + A \otimes h J = \bigcap \{ P \otimes h A + A \otimes h Q : P, Q \in \text{Prim}(A/J) \} \) ([6], 1.3). Thus Theorem 1 shows that \( J_A \) is an upper ideal, or in other words, that \( A \otimes Z A \) is a semisimple Banach algebra.

An ideal in \( A \otimes h A \) is lower ([2], Section 6) if it is generated by the elementary tensors that it contains, and \( J_A \) looks a good candidate, being generated by differences of elementary tensors. Since \( J_A \) is generated by elements of the form \( z \otimes 1 - 1 \otimes z, z \in Z(A) \), it is enough to consider the case when \( A \) is an abelian C*-algebra, but even here the answer seems to be unknown.
(iii) Let $I(A, A)$ denote the ideal in $A \otimes A$ (the algebraic tensor product) generated by elements of the form $az \otimes b - a \otimes zb$, $a, b \in A$, $z \in Z(A)$, and let $J(A, A)$ be the ideal $\bigcap \{G \otimes A + A \otimes G : G \in \text{Glimm}(A)\} \subseteq A \otimes A$. Clearly $I(A, A) \subseteq J(A, A)$. It is known that $I(A, A) = J(A, A)$ if $A$ is a continuous field of $C^*$-algebras over Glimm($A$), see [9]. Theorem 1 implies that for any unital $C^*$-algebra $I(A, A)$ and $J(A, A)$ have the same closure, namely $J_A$, in $A \otimes_A A$ (and hence the same closure, namely the closure of $J_A$, in $A \otimes_{\min} A$, the minimal $C^*$-tensor product).

The next result is a combination of Lemma 3.1 and Theorem 3.4 of [6].

**Proposition 2.** Let $A$ be a $C^*$-algebra. For each $u \in A \otimes_h A$ the map

$$(I, J) \mapsto \|u + (I \otimes_h A + A \otimes_h J)\|_h, \ (I, J) \in \text{Id}(A) \times \text{Id}(A),$$

is continuous for the product $\tau_s$-topology on $\text{Id}(A) \times \text{Id}(A)$.

The next result generalizes [18], 2.6, using the same method of proof.

**Proposition 3.** Let $A$ be a $C^*$-algebra and let $u \in A \otimes_h A$. Then

$$\|\theta(u)\|_{cb} = \sup\{\|u^P\|_h : P \in \text{MinPrimal}(A)\}.$$

**Proof.** Let $D$ denote the diagonal of Primal($A$) × Primal($A$), in the product $\tau_s$-topology. Then $D$ is a compact set, and the norm function $(P, P) \mapsto \|u^P\|_h$ ($(P, P) \in D$) is continuous on $D$, by Proposition 2, so it attains its supremum, clearly at some $(R, R)$ with $R \in \text{MinPrimal}(A)$. But $R$ is in the $\tau_s$-closure of Primal($A$) ([4], 4.3), so $\|u^R\|_h = \sup\{\|u^P\|_h : P \in \text{Primal}(A)\}$. But $\|\theta(u)\|_{cb} = \sup\{\|u^P\|_h : P \in \text{Primal}(A)\}$, by [3], 3.6, and the result follows. \[\square\]

For $a \in A$, let $D_a$ denote the inner derivation induced by $a$. Then $D_a = \theta(a \otimes 1 - 1 \otimes a)$, and $\|D_a\| = \|D_a\|_{cb}$, see [11], 4.1. Now $\|a \otimes 1 - 1 \otimes a\|_h$ is equal to twice the distance from $a$ to the scalars [14], 3.3, so it follows from Theorem 1 and [18], 2.3 that $\|a \otimes 1 - 1 \otimes a\|_Z = 2 \, d(a, Z(A))$, where $d(a, Z(A))$ is the distance from $a$ to the centre of $A$. But it was shown in [18], 3.2, 3.3 that a necessary and sufficient condition for $\|D_a\|$ to equal $2 \, d(a, Z(A))$ for all $a \in A$ is that every Glimm ideal of $A$ should be 3-primal. Thus a necessary condition for $\theta_Z$ to be an isometry is that every Glimm ideal of $A$ should be 3-primal. Whether this is also a sufficient condition, we do not know. Our main result, however, is a partial converse. It follows from Theorem 1 and Proposition 3.
Theorem 4. Let $A$ be a $C^*$-algebra with an identity. If every Glimm ideal of $A$ is primal then the map $\theta_Z : A \otimes_Z A \to CB(A)$ is an isometry.

Thus $\theta_Z$ is an isometry if $A$ is a unital quasi-standard $C^*$-algebra, or a quotient of an AW*-algebra. It seems worth remarking that this is very easy to show that every Glimm ideal of a von Neumann algebra is primal ([5], 4.1).

Since $A \otimes_Z A$ and $CB(A)$ are both not only Banach spaces but operator spaces, it would be interesting to know whether $\theta_Z$ is, in fact, a complete isometry.

Finally we show that the injectivity of $\theta_Z$ has a simple characterization in terms of Glimm and 2-primal ideals.

Lemma 5. Let $A$ be a unital $C^*$-algebra, and let $R \in \Id(A)$. Then $R$ is 2-primal if and only if $R \otimes_h A + A \otimes_h R \supseteq \ker \theta$.

Proof. Suppose that $R$ is not 2-primal. Then there exist orthogonal ideals $I$ and $J$ with $I, J \not\subseteq R$. If $a \in I \setminus R$ and $b \in J \setminus R$ then $a \otimes b \notin R \otimes_h A + A \otimes_h R$ but $\theta(a \otimes b) = 0$. Hence $R \otimes_h A + A \otimes_h R \not\supseteq \ker \theta$.

Conversely, suppose that $R$ is 2-primal, and that $c \in A \otimes_h A$ with $c \notin R \otimes_h A + A \otimes_h R$. Then by [6], 1.3 there exist $P, Q \in \Prim(A/R)$ such that $c \notin P \otimes_h A + A \otimes_h Q$. But $S = P \cap Q$ is primal, since $R$ is 2-primal, and $c \notin S \otimes_h A + A \otimes_h S$. This means, by Proposition 3, that $\theta(c)$ is non-zero. Hence $R \otimes_h A + A \otimes_h R \geq \ker \theta$. $\blacksquare$

Corollary 6. Let $A$ be a unital $C^*$-algebra. Then

(i) $\ker \theta = \bigcap \{ R \otimes_h A + A \otimes_h R : R \in 2\text{-Primal}(A) \}$;

(ii) $\theta_Z$ is injective if and only if every Glimm ideal of $A$ is 2-primal.

Proof. Set $I = \bigcap \{ R \otimes_h A + A \otimes_h R : R \in 2\text{-Primal}(A) \}$.

(i) It is clear from Lemma 5 that $\ker \theta \subseteq I$. On the other hand, if $c \in I$ then $\theta(c) = 0$, by Proposition 3. Thus $I = \ker \theta$.

(ii) If every Glimm ideal of $A$ is 2-primal then $I = J_A$, by Theorem 1, so $\theta_Z$ is injective. Conversely, if $G$ is a Glimm ideal of $A$ which is not 2-primal then $G \otimes_h A + A \otimes_h G \not\supseteq \ker \theta$ by Lemma 5, so $\theta_Z$ is not injective. $\blacksquare$

The condition of every Glimm ideal being 2-primal has a number of equivalent formulations. For $P, Q \in \Prim(A)$, let $P \sim Q$ if $P$ and $Q$ cannot be separated by disjoint open sets, and $P \approx Q$ if $P$ and $Q$ cannot be separated by continuous, complex functions on $\Prim(A)$. Define a graph structure on $\Prim(A)$ by saying that $P$ and $Q$ are adjacent if $P \sim Q$, and let $\Orc(A)$ be the supremum of the diameters of the connected components of $\Prim(A)$ in this graph structure (with the convention that a singleton has diameter 1). The work in [17] shows that for a unital $C^*$-algebra $A$ the following are equivalent:
(i) Orc(A) = 1;
(ii) ∼ is an equivalence relation on Prim(A);
(iii) the relations ∼ and ≈ coincide on Prim(A);
(iv) every Glimm ideal of A is 2-primal.

One of the main results of [17] is that Orc(A) = 1 if and only if ∥Da∥ = 2d(a, Z(A)) for all self-adjoint a ∈ A.

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