α-LIPSCHITZ ALGEBRAS
ON THE NONCOMMUTATIVE TORUS

NIK WEAVER

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Abstract. We define deformed, noncommutative versions of the Lipschitz algebras Lip^α(T^2) and lip^α(T^2). Deformation preserves the property that the former is isometrically isomorphic to the second dual of the latter.

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The algebra Lip(X) of Lipschitz functions on a complete metric space X plays a role in noncommutative metric theory similar to that played by the algebra C(K) in noncommutative topology. For instance, there is a robust duality between metric properties of X and algebraic properties of Lip(X) ([24]) which matches closed subsets with weak*-closed ideals etc. Furthermore, one has an abstract characterization of Lipschitz algebras in terms of derivations of abelian von Neumann algebras into abelian operator bimodules ([26]) which admits a natural extension to the noncommutative setting. For more on noncommutative metrics see [4], [5], [6], [7], [15], [17] and for more on the particular approach described above see [26], [27], [28]. The abstract commutative theory of Lipschitz algebras is considered in [1], [2], [10], [12], [19], [20], [21], [22], [23], [24], [25], [29], among other places.

For 0 < α ≤ 1 one calls a function f : X → C α-Lipschitz (or Hölder) if it is Lipschitz with respect to the original metric on X raised to the power α. The space of α-Lipschitz functions on X is denoted Lip^α(X). This concept
is of interest in connection with little Lipschitz functions. A Lipschitz function on $X$ is little if its slopes are locally null, i.e. every point has neighborhoods the restrictions of $f$ to which have arbitrarily small Lipschitz number. The space of little Lipschitz functions (respectively, little $\alpha$-Lipschitz functions) is denoted $\text{lip}(X)$ (resp. $\text{lip}^\alpha(X)$). In general, there may be no nonconstant little Lipschitz functions, but for $\alpha < 1$ little $\alpha$-Lipschitz functions always exist in abundance. These notions have long been important in harmonic analysis, and have also played a special role in the abstract theory of Lipschitz algebras, going back to the seminal paper [8] which initiated this theory.

At the moment we have no general noncommutative versions of $\alpha$-Lipschitz or little Lipschitz functions. However, we wish to show here that there are reasonable versions of both concepts in relation to the noncommutative torus ([16]). Our definitions are based on an approach to $\alpha$-Lipschitz functions on the unit circle developed in [13]. Thus, we define and study deformed, noncommutative versions $\text{Lip}_\theta^\alpha(T^2)$ and $\text{lip}_\theta^\alpha(T^2)$ of the classical algebras $\text{Lip}^\alpha(T^2)$ and $\text{lip}^\alpha(T^2)$. Among our results is the fact that for $\alpha < 1$ the space $\text{Lip}_\theta^\alpha(T^2)$ is isometrically isomorphic to the second dual of $\text{lip}_\theta^\alpha(T^2)$. This holds in the commutative case by [2].

Our main interest in this material is that it provides a class of examples of noncommutative metrics which are not differential geometric in nature. For instance, the operator bimodule in Theorem 2.3 (ii) is not a Hilbert module; also, the derivation discussed there is not an actual differentiation. Much of what is done here generalizes immediately to the setting of an arbitrary Lie group acting on a von Neumann algebra. Another class of noncommutative metrics which are not Riemannian was given in [28].

Lipschitz functions on the noncommutative torus were discussed in [26] and some of our results here generalize work done there in the $\alpha = 1$ case.

1. THE NONCOMMUTATIVE TORUS

We begin with a review of the noncommutative torus, as described in [16] (we use different notation here). Fix a real number $\theta \in [0,1)$ and define unitary operators $U, V \in B(l^2(\mathbb{Z}^2))$ by setting

$$Uv_{mn} = v_{(m+1)n} \quad \text{and} \quad Vv_{mn} = e^{2\pi i \theta m}v_{m(n+1)},$$

where $v_{mn}$ is the canonical basis of $l^2(\mathbb{Z}^2)$. Let $C_\theta(T^2)$ and $L^\infty_\theta(T^2)$ respectively be the $C^*$-algebra and von Neumann algebra generated by $U$ and $V$. In the $\theta = 0$ case the Fourier transform identifies $C_\theta(T^2)$ and $L^\infty_\theta(T^2)$ with $C(T^2)$ and $L^\infty(T^2)$,
respectively. However, for $\theta \neq 0$ these algebras are noncommutative and our “function space” notation is merely symbolic.

For $x \in L^\infty_\theta(T^2)$ and $N \geq 0$ define

$$s_N(x) = \sum_{|m|,|n| \leq N} a_{mn} U^m V^n$$

where $a_{mn} = \langle xv_{00}, v_{mn} \rangle$, and set

$$\sigma_N(x) = \frac{s_0 + \cdots + s_N}{N + 1}.$$ These are respectively the partial sums and Cesaro means of the Fourier series of $x$. (For basic material on harmonic analysis see [9], [11], or [30].)

Define unbounded self-adjoint operators $D_1, D_2$ on $l^2(\mathbb{Z}^2)$ by

$$D_1 v_{mn} = mv_{mn} \quad \text{and} \quad D_2 v_{mn} = nv_{mn}.$$ For $\theta = 0$ these correspond via the Fourier transform to $i\partial/\partial x$ and $i\partial/\partial y$. Then we have two actions $\gamma^1, \gamma^2$ of $\mathbb{R}$ by automorphisms of $L^\infty_\theta(T^2)$, given by

$$\gamma^k_t(x) = e^{-itD_k}xe^{itD_k}$$

for $k = 1, 2$. For $\theta = 0$ these correspond to translations of $L^\infty(T^2)$ in the two variables.

The following was noted in [26], and is probably well-known.

**Proposition 1.1.** (i) $\gamma^1$ and $\gamma^2$ are ultraweakly continuous actions of $\mathbb{R}$ on $L^\infty_\theta(T^2)$.

(ii) $C_\theta(T^2)$ is stable for the actions of $\gamma^1$ and $\gamma^2$, and consists of precisely those elements of $L^\infty_\theta(T^2)$ for which both actions are continuous in operator norm.

(iii) For any $x \in L^\infty_\theta(T^2)$, $s_N(x) \to x$ ultraweakly.

(iv) For any $x \in C_\theta(T^2)$, $\sigma_N(x) \to x$ in operator norm.

In [26] we defined a $\theta$-deformed version of the algebra of Lipschitz functions on $T^2$ by $\text{Lip}_\theta(T^2) = \text{dom}(\delta_1) \cap \text{dom}(\delta_2)$, where $\delta_k (k = 1, 2)$ is the generator of the flow $\gamma^k$, i.e. $\delta_k(x) = i[D_k, x]$. This is a variation on a definition in [4]. In the $\theta = 0$ case it corresponds to precisely the algebra of Lipschitz functions on $T^2$.

The following is also from [26].

**Theorem 1.2.** (i) $\text{Lip}_\theta(T^2)$ is a dual Banach space.

(ii) $\text{Lip}_\theta(T^2) \subset C_\theta(T^2)$, densely in operator norm.

(iii) For any $x \in \text{Lip}_\theta(T^2)$, $s_N(x) \to x$ in operator norm.

$\text{Lip}_\theta(T^2)$ can also be viewed in the following way. Consider $E = L^\infty_\theta(T^2) \oplus L^\infty_\theta(T^2)$ as a Hilbert $L^\infty_\theta(T^2)$-bimodule in the natural way. Then one has an unbounded derivation $\delta : L^\infty_\theta(T^2) \to E$ defined by $\delta(x) = \delta(x) \oplus \delta_2(x)$. This exhibits $\text{Lip}_\theta(T^2)$ as the domain of a natural “exterior derivative” on the noncommutative torus.
2. NONCOMMUTATIVE $\alpha$-LIPSCHITZ ALGEBRAS

We retain the notation of the previous section.

**Definition 2.1.** Let $0 < \alpha \leq 1$. Then we define $\text{Lip}_\alpha^\theta(T^2)$ to be the set of $x \in L_0^\infty(T^2)$ for which there exists a constant $C \geq 0$ such that

$$\|x - \gamma_k^t(x)\| \leq Ct^\alpha$$

for $k = 1, 2$ and all $t > 0$. We let $L^\alpha(x)$ be the least such value of $C$ and norm $\text{Lip}_\alpha^\theta(T^2)$ by

$$\|x\|_\alpha = \max(\|x\|, L^\alpha(x)),$$

which we call the Lipschitz norm. We define $\text{lip}_\alpha^\theta(T^2)$ to be the set of $x \in \text{Lip}_\alpha^\theta(T^2)$ such that

$$\frac{\|x - \gamma_k^t(x)\|}{t^\alpha} \to 0$$

for $k = 1, 2$ as $t \to 0$.

**Proposition 2.2.** (i) $\text{Lip}_\alpha^\theta(T^2)$ and $\text{lip}_\alpha^\theta(T^2)$ are involutive Banach algebras for the Lipschitz norm $\| \cdot \|_\alpha$.

(ii) For $\theta = 0$, $\text{Lip}_\alpha^\theta(T^2)$ and $\text{lip}_\alpha^\theta(T^2)$ are identified by means of the Fourier transform with the classical $\alpha$-Lipschitz and little $\alpha$-Lipschitz algebras on $T^2$, respectively.

**Proof.** (i) Checking that $\text{Lip}_\alpha^\theta(T^2)$ and $\text{lip}_\alpha^\theta(T^2)$ are involutive algebras is a straightforward calculation. For instance, if $x$ and $y$ belong to $\text{Lip}_\alpha^\theta(T^2)$ then

$$\|xy - \gamma_k^t(xy)\| \leq \|xy - x\gamma_k^t(y)\| + \|x\gamma_k^t(y) - \gamma_k^t(x))\gamma_k^t(y)\|$$

$$\leq \|x\| \|y - \gamma_k^t(y)\| + \|x - \gamma_k^t(x)\| \|\gamma_k^t(y)\|$$

$$\leq (\|x\|L^\alpha(y) + \|y\|L^\alpha(x))t^\alpha$$

$$\leq 2\|x\|_\alpha \|y\|_\alpha t^\alpha$$

shows that $xy \in \text{Lip}_\alpha^\theta(T^2)$. This also shows that $\|xy\|_\alpha \leq 2\|x\|_\alpha \|y\|_\alpha$, hence multiplication is continuous for the Lipschitz norm, although note that $\text{Lip}_\alpha^\theta(T^2)$ is not a Banach algebra in the stricter sense of satisfying $\|xy\| \leq \|x\| \|y\|$.

To see that $\text{Lip}_\alpha^\theta(T^2)$ is complete for the Lipschitz norm, let $(x_n) \subset \text{Lip}_\alpha^\theta(T^2)$ be Cauchy. It follows that $(x_n)$ is Cauchy in operator norm, hence converges in this sense to some $x \in L_0^\infty(T^2)$. For any $t > 0$ choose $n$ such that $\|x - x_n\| \leq t^\alpha$; then

$$\|x - \gamma_k^t(x)\| \leq \|x - x_n\| + \|x_n - \gamma_k^t(x_n)\| + \|\gamma_k^t(x_n - x)\|$$

$$\leq t^\alpha + Ct^\alpha + t^\alpha = (C + 2)t^\alpha$$
where $C = \sup \|x_n\|_\alpha$. This shows that $x \in \text{Lip}_\alpha^0(T^2)$. Furthermore, given $\varepsilon > 0$ choose $n$ large enough that $\|x_m - x_n\|_\alpha < \varepsilon$ for all $m > n$. Then for any $t > 0$ we can find $m > n$ so that $\|x - x_m\| \leq t^\alpha$, and then

$$\|x - x_n\| - \gamma^k_t(x - x_n)\| \leq \|x - x_m\| - \gamma^k_t(x - x_m)\| + \|x_m - x_n\| - \gamma^k_t(x_m - x_n)\| \leq 2rt^\alpha + \varepsilon t^\alpha = 3t^\alpha.$$ 

This shows that $L^\alpha(x_n - x) \to 0$, and as we already know $\|x_n - x\| \to 0$, it follows that $\|x_n - x\| = 0$. Thus, Lip$_\alpha$($T^2$) is complete for the Lipschitz norm.

For completeness of lip$_\alpha$($T^2$) let $(x_n) \subseteq$ lip$_\alpha$($T^2$) be Cauchy, so that by the above $x_n$ converges in Lipschitz norm to some $x \in$ Lip$_\alpha$($T^2$). We must show $x \in$ lip$_\alpha$($T^2$). Given $\varepsilon > 0$ choose $n$ such that $\|x_m - x_n\|_\alpha \leq \varepsilon$ for $m > n$. Then since $x_n \in$ lip$_\alpha$($T^2$) there exists $\delta > 0$ such that $t \leq \delta$ implies $\|x_n - \gamma^k_t(x_n)\| \leq \varepsilon t^\alpha$.

For any $t \leq \delta$ we can find $m > n$ so that $\|x - x_m\|_\alpha \leq t^\alpha$, and then

$$\|x - \gamma^k_t(x)\| \leq \|x - x_m\| + \|x_m - \gamma^k_t(x_n)\| + \|x_n - \gamma^k_t(x_m - x_n)\| \leq \varepsilon t^\alpha + \varepsilon t^\alpha + \varepsilon t^\alpha = 4\varepsilon t^\alpha.$$ 

This shows that $\|x - \gamma^k_t(x)\|/t^\alpha \to 0$ as $t \to 0$, so $x \in$ lip$_\alpha$($T^2$).

(ii) In the $\theta = 0$ case, Lip$_\alpha$($T^2$) is identified with the set of functions $f \in L^\infty(T^2)$ which satisfy

$$\|f - \gamma^k_t(f)\|_\infty \leq Ct^\alpha$$

for $k = 1, 2$ and all $t$. That is, these are the functions which satisfy

$$\sup \{|f(x, y) - f(x + t, y)|, |f(x, y) - f(x, y + t)| : (x, y) \in T^2\} \leq Ct^\alpha$$

for all $t > 0$. This condition is automatically satisfied by any $\alpha$-Lipschitz function on $T^2$; conversely, for any function $f$ which satisfies this condition we have

$$|f(x_1, y_2) - f(x_2, y_2)| \leq |f(x_1, y_1) - f(x_2, y_1)| + |f(x_2, y_1) + f(x_2, y_2)| \leq C(d^\alpha(x_1, x_2) + d^\alpha(y_1, y_2)) \leq 2Cd^\alpha((x_1, y_1), (x_2, y_2))$$

where $d$ denotes the ordinary Euclidean distance on $T$ and $T^2$, hence $f$ is $\alpha$-Lipschitz. Thus, for $\theta = 0$ we may identify Lip$_\alpha$($T^2$) with the $\alpha$-Lipschitz functions on $T^2$.

To see that lip$_\alpha$($T^2$) is identified with the little $\alpha$-Lipschitz functions, suppose that $t \leq \delta$ implies

$$|f(x, y) - f(x + t, y)|, |f(x, y) - f(x, y + t)| \leq \varepsilon t^\alpha$$
for all \((x, y) \in \mathbb{T}^2\); then \(d((x_1, y_1), (x_2, y_2)) \leq \delta\) implies

\[
|f(x_1, y_1) - f(x_2, y_2)| \leq |f(x_1, y_1) - f(x_2, y_1)| + |f(x_2, y_1) - f(x_2, y_2)| \\
\leq \varepsilon d^\alpha((x_1, x_2)) + \varepsilon d^\alpha((y_1, y_2)) \\
\leq 2\varepsilon d^\alpha((x_1, y_1), (x_2, y_2)).
\]

Conversely, if \(f\) is a little \(\alpha\)-Lipschitz function then for every \(\varepsilon > 0\) we can find \(\delta > 0\) such that for all \((x_1, y_1), (x_2, y_2) \in \mathbb{T}^2, d((x_1, y_1), (x_2, y_2)) \leq \delta\) implies

\[
|f(x_1, y_1) - f(x_2, y_2)| \leq \varepsilon d^\alpha((x_1, y_1), (x_2, y_2)).
\]

(Each point has a neighborhood in which this is true, and then by compactness we can take \(\delta\) to be the Lebesgue number of the resulting covering of \(\mathbb{T}^2\).) In particular,

\[
|f(x, y) - f(x + t, y)|, |f(x, y) - f(x, y + t)| \leq \varepsilon t^\alpha
\]

for \(t \leq \delta\), i.e. \(\|f - \gamma^k_t(f)\|_\infty \leq \varepsilon t^\alpha\) for \(t \leq \delta\).

We now wish to demonstrate that the definitions given in this paper match up with our previous work, specifically, that \(\text{Lip}_1^\theta(\mathbb{T}^2)\) equals the Lipschitz algebra \(\text{Lip}_0(\mathbb{T}^2)\) defined in [26] (and above in Section 1), and that each \(\text{Lip}_\alpha^\theta(\mathbb{T}^2)\) is a Lipschitz algebra in the sense of [26], i.e. is the domain of a von Neumann algebra derivation. For the latter, let

\[
E = \bigoplus_{t > 0}^{\infty}(L_0^\infty(\mathbb{T}^2) \oplus L_0^\infty(\mathbb{T}^2))
\]

be the \(l^\infty\) direct sum of von Neumann algebras. It is a von Neumann algebra, and it is also a dual operator \(L_0^\infty(\mathbb{T}^2)\)-bimodule with left action given by the diagonal embedding of \(L_0^\infty(\mathbb{T}^2)\) in \(E\) and right action given by the embedding

\[
x \mapsto \bigoplus_{t > 0}(\gamma^1_t(x) \oplus \gamma^2_t(x)).
\]

Define an unbounded map \(\delta : L_0^\infty(\mathbb{T}^2) \to E\) with domain \(\text{Lip}_0^\theta(\mathbb{T}^2)\) by \(\delta = \bigoplus(\delta^1_t \oplus \delta^2_t)\) with

\[
\delta^k_t(x) = \frac{x - \gamma^k_t(x)}{t^\alpha}.
\]

Notice that indeed \(\delta(x) \in E\) if \(x \in \text{Lip}_0^\theta(\mathbb{T}^2)\) since \(\sup_{t, k} \|\delta^k_t(x)\| = L^\alpha(x) < \infty\).
Theorem 2.3. (i) Lip$_{\phi}^1(\mathbb{T}^2) = \text{Lip}_\phi(\mathbb{T}^2)$ as sets.
(ii) Lip$_{\phi}^\alpha(\mathbb{T}^2)$ is the domain of an unbounded von Neumann algebra derivation with weak*-closed graph.

Proof. (i) Let $x \in L^\infty_\phi(\mathbb{T}^2)$. Then $x \in$ Lip$_{\phi}^1(\mathbb{T}^2)$ if and only if
\[
\sup_{t>0} \left\{ \frac{\|x - \gamma_1^t(x)\|}{t}, \frac{\|x - \gamma_2^t(x)\|}{t} \right\} < \infty,
\]
while $x \in$ Lip$_\phi(\mathbb{T}^2)$ if and only if it belongs to the domains of the generators of $\gamma^1$ and $\gamma^2$. According to [3], Proposition 3.1.23, these two conditions are equivalent. (Note however that the norm $\|x\|_1$ defined here on Lip$_{\phi}^1(\mathbb{T}^2)$ does not agree with the norm $\|x\|_L$ given in [26] on Lip$_\phi(\mathbb{T}^2)$, although the two are equivalent.)

(ii) An easy calculation shows that the map $\delta$ defined before the theorem is linear and self-adjoint and satisfies the derivation identity (with respect to the bimodule structure described above), and its domain is Lip$_{\phi}^\alpha(\mathbb{T}^2)$ by definition. To check ultraweak closure of the graph of $\delta$, suppose $x \delta \oplus \delta(x)$ is a bounded net in the graph which converges ultraweakly to some element $x \oplus y \in L^\infty_\phi(\mathbb{T}^2) \oplus E$. (By the Krein-Smulian theorem, it is sufficient to consider bounded nets.) Write $y = \bigoplus (y_1^k \oplus y_2^k)$. Then for each $t > 0$ we have
\[
y_t^k = \lim_{\lambda} \delta_t^k(x_\lambda) = \lim_{\lambda} \frac{(x_\lambda - \gamma_t^k(x_\lambda))}{t^\alpha} = \frac{(x - \gamma_t^k(x))}{t^\alpha}.
\]
$(k = 1, 2)$. As this holds for all $t$ and
\[
\sup_{t>0} \{ \|y_1^\lambda\|, \|y_2^\lambda\| \} = \|y\| < \infty,
\]
it follows that $x \in$ Lip$_{\phi}^\alpha(\mathbb{T}^2)$ and $\delta(x) = y$. Thus, the graph of $\delta$ is weak*-closed.}

Corollary 2.4. Lip$_{\phi}^\alpha(\mathbb{T}^2)$ is a dual Banach space.

Proof. For any $x \in$ Lip$_{\phi}^\alpha(\mathbb{T}^2)$ we have
\[
\|x\|_\alpha = \max(\|x\|, L^\alpha(x)) = \max(\|x\|, \sup_{t,k} \|x - \gamma_t^k(x)\|) = \max(\|x\|, \|\delta(x)\|) = \|x \oplus \delta(x)\|.
\]
Thus, Lip$_{\phi}^\alpha(\mathbb{T}^2)$ is linearly isometric to the graph of $\delta$. But the latter is an ultraweakly closed subspace of $L^\infty_\phi(\mathbb{T}^2) \oplus E$, hence a dual Banach space.

In consequence of this corollary Lip$_{\phi}^\alpha(\mathbb{T}^2)$ has a weak*-topology. In general it is distinct from the restriction of the ultraweak topology on $L^\infty_\phi(\mathbb{T}^2)$, which of course is itself a weak*-topology. To avoid confusion we shall always refer to the latter topology with the term “ultraweak” rather than “weak*”.

\[\alpha-Lipschitz algebras on the noncommutative torus\]
3. RELATIONS BETWEEN $\alpha$-LIPSCHITZ SPACES

In this section we investigate the various containments that obtain among the big and little $\alpha$-Lipschitz spaces, the algebra of polynomials in $U$ and $V$, $C_0(T^2)$, and $L^\infty_0(T^2)$. Corresponding statements for classical Lipschitz algebras were proved in [13] and [14] (for the unit circle) and [2] and [25] (for any compact metric space).

Our first lemma provides basic tools that we will use repeatedly. It is a noncommutative version of basic facts from harmonic analysis and was proved in [26]. Let $K_N$ be the Fejér kernel,

$$K_N(t) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) e^{int} = \frac{1}{N+1} \left(\frac{\sin((N+1)t/2)}{\sin(t/2)}\right)^2.$$

It has the properties that:

1. $K_N(t) \geq 0$ for all $t \in [-\pi, \pi]$;
2. $\pi \int_{-\pi}^{\pi} K_N(t) \, dt = 1$; and
3. for any $\varepsilon > 0$, $\int_{|t| \geq \varepsilon} K_N(t) \, dt \to 0$ as $N \to \infty$.

**Lemma 3.1.** Let $x \in L^\infty_0(T^2)$. Then

$$\sigma_N(x) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \gamma_1^1(x) \gamma_1^2(x) K_N(s) K_N(t) \, ds \, dt$$

and

$$x - \sigma_N(x) = \int_{-\pi}^{\pi} \left( x - \gamma_1^1(x) \right) K_N(s) \, ds$$

$$+ \int_{-\pi}^{\pi} \gamma_1^1 \left( \int_{-\pi}^{\pi} \left( x - \gamma_1^2(x) \right) K_N(t) \, dt \right) K_N(s) \, ds,$$

where all operator integrals are taken in the ultraweak sense.

**Lemma 3.2.** For any $\varepsilon > 0$ there exists $N$ large enough that $\|x - \sigma_n(x)\| \leq \varepsilon$ for all $x \in \text{ball}(L^\infty_0(T^2))$ and $n \geq N$.

**Proof.** Consider the second formula in Lemma 3.1. For any $x \in \text{ball}(L^\infty_0(T^2))$ we have

$$\left\| \int_{-\pi}^{\pi} \left( x - \gamma_1^1(x) \right) K_N(s) \, ds \right\| \leq \int_{-\pi}^{\pi} \|x - \gamma_1^1(x)\| K_N(s) \, ds \leq \int_{-\pi}^{\pi} |s|^{\alpha} K_N(s) \, ds$$
and

\[
\left\| \int_{-\pi}^{\pi} \gamma_k \left( \int_{-\pi}^{\pi} (x - \gamma_k^2(t))K_N(t) \, dt \right) K_N(s) \, ds \right\| \leq \int_{-\pi}^{\pi} \left\| \int_{-\pi}^{\pi} (x - \gamma_k^2(t))K_N(t) \, dt \right\| K_N(s) \, ds = \left\| \int_{-\pi}^{\pi} (x - \gamma_k^2(t))K_N(t) \, dt \right\| \leq \int_{-\pi}^{\pi} |t|^\alpha K_N(t) \, dt.
\]

Since the function \( t \mapsto |t|^\alpha \) is continuous on \([-\pi, \pi]\) and vanishes at \( t = 0 \), it follows that \( \int_{-\pi}^{\pi} |t|^\alpha K_N(t) \, dt \to 0 \) as \( N \to \infty \). The second formula given in Lemma 3.1 then implies that for any \( \varepsilon > 0 \) we can choose \( N \) large enough that \( \|x - \sigma_n(x)\| \leq \varepsilon \) for all \( x \in \text{ball}(\text{Lip}_\alpha^\theta(T^2)) \) and \( n \geq N \).

The next lemma was proved for \( \text{Lip}_\theta(T^2) \) in [26]. The proof for \( \text{Lip}_\alpha^\theta(T^2) \) given here is essentially the same. The result in [26] can also be generalized in a different direction, in the broad setting of compact groups acting on \( C^* \)-algebras ([18]).

**Lemma 3.3.** On the unit ball of \( \text{Lip}_\alpha^\theta(T^2) \) the weak*-topology agrees with the operator norm topology.

**Proof.** Both topologies are Hausdorff on \( \text{ball}(\text{Lip}_\alpha^\theta(T^2)) \), and the weak*-topology is compact. Furthermore, the weak*-topology is weaker than the operator norm topology; for if \( x, x_\lambda \in \text{ball}(\text{Lip}_\theta^\alpha(T^2)) \) and \( x_\lambda \to x \) in operator norm, then in the notation of Section 2 we have \( \delta_k^t(x_\lambda) \to \delta_k^t(x) \) in operator norm for each \( k = 1, 2 \) and \( t > 0 \), hence (by boundedness) \( x_\lambda \oplus \delta(x_\lambda) \to x \oplus \delta(x) \) ultraweakly, i.e. \( x_\lambda \to x \) weak*. Thus, it will suffice to show that the unit ball of \( \text{Lip}_\theta^\alpha(T^2) \) is compact in operator norm.

To see this let \( (x_k) \subset \text{ball}(\text{Lip}_\theta^\alpha(T^2)) \); we will find a subsequence which converges in operator norm. (Since the topology is metric, we may use sequences rather than nets.) Recalling the representation on \( l^2(\mathbb{Z}^2) \) described in Section 1, let \( a_{km}^k = \langle x_k v_{00}, v_{mn} \rangle \) be the Fourier coefficients of \( x_k \). Since \( ||x_k|| \leq ||x_k||_\alpha \leq 1 \) it follows that \( |a_{km}^k| \leq 1 \) for all \( k, m, n \) and so we may choose a subsequence \( x_{k_\lambda} \) such that the coefficients \( (a_{km}^{k_\lambda}) \) converge for each index \( m, n \).
Let \( x \) be an ultraweak cluster point of \( (x_{jk}) \) and let \( a_{mn} \) be its Fourier coefficients; then \( a_{mn} \) is a cluster point of \( (a_{mn}^k) \) for each \( m, n \). But the latter sequences have been chosen to converge, so we must have \( a_{mn}^k \to a_{mn} \) for each \( m, n \). We will show that \( m, n \) for all \( t > 0 \). Similar statements hold for \( \|x \| \) to \( \|x \| \) and \( \|x \| \) weak to \( \|x \| \) in operator norm.

Given \( \varepsilon > 0 \), by Lemma 3.2 we can choose \( N \) so that

\[
\|x \| - \|x \| + \|x \| - \|x \| \| \leq \varepsilon
\]

for all \( k \). By the last paragraph we can then choose \( M \) so that \( k \geq M \) implies

\[
|a_{mn}^k - a_{mn}^i| \leq \frac{\varepsilon}{(2N + 1)^2}
\]

for all \( |m|, |n| \leq N \). This implies that \( \|s_n(x) - s_n(x_{jk})\| \leq \varepsilon \) for \( n \leq N \) hence \( \|\sigma_N(x) - \sigma_N(x_{jk})\| \leq \varepsilon \). We conclude that

\[
\|x \| - x_{jk} \| \leq \|x \| + \|x \| - \|x \| - \|x \| - x_{jk} \| \leq 3\varepsilon
\]

for \( k \geq M \). So \( x_{jk} \to x \) in operator norm, as desired. \( \Box \)

**Lemma 3.4.** (i) Any polynomial formed from \( U \) and \( V \) and their adjoints belongs to \( \text{Lip}_0^\alpha(T^2) \) for all \( \alpha \leq 1 \) and to \( \text{lip}_0^\alpha(T^2) \) for all \( \alpha < 1 \).

(ii) Let \( x \in \text{Lip}_0^\alpha(T^2) \) (\( \alpha \leq 1 \)). Then \( \|\sigma_N(x)\|_\alpha \leq \|x\|_\alpha \) for all \( N \) and \( \sigma_N(x) \to x \) weak*.

(iii) Let \( x \in \text{lip}_0^\alpha(T^2) \) (\( \alpha < 1 \)). Then \( \sigma_N(x) \to x \) in Lipschitz norm.

**Proof.** (i) The operators \( U \) and \( V \) were defined in Section 1. Now \( U \) belongs to \( \text{lip}_0^\alpha(T^2) \) for \( \alpha < 1 \) since \( \gamma_t^\alpha(U) = U \) and

\[
\frac{\|U - \gamma_t^\alpha(U)\|}{t^\alpha} = \frac{\|U - e^{-it}U\|}{t^\alpha} = \frac{|1 - e^{-it}|}{t^\alpha} \to 0
\]

as \( t \to 0 \). For \( \alpha = 1 \) we still have \( U \in \text{Lip}_0^1(T^2) \) since \( |1 - e^{-it}|/t \) is bounded for \( t > 0 \). Similar statements hold for \( V \), and so the polynomials formed from \( U \) and \( V \) and their adjoints belong to \( \text{lip}_0^\alpha(T^2) \) for \( \alpha < 1 \) and to \( \text{Lip}_0^\alpha(T^2) \) for \( \alpha \leq 1 \) by Proposition 2.2 (i).

(ii) First of all, \( \sigma_N(x) \in \text{Lip}_0^\alpha(T^2) \) by part (i). The sequence is bounded because, using the first formula in Lemma 3.1,

\[
\|\sigma_N(x)\|_\alpha = \left\| \int \int \gamma_s^\alpha(\gamma_t^\alpha(x))K_N(s)K_N(t) \|x\|_\alpha dt \right\|_\alpha \leq \int \int \|x\|_\alpha K_N(s)K_N(t) \|x\|_\alpha dt = \|x\|_\alpha.
\]
Weak*-convergence then follows from Lemmas 3.2 and 3.3.

(iii) We have \( \sigma_N(x) \in \text{lip}_{\theta}^\alpha(T^2) \) by part (i). Given \( \varepsilon > 0 \), find \( \delta > 0 \) such that \( t \leq \delta \) implies \( \|x - \gamma^k_t(x)\| \leq \varepsilon t^\alpha \). Then choose \( N \) large enough that \( n \geq N \) implies

\[
\int_{|s| \geq \delta} K_n(s) \, ds \leq \frac{\varepsilon \delta^\alpha}{\|x\|}
\]

We are going to estimate \( \|(x - \sigma_n(x)) - \gamma^k_t(x - \sigma_n(x))\| \) (hence \( L^\alpha(x - \sigma_n(x)) \)) for \( n \geq N \) by using the second formula in Lemma 3.1.

For \( t \leq \delta \) and \( n \geq N \), we have

\[
\left\| \int ((x - \gamma^1_s(x)) - \gamma^k_t(x - \gamma^1_s(x))) K_n(s) \, ds \right\| 
= \left\| \int ((x - \gamma^k_t(x)) - \gamma^1_s(x - \gamma^k_t(x))) K_n(s) \, ds \right\|
\leq \int (\|x - \gamma^k_t(x)\| + \|\gamma^1_s(x - \gamma^k_t(x))\|) K_n(s) \, ds \leq 2\varepsilon t^\alpha.
\]

For \( t \geq \delta \), our choice of \( N \) implies that

\[
\left\| \int_{|s| \geq \delta} ((x - \gamma^1_s(x)) - \gamma^k_t(x - \gamma^1_s(x))) K_n(s) \, ds \right\| \leq \int_{|s| \geq \delta} 4\|x\| K_n(s) \, ds \leq 4\varepsilon \delta^\alpha \leq 4\varepsilon t^\alpha
\]

for \( n \geq N \), while

\[
\left\| \int_{|s| \leq \delta} ((x - \gamma^1_s(x)) - \gamma^k_t(x - \gamma^1_s(x))) K_n(s) \, ds \right\|
\leq \int_{|s| \leq \delta} (\|x - \gamma^1_s(x)\| + \|\gamma^k_t(x - \gamma^1_s(x))\|) K_n(s) \, ds
\leq \int_{|s| \leq \delta} 2\varepsilon |s|^\alpha K_n(s) \, ds \leq 2\varepsilon \delta^\alpha \leq 2\varepsilon t^\alpha
\]

for \( n \geq N \). Thus, for any \( t > 0 \) we have a bound of \( 6\varepsilon t^\alpha \) on the first integral in the second formula in Lemma 3.1 as applied to

\[
\|(x - \sigma_n(x)) - \gamma^k_t(x - \sigma_n(x))\|;
\]

the second integral is bounded similarly. We conclude that \( L^\alpha(x - \sigma_N(x)) \rightarrow 0 \), and as we already know that \( \|x - \sigma_N(x)\| \rightarrow 0 \) by Lemma 3.2, it follows that \( \|x - \sigma_N(x)\|_{\alpha} \rightarrow 0 \).
Theorem 3.5. (i) $\text{lip}_1^1(\mathbb{T}^2) = \mathbb{C}$.

(ii) The space of polynomials formed from $U$ and $V$ and their adjoints is Lipschitz norm dense in $\text{lip}_0^0(\mathbb{T}^2)$ for $\alpha < 1$ and weak*-dense in $\text{Lip}_0^0(\mathbb{T}^2)$ for $\alpha \leq 1$.

(iii) $\text{Lip}_0^0(\mathbb{T}^2) \subset C_0(\mathbb{T}^2)$ for all $\alpha \leq 1$. If $\alpha < 1$ then $\text{lip}_0^0(\mathbb{T}^2)$ is operator norm (ultraweakly) dense in $C_0(\mathbb{T}^2)$ ($L_0^\infty(\mathbb{T}^2)$), and if $\alpha \leq 1$ then $\text{Lip}_0^0(\mathbb{T}^2)$ is operator norm (ultraweakly) dense in $C_0(\mathbb{T}^2)$ ($L_0^\infty(\mathbb{T}^2)$).

(iv) For $\alpha < \beta \leq 1$ we have $\text{Lip}_0^\beta(\mathbb{T}^2) \subset \text{lip}_0^\alpha(\mathbb{T}^2)$, densely in Lipschitz norm.

Proof. (i) It is clear that $\text{lip}_1^1(\mathbb{T}^2)$ contains the constants. Conversely, for any $x \in \text{Lip}_1^1(\mathbb{T}^2)$ we have

\[
\frac{(x - \gamma_k^t(x))}{t} \to i[D_k, x]
\]

ultraweakly. It follows that $x \in \text{lip}_1^1(\mathbb{T}^2)$, i.e. $\|x - \gamma_k^t(x)/t \to 0$, only if $[D_1, x] = [D_2, x] = 0$. But then

\[0 = \langle [D_1, x]|v_{00}, v_{mn} \rangle = m(xv_{00}, v_{mn})\]

implies that the Fourier coefficient $a_{mn}$ vanishes for $m \neq 0$, and similarly $a_{mn}$ vanishes for $n \neq 0$. Thus the Fourier series of $x$ consists of simply a constant term, and convergence of Fourier series (Lemma 3.4 (ii)) implies that $x$ is a constant.

(ii) Containment was proved in Lemma 3.4 (i), and density follows from Lemma 3.4 (ii) and (iii).

(iii) For any $x \in \text{Lip}_0^\alpha(\mathbb{T}^2)$ we have $\|x - \gamma_k^t(x)\| \leq L^\alpha(x)t^\alpha \to 0$ as $t \to 0$, so $x \in C_0(\mathbb{T}^2)$ by Proposition 1.1 (ii). This shows that $\text{Lip}_0^\alpha(\mathbb{T}^2) \subset C_0(\mathbb{T}^2)$. The density assertions follow from Lemma 3.4 (i).

(iv) Suppose $x \in \text{Lip}_0^\beta(\mathbb{T}^2)$. Then

\[\|x - \gamma_k^t(x)\| \leq L^\beta(x)t^\beta = (L^\beta(x)t^{\beta-\alpha})t^\alpha.\]

As $t^{\beta-\alpha} \to 0$ as $t \to 0$, this shows that $x \in \text{lip}_0^\alpha(\mathbb{T}^2)$. Density follows from Lemma 3.4 (i) and (iii).
4. DOUBLE DUALITY

We now aim to prove for any $\alpha < 1$ that Lip$_{\theta}^{\alpha}(T^2)$ is naturally isometrically isomorphic to the double dual of lip$_{\theta}^{\alpha}(T^2)$. This was established for $\alpha$-Lipschitz functions on the unit circle in [8] and later generalized to a large class of spaces by many people, most notably in [2] and [10] (see also [29]).

For $n \in \mathbb{N}$ define
\[ A_n = C\left(\left[\frac{\pi}{n+1}, \frac{\pi}{n}\right], C_{\theta}(T^2)\right), \]
the $C^*$-algebra of continuous functions from the interval $[\pi/(n+1), \pi/n]$ into $C_{\theta}(T^2)$. By Proposition 1.1 (ii), for any $x \in C_{\theta}(T^2)$ the function
\[ \delta_n^k : t \mapsto \frac{(x - \gamma_k(t))}{t^\alpha} \]
(with domain $[\pi/(n+1), \pi/n]$) belongs to $A_n$, and if $x \in$ Lip$_{\theta}^{\alpha}(T^2)$ then these functions have uniformly bounded norms. Thus, we have a map
\[ \delta : \text{Lip}_{\theta}^{\alpha}(T^2) \to \bigoplus_n C_{\theta}(T^2) \]
into the $\ell^\infty$ direct sum, defined by $\delta = \bigoplus (\delta_1 \oplus \delta_2)$. Note that $x \in$ lip$_{\theta}^{\alpha}(T^2)$ precisely if $\|\delta_n^k(x)\| \to 0$ for $k = 1, 2$ as $n \to \infty$, so that $\delta$ takeslip$_{\theta}^{\alpha}(T^2)$ into the $c_0$ direct sum $\bigoplus \delta^0(A_n \oplus A_n)$.

Now define
\[ A = C_{\theta}(T^2) \oplus \bigoplus_n C_{\theta}(T^2) \]
and
\[ B = C_{\theta}(T^2) \oplus \bigoplus_n C_{\theta}(T^2) \]
(the $l^\infty$ and $c_0$ direct sums, respectively). The map $\Gamma : x \mapsto x \oplus \delta(x)$ defines an isometric embedding of Lip$_{\theta}^{\alpha}(T^2)$ in $A$ and of lip$_{\theta}^{\alpha}(T^2)$ in $B \subset A$.

**Theorem 4.1.** Let $0 < \alpha < 1$. Then Lip$_{\theta}^{\alpha}(T^2) \cong$ lip$_{\theta}^{\alpha}(T^2)^{**}$.

**Proof.** We already know from Corollary 2.4 that Lip$_{\theta}^{\alpha}(T^2)$ is a dual space. We begin by defining a map from the dual of lip$_{\theta}^{\alpha}(T^2)$ into the predual of Lip$_{\theta}^{\alpha}(T^2)$. Given a bounded linear functional $f \in$ lip$_{\theta}^{\alpha}(T^2)^*$, we can extend it to a bounded linear functional $F \in B^*$ via the embedding $\Gamma$ of lip$_{\theta}^{\alpha}(T^2)$ in $B$. Since $B$ is a $c_0$ direct sum its dual space is an $l^1$ direct sum of the dual summands, i.e.
\[ B^* = C_{\theta}(T^2)^* \oplus \bigoplus_n (A_n^* \oplus A_n^*). \]
Therefore $F$ has a natural action on $\mathcal{A}$, i.e. we may consider $F \in \mathcal{A}^*$, hence $F \circ \Gamma \in \text{Lip}_0^0(\mathbb{T}^2)^*$. We now must show that $F \circ \Gamma$ is weak*-continuous on Lip$_0^0(\mathbb{T}^2)$.

It will suffice to show that $F \circ \Gamma$ is weak*-continuous on the unit ball of Lip$_0^0(\mathbb{T}^2)$. We will apply Lemma 3.3. Thus, let $x, x_m \in \text{ball}(\text{Lip}_0^0(\mathbb{T}^2))$ and suppose $x_m \to x$ in operator norm. Let $\varepsilon > 0$. Writing

$$F = F_0 \oplus \bigoplus_n (F_n^1 \oplus F_n^2) ,$$

we may choose $N$ large enough that $\sum_{n>N} \|F_n^k\| \leq \varepsilon$ for $k = 1, 2$. Also, from the definition of $\delta_n^k$ we have $\delta_n^k(x_m) \to \delta_n^k(x)$ in $\mathcal{A}_n$ for each $k$ and $n$, so we may then choose $M$ large enough that $m \geq M$ implies

$$\|\delta_n^k(x_m) - \delta_n^k(x)\| \leq \frac{\varepsilon}{N\|F_n^k\|}$$

for $k = 1, 2$ and all $n \leq N$. We may also take $M$ large enough that $\|x_m - x\| \leq \varepsilon/\|F_0\|$ for $m \geq M$. It follows that $m \geq M$ implies

$$|F(\Gamma(x_m)) - F(\Gamma(x))| \leq |F_0(x_m) - F_0(x)| + \sum_{n,k} |F_n^k(\delta_n^k(x_m)) - F_n^k(\delta_n^k(x))|$$

$$\leq \|F_0\|\|x_m - x\| + \sum_{n \leq N,k} \|F_n^k\| \|\delta_n^k(x_m) - \delta_n^k(x)\|$$

$$+ \sum_{n>N,k} \|F_n^k\| \|\delta_n^k(x_m) - \delta_n^k(x)\|$$

$$\leq \varepsilon + 2N \left( \frac{\varepsilon}{N} \right) + 4\varepsilon = 7\varepsilon .$$

We conclude that $F(\Gamma(x_m)) \to F(\Gamma(x))$, and this completes the proof that $F \circ \Gamma$ is weak*-continuous on Lip$_0^0(\mathbb{T}^2)$.

We have seen that every bounded linear functional on lip$_0^0(\mathbb{T}^2)$ extends to a weak*-continuous linear functional on Lip$_0^0(\mathbb{T}^2)$. The extension is unique by weak*-density of lip$_0^0(\mathbb{T}^2)$ in Lip$_0^0(\mathbb{T}^2)$ (Theorem 3.5 (ii)). Thus we may define a map $T : \text{lip}_0^0(\mathbb{T}^2)^* \to \text{Lip}_0^0(\mathbb{T}^2)^*$ by setting $Tf = F \circ \Gamma$. This map is obviously $1$-1, and it is onto since every weak*-continuous linear functional on Lip$_0^0(\mathbb{T}^2)$ restricts to a bounded linear functional on lip$_0^0(\mathbb{T}^2)$, of which it is then an extension. Also it is clear that $\|Tf\| \geq \|f\|$, since $Tf$ is an extension of $f$.

To complete the proof that lip$_0^0(\mathbb{T}^2)^* \cong \text{Lip}_0^0(\mathbb{T}^2)^*$, we must show that $\|Tf\| \leq \|f\|$ for any $f \in \text{lip}_0^0(\mathbb{T}^2)^*$. To see this let $x \in \text{Lip}_0^0(\mathbb{T}^2)$. Then for each $N$, $\sigma_N(x) \in \text{lip}_0^0(\mathbb{T}^2)$ by Lemma 3.4 (i) and $\|\sigma_N(x)\| \leq \|x\|$ by Lemma 3.4 (ii), so $|f(\sigma_N(x))| \leq \|f\| \|x\|$. But $f(\sigma_N(x)) \to (Tf)(x)$ by weak*-continuity of $Tf$ and Lemma 3.4 (ii), so we conclude that $|(Tf)(x)| \leq \|f\| \|x\|$ for all $x \in \text{Lip}_0^0(\mathbb{T}^2)$. Thus $\|Tf\| \leq \|f\|$. \qed
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NIK WEAVER  
Department of Mathematics  
UCLA  
Los Angeles, CA 90024  
U.S.A.  
E-mail: nweaver@math.ucla.edu

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