J. OPERATOR THEORY
39(1998), 151–176

CROSSED PRODUCTS BY DUAL COACTIONS OF GROUPS AND HOMOGENEOUS SPACES

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Communicated by Norberto Salinas

Abstract. Mansfield showed how to induce representations of crossed products of C^*-algebras by coactions from crossed products by quotient groups and proved an imprimitivity theorem characterising these induced representations. We give an alternative construction of his bimodule in the case of dual coactions, based on the symmetric imprimitivity theorem of the third author; this provides a more workable way of inducing representations of crossed products of C^*-algebras by dual coactions. The construction works for homogeneous spaces as well as quotient groups, and we prove an imprimitivity theorem for these induced representations.

Keywords: *C^*-algebra, coaction, crossed product, imprimitivity, homogeneous space.*

AMS Subject Classification: Primary 46L55; Secondary 22D25.

Coactions of groups on C^*-algebras, and their crossed products, were introduced to make duality arguments available for the study of dynamical systems involving actions of nonabelian groups. For these to be effective, one needs to understand the representation theory of crossed products by coactions. The most powerful tool we have was provided by Mansfield ([13]): he showed how to induce representations from crossed products by quotient groups, and proved an imprimitivity theorem which characterises these induced representations. Unfortunately, Mansfield's construction is complicated and technical. The Hilbert bimodule with which he defines induced representations is difficult to manipulate, and one is tempted to seek other realisations of this bimodule and the induced representations. Here we show that, at least for the dual coactions arising in the study of ordinary dynamical systems, there is an alternative bimodule built along more conventional lines from spaces of continuous functions with values in C^*-algebras. This bimodule will be easier to work with, and will allow us to induce representations from quotient homogeneous spaces as well as quotient groups.

The core of our construction is a special case of the symmetric imprimitivity theorem of [16]. Suppose α is an action of a locally compact group G on a C^*-algebra A. For each closed subgroup H of G, there is a diagonal action $\alpha \otimes \tau$ of G on $A \otimes C_0(G/H)$: if we identify $A \otimes C_0(G/H)$ with $C_0(G/H, A)$ in the usual way, then $(\alpha\otimes\tau)_t(f)(sH) = \alpha_t(f(t^{-1}sH))$. We show in Section 1 that there is a natural Morita equivalence between an iterated crossed product $(C_0(G, A)\times_{\alpha\otimes\tau} G)\times H$ and the imprimitivity algebra $C_0(G/H, A) \times_{\alpha\otimes\tau} G$ of Green ([7]). If H is normal, this imprimitivity algebra can be identified with the crossed product $(A\times_\alpha G)\times_{\hat\alpha|} G/H$ by the restriction of the dual coaction, and the iterated crossed product with $((A\times_\alpha G)\times_{\widehat{\alpha}} G)\times_{\widehat{\widehat{\alpha}}|} H$; the existence of our Morita equivalence is therefore predicted by Mansfield's imprimitivity theorem, although his construction gives no hint that the bimodule can be realised as a completion of $C_{\mathrm{c}}(G \times G, A)$. In Section 2, we shall discuss these isomorphisms in detail, and show how our bimodule can be used to induce representations from G/H to G even when H is not normal.

Although it is not clear in general how to define coactions of homogeneous spaces, let alone their crossed products (see the discussion at the start of Section 2), there is considerable evidence that our inducing process is a step in the right direction. There is an appropriate imprimitivity theorem (Proposition 2.11), the induction process interacts with Green induction and duality as one would expect from the results of [3] and [9] (Theorem 3.1 and Corollary 3.3), and our bimodule is isomorphic to Mansfield's when the subgroup H is normal and amenable (Theorem 4.1).

When the subgroup H is normal but not amenable, the relationship between our bimodule and the extension of Mansfield's in Section 3 of [8] becomes quite subtle. There are two candidates for the crossed product $(A \times G) \times G/H$: the spatial version on $\mathcal{H} \otimes L^2(G)$ used in [8], and the imprimitivity algebra $C_0(G/H, A) \times G$. We believe that one can usefully view the former as a reduced crossed product by the homogeneous space, and the latter as a full crossed product. We discuss this in detail in Section 2. However, that the two can be different has an interesting consequence: the bimodule used in [8] can be a proper quotient of the one we construct in Section 1. Thus for nonamenable subgroups, our Morita equivalence is analogous to Green's equivalence of $A \times H$ and $C_0(G/H, A) \times G$, whereas Theorem 3.3 of [8] is analogous to that of the reduced crossed products $A \times_{\mathrm{r}} H$ and $C_0(G/H, A) \times_{\mathrm{r}} G$.

While we are discussing crossed products by homogeneous spaces, it is worth pointing out that for any coaction (B, G, δ) and any closed subgroup H, the spatially defined algebra $B \times G/H$ is Morita equivalent to $(B \times_\delta G) \times_{\widehat{\delta},\mathrm{r}} H$; however, this equivalence is obtained as a composition of other equivalences, and is not obviously implemented by any one concretely defined bimodule. We discuss this weak version of Mansfield's Imprimitivity Theorem in an appendix.

PRELIMINARIES

Let G be a locally compact group; we always use left Haar measure on G. We denote by λ the left regular representation of G on $L^2(G)$, and by M the representation of $C_0(G)$ by multiplication operators on $L^2(G)$. We extend representations and nondegenerate homomorphisms to multiplier algebras without comment or change of notation; thus, for example, M also denotes the representation of $C_{\mathrm{b}}(G) = M(C_0(G))$ by multiplication operators.

An action of G on a C^*-algebra A is a homomorphism α of G into $\operatorname{Aut} A$ such that $s \mapsto \alpha_s(a)$ is continuous for every $a \in A$. The crossed product $(A \times_\alpha G, i_A, i_G)$ is the universal object for covariant representations of (A, G, α), as in [17]; the set $C_{\mathrm{c}}(G, A)$ of continuous, compactly supported functions from G into A embeds as a dense $*$-subalgebra of $A \times_\alpha G$, with

$$f * g(s) = \int_G f(t)\alpha_t(g(t^{-1}s))\,\mathrm{d}t \quad \text{and} \quad f^*(s) = \alpha_s(f(s^{-1})^*)\Delta_G(s)^{-1}.$$

If π is a nondegenerate representation of A, the induced representation $\operatorname{Ind}\pi$ of the system (A, G, α) is the covariant representation $(\widetilde{\pi}, 1 \otimes \lambda)$, in which $\widetilde{\pi}(a)\xi(s) := \pi(\alpha_s^{-1}(a))(\xi(s))$ for $\xi \in L^2(G, \mathcal{H}_\pi) = \mathcal{H}_\pi \otimes L^2(G)$. If H is a closed subgroup of G, we identify $A \otimes C_0(G/H)$ with $C_0(G/H, A)$; we write $\alpha \otimes \tau$ for the diagonal action of G on either algebra, so that $(\alpha \otimes \tau)_t(f)(sH) = \alpha_t(f(t^{-1}sH))$ for $f \in C_0(G/H, A)$. We use σ to denote the action of G on $C_0(G)$ by right translation: $\sigma_t(f)(s) := f(st)$.

We use the full coactions of [18], as modified in [14]: we use minimal tensor products throughout. Thus a coaction δ of G on a C^*-algebra B is a nondegenerate homomorphism $\delta : B \to M(B \otimes C^*(G))$ such that

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta_G) \circ \delta \quad \text{and} \quad \delta(b)(1 \otimes z) \in B \otimes C^*(G)$$

for all $b \in B$ and $z \in C^*(G)$, where $\delta_G : C^*(G) \to M(C^*(G) \otimes C^*(G))$ is the comultiplication on $C^*(G)$ characterised by $\delta_G(i_G(s)) = i_G(s) \otimes i_G(s)$. If N is a closed normal subgroup of G and $q : C^*(G) \to M(C^*(G/N))$ is characterised by $q(i_G(s)) = i_{G/N}(sN)$, then $(\mathrm{id} \otimes q) \circ \delta$ is a coaction of G/N on B, called the restriction of δ to G/N, and denoted $\delta|$. The crossed product $(B \times_\delta G, j_B, j_{C(G)})$ is the universal object for covariant representations of (B, G, δ); in particular

$$B \times_\delta G = \overline{\operatorname{span}}\{j_B(b) j_{C(G)}(f) \mid b \in B, f \in C_0(G)\}.$$

If π is a nondegenerate representation of B, the induced representation $\operatorname{Ind}\pi$ of (B, G, δ) is the covariant representation $((\pi \otimes \lambda) \circ \delta, 1 \otimes M)$ on $\mathcal{H}_\pi \otimes L^2(G)$. We shall follow the conventions of [18] concerning dual actions and coactions.

1. THE SYMMETRIC IMPRIMITIVITY THEOREM

We begin by recalling the symmetric imprimitivity theorem of [16]. Our conventions will be slightly different from those used there: here the second group L acts on the right of the locally compact space P. To convert to the two-left-actions situation of [16], just let $l \cdot p = p \cdot l^{-1}$.

Consider a C^*-algebra D, two locally compact groups K and L, and a locally compact space P; suppose that K acts freely and properly on the left of P, and that L acts likewise on the right, and that these actions commute (i.e., $k \cdot (p \cdot l) = (k \cdot p) \cdot l$). Suppose also that we have commuting actions σ of K and ρ of L on D. Recall that for the left action of K we define the induced C^*-algebra $\operatorname{Ind}\sigma$ to be the set of continuous bounded functions $f : P \to D$ such that $f(k \cdot p) = \sigma_k(f(p))$ for all $k \in K$ and $p \in P$, and such that the function $Kp \mapsto \|f(p)\|$ vanishes at infinity on $K \setminus P$. For the right action of L we define the induced C^*-algebra $\operatorname{Ind}\rho$ to be the set of continuous bounded functions $f : P \to D$ such that $f(p \cdot l) = {\rho_l}^{-1}(f(p))$ for all $p \in P$ and $l \in L$, and such that the function $pL \mapsto \|f(p)\|$ vanishes at infinity on P/L. The induced algebras are C^*-algebras with pointwise operations, and carry actions $\gamma : K \to \operatorname{Aut}(\operatorname{Ind}\rho)$ and $\delta : L \to \operatorname{Aut}(\operatorname{Ind}\sigma)$ given by

$$\gamma_k(f)(p) = \sigma_k(f(k^{-1} \cdot p)) \quad \text{and} \quad \delta_l(f)(p) = \rho_l(f(p \cdot l)).$$

Then Theorem 1.1 of [16] states that $C_c(P, D)$ can be given a pre-imprimitivity bimodule structure which completes to give a Morita equivalence between $\operatorname{Ind}\rho \times_\gamma K$ and $\operatorname{Ind}\sigma \times_\delta L$. The actions and inner products are given for

$b \in C_c(K, \operatorname{Ind}\rho) \subseteq \operatorname{Ind}\rho \times_\gamma K$, x and y in $C_c(P, D)$, and $c \in C_c(L, \operatorname{Ind}\sigma) \subseteq \operatorname{Ind}\sigma \times_\delta L$ as follows:

$$\begin{aligned}
b \cdot x(p) &= \int_K b(t,p)\sigma_t(x(t^{-1}\cdot p))\,\Delta_K(t)^{\frac{1}{2}}\,\mathrm{d}t \\
x \cdot c(p) &= \int_L \rho_s\big(x(p\cdot s)c(s^{-1}, p\cdot s)\big)\,\Delta_L(s)^{-\frac{1}{2}}\,\mathrm{d}s \\
{}_{\operatorname{Ind}\rho\times_\gamma K}\langle x, y\rangle\,(k,p) &= \int_L \rho_s\big(x(p\cdot s)\sigma_k(y(k^{-1}\cdot p\cdot s)^*)\big)\,\mathrm{d}s\,\Delta_K(k)^{-\frac{1}{2}} \\
\langle x, y\rangle_{\operatorname{Ind}\sigma\times_\delta L}\,(l,p) &= \int_K \sigma_t\big(x(t^{-1}\cdot p)^*\rho_l(y(t^{-1}\cdot p\cdot l))\big)\,\mathrm{d}t\,\Delta_L(l)^{-\frac{1}{2}}.
\end{aligned}$$

If $\alpha : G \to \operatorname{Aut} A$ is an action, we denote by $\widehat{\widehat{\alpha}}$ the action of G on $C_0(G, A) \times_{\alpha\otimes\tau} G$ given for $f \in C_c(G \times G, A)$ by

$$\widehat{\widehat{\alpha}}_t(f)(r,s) = f(r, st).$$

(This action is carried into the usual second dual action on $(A \times_\alpha G) \times_{\widehat{\alpha}} G \cong C_0(G, A) \times_{\alpha\otimes\tau} G$ under the isomorphism of Lemma 2.4 below.)

PROPOSITION 1.1. *Let $\alpha : G \to \operatorname{Aut} A$ be an action, and let H be a closed subgroup of G. Then there exists a pre-imprimitivity bimodule structure on $C_c(G \times G, A)$ which completes to give an $(C_0(G, A) \times_{\alpha\otimes\tau} G) \times_{\widehat{\widehat{\alpha}}|} H$ – $C_0(G/H, A) \times_{\alpha\otimes\tau} G$ imprimitivity bimodule.*

Proof. We apply the symmetric imprimitivity theorem, with $P = G \times G$, $K = H \times G$, $L = G$, and $D = A$. Define a left action of $H \times G$, and a right action of G on $G \times G$ by

$$(h,t)\cdot(r,s) = (hr, ts) \quad \text{and} \quad (r,s)\cdot t = (rt, st). \tag{1.1}$$

Both these actions are free and proper, and they commute with one another. Define actions σ and ρ of $H \times G$ and G on A as follows:

$$\sigma_{(h,t)}(a) = \alpha_t(a) \quad \text{and} \quad \rho_t(a) = a.$$

It is clear that these actions also commute; thus by the symmetric imprimitivity theorem ([16], Theorem 1.1), $C_c(G \times G, A)$ completes to give an $\operatorname{Ind}\rho \times_\gamma (H \times G)$ – $\operatorname{Ind}\sigma \times_\delta G$ imprimitivity bimodule.

It only remains to identify $\operatorname{Ind}\rho \times_\gamma (H \times G)$ with $(C_0(G,A) \times_{\alpha\otimes\tau} G) \times_{\widehat{\widehat{\alpha}}} H$, and $\operatorname{Ind}\sigma \times_\delta G$ with $C_0(G/H,A) \times_{\alpha\otimes\tau} G$. To this end, we first remark that $H \times G$ acts on $C_0(G,A)$ by $\tilde{\alpha}_{(h,t)}(f)(s) = \alpha_t(f(t^{-1}sh))$, and then that the identity map of $C_c(H\times G\times G, A)$ onto itself extends to an isomorphism of $C_0(G,A) \times_{\tilde{\alpha}} (H \times G)$ onto $(C_0(G,A) \times_{\alpha\otimes\tau} G) \times_{\widehat{\widehat{\alpha}}} H$.

Next, note that $(G \times G)/G$ (with the action of (1.1)) is homeomorphic to G via the map $(r,s) \mapsto sr^{-1}$, so we have a bijection $\Theta : C_0(G,A) \to \operatorname{Ind}\rho$ given by

$$\Theta(f)(r,s) = f(sr^{-1}), \quad \Theta^{-1}(g)(s) = g(e,s).$$

Since the operations on both $C_0(G,A)$ and $\operatorname{Ind}\rho$ are pointwise, Θ gives an isomorphism of the C^*-algebras.

Now Θ is $\tilde{\alpha} - \gamma$ equivariant:

$$\begin{aligned}\Theta(\tilde{\alpha}_{(h,t)}(f))(r,s) &= \tilde{\alpha}_{(h,t)}(f)(sr^{-1}) = \alpha_t(f(t^{-1}sr^{-1}h)) = \alpha_t(\Theta(f)(h^{-1}r, t^{-1}s)) \\ &= \sigma_{(h,t)}\big(\Theta(f)((h,t)^{-1} \cdot (r,s))\big) = \gamma_{(h,t)}(\Theta(f))(r,s).\end{aligned}$$

Thus Θ induces an isomorphism of $C_0(G,A) \times_{\tilde{\alpha}} (H \times G)$ onto $\operatorname{Ind}\rho \times_\gamma (H \times G)$. Combined with the previous isomorphism, this completes the first identification.

For the second identification, note that $(H\times G) \backslash (G \times G)$ (with the action of (1.1)) is homeomorphic to G/H via the map $(r,s) \mapsto r^{-1}H$. So we have a bijection $\Omega : C_0(G/H, A) \to \operatorname{Ind}\sigma$ given by

$$\Omega(f)(r,s) = \alpha_s(f(r^{-1}H)), \quad \Omega^{-1}(g)(tH) = g(t^{-1}, e).$$

As above, since the operations on both algebras are pointwise, Ω is an isomorphism. Moreover, Ω is $\alpha \otimes \tau - \delta$ equivariant:

$$\begin{aligned}\Omega(\alpha_t \otimes \tau_t(f))(r,s) &= \alpha_s(\alpha_t \otimes \tau_t(f)(r^{-1}H)) = \alpha_s(\alpha_t(f(t^{-1}r^{-1}H))) \\ &= \alpha_{st}(f((rt)^{-1}H)) = \Omega(f)(rt, st) = \delta_t(\Omega(f))(r,s).\end{aligned}$$

Thus Ω induces the second identification of crossed products. ∎

The isomorphisms of the proof of Proposition 1.1 can be used to make $C_c(G\times G, A)$ explicitly a $C_c(H\times G\times G, A) - C_c(G\times G/H, A)$ pre-imprimitivity bimodule. However, for technical reasons, we shall combine these with the automorphism Υ of $C_c(G \times G, A)$ defined by

$$\Upsilon(x)(r,s) = x(r, rs^{-1})\Delta_G(r)^{\frac{1}{2}}.$$

This gives a bimodule structure which is more natural for our considerations in Section 4. The resulting actions and inner products are given, for $f \in C_c(H \times G \times G, A)$, x and y in $C_c(G \times G, A)$, and $g \in C_c(G \times G/H, A)$ as follows:

$$f \cdot x(r,s) = \int_G \int_H f(h,t,s)\alpha_t(x(t^{-1}r, t^{-1}sh))\, \Delta_H(h)^{\frac{1}{2}}\, dh\, dt \tag{1.2}$$

$$x \cdot g(r,s) = \int_G x(t,s)\alpha_t(g(t^{-1}r, t^{-1}sH))\, dt \tag{1.3}$$

$$_L\langle x,y\rangle\,(h,r,s) = \int_G x(t,s)\alpha_r(y(r^{-1}t, r^{-1}sh)^*)\, \Delta_H(h)^{-\frac{1}{2}}\, \Delta_G(r^{-1}t)\, dt \tag{1.4}$$

$$\langle x,y\rangle_R\,(r,sH) = \int_G \int_H \alpha_t(x(t^{-1}, t^{-1}sh)^* y(t^{-1}r, t^{-1}sh))\, \Delta_G(t^{-1})\, dh\, dt. \tag{1.5}$$

2. Inducing representations from homogeneous spaces

It is a major defect of the current theory of crossed products by coactions that we do not know how to best define coactions of homogeneous spaces and their crossed products. However, if we start with a coaction of G on B, and H is a closed subgroup of G, we can obtain what should be covariant representations of $(B, G/H, \delta)$ by restricting covariant representations (π, μ) of (B, G, δ): just extend μ to the multiplier algebra $M(C_0(G)) = C_b(G)$ and restrict it to the subalgebra $C_0(G/H)$ of functions constant on H-cosets. In particular, we can restrict a regular representation $((\pi \otimes \lambda) \circ \delta, 1 \otimes M)$, which motivates the following definition.

Definition 2.1. Let (B, G, δ) be a coaction, let H be a closed subgroup of G, and let π be a representation of B such that $\operatorname{Ind}\pi$ is faithful on $B \times_\delta G$. We define the *reduced crossed product* $B \times_{\delta,\mathrm{r}} G/H$ to be the C^*-subalgebra of $B(\mathcal{H}_\pi \otimes L^2(G))$ generated by the operators

$$\{(\pi \otimes \lambda) \circ \delta(b)(1 \otimes M_f) \mid b \in B, f \in C_0(G/H)\}.$$

The proviso that $\operatorname{Ind}\pi$ be faithful on $B \times_\delta G$ implies that $M(B \times_\delta G)$ is represented faithfully on $\mathcal{H}_\pi \otimes L^2(G)$, so $B \times_{\delta,\mathrm{r}} G/H$ is actually a subalgebra of $M(B \times_\delta G)$; this ensures that the isomorphism class of $B \times_{\delta,\mathrm{r}} G/H$ is independent of the choice of π.

Remark 2.2. (i) Implicit in the above definition (since $\operatorname{Ind}\pi$ is required to be faithful), we have $B \times_{\delta,\mathrm{r}} G \cong B \times_\delta G$ for any coaction δ of G on B. In particular, if N is a closed *normal* subgroup of G, then $\delta|$ is a coaction of G/N on B, so $B \times_{\delta|,\mathrm{r}} G/N \cong B \times_{\delta|} G/N$. We emphasise that this crossed product is not necessarily isomorphic to $B \times_{\delta,\mathrm{r}} G/N$: restricting the regular representation of (B, G, δ) gives a covariant representation of $(B, G/N, \delta|)$ onto $B \times_{\delta,\mathrm{r}} G/N \subseteq B(\mathcal{H}_\pi \otimes L^2(G))$, which is known to be faithful if N is amenable ([8], Lemma 3.2; see also Corollary 2.9 below), but is not faithful in general (Remark 2.10 below).

We have chosen the notation $B \times_{\delta,\mathrm{r}} G/H$ to stress that the reduced crossed product depends on the coaction δ, and, implicitly, on the group G. (A given space may be realisable in several different ways as a homogeneous space.) This notation is consistent with that used by Mansfield to distinguish the subalgebra $B \times_\delta G/H$ of $B(\mathcal{H}_\pi \otimes L^2(G))$ from his spatially defined crossed product $B \times_{\delta|} G/H$ on $\mathcal{H}_\pi \otimes L^2(G/H)$. We mention in passing that, for arbitrary H, it follows from Proposition 8 of [13] that

$$B \times_{\delta,\mathrm{r}} G/H = \overline{\operatorname{span}}\{(\pi \otimes \lambda) \circ \delta(b)(1 \otimes M_f) \mid b \in B, f \in C_0(G/H)\}.$$

(ii) Since $M(B \times_\delta G)$ is faithfully represented on $\mathcal{H}_\pi \otimes L^2(G)$, for normal N the algebra $B \times_{\delta,\mathrm{r}} G/N$ is the algebra $\operatorname{im}(j_B \times j_G|)$ appearing in Theorem 3.3 of [8], and hence that theorem establishes a Morita equivalence between $B \times_{\delta,\mathrm{r}} G/N$ and the reduced crossed product $(B \times_\delta G) \times_\mathrm{r} N$. This bimodule can be used to define induction of representations from $B \times_{\delta,\mathrm{r}} G/N$ to $B \times_\delta G$. As we shall see, this is not necessarily the same as the induction process we shall construct for B of the form $A \times_\alpha G$.

When δ is the dual coaction $\widehat{\alpha}$ of an action $\alpha : G \to \operatorname{Aut} A$, there is also a natural candidate for a full crossed product $B \times_\delta G/H$, whose representations are given by certain covariant pairs (π, μ) of representations of B and $C_0(G/H)$. To motivate this, we recall that for normal N, the crossed product $(A \times_\alpha G) \times_{\widehat{\alpha}|} G/N$ is one realisation of Green's imprimitivity algebra $(A \otimes C_0(G/N)) \times_{\alpha\otimes\tau} G$; indeed, the resulting interpretation of Green's Imprimitivity Theorem motivated Mansfield's theorem (see [12]). We digress to establish this realisation in the context of full coactions and nonamenable subgroups.

Lemma 2.3. *Let $\alpha : G \to \operatorname{Aut} A$ be an action, and let N be a closed normal subgroup of G. Consider representations π, U, and μ of A, G and $C_0(G/N)$, respectively, on a Hilbert space $\mathcal{H}$. Then (π, U) is a covariant representation of (A, G, α) and $(\pi \times U, \mu)$ is a covariant representation of $(A \times_\alpha G, G/N, \widehat{\alpha})$ if and* *only if* π *and* μ *have commuting ranges and* $(\pi\otimes\mu, U)$ *is a covariant representation of* $(C_0(G/N, A), G, \alpha\otimes\tau)$.

Proof. The proof is sketched in Example 2.9 of [18]. ∎

LEMMA 2.4. *Let* $\alpha : G \to \operatorname{Aut} A$ *be an action of a locally compact group, and let* N *be a closed normal subgroup of* G. *Then there is an isomorphism*

$$\Psi : C_0(G/N, A) \times_{\alpha\otimes\tau} G \to (A \times_\alpha G) \times_{\hat{\alpha}|} G/N$$

which is natural in the sense that

$$\Psi \circ k_A = j_{A\times G} \circ i_A, \quad \Psi \circ k_G = j_{A\times G} \circ i_G, \quad \textit{and} \quad \Psi \circ k_{C(G/N)} = j_{C(G/N)},$$

where $(k_A \otimes k_{C(G/N)}, k_G)$ *are the canonical maps of* $(C_0(G/N, A), G, \alpha\otimes\tau)$ *into the crossed product. The induced map on representations takes* $(\pi\times U)\times\mu$ *to* $(\pi\otimes\mu)\times U$.

Proof. Realise $C_0(G/N, A) \times_{\alpha\otimes\tau} G$ on $\mathcal{H}$; then k_A and $k_{C(G/N)}$ are commuting representations on $\mathcal{H}$, and $(k_A \otimes k_{C(G/N)}, k_G)$ is a covariant representation of $(C_0(G/N, A), G, \alpha\otimes\tau)$. By Lemma 2.3, (k_A, k_G) is covariant for (A, G, α), and $(k_A \times k_G, k_{C(G/N)})$ is covariant for $(A\times_\alpha G, G/N, \widehat{\alpha}|)$. It follows that there is a nondegenerate representation $\Phi = (k_A \times k_G) \times k_{C(G/N)}$ of $(A\times_\alpha G)\times_{\hat{\alpha}|} G/N$ on $\mathcal{H}$ such that

$$\Phi \circ j_{A\times G} \circ i_A = k_A, \quad \Phi \circ j_{A\times G} \circ i_G = k_G, \quad \Phi \circ j_{C(G/N)} = k_{C(G/N)}. \tag{2.1}$$

Now suppose $(A\times_\alpha G)\times_{\widehat{\alpha}|} G/N$ acts on $\mathcal{K}$. Then

$$(j_{A\times_\alpha G}, j_{C(G/N)}) = ((j_{A\times G}\circ i_A)\times(j_{A\times G}\circ i_G), j_{C(G/N)})$$

is a covariant representation of $(A\times_\alpha G, G/N, \widehat{\alpha}|)$. Thus we deduce from Lemma 2.3 that $((j_{A\times G}\circ i_A)\otimes j_{C(G/N)}, j_{A\times G}\circ i_G)$ is covariant for $(C_0(G/N, A), G, \alpha\otimes\tau)$, and hence there is a representation $\Psi = ((j_{A\times G}\circ i_A)\otimes j_{C(G/N)})\times(j_{A\times G}\circ i_G)$ of $C_0(G/N, A)\times_{\alpha\otimes\tau} G$ on $\mathcal{K}$ such that

$$\Psi \circ k_A = j_{A\times G} \circ i_A, \quad \Psi \circ k_G = j_{A\times G} \circ i_G, \quad \Psi \circ k_{C(G/N)} = j_{C(G/N)}. \tag{2.2}$$

Equations (2.1) and (2.2) imply that Ψ is an inverse for Φ.

For the last statement, let $(\pi\times U)\times\mu$ be a representation of $(A\times_\alpha G)\times_{\widehat{\alpha}|} G/N$. With $a\in A$, $z\in C_c(G)$, and $f\in C_c(G/N)$, $k_A\otimes k_{C(G/N)}(a\otimes f)k_G(z)$ is a typical enough element of $C_0(G/N, A)\times_{\alpha\otimes\tau} G$, and we have:

$$\begin{aligned}
&((\pi\times U)\times\mu)\circ\Psi\left(k_A\otimes k_{C(G/N)}(a\otimes f)k_G(z)\right)\\
&\quad= ((\pi\times U)\times\mu)\left(j_{C(G/N)}(f)j_{A\times G}(i_A(a)i_G(z))\right) = \mu(f)\pi(a)U(z)\\
&\quad= \pi\otimes\mu(a\otimes f)U(z) = ((\pi\otimes\mu)\times U)\left(k_A\otimes k_{C(G/N)}(a\otimes f)k_G(z)\right). \qquad ∎
\end{aligned}$$

For the rest of this section, $\alpha : G \to \operatorname{Aut} A$ will be an action of a locally compact group, and H an arbitrary closed subgroup of G. Lemma 2.3 suggests that it is reasonable to make the following definition.

DEFINITION 2.5. A pair of representations $(\pi \times U, \mu)$ of $(A \times_\alpha G, C_0(G/H))$ is a *covariant representation of* $(A \times_\alpha G, G/H, \widehat{\alpha})$ if the ranges of π and μ commute and $(\pi \otimes \mu, U)$ is a covariant representation of $(C_0(G/H, A), G, \alpha \otimes \tau)$.

As anticipated at the beginning of this section (for arbitrary coactions), covariant representations of $(A \times_\alpha G, G/H, \widehat{\alpha})$ arise by restricting covariant representations of $(A \times_\alpha G, G, \widehat{\alpha})$.

LEMMA 2.6. *Let (A, G, α) be an action, and let H be a closed subgroup of G. If $(\pi \times U, \mu)$ is a covariant representation of $(A \times_\alpha G, G, \widehat{\alpha})$, then $(\pi \times U, \mu|)$ is a covariant representation of $(A \times_\alpha G, G/H, \widehat{\alpha})$.*

Proof. We need to show that $\pi(A)$ and $\mu|(C_0(G/H))$ commute, and that $(\pi \otimes \mu|, U)$ is covariant for $(C_0(G/H, A), G, \alpha \otimes \tau)$. For the first, fix $f \in C_0(G/H)$, $a \in A$, $g \in C_0(G)$, and $\xi \in \mathcal{H}_\pi$. Then:

$$\begin{aligned}(\mu|(f)\pi(a))(\mu(g)\xi) &= \mu|(f)\mu(g)\pi(a)\xi = \mu(fg)\pi(a)\xi \\ &= \pi(a)\mu(fg)\xi = (\pi(a)\mu|(f))(\mu(g)\xi).\end{aligned}$$

Since μ is nondegenerate, this implies $\mu|(f)\pi(a) = \pi(a)\mu|(f)$ in $B(\mathcal{H}_\pi)$. For the second, for each $s \in G$ we have

$$\begin{aligned}&\pi \otimes \mu|(\alpha_s \otimes \tau_s(a \otimes f))(\mu(g)\xi) \\ &\qquad = \pi(\alpha_s(a))\mu|(\tau_s(f))\mu(g)\xi = \pi(\alpha_s(a))\mu(\tau_s(f)g)\xi \\ &\qquad = \pi \otimes \mu(\alpha_s \otimes \tau_s(a \otimes f\tau_{s^{-1}}(g)))\xi = U_s\pi \otimes \mu(a \otimes f\tau_{s^{-1}}(g))U_s^*\xi \\ &\qquad = U_s\pi(a)\mu|(f)\mu(\tau_{s^{-1}}(g))U_s^*\xi = U_s\pi(a)\mu|(f)U_s^*\mu(g)\xi \\ &\qquad = U_s\pi \otimes \mu|(a \otimes f)U_s^*(\mu(g)\xi),\end{aligned}$$

which implies covariance of $(\pi \times \mu|, U)$. ∎

Because (by definition) the covariant representations of $(A \times_\alpha G, G/H, \widehat{\alpha})$ correspond to the covariant representations of $(C_0(G/H, A), G, \alpha \otimes \tau)$, we view $C_0(G/H, A) \times_{\alpha\otimes\tau} G$ as a full crossed product of $A \times_\alpha G$ by the "coaction" $\widehat{\alpha}$ of the homogeneous space G/H. We now want to discuss the "regular representations" of this full crossed product. But first we need to know that certain representations of $A \times_\alpha G$ induce to faithful representations of $(A \times_\alpha G) \times_{\widehat{\alpha}} G$, so that we can use them to realise the reduced crossed product $(A \times_\alpha G) \times_{\widehat{\alpha},\mathrm{r}} G/H$.

LEMMA 2.7. *Let* (π, U) *be a covariant representation of* (A, G, α) *such that* π *is faithful. Then the representation* $\operatorname{Ind}(\pi \times U)$ *of* $(A \times_\alpha G) \times_{\widehat{\alpha}} G$ *is faithful; so is the corresponding representation* $(\pi \otimes M) \times (U \otimes \lambda)$ *of* $(A \otimes C_0(G)) \times_{\alpha\otimes\tau} G$.

Proof. Since

$$\operatorname{Ind}(\pi \times U) = (((\pi \times U) \otimes \lambda) \circ \widehat{\alpha}) \times (1 \otimes M) = ((\pi \otimes 1) \times (U \otimes \lambda)) \times (1 \otimes M), \tag{2.3}$$

it follows from Lemma 2.4 that it is enough to show that the representation $(\pi \otimes M) \times (U \otimes \lambda)$ of $C_0(G, A) \times_{\alpha\otimes\tau} G$ is faithful. The automorphism ϕ of $C_0(G, A)$ defined by $\phi(f)(t) = \alpha_{t^{-1}}(f(t))$ induces an isomorphism of $C_0(G, A) \times_{\alpha\otimes\tau} G$ onto

$$C_0(G, A) \times_{\mathrm{id}\otimes\tau} G \cong A \otimes (C_0(G) \times_\tau G) \cong A \otimes \mathcal{K}(L^2(G)).$$

If we now define V on $L^2(G, \mathcal{H})$ by $V\xi(t) = U_t(\xi(t))$, then one can verify that

$$V^*(\pi \otimes M)(\phi^{-1}(f))V = \pi \otimes M(f), \quad \text{and } V^*(U \otimes \lambda)V = 1 \otimes \lambda.$$

Since the representation $(\pi \otimes M) \times (1 \otimes \lambda) = \pi \otimes (M \times \lambda)$ is certainly faithful on $A \otimes \mathcal{K}(L^2(G))$, the result follows. ∎

Let π and U be as above, so that by (2.3), $((\pi \otimes 1) \times (U \otimes \lambda), 1 \otimes M)$ is covariant for $(A \times_\alpha G, G, \widehat{\alpha})$. By Lemma 2.6, restricting $1 \otimes M$ to $C_0(G/H)$ gives a covariant representation $((\pi \otimes 1) \times (U \otimes \lambda), 1 \otimes M|)$ of $(A \times_\alpha G, G/H, \widehat{\alpha})$, and hence we have a representation $(\pi \otimes M|) \times (U \otimes \lambda)$ of the full crossed product $C_0(G/H, A) \times_{\alpha\otimes\tau} G$ on $\mathcal{H}_\pi \otimes L^2(G)$. Because we know from Lemma 2.7 that $\operatorname{Ind}(\pi \times U)$ is faithful on $(A \times_\alpha G) \times_{\widehat{\alpha}} G$, the image of $C_0(G/H, A) \times_{\alpha\otimes\tau} G$ is precisely (one realisation of) the reduced crossed product $(A \times_\alpha G) \times_{\widehat{\alpha},\mathrm{r}} G/H$. Just as we think of $C_0(G/H, A) \times_{\alpha\otimes\tau} G$ as a full crossed product for $(A \times_\alpha G, G/H, \widehat{\alpha})$, we shall think of $(\pi \otimes M|) \times (U \otimes \lambda)$ as the regular representation of $C_0(G/H, A) \times_{\alpha\otimes\tau} G$ induced from (π, U). As we shall see, this representation is not always faithful.

PROPOSITION 2.8. *Suppose* (π, U) *is a covariant representation of* (A, G, α) *on* $\mathcal{H}$ *and* π *is faithful. Then the representation* $(\pi \otimes M|) \times (U \otimes \lambda)$ *induces an isomorphism of* $(A \otimes C_0(G/H)) \times_{\alpha\otimes\tau,\mathrm{r}} G$ *onto* $(A \times_\alpha G) \times_{\widehat{\alpha},\mathrm{r}} G/H$.

Proof. In view of the preceding remarks, it is enough to prove that the kernel of $(\pi \otimes M|) \times (U \otimes \lambda)$ is precisely the kernel of a regular representation of $(A \otimes C_0(G/H)) \times_{\alpha\otimes\tau} G$.

The inclusion of $C_0(G/H)$ in $M(C_0(G))$ induces a homomorphism ϕ of the crossed product $(A \otimes C_0(G/H)) \times_{\alpha\otimes\tau} G$ into $M((A \otimes C_0(G)) \times_{\alpha\otimes\tau} G)$. The regular representation $\operatorname{Ind}(\pi \otimes M)$ is faithful on $(A \otimes C_0(G)) \times_{\alpha\otimes\tau} G$, and the composition $\mathrm{Ind}\,(\pi \otimes M) \circ \phi$ is the regular representation induced by the faithful representation $\pi \otimes M|$ of $A \otimes C_0(G/H)$. Thus the kernel of ϕ is the kernel of the regular representation, and ϕ induces an injection of $(A \otimes C_0(G/H)) \times_{\alpha\otimes\tau,\mathrm{r}} G$ into $M((A \otimes C_0(G)) \times_{\alpha\otimes\tau} G)$. Composing this injection with the faithful (by Lemma 2.7) representation $(\pi \otimes M) \times (U \otimes \lambda)$ gives a faithful representation of $(A \otimes C_0(G/H)) \times_{\alpha\otimes\tau,\mathrm{r}} G$; but $((\pi \otimes M) \times (U \otimes \lambda)) \circ \phi = (\pi \otimes M|) \times (U \otimes \lambda)$, so the result follows. ∎

COROLLARY 2.9. *We have $A \times_\alpha H = A \times_{\alpha,\mathrm{r}} H$ if and only if, whenever (π, U) is a covariant representation of (A, G, α) with π faithful, the representation $(\pi \otimes M|) \times (U \otimes \lambda)$ of $(A \otimes C_0(G/H)) \times_{\alpha\otimes\tau} G$ induced from (π, U) is faithful.*

Proof. Recall from [15] that $A \times_\alpha H = A \times_{\alpha,\mathrm{r}} H$ if and only if the imprimitivity algebra $(A \otimes C_0(G/H)) \times_{\alpha\otimes\tau} G$ is canonically isomorphic to $(A \otimes C_0(G/H)) \times_{\alpha\otimes\tau,\mathrm{r}} G$. ∎

REMARK 2.10. Applying this result with H normal and amenable gives Lemma 3.2 of [13], albeit only for dual coactions (cf. also [13], Proposition 7). Taking $H = G$, $A = \mathbb{C}$ and G nonamenable shows that the representation $(\pi \otimes M|) \times (U \otimes \lambda)$ in Proposition 2.8 is not always faithful.

Restricting the action on the left of the bimodule of Proposition 1.1 gives a right-Hilbert $C_0(G, A) \times G - C_0(G/H, A) \times G$ bimodule, which by Lemma 2.4 we can view as a right-Hilbert $A \times G \times G - C_0(G/H, A) \times G$ bimodule $Z^G_{G/H}(A \times G)$. Using this, we can induce a covariant representation $(\pi \times U, \mu)$ of $(A \times_\alpha G, G/H, \widehat{\alpha})$ to a representation $\mathrm{Ind}^G_{G/H}(\pi \times U, \mu)$ of $(A \times_\alpha G) \times_{\widehat{\alpha}} G$, acting in a completion of $Z^G_{G/H} \otimes \mathcal{H}_\pi$. Since the isomorphism of $(A \times_\alpha G) \times_{\widehat{\alpha}} G$ with $C_0(G, A) \times_{\alpha\otimes\tau} G$ carries the double dual action into the action of G used in Section 1, we deduce from Proposition 1.1 the following representation-theoretic imprimitivity theorem:

PROPOSITION 2.11. *Suppose $\alpha : G \to \mathrm{Aut}\, A$ is an action of a locally compact group on a C^*-algebra A and H is a closed subgroup of G. A representation $(\rho \times V) \times \nu$ of $(A \times_\alpha G) \times_{\widehat{\alpha}} G$ is induced from a covariant representation of $(A \times_\alpha G, G/H, \widehat{\alpha})$ if and only if there is a representation U of H on $\mathcal{H}_\rho$ such that $((\rho \times V) \times \nu, U)$ is a covariant representation of $((A \times_\alpha G) \times_{\widehat{\alpha}} G, H, \widehat{\widehat{\alpha}}|)$. (That is, if and only if the range of U commutes with the ranges of V and ρ, and $\nu(\sigma_s(f)) = U_s \nu(f) U_s^*$ for $s \in H$, $f \in C_0(G)$.)*

REMARK 2.12. From [15], we know that the imprimitivity bimodule of Proposition 1.1 has as a (possibly proper) quotient a $C_0(G, A) \times_{\mathrm{r}} (H \times G)$ –

$C_0(G/H, A) \times_{\mathrm{r}} G$ imprimitivity bimodule Z_{r}. Since

$$\begin{aligned} C_0(G, A) \times_{\mathrm{r}} (H \times G) &\cong (C_0(G, A) \times_{\mathrm{r}} G) \times_{\mathrm{r}} H \\ &\cong (C_0(G, A) \times G) \times_{\mathrm{r}} H \\ &\cong (A \times_\alpha G \times_{\widehat{\alpha}} G) \times_{\mathrm{r}} H, \end{aligned}$$

we can by Proposition 2.8 realise Z_{r} as a right-Hilbert $(A \times_\alpha G) \times_{\widehat{\alpha}} G$ – $(A \times_\alpha G) \times_{\widehat{\alpha},\mathrm{r}} G/H$ bimodule, and use it to induce representations from the reduced crossed product. We shall see in Theorem 4.1 that this induction process agrees with the one studied in [13] and [8] for normal H.

3. INDUCTION AND DUALITY

In this section we show that, modulo duality, our induction process for dual systems is the inverse of Green induction. Before stating our theorem, we describe the three bimodules involved.

Consider an action $\alpha : G \to \operatorname{Aut} A$ and a closed, not-necessarily-normal subgroup H of G. Recall from [7] that $C_{\mathrm{c}}(G, A)$ can be completed to a $C_0(G/H, A) \times_{\alpha\otimes\tau} G$ – $A \times_\alpha H$ imprimitivity bimodule $X_H^G(A)$. We use the pre-imprimitivity bimodule structure on $C_{\mathrm{c}}(G, A)$ given for $f \in C_{\mathrm{c}}(G \times G/H, A)$, x and y in $C_{\mathrm{c}}(G, A)$, and $g \in C_{\mathrm{c}}(H, A)$ as follows:

$$f \cdot x(r) = \int_G f(s, rH)\alpha_s(x(s^{-1}r))\, \Delta_G(s)^{\frac{1}{2}}\, \mathrm{d}s \tag{3.1}$$

$$x \cdot g(r) = \int_H x(rt)\alpha_{rt}(g(t^{-1}))\, \Delta_H(t)^{-\frac{1}{2}}\, \mathrm{d}t \tag{3.2}$$

$${}_{C_0(G/H,A)\times G}\langle x, y\rangle\, (s, rH) = \int_H x(rt)\alpha_s(y(s^{-1}rt)^*)\, \Delta_G(s)^{-\frac{1}{2}}\, \mathrm{d}t \tag{3.3}$$

$$\langle x, y\rangle_{A\times H}\, (t) = \int_G \alpha_s(x(s^{-1})^* y(s^{-1}t))\, \Delta_H(t)^{-\frac{1}{2}}\, \mathrm{d}s. \tag{3.4}$$

These actions and inner products, and in particular the modular functions, come straight from the symmetric imprimitivity theorem (see Section 1), with $K = G$ and $L = H$ acting on $P = G$ by left and right multiplication, $\sigma = \alpha$, and $\rho = \mathrm{id}$.

Recall from Section 1 that, in the case $H = \{e\}$, the action $\widehat{\widehat{\alpha}}$ of G on Green's imprimitivity algebra $C_0(G, A) \times G$ is given for $f \in C_{\mathrm{c}}(G \times G, A)$ by $\widehat{\widehat{\alpha}}_t(f)(r, s) = f(r, st)$. The imprimitivity bimodule $X_{\{e\}}^G(A)$ also admits an action γ of G, given for $x \in C_{\mathrm{c}}(G, A)$ by $\gamma_t(x)(s) = x(st)$, and by Theorem 1 of [4], this gives an equivariant Morita equivalence $(X^G_{\{e\}}(A),\gamma)$ between $(C_0(G,A)\times G,G,\widehat{\widehat{\alpha}})$ and (A,G,α). Thus for any closed subgroup H of G we have a $(C_0(G,A)\times_{\alpha\otimes\tau}G)\times_{\widehat{\widehat{\alpha}}}H$ – $A\times_\alpha H$ imprimitivity bimodule $X^G_{\{e\}}(A)\times H$, with dense submodule $C_c(H\times G,A)$ ([2], Section 6 of [1]). For $f\in C_c(H\times G\times G,A)$, x and y in $C_c(H\times G,A)$, and $g\in C_c(H,A)$, the actions and inner products are as follows:

$$f\cdot x(h,r)=\int_G\int_H f(k,u,r)\alpha_u(x(k^{-1}h,u^{-1}rk))\,\Delta_G(u)^{\frac12}\,\mathrm{d}k\,\mathrm{d}u \tag{3.5}$$

$$x\cdot g(h,r)=\int_H x(k,r)\alpha_{rk}(g(k^{-1}h))\,\mathrm{d}k \tag{3.6}$$

$${}_L\langle x,y\rangle\,(h,r,s)=\int_H x(k,s)\alpha_r(y(h^{-1}k,r^{-1}sh)^*)\,\Delta_H(h^{-1}k)\Delta_G(r)^{-\frac12}\,\mathrm{d}k \tag{3.7}$$

$$\langle x,y\rangle_R\,(h)=\int_G\int_H \alpha_s(x(k^{-1},s^{-1}k)^*y(k^{-1}h,s^{-1}k))\,\Delta_H(k)^{-1}\,\mathrm{d}k\,\mathrm{d}s. \tag{3.8}$$

As in the previous section, we denote by $Z^G_{G/H}(A\times G)$ the bimodule of Proposition 1.1 viewed as an $(A\times_\alpha G\times_{\widehat{\alpha}}G)\times_{\widehat{\widehat{\alpha}}}H - C_0(G/H,A)\times_{\alpha\otimes\tau}G$ imprimitivity bimodule.

THEOREM 3.1. *Let $\alpha : G\to \operatorname{Aut} A$ be an action of a locally compact group G on a C^*-algebra A, and let H be a closed subgroup of G. Then*

$$Z^G_{G/H}(A\times G)\otimes_{C_0(G/H,A)\times G}X^G_H(A)\cong X^G_{\{e\}}(A)\times H$$

as $(A\times_\alpha G\times_{\widehat{\alpha}}G)\times_{\widehat{\widehat{\alpha}}}H - A\times_\alpha H$ imprimitivity bimodules.

For the proof, we shall need the special case of the following lemma in which ψ_A and ψ_B are the identity; the general case will be used in Section 4.

LEMMA 3.2. *Suppose that ${}_AX_B$ and ${}_CY_D$ are imprimitivity bimodules, let $\psi_A : A\to C$, $\psi_B : B\to D$ be surjective homomorphisms, and let $J=\ker\psi_A$, $I=\ker\psi_B$. If $\psi_X : X\to Y$ is a linear map satisfying*

$$\psi_X(a\cdot x)=\psi_A(a)\cdot\psi_X(x)$$
$$\psi_X(x\cdot b)=\psi_X(x)\cdot\psi_B(b)$$
$$\langle\psi_X(x),\psi_X(y)\rangle_D=\psi_B(\langle x,y\rangle_B),$$

then $\ker\psi_X=X\cdot I$, and (ψ_A,ψ_X,ψ_B) factors through an imprimitivity bimodule isomorphism of ${}_{A/J}(X/X\cdot I)_{B/I}$ onto ${}_CY_D$.

Proof. We have

$$\begin{aligned} {}_C\langle \psi_X(x), \psi_X(y)\rangle \cdot \psi_X(z) &= \psi_X(x) \cdot \langle \psi_X(y), \psi_X(z)\rangle_D = \psi_X(x) \cdot \psi_B(\langle y, z\rangle_B) \\ &= \psi_X(x \cdot \langle y, z\rangle_B) = \psi_X({}_A\langle x, y\rangle \cdot z) \\ &= \psi_A({}_A\langle x, y\rangle) \cdot \psi_X(z). \end{aligned}$$

Since ψ_A and ψ_B are surjective, it follows that $\overline{\psi_X(X)}$ is a full $C-D$ sub-bimodule of ${}_CY_D$. Thus, by Theorem 3.1 of [19], $\psi_X(X)$ is dense in Y. Then the above computations imply that (ψ_A, ψ_X, ψ_B) is an imprimitivity bimodule homomorphism which factors through an injective imprimitivity bimodule homomorphism $(\psi_{A/J}, \psi_{X/X\cdot I}, \psi_{A/I})$ of ${}_{A/J}(X/X\cdot I)_{B/I}$ into ${}_CY_D$ by [6], Lemma 2.7. Since $\psi_{X/X\cdot I}$ is isometric, it follows that $\psi_X(X)$ is complete. Hence $\psi_X(X) = Y$. ∎

Proof of Theorem 3.1. We work with the dense subalgebras

$$C_c(H \times G \times G, A) \subseteq (A \times_\alpha G \times_{\widehat{\alpha}} G) \times_{\widehat{\widehat{\alpha}}} H \quad \text{and} \quad C_c(H, A) \subseteq A \times_\alpha H,$$

and the dense submodules

$$C_c(G \times G, A) \subseteq Z^G_{G/H}(A \times G), \quad C_c(G, A) \subseteq X^G_H(A),$$

and

$$C_c(H \times G, A) \subseteq X^G_{\{e\}}(A) \times H.$$

Fix (f, x) in $C_c(G \times G, A) \times C_c(G, A)$ and suppose E_{f_1}, E_{f_2}, and E_x are compact sets such that $\operatorname{supp}(f) \subseteq E_{f_1} \times E_{f_2}$ and $\operatorname{supp}(x) \subseteq E_x$; then the map $F_{f,x} : H \times G \times G \to A$ defined by

$$F_{f,x}(h, s, t) = f(t, s)\alpha_t(x(t^{-1}sh))\Delta_H(h)^{-\frac{1}{2}}\Delta_G(t)^{\frac{1}{2}}$$

is continuous and has support in $(E_{f_2}^{-1}E_{f_1}E_x) \cap H \times E_{f_2} \times E_{f_1}$. It follows that the map $(h, s) \mapsto \int_G F_{f,x}(h, s, t)\,\mathrm{d}t$ is in $C_c(H \times G, A)$. The pairing which sends (f, x) to this element of $C_c(H \times G, A)$ is bilinear, and so we have a well-defined map ψ of $C_c(G \times G, A) \odot C_c(G, A)$ into $C_c(H \times G, A)$ given by

$$\psi(f \otimes x)(h, s) = \int_G f(t, s)\alpha_t(x(t^{-1}sh))\Delta_H(h)^{-\frac{1}{2}}\Delta_G(t)^{\frac{1}{2}}\,\mathrm{d}t.$$

The following calculations verify that ψ preserves both actions and the right inner product. For $g \in C_c(H \times G \times G, A)$ and $f \otimes x \in C_c(G \times G, A) \odot C_c(G, A)$:

$$\begin{aligned}
\psi(g \cdot f \otimes x)(h,s) &= \int_G g \cdot f(t,s)\alpha_t(x(t^{-1}sh))\Delta_H(h)^{-\frac{1}{2}}\Delta_G(t)^{\frac{1}{2}}\,\mathrm{d}t \\
&\overset{t\mapsto ut}{=} \int_G\int_G\int_H g(k,u,s)\alpha_u\big(f(t,u^{-1}sk)\alpha_t(x(t^{-1}u^{-1}sh)) \\
&\qquad\qquad \Delta_H(k^{-1}h)^{-\frac{1}{2}}\Delta_G(t)^{\frac{1}{2}}\big)\Delta_G(u)^{\frac{1}{2}}\,\mathrm{d}k\,\mathrm{d}u\,\mathrm{d}t \\
&= \int_G\int_H g(k,u,s)\alpha_u(\psi(f\otimes x)(k^{-1}h, u^{-1}sk))\Delta_G(u)^{\frac{1}{2}}\,\mathrm{d}k\,\mathrm{d}u \\
&\overset{(3.5)}{=} g\cdot\psi(f\otimes x)(h,s).
\end{aligned}$$

For $f \otimes x \in C_c(G \times G, A) \odot C_c(G, A)$ and $g \in C_c(H, A)$:

$$\begin{aligned}
\psi(f\otimes x\cdot g)(h,s) &= \int_G f(t,s)\alpha_t(x\cdot g(t^{-1}sh))\Delta_H(h)^{-\frac{1}{2}}\Delta_G(t)^{\frac{1}{2}}\,\mathrm{d}t \\
&= \int_G\int_H f(t,s)\alpha_t(x(t^{-1}shk))\alpha_{shk}(g(k^{-1}))\Delta_H(hk)^{-\frac{1}{2}}\Delta_G(t)^{\frac{1}{2}}\,\mathrm{d}k\,\mathrm{d}t \\
&\overset{k\mapsto h^{-1}k}{=} \int_G\int_H f(t,s)\alpha_t(x(t^{-1}sk))\alpha_{sk}(g(k^{-1}h))\Delta_H(k)^{-\frac{1}{2}}\Delta_G(t)^{\frac{1}{2}}\,\mathrm{d}k\,\mathrm{d}t \\
&= \int_H \psi(f\otimes x)(k,s)\alpha_{sk}(g(k^{-1}h))\,\mathrm{d}k \\
&\overset{(3.6)}{=} \psi(f\otimes x)\cdot g\,(h,s).
\end{aligned}$$

For $f \otimes x$ and $g \otimes y$ in $C_c(G \times G, A) \odot C_c(G, A)$:

$$\begin{aligned}
&\langle f\otimes x, g\otimes y\rangle_{A\times H}(h) = \langle x, \langle f,g\rangle_{C_0(G/H,A)\times G}\cdot y\rangle_{A\times H}(h) \\
&\overset{(3.4)}{=} \int_G \alpha_s(x(s^{-1})^*\langle f,g\rangle_R\cdot y(s^{-1}h))\Delta_H(h)^{-\frac{1}{2}}\,\mathrm{d}s \\
&\overset{(3.1)}{=} \int_G\int_G \alpha_s\big(x(s^{-1})^*\langle f,g\rangle_R(t,s^{-1}H)\alpha_t(y(t^{-1}s^{-1}h))\Delta_G(t)^{\frac{1}{2}}\big)\Delta_H(h)^{-\frac{1}{2}}\,\mathrm{d}t\,\mathrm{d}s \\
&\overset{(1.5)}{=} \int_G\int_G\int_G\int_H \alpha_s\big(x(s^{-1})^*\alpha_u(f(u^{-1},u^{-1}s^{-1}k)^*g(u^{-1}t,u^{-1}s^{-1}k)) \\
&\qquad\qquad \cdot\Delta_G(u^{-1})\alpha_t(y(t^{-1}s^{-1}h))\Delta_G(t)^{\frac{1}{2}}\big)\Delta_H(h)^{-\frac{1}{2}}\,\mathrm{d}k\,\mathrm{d}u\,\mathrm{d}t\,\mathrm{d}s
\end{aligned}$$

$$\overset{u\mapsto u^{-1}}{=} \int\limits_G \int\limits_G \int\limits_G \int\limits_H \alpha_s(x(s^{-1})^*)\alpha_{su^{-1}}(f(u, us^{-1}k)^*)\alpha_{su^{-1}}(g(ut, us^{-1}k)) \cdot \alpha_{st}(y(t^{-1}s^{-1}h))\Delta_G(t)^{\frac{1}{2}}\Delta_H(h)^{-\frac{1}{2}}\,\mathrm{d}k\,\mathrm{d}u\,\mathrm{d}t\,\mathrm{d}s$$

$$\overset{\substack{t\mapsto u^{-1}t\\ s\mapsto su}}{=} \int\limits_G \int\limits_G \int\limits_G \int\limits_H \alpha_s\big((f(u, s^{-1}k)\alpha_u(x(u^{-1}s^{-1}))\Delta_G(u)^{\frac{1}{2}})^* g(t, s^{-1}k) \cdot \alpha_t(y(t^{-1}s^{-1}h))\Delta_H(h^{-1})^{\frac{1}{2}}\Delta_G(t)^{\frac{1}{2}}\big)\,\mathrm{d}k\,\mathrm{d}u\,\mathrm{d}t\,\mathrm{d}s$$

$$= \int\limits_G \int\limits_H \alpha_s(\psi(f\otimes x)(k^{-1}, s^{-1}k)^*\psi(g\otimes y)(k^{-1}h, s^{-1}k))\Delta_H(k^{-1})\,\mathrm{d}k\,\mathrm{d}s$$

$$\overset{(3.8)}{=} \langle \psi(f\otimes x), \psi(g\otimes y)\rangle_{A\times H}(h).$$

It follows that ψ extends to a linear map of $Z^G_{G/H}(A\times G)\otimes_{C_0(G/H,A)\times G} X^G_H(A)$ into $X^G_{\{e\}}(A)\times H$ which also preserves the actions and right inner product, and which therefore by Lemma 3.2 is actually an isomorphism of the imprimitivity bimodules. ∎

Corollary 3.3. *Let $\alpha : G \to \operatorname{Aut} A$ be an action of a locally compact group on a C^*-algebra, and let H be a closed subgroup of G. Then we have a commutative diagram*

$$\begin{array}{ccc} \operatorname{Rep} A\times_\alpha H & \longrightarrow & \operatorname{Rep} C_0(G/H, A)\times_{\alpha\otimes\tau} G \\ \operatorname{Res}^H_{\{e\}}\Big\downarrow & & \Big\downarrow \operatorname{Ind}^G_{G/H} \\ \operatorname{Rep} A & \longrightarrow & \operatorname{Rep}(A\times_\alpha G)\times_{\widehat{\alpha}} G \end{array}$$

in which the horizontal arrows are the bijections induced by the Green bimodules $X^G_H(A)$ and $X^G_{\{e\}}(A)$.

Proof. We shall show rather more: each arrow is implemented by a right-Hilbert bimodule, so the two compositions are implemented by tensor products of these bimodules, and we shall show that

$$Z^G_{G/H}(A\times G)\otimes_{C_0(G/H,A)\times G} X^G_H(A) \cong X^G_{\{e\}}(A)\otimes_A (A\times_\alpha H)$$

as right-Hilbert $(A\times_\alpha G)\times_{\widehat{\alpha}} G - A\times_\alpha H$ bimodules. But the bimodule on the right-hand side is isomorphic to the right-Hilbert $(A\times_\alpha G)\times_{\widehat{\alpha}} G - A\times_\alpha H$ bimodule $X^G_{\{e\}}(A)\times H$ by a special case of Lemma 5.7 of [9], so the isomorphism follows from Theorem 3.1. ∎

4. COMPARISON WITH MANSFIELD'S BIMODULE

Here we compare our inducing process for dual coactions with that of [8], which extends Mansfield's process to nonamenable subgroups. For each coaction $\delta : B \to M(B \otimes C^*(G))$ and normal subgroup N, [8], Theorem 3.3 provides an imprimitivity bimodule $Y^G_{G/N}$ between the reduced crossed products $(B \times_\delta G) \times_{\widehat{\delta},\mathrm{r}} N$ and

$$B \times_{\delta,\mathrm{r}} G/N := j_B \times j_G|(B \times_{\delta|} G/N)$$

(see Remark 2.2 (ii)).

We consider an action $\alpha : G \to \operatorname{Aut} A$, the dual coaction $\widehat{\alpha}$ on $A \times_\alpha G$, and a closed normal subgroup N of G. To define the reduced crossed product, we fix a faithful representation π of A on $\mathcal{H}$, and use the covariant representation $\operatorname{Ind} \pi := \widetilde{\pi} \times (1 \otimes \lambda)$ of $A \times_\alpha G$ on $\mathcal{H} \otimes L^2(G)$. Note that $\widetilde{\pi}$ is faithful, so Proposition 2.8 gives us a faithful representation $(\widetilde{\pi} \otimes M|) \times (1 \otimes \lambda \otimes \lambda)$ of $(A \otimes C_0(G/N)) \times_{\alpha\otimes\tau,\mathrm{r}} G$ onto $(A \times_\alpha G) \times_{\widehat{\alpha},\mathrm{r}} G/N$.

We now recall the construction of the bimodule from [13], [8]. Consider the map $\varphi : C_\mathrm{c}(G) \to C_\mathrm{c}(G/N)$ defined by

$$\varphi(f)(sN) = \int_N f(sn)\,\mathrm{d}n.$$

Then $\mathcal{D}_N$ is a $*$-subalgebra of $B(\mathcal{H} \otimes L^2(G) \otimes L^2(G))$ containing in particular the elements of the form

$$\operatorname{Ind} \pi \otimes \lambda(\widehat{\alpha}(\widehat{\alpha}_u(b)))(1 \otimes 1 \otimes M(\varphi(f))) \quad \text{and} \quad (1 \otimes 1 \otimes M(\varphi(f)))\operatorname{Ind} \pi \otimes \lambda(\widehat{\alpha}(\widehat{\alpha}_u(b))),$$

for $b \in A \times_\alpha G$, $u \in A_\mathrm{c}(G)$, and $f \in C_\mathrm{c}(G)$. (By definition, $\widehat{\alpha}_u$ is the composition of $\widehat{\alpha}$ with the slice map $S_u := \operatorname{id} \otimes u : M(A \times_\alpha G \otimes C^*(G)) \to M(A \times_\alpha G)$.) $\mathcal{D}$ is by definition $\mathcal{D}_{\{e\}}$. Mansfield shows that there is a well-defined map $\Psi : \mathcal{D} \to \mathcal{D}_N$ such that

(4.1)

$$\Psi\left(\operatorname{Ind} \pi \otimes \lambda(\widehat{\alpha}(\widehat{\alpha}_u(b)))(1 \otimes 1 \otimes M(f))\right) = \operatorname{Ind} \pi \otimes \lambda(\widehat{\alpha}(\widehat{\alpha}_u(b)))(1 \otimes 1 \otimes M(\varphi(f))).$$

Then $\mathcal{D}$ has a $\mathcal{D}_N$-valued pre-inner product given by

$$\langle c, d \rangle_{\mathcal{D}_N} = \Psi(c^* d).$$

With the left action of Mansfield's dense $*$-subalgebra $\mathcal{I}_N \subseteq C_\mathrm{c}(N, \mathcal{D})$ of $(A \times_\alpha G \times_{\widehat{\alpha}} G) \times_{\widehat{\widehat{\alpha}},\mathrm{r}} N$ given by

$$f \cdot d = \int_N f(n)\widehat{\widehat{\alpha}}_n(d)\,\Delta_N(n)^{\frac{1}{2}}\,\mathrm{d}n \tag{4.2}$$

and the right action of $\mathcal{D}_N$ given by $d \cdot z = dz$, $\mathcal{D}$ becomes an $\mathcal{I}_N$ – $\mathcal{D}_N$ pre-imprimitivity bimodule, whose completion $Y^G_{G/N}(A\times G)$ is an $(A\times_\alpha G\times_{\widehat{\alpha}} G)\times_{\widehat{\widehat{\alpha}},\mathrm{r}} N$ – $(A\times_\alpha G)\times_{\widehat{\alpha},\mathrm{r}} G/N$ imprimitivity bimodule ([8], Theorem 3.3; [13], Theorem 27). Recall that our bimodule $Z^G_{G/N}(A\times_\alpha G)$ is an imprimitivity bimodule between the full crossed products $(A\times_\alpha G\times_{\widehat{\alpha}} G)\times_{\widehat{\widehat{\alpha}}} N$ and $(A\otimes C_0(G/N))\times_{\alpha\otimes\tau} G$.

THEOREM 4.1. *Let $\alpha : G \to \operatorname{Aut} A$ be an action of a locally compact group G on a C^*-algebra A, and let N be a closed normal subgroup of G. Let*

$$\Upsilon := (\widetilde{\pi}\times M|)\times(1\otimes\lambda\otimes\lambda) : (A\otimes C_0(G/N))\times_{\alpha\otimes\tau} G \to (A\times_\alpha G)\times_{\widehat{\alpha},\mathrm{r}} G/N,$$

and let $\Phi : (A\times_\alpha G\times_{\widehat{\alpha}} G)\times_{\widehat{\widehat{\alpha}}} N \to \mathcal{L}(Y^G_{G/N})$ be the extension of the left action (4.2). *Then there exists a linear map Θ of $Z^G_{G/N}(A\times G)$ onto $Y^G_{G/N}(A\times G)$ such that (Φ,Θ,Υ) is a surjective imprimitivity bimodule homomorphism. In particular, if $I = \ker\Upsilon$, then*

$$Z^G_{G/N,\mathrm{r}}(A\times G) := Z^G_{G/N}(A\times G)/\big(Z^G_{G/N}(A\times G)\cdot I\big) \cong Y^G_{G/N}(A\times G)$$

as $(A\times_\alpha G\times_{\widehat{\alpha}} G)\times_{\widehat{\widehat{\alpha}},\mathrm{r}} N$ – $(A\times_\alpha G)\times_{\widehat{\alpha},\mathrm{r}} G/N$ imprimitivity bimodules.

Proof. It is sufficient to produce a linear map Θ of a dense subspace $Z_0 \subset Z$ into $\mathcal{D}$ such that

$$\Theta(f\cdot x) = \Phi(f)\cdot\Theta(x), \tag{4.3}$$

$$\Theta(x\cdot g) = \Theta(x)\cdot\Upsilon(g), \tag{4.4}$$

and

$$\langle\Theta(x),\Theta(y)\rangle_{(A\times_\alpha G)\times_{\hat{\alpha},\mathrm{r}} G/N} = \Upsilon\big(\,\langle x,y\rangle_{C_0(G/N,A)\times_{\alpha\otimes\tau} G}\,\big), \tag{4.5}$$

for $f\in C_{\mathrm{c}}(N\times G\times G, A)$, $g\in C_{\mathrm{c}}(G\times G/N, A)$, and $x,y\in Z_0$; for then Θ extends to a linear map of $Z^G_{G/N}(A\times G)$ into $Y^G_{G/N}(A\times G)$ which also satisfies (4.3)–(4.5), and hence factors through an imprimitivity bimodule isomorphism of $Z^G_{G/N,\mathrm{r}}(A\times G)$ onto $Y^G_{G/N}(A\times G)$ by Lemma 3.2.

Let Θ be the restriction of $(\tilde{\pi}\otimes M)\times(1\otimes\lambda\otimes\lambda)$ to the subalgebra $C_{\mathrm{c}}(G\times G, A)$ of $(A\otimes C_0(G))\times_{\alpha\otimes\tau} G$, and let

$$Z_0 = \operatorname{span}\{a\otimes z\otimes f \mid a\in A;\ z,f\in C_{\mathrm{c}}(G)\} \subseteq C_{\mathrm{c}}(G\times G, A).$$

By Lemma 2.4 we have

$$\Theta(a\otimes z\otimes f)=(1\otimes 1\otimes M_f)\mathrm{Ind}\,\pi\otimes\lambda(\widehat{\alpha}(a\otimes z)). \tag{4.6}$$

Choosing $u\in A_{\mathrm{c}}(G)$ to be identically 1 on $\mathrm{supp}(z)$, we have

$$\widehat{\alpha}(a\otimes z)=\widehat{\alpha}(a\otimes uz)=\widehat{\alpha}(\widehat{\alpha}_u(a\otimes z)),$$

because $S_u(\widehat{\alpha}(g))$ is the pointwise product ug ([18], Lemma 1.3). Thus Θ maps Z_0 into $\mathcal{D}$. To see that Z_0 is dense in $Z_{G/N}^G(A\times G)$, note that the inductive limit topology dominates the imprimitivity bimodule norm topology on $C_{\mathrm{c}}(G\times G,A)$ ([16], p. 374).

To verify (4.3), notice that for $f\in C_{\mathrm{c}}(N\times G\times G,A)=C_{\mathrm{c}}(N,C_{\mathrm{c}}(G\times G,A))$ and $x\in Z_0$, (1.2) can be re-written in terms of the multiplication $*$ on $C_{\mathrm{c}}(G\times G,A)\subseteq(A\times_\alpha G)\times_{\widehat{\alpha}}G$ as

$$f\cdot x=\int_N f(n)*\widehat{\widehat{\alpha}}_n(x)\,\Delta_N(n)^{\frac12}\,\mathrm{d}n.$$

If we identify $(A\otimes C_0(G))\times_{\alpha\otimes\tau}G$ with $(A\times_\alpha G)\times_{\widehat{\alpha}}G$ as in Lemma 2.4, then (4.6) says that Θ is the restriction of the regular representation $((\mathrm{Ind}\,\pi\otimes\lambda)\circ\widehat{\alpha})\times(1\otimes M)$ to the dense subalgebra $C_{\mathrm{c}}(G\times G,A)$. This is a $*$-homomorphism of $C_{\mathrm{c}}(G\times G,A)$ into $\mathcal{D}$, which implements the action of $(A\times_\alpha G)\times_{\widehat{\alpha}}G$ on Mansfield's bimodule, so the action Φ of $((A\times_\alpha G)\times_{\widehat{\alpha}}G)\times_{\widehat{\widehat{\alpha}}}N$ is given in terms of the action (4.2) by

$$\Phi(f)\cdot d=\int_N \Theta(f(n))\widehat{\widehat{\alpha}}_n(d)\,\Delta_N(n)^{\frac12}\,\mathrm{d}n.$$

Thus

$$\begin{aligned}\Phi(f)\cdot\Theta(x)&=\int_N \Theta(f(n))\widehat{\widehat{\alpha}}_n(\Theta(x))\,\Delta_N(n)^{\frac12}\,\mathrm{d}n=\int_N \Theta(f(n)*\widehat{\widehat{\alpha}}_n(x))\,\Delta_N(n)^{\frac12}\,\mathrm{d}n\\&=\Theta\left(\int_N f(n)*\widehat{\widehat{\alpha}}_n(x)\,\Delta_N(n)^{\frac12}\,\mathrm{d}n\right)=\Theta(f\cdot x),\end{aligned}$$

which gives (4.3).

To verify (4.4) and (4.5), we first let $a\otimes z\otimes f\in Z_0$ and $\xi\in L^2(G\times G,\mathcal{H})\cong\mathcal{H}\otimes L^2(G)\otimes L^2(G)$, and compute:

$$\begin{aligned}(\Theta(a\otimes z\otimes f)\xi)(r,s)&\overset{(4.6)}{=}\big((1\otimes 1\otimes M_f)\mathrm{Ind}\,\pi\otimes\lambda(\widehat{\alpha}(a\otimes z))\xi\big)(r,s)\\&=\big((1\otimes 1\otimes M_f)(\tilde{\pi}\otimes 1)(a)(1\otimes\lambda\otimes\lambda)(z)\xi\big)(r,s)\\&=\int_G \pi\big(\alpha_{r^{-1}}(az(t)f(s))\big)\xi(t^{-1}r,t^{-1}s)\,\mathrm{d}t.\end{aligned}$$

Thus, since Z_0 is inductive-limit dense in $C_c(G \times G, A)$, it follows that for all g in the subalgebra $C_c(G \times G, A)$ of $(A \otimes C_0(G)) \times_{\alpha\otimes\tau} G$, we have

$$\Theta(g)\xi(s,t) = \int_G \pi\big(\alpha_{r^{-1}}(g(t,s))\big)\xi(t^{-1}r, t^{-1}s)\,\mathrm{d}t. \tag{4.7}$$

Since Υ is the restriction of Θ to the image of $(A \otimes C_0(G/N)) \times_{\alpha\otimes\tau} G$ in $M((A \otimes C_0(G)) \times_{\alpha\otimes\tau} G)$, where for the moment we identify Θ with its extension to $(A \otimes C_0(G)) \times_{\alpha\otimes\tau} G$, it also follows that

$$\Upsilon(g)\xi(s,t) = \int_G \pi\big(\alpha_{r^{-1}}(g(t,sN))\big)\xi(t^{-1}r, t^{-1}s)\,\mathrm{d}t. \tag{4.8}$$

Notice that for $x \in Z_0$ and $g \in C_c(G \times G/N, A)$, Equation (1.3) can be re-written as $x \cdot g = x * g$, where $x * g$ denotes convolution of $x \in Z_0$ with $g \in C_c(G \times G/N, A)$. Thus (4.4) follows from

$$\Theta(x \cdot g) = \Theta(x * g) = \Theta(x)\Theta(g) = \Theta(x) \cdot \Upsilon(g).$$

Before checking (4.5), we need to do some background calculations. First, since Θ is involutive on $C_c(G \times G, A)$, we have for x and y in Z_0

$$\langle \Theta(x), \Theta(y)\rangle_{(A\times_\alpha G)\times_{\hat\alpha,\mathrm{r}} G/N} = \Psi(\Theta(x)^*\Theta(y)) = \Psi(\Theta(x^* * y));$$

thus to establish (4.5), it is enough to verify that

$$\Psi(\Theta(x^* * y)) = \Upsilon\left(\langle x, y\rangle_{C_0(G/N,A)\times_{\alpha\otimes\tau} G}\right).$$

Next, note that by (4.6) and (4.1) for $a \otimes z \otimes f \in Z_0$ we have

$$\Psi(\Theta(a \otimes z \otimes f)) = \Theta(a \otimes z \otimes \varphi(f));$$

thus we can compute:

$$\begin{aligned}
\big(\Psi(\Theta(a \otimes z \otimes f))\xi\big)(r,s) &= \big(\Theta(a \otimes z \otimes \varphi(f))\xi\big)(r,s) \\
&\overset{(4.7)}{=} \int_G \pi(\alpha_{r^{-1}}(az(t)\varphi(f)(sN)))\xi(t^{-1}r, t^{-1}s)\,\mathrm{d}t \\
&= \int_G \int_N \pi(\alpha_{r^{-1}}(az(t)f(sn)))\xi(t^{-1}r, t^{-1}s)\,\mathrm{d}n\,\mathrm{d}t \\
&= \int_G \int_N \pi(\alpha_{r^{-1}}(a \otimes z \otimes f(t, sn)))\xi(t^{-1}r, t^{-1}s)\,\mathrm{d}n\,\mathrm{d}t.
\end{aligned}$$

Hence by continuity we have

$$(\Psi(\Theta(g))\xi)(r,s) = \int_G \int_N \pi(\alpha_{r^{-1}}(g(t,sn)))\xi(t^{-1}r, t^{-1}s)\,\mathrm{d}n\,\mathrm{d}t \tag{4.9}$$

for $g \in C_c(G \times G, A)$.

Now to check (4.5), we fix $x, y \in Z_0$ and compute:

$$\begin{aligned}
&\big(\Upsilon\big(\langle x,y\rangle_{C_0(G/N,A)\times_{\alpha\otimes\tau}G}\big)\xi\big)(r,s)\\
&\overset{(4.8)}{=} \int_G \pi\big(\alpha_{r^{-1}}(\langle x,y\rangle_{C_0(G/N,A)\times_{\alpha\otimes\tau}G}(t,sN))\big)\xi(t^{-1}r,t^{-1}s)\,\mathrm{d}t\\
&\overset{(1.5)}{=} \int_G\int_G\int_N \pi\left(\alpha_{r^{-1}u}(x(u^{-1},u^{-1}sn)^* y(u^{-1}t,u^{-1}sn))\right)\\
&\qquad\qquad \cdot\xi(t^{-1}r,t^{-1}s)\Delta_G(u^{-1})\,\mathrm{d}n\,\mathrm{d}u\,\mathrm{d}t\\
&= \int_G\int_N \pi\left(\alpha_{r^{-1}}\left(\int_G x^*(u,sn)\alpha_u(y(u^{-1}t,u^{-1}sn))\,\mathrm{d}u\right)\right)\xi(t^{-1}r,t^{-1}s)\,\mathrm{d}n\,\mathrm{d}t\\
&= \int_G\int_N \pi\big(\alpha_{r^{-1}}(x^* * y(t,sn))\big)\xi(t^{-1}r,t^{-1}s)\,\mathrm{d}n\,\mathrm{d}t\\
&\overset{(4.9)}{=} \big(\Psi(\Theta(x^* * y))\big)\xi(r,s).
\end{aligned}$$

This completes the proof. ∎

5. APPENDIX

We prove the following weak version of Mansfield's imprimitivity theorem for the reduced crossed product $B \times_{\delta,\mathrm{r}} G/H$ of Section 2:

THEOREM 5.1. *Let $\delta : B \to M(B \otimes C^*(G))$ be a nondegenerate coaction of G on B and H a closed subgroup of G. Then the reduced crossed product $B \times_{\delta,\mathrm{r}} G/H$ is Morita equivalent to $(B \times_\delta G) \times_{\widehat{\delta},\mathrm{r}} H$.*

We saw at the end of Section 2 that the theorem is true for dual coactions, so we use the Morita equivalence of δ and $\widehat{\widehat{\delta}}$ to reduce to this case. There is one subtlety involved: if $\delta : B \to M(B \otimes C^*(G))$ is an arbitrary full coaction, it may not be true that δ is Morita equivalent to $\widehat{\widehat{\delta}}$. However, from Katayama's Duality Theorem ([10]) we can deduce that this is true for nondegenerate reduced coactions (see Proposition 5.4 below).

We recall the definition of the reduction of a coaction $\delta : B \to M(B \otimes C^*(G))$ from [18], [14]. Let $p : B \to B^{\mathrm{r}} := B/\ker j_B$ denote the quotient map. Then there is a well-defined homomorphism $\delta^{\mathrm{r}} : B^{\mathrm{r}} \to M(B^{\mathrm{r}} \otimes C^*_{\mathrm{r}}(G))$ such that $\delta^{\mathrm{r}} \circ p = (p \otimes \lambda) \circ \delta$, and δ^{r} is a reduced coaction of G on B^{r} which is nondegenerate if δ is ([18], Lemma 3.1; [14], Corollary 3.4). The canonical map j_B factors through an embedding $j_{B^{\mathrm{r}}}$ of B^{r} in $M(B \times_\delta G)$, and then $(B \times_\delta G, j_{B^{\mathrm{r}}}, j_{C(G)})$ is a crossed product for the reduced system $(B^{\mathrm{r}}, G, \delta^{\mathrm{r}})$. Thus both reduced crossed products in Theorem 5.1 depend only on the reduced system, and Theorem 5.1 will be a corollary of:

THEOREM 5.2. *Let $\delta : B \to M(B \otimes C^*_{\mathrm{r}}(G))$ be a nondegenerate reduced coaction of G on B and assume that B is represented faithfully and nondegenerately on a Hilbert space $\mathcal{H}$. Then*

$$B \times_{\delta,\mathrm{r}} G/H = \overline{\operatorname{span}}\{\delta(b)(1 \otimes M_f) : b \in B, f \in C_0(G/H)\}$$

is Morita equivalent to $(B \times_\delta G) \times_{\widehat{\delta},\mathrm{r}} H$.

From now on, all coactions will be reduced. Recall that a Morita equivalence (X, δ_X) between two cosystems (A, G, δ_A) and (B, G, δ_B) consists of an $A - B$ imprimitivity bimodule X together with a linear map

$$\delta_X : X \to M({}_{A \otimes C^*_{\mathrm{r}}(G)}(X \otimes C^*_{\mathrm{r}}(G))_{B \otimes C^*_{\mathrm{r}}(G)})$$

such that $(\delta_A, \delta_X, \delta_B)$ is an imprimitivity bimodule homomorphism, and such that δ_X satisfies the coaction identity $(\delta_X \otimes \mathrm{id}_G) \circ \delta_X = (\mathrm{id}_X \otimes \delta_G) \circ \delta_X$ (see [5] for more details).

EXAMPLE 5.3. (i) *Stabilised coactions.* Suppose that $\delta : B \to M(B \otimes C^*_{\mathrm{r}}(G))$ is a coaction. Let $\sigma : C^*_{\mathrm{r}}(G) \otimes \mathcal{K}(\mathcal{H}) \to \mathcal{K}(\mathcal{H}) \otimes C^*_{\mathrm{r}}(G)$ denote the flip map. Then $\delta^{\mathrm{s}} = (\mathrm{id}_B \otimes \sigma) \circ (\delta \otimes \mathrm{id}_{\mathcal{K}})$ is a coaction of G on $B \otimes \mathcal{K}(\mathcal{H})$, called the *stabilised coaction* of δ. Let $X := B \otimes \mathcal{H}$ viewed as an $B \otimes \mathcal{K}(\mathcal{H}) - B$ imprimitivity bimodule. Then the map $\delta_X := (\mathrm{id}_B \otimes \sigma_{\mathcal{H}}) \circ (\delta \otimes \mathrm{id}_{\mathcal{H}})$ of X into $M(X \otimes C^*_{\mathrm{r}}(G))$ is a Morita equivalence for δ^{s} and δ, where now $\sigma_{\mathcal{H}}$ denotes the flip map between the imprimitivity bimodules ${}_{C^*_{\mathrm{r}}(G) \otimes \mathcal{K}(\mathcal{H})}(C^*_{\mathrm{r}}(G) \otimes \mathcal{H})_{C^*_{\mathrm{r}}(G)}$ and ${}_{\mathcal{K}(\mathcal{H}) \otimes C^*_{\mathrm{r}}(G)}(\mathcal{H} \otimes C^*_{\mathrm{r}}(G))_{C^*_{\mathrm{r}}(G)}$.

(ii) *Exterior equivalent coactions.* A δ-one cocycle for a coaction $\delta : B \to M(B \otimes C^*_{\mathrm{r}}(G))$ is a unitary $V \in UM(B \otimes C^*_{\mathrm{r}}(G))$ satisfying $(\mathrm{id}_B \otimes \delta_G)(V) = (V \otimes 1)\big((\delta \otimes \mathrm{id}_G)(V)\big)$ and $V\delta(b)V^*(1 \otimes z) \in B \otimes C^*_{\mathrm{r}}(G)$ for all $b \in B, z \in C^*_{\mathrm{r}}(G)$ (see [11], Definition 2.7). Then $\varepsilon = \operatorname{Ad} V \circ \delta$ is a coaction of G on B. If $X = B$ is the trivial $B - B$ imprimitivity bimodule, then $\delta_X : b \mapsto V\delta(b)$ is a Morita equivalence between ε and δ.

PROPOSITION 5.4. *Suppose that $\delta : B \to M(B \otimes C^*_{\mathrm{r}}(G))$ is a nondegenerate reduced coaction. Then δ is Morita equivalent to the double dual coaction $\widehat{\widehat{\delta}}$ of G on $(B \times_\delta G) \times_{\widehat{\delta},\mathrm{r}} G$.*

Proof. It follows from Theorem 8 of [10] that there is an isomorphism of $(B \times_\delta G) \times_{\widehat{\delta},\mathrm{r}} G$ onto $B \otimes \mathcal{K}(L^2(G))$ carrying $\widehat{\widehat{\delta}}$ to the coaction $\operatorname{Ad} V \circ \delta^{\mathrm{s}}$, where $V = 1 \otimes W_G^* \in UM(B \otimes \mathcal{K}(L^2(G)) \otimes C^*_{\mathrm{r}}(G))$ is a δ^{s}-one cocycle. Thus δ is Morita equivalent to $\widehat{\widehat{\delta}}$ by Example 5.3. ∎

PROPOSITION 5.5. *If (X, δ_X) is a Morita equivalence for the cosystems (A, G, δ_A) and (B, G, δ_B), and H is a closed subgroup of G, then there is an $A \times_{\delta_A,\mathrm{r}} G/H$ – $B \times_{\delta_B,\mathrm{r}} G/H$ imprimitivity bimodule $X \times_{\delta_X,\mathrm{r}} G/H$.*

Proof. Let $L = \begin{pmatrix} A & X \\ \widetilde{X} & B \end{pmatrix}$ denote the linking algebra for ${}_AX_B$, and let $\delta_L = \begin{pmatrix} \delta_A & \delta_X \\ \delta_{\widetilde{X}} & \delta_B \end{pmatrix}$ denote the corresponding coaction of G on L (see [5], Appendix). We can represent L faithfully on $\mathcal{H} \oplus \mathcal{K}$ in such a way that the corners $A = pLp$ and $B = qLq$, $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, act faithfully and nondegenerately on $\mathcal{H}$ and $\mathcal{K}$. Then

$$L \times_{\delta_L,\mathrm{r}} G/H = \overline{\operatorname{span}}\{\delta_L(l)(1 \otimes M_f) : l \in L, f \in C_0(G/H)\},$$

and if $p \otimes 1, q \otimes 1$ denote the projections of $(\mathcal{H} \oplus \mathcal{K}) \otimes L^2(G) \cong (\mathcal{H} \otimes L^2(G)) \oplus (\mathcal{K} \otimes L^2(G))$ onto its factors, then

$$(p \otimes 1)\big(L \times_{\delta_L,\mathrm{r}} G/H\big)(p \otimes 1) = A \times_{\delta_A,\mathrm{r}} G/H,$$

and

$$(q \otimes 1)\big(L \times_{\delta_L,\mathrm{r}} G/H\big)(q \otimes 1) = B \times_{\delta_B,\mathrm{r}} G/H.$$

We claim that

$$X \times_{\delta_X,\mathrm{r}} G/H := (p \otimes 1)\big(L \times_{\delta_L,\mathrm{r}} G/H\big)(q \otimes 1)$$

is an $A \times_{\delta_A,\mathrm{r}} G/H - B \times_{\delta_B,\mathrm{r}} G/H$ imprimitivity bimodule. For this we only have to check that $A \times_{\delta_A,\mathrm{r}} G/H$ and $B \times_{\delta_B,\mathrm{r}} G/H$ are full corners in $L \times_{\delta_L,\mathrm{r}} G/H$. But since $p \otimes 1 = \delta_L(p)$ it follows that

$$\begin{aligned}
\big(L \times_{\delta_L,\mathrm{r}} G/H\big)&(p \otimes 1)\big(L \times_{\delta_L,\mathrm{r}} G/H\big) \\
&= \big((1 \otimes M(C_0(G/H))\delta_L(L)\big)(p \otimes 1)\big(\delta_L(L)(1 \otimes M(C_0(G/H))\big) \\
&= (1 \otimes M(C_0(G/H))\delta_L(LpL)(1 \otimes M(C_0(G/H))
\end{aligned}$$

which is dense in $L \times_{\delta_L,\mathrm{r}} G/H$ because LpL is dense in L. The argument for $B \times_{\delta_B,\mathrm{r}} G/H$ is the same. ∎

Proof of Theorem 5.2. Let (X, δ_X) be the Morita equivalence between $\widehat{\widehat{\delta}}$ and δ of Proposition 5.4. Then Proposition 5.5 provides a Morita equivalence $X \times_{\delta_X,\mathrm{r}} G/H$ between $B \times_{\delta,\mathrm{r}} G/H$ and $(B \times_\delta G \times_{\widehat{\delta},\mathrm{r}} G) \times_{\widehat{\widehat{\delta}},\mathrm{r}} G/H$. Now Green's imprimitivity theorem together with [15] provides a Morita equivalence X_H^G between $(B \times_\delta G) \times_{\widehat{\delta},\mathrm{r}} H$ and $(B \times_\delta G \times_{\widehat{\delta},\mathrm{r}} G) \times_{\widehat{\widehat{\delta}},\mathrm{r}} G/H$. Thus

$$\widetilde{X}_H^G \otimes_{B \times G \times_{\mathrm{r}} G \times_{\mathrm{r}} G/H} (X \times_{\delta_X,\mathrm{r}} G/H)$$

is a $(B \times_\delta G) \times_{\widehat{\delta},\mathrm{r}} H - B \times_{\delta,\mathrm{r}} G/H$ imprimitivity bimodule. ∎

REMARK 5.6. As we pointed out in the introduction, it would be preferable to have a more concrete bimodule implementing the equivalence. We do not know whether the original construction of Mansfield can be modified to avoid the assumption of normality.

This research was supported by the Australian Research Council.

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Received September 7, 1996; revised April 7, 1997.