CERTAIN STRUCTURE OF SUBDIAGONAL ALGEBRAS

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Abstract. Let $A$ be a maximal subdiagonal algebra of a von Neumann algebra $M$ with respect to a faithful normal expectation $\Phi$. Then we show that if $\varphi$ is a faithful normal state of $M$ such that $\varphi \circ \Phi = \varphi$, then $A$ is $\sigma_\varphi^t$-invariant, where $\{\sigma_\varphi^t\}_{t \in \mathbb{R}}$ is the modular automorphism group associated with $\varphi$. As an application, we prove that every $\sigma$-weakly closed subdiagonal algebra of $B(H)$ is a nest algebra with an atomic nest.

Keywords: von Neumann algebra, subdiagonal algebra.


1. INTRODUCTION AND PRELIMINARIES

In [1], Arveson introduced the notion of subdiagonal algebras to study the analyticity in operator algebras. At first, we start by given the definition of subdiagonal algebras. Let $M$ be a von Neumann algebra on a separable complex Hilbert space $H$, and let $\Phi$ be a faithful normal positive idempotent linear map from $M$ onto a von Neumann subalgebra $D$ of $M$. A $\sigma$-weakly closed subalgebra $A$ of $M$, containing $D$, is called subdiagonal algebra in $M$ with respect to $\Phi$ if

(i) $A \cap A^* = D$,

(ii) $\Phi$ is multiplicative on $A$, and

(iii) $A + A^*$ is $\sigma$-weakly dense in $M$.

The algebra $D$ is called the diagonal of $A$. We say that $A$ is a maximal subdiagonal algebra in $M$ with respect to $\Phi$ in case $A$ is not properly contained in any other subalgebra of $M$ which is subdiagonal with respect to $\Phi$. 
Although subdiagonal algebras are not assumed to be \( \sigma \)-weakly closed in [1], the \( \sigma \)-weak closure of a subdiagonal algebra is again a subdiagonal algebra ([1], Remark 2.1.2). Thus we assume that our subdiagonal algebras are always \( \sigma \)-weakly closed.

In [1], Arveson asked whether all (\( \sigma \)-weakly closed) subdiagonal algebras are maximal subdiagonal algebras. In [3], Exel gave an affirmative answer to this problem in finite case. That is, Exel showed that if \( A \) is finite in the sense that there exists a faithful normal finite trace \( \tau \) on \( M \) such that \( \tau \circ \Phi = \tau \), then \( A \) is automatically maximal subdiagonal.

In general, let \( A \) be a subdiagonal algebra of \( M \) with respect to \( \Phi \). Then there exists a faithful normal state \( \varphi \) of \( M \) such that \( \varphi \circ \Phi = \varphi \) by the separability of \( H \).

First, we shall prove that, if \( A \) is maximal, then \( A \) is invariant under the modular automorphism group \( \{ \sigma^t_\varphi \}_{t \in \mathbb{R}} \) of \( \varphi \) (cf. [5]). Further, as an application, we shall show that every subdiagonal algebra \( A \) of \( B(H) \) is a nest algebra with an atomic nest.

2. \( \sigma^t_\varphi \)-INVARIANCE OF SUBDIAGONAL ALGEBRAS

Let \( M \) be a von Neumann algebra, acting on a separable Hilbert space \( H \), and let \( A \) be a subdiagonal algebra of \( M \) with respect to \( \Phi \). Put \( D = A \cap A^* \). Then there exists a faithful normal state \( \varphi \) on \( M \) such that \( \varphi \circ \Phi = \varphi \). Without loss of generality, we may assume that \( M \) has a cyclic and separating vector \( \xi_0 \) in \( H \) such that \( \varphi(T) = (T \xi_0, \xi_0) \) for any \( T \in M \).

Put \( A_0 = \{ X \in A : \Phi(X) = 0 \} \), and we define the closed subspaces \( H_1, H_2 \) and \( H_3 \) by \( H_1 = [A_0 \xi_0] \), \( H_2 = [D \xi_0] \) and \( H_3 = [A^*_0 \xi_0] \) respectively, where \( [S] \) is the closed linear span of a subset \( S \) of \( H \). Let \( P_i \) be the orthogonal projection from \( H \) onto \( H_i \) for every \( i = 1, 2, 3 \). Let \( \mathfrak{A}_m \) be the set of all \( A \in M \) such that \( \Phi(A \mathfrak{A}_0) = \Phi(\mathfrak{A}_0 A) = 0 \). By [1], Theorem 2.2.1, we recall that \( \mathfrak{A}_m \) is a maximal subdiagonal algebra in \( M \) with respect to \( \Phi \) containing \( \mathfrak{A} \). Then we easily have the following lemma.

**Lemma 2.1.** Keep the notation as above. Then

(i) \( H = H_1 \oplus H_2 \oplus H_3 \);
(ii) \( D H_i \subseteq H_i \) (\( i = 1, 2, 3 \));
(iii) \( H_1 = [(\mathfrak{A}_m) \xi_0] \) and \( H_3 = [(\mathfrak{A}_m) \xi_0] \);
(iv) \( (\mathfrak{A}_m)_0 (H_1 \oplus H_2) \subseteq H_1 \) and \( (\mathfrak{A}_m)_0 (H_2 \oplus H_3) \subseteq H_3 \).

Considering the Hilbert space decomposition in (i), we have the following lemma.
Lemma 2.2. Keep the assumptions and the notation as above. Then

\[ \mathcal{D} = \left\{ D \in \mathcal{M} : D = \begin{pmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{pmatrix} \right\} \]

and

\[ (\mathfrak{A}_m)_0 = \left\{ X \in \mathcal{M} : X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ 0 & 0 & X_{23} \\ 0 & 0 & X_{33} \end{pmatrix} \right\}. \]

Proof. Put

\[ \mathfrak{B} = \left\{ D \in \mathcal{M} : D = \begin{pmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{pmatrix} \right\} \]

and

\[ \mathfrak{C} = \left\{ X \in \mathcal{M} : X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ 0 & 0 & X_{23} \\ 0 & 0 & X_{33} \end{pmatrix} \right\} \]

respectively. Then it is clear that \( \mathcal{D} \subseteq \mathfrak{B} \) and \( (\mathfrak{A}_m)_0 \subseteq \mathfrak{C} \).

If \( D \in \mathfrak{B} \), then \( \Phi(D) \in \mathcal{D} \subseteq \mathfrak{B} \) and so \( \Phi(D) \) has the matrix form as follows:

\[ \Phi(D) = \begin{pmatrix} V_{11} & 0 & 0 \\ 0 & V_{22} & 0 \\ 0 & 0 & V_{33} \end{pmatrix}. \]

Since \( \mathfrak{A}_0 + \mathcal{D} + \mathfrak{A}_0^* \) is \( \sigma \)-weakly dense in \( \mathcal{M} \), \( \mathfrak{A}_0 + \mathfrak{A}_0^* \) is \( \sigma \)-weakly dense in \( \text{Ker}(\Phi) \). Let \( P_2 \) be the orthogonal projection from \( \mathcal{H} \) onto \( \mathcal{H}_2 \). Then it is clear that \( P_2(X\xi_0) = \Phi(X)\xi_0 \), \( X \in \mathcal{M} \). Thus \( P_2XP_2 = 0 \) for every \( X \in \text{Ker}(\Phi) \). Since \( D - \Phi(D) \in \text{Ker}(\Phi) \), we have \( D_{22} - V_{22} = P_2(D - \Phi(D))P_2 = 0 \). Hence we have

\[ (D - \Phi(D))\xi_0 = (D_{22} - V_{22})\xi_0 = 0. \]

Since \( \xi_0 \) is a separating vector for \( \mathcal{M} \) and \( \xi_0 \in \mathcal{H}_2 \), we have \( D = \Phi(D) \in \mathcal{D} \) and so \( \mathcal{D} = \mathfrak{B} \).

On the other hand, take any \( X \in \mathfrak{C} \). Since \( \Phi(X) \in \mathcal{D} \), \( \Phi(X) \) is of the form

\[ \Phi(X) = \begin{pmatrix} V_{11} & 0 & 0 \\ 0 & V_{22} & 0 \\ 0 & 0 & V_{33} \end{pmatrix}. \]

Then we similarly have \( P_2(\Phi(X) - X)P_2 = V_{22} = 0 \). Thus \( \Phi(X) = 0 \).

It is trivial that \( \mathfrak{C} \) is a \( \mathcal{D} \)-bimodule and \( (\mathfrak{A}_m)_0 \subseteq \mathfrak{C} \). Hence it is easy to check that \( \mathcal{D} + \mathfrak{C} \) is a subdiagonal algebra of \( \mathcal{M} \) with respect to \( \Phi \), containing \( \mathfrak{A}_m \). By the maximality of \( \mathfrak{A}_m \), we have \( \mathfrak{A}_m = \mathcal{D} + \mathfrak{C} \). This implies that \( (\mathfrak{A}_m)_0 = \mathfrak{C} \). This completes the proof. \( \blacksquare \)
From Tomita-Takesaki theory, we define conjugate-linear operators \( S_0 \) and \( F_0 \), with dense domains \( \{ \mathcal{M}\xi_0 \} \) and \( \{ \mathcal{M}'\xi_0 \} \) respectively by

\[
S_0 A \xi_0 = A^* \xi_0 \quad \text{and} \quad F_0 B \xi_0 = B^* \xi_0 \quad (A \in \mathcal{M}, \ B \in \mathcal{M}').
\]

By [5], Lemma 9.2.1, the operator \( S_0 \) is preclosed, and the adjoint \( F \) is an extension of \( F_0 \). Further, if \( S \) is the closure of \( S_0 \), then \( S^* = F \). Let \( \mathcal{D}(S) \) be the domain of \( S \), and let \( G(S) \) be the graph of \( S \).

**Lemma 2.3.** Keep the notation as above. Then the closed operator \( S \) has the following matrix decomposition with respect to Lemma 2.1 (i):

\[
S = \begin{pmatrix}
0 & 0 & S_3 \\
0 & S_2 & 0 \\
S_1 & 0 & 0
\end{pmatrix}
\]

where for \( i = 1, 2, 3 \), \( S_i \) is a closed operator with domain \( \mathfrak{F}_i \) in \( \mathcal{H}_i \) such that \( S_1 \mathfrak{F}_1 = \mathfrak{F}_3 \), \( S_2 \mathfrak{F}_2 = \mathfrak{F}_2 \) and \( S_3 \mathfrak{F}_3 = \mathfrak{F}_1 \).

**Proof.** Since \( \{ \mathfrak{A}_0 \xi_0 \} \oplus \{ \mathcal{D} \xi_0 \} \oplus \{ \mathfrak{A}_0^* \xi_0 \} \subseteq \mathcal{D}(S) \), we can define a preclosed operator \( V_0 \) by

\[
V_0(A + D + B^*)\xi_0 = (A^* + D^* + B)\xi_0
\]

for \( A, B \in \mathfrak{A}_0 \) and \( D \in \mathcal{D} \). Since \( S_0 \) is an extension of \( V_0 \), \( S \) is an extension of the closure \( V \) of \( V_0 \). Since \( G(S) \) is the norm closure of \( \{ X\xi_0 \oplus X^*\xi_0 : X \in \mathcal{M} \} \) and \( \mathfrak{A}_0 + \mathcal{D} + \mathfrak{A}_0^* \) is \( \sigma \)-weakly dense in \( \mathcal{M} \), it is easy to prove that \( S = V \).

Let \( \zeta \oplus S\zeta \in G(S) \). Since \( S \) is the closure of \( V_0 \), there exist \( \{ A_n, B_n \}_{n=1}^{\infty} \) in \( \mathfrak{A}_0 \) and \( \{ D_n \}_{n=1}^{\infty} \) in \( \mathcal{D} \) such that

\[
\lim_{n \to \infty} \left( \|(A_n + D_n + B_n^*)\xi_0 - \zeta\|^2 + \|(A_n^* + D_n^* + B_n)\xi_0 - S\zeta\|^2 \right) = 0.
\]

Then, we have

\[
\lim_{n \to \infty} \left( \|A_n\xi_0 - P_1\zeta\|^2 + \|D_n\xi_0 - P_2\zeta\|^2 + \|B_n^*\xi_0 - P_3\zeta\|^2 \right)
+ \|A_n^*\xi_0 - P_1 S\zeta\|^2 + \|D_n^*\xi_0 - P_2 S\zeta\|^2 + \|B_n\xi_0 - P_3 S\zeta\|^2 = 0,
\]

where \( P_i \) is the projection from \( \mathcal{H} \) onto \( \mathcal{H}_i \) for \( i = 1, 2, 3 \). This implies that \( P_i \zeta \oplus SP_i \zeta \in G(S) \) and \( P_i \mathcal{D}(S) \subset \mathcal{D}(S) \) for \( i = 1, 2, 3 \). Put \( \mathfrak{F}_i = P_i \mathcal{D}(S) \). Since \( S\mathfrak{F}_1 = \mathfrak{F}_3, S\mathfrak{F}_2 = \mathfrak{F}_2 \) and \( S\mathfrak{F}_3 = \mathfrak{F}_1 \), we have the desired matrix form of \( S \) with respect to Lemma 2.1 (i). This completes the proof.
Put $\Delta = S^* S$. We recall that the modular automorphism group $\{\sigma^\varphi_t\}_{t \in \mathbb{R}}$ of $\mathcal{M}$ associated with $\varphi$ has the following form:

$$\sigma^\varphi_t(X) = \Delta^{it} X \Delta^{-it} \quad (\forall t \in \mathbb{R}, X \in \mathcal{M}).$$

Then we have the following theorem.

**Theorem 2.4.** Let $\mathfrak{A}$ be a maximal subdiagonal algebra of $\mathcal{M}$ with respect to $\Phi$ and let $\varphi$ is a faithful normal state of $\mathcal{M}$ such that $\varphi \circ \Phi = \varphi$. Then $\mathfrak{A}$ is $\sigma^\varphi_t$-invariant, that is $\sigma^\varphi_t(\mathfrak{A}) = \mathfrak{A}$.

**Proof.** From Lemma 2.3, the adjoint $S^*$ of $S$ has the matrix form

$$S^* = \begin{pmatrix} 0 & 0 & S_1^* \\ 0 & S_2^* & 0 \\ S_3^* & 0 & 0 \end{pmatrix}$$

where $S_i^*$ is the adjoint operator of $S_i$ with domain $\mathfrak{A}_i^*$ $(i = 1, 2, 3)$. Then the modular operator $\Delta$ has the matrix form

$$\Delta = \begin{pmatrix} S_1^* S_1 & 0 & 0 \\ 0 & S_2^* S_2 & 0 \\ 0 & 0 & S_3^* S_3 \end{pmatrix}.$$ 

By Lemmas 2.2 and 2.3, it is easy to prove that, for every $t \in \mathbb{R}$,

$$\sigma^\varphi_t(\mathfrak{D}) = \mathfrak{D} \quad \text{and} \quad \sigma^\varphi_t(\mathfrak{A}_0) = \mathfrak{A}_0 \quad (t \in \mathbb{R}).$$

Thus we have the theorem. $\blacksquare$

Let $\mathcal{M}^\varphi$ be the centralizer of $\mathcal{M}$ associated with $\varphi$, that is, $\mathcal{M}^\varphi = \{ A \in \mathcal{M} : \varphi(AB) = \varphi(BA), \forall B \in \mathcal{M} \}$. Recall that $\mathcal{M}^\varphi$ is the fixed point algebra of $\mathcal{M}$ with respect to $\{\sigma^\varphi_t\}_{t \in \mathbb{R}}$ and there exists a faithful normal expectation $\mathcal{E}$ from $\mathcal{M}$ onto $\mathcal{M}^\varphi$ (cf. [6], Theorem 1.2). We remark that $\mathcal{M}^\varphi$ is a finite von Neumann algebra.

**Corollary 2.5.** Keep the notation and assumptions as above. Then $\mathcal{E}(\mathfrak{A})$ is a maximal finite subdiagonal algebra of $\mathcal{M}^\varphi$ with respect to $\Phi|_{\mathcal{M}^\varphi}$.

**Proof.** Let $X \in \mathcal{M}$. By [6], Theorem 1.2, $\mathcal{E}(X)$ is in the $\sigma$-weak closure of convex hull of $\{\sigma^\varphi_t(X) : t \in \mathbb{R}\}$. Since $\mathfrak{A}$ is $\sigma^\varphi_t$-invariant by Theorem 2.4, we have $\mathcal{E}(\mathfrak{A}_0) \subset \mathfrak{A}_0$ and $\mathcal{E}(\mathfrak{D}) \subset \mathfrak{D}$. Then we easily prove that $\mathcal{E}(\mathfrak{A})$ is a subdiagonal algebra of $\mathcal{M}^\varphi$ with respect to $\Phi|_{\mathcal{M}^\varphi}$. By [5], Proposition 9.2.14, $\varphi|_{\mathcal{M}^\varphi}$ is a faithful normal trace on $\mathcal{M}^\varphi$. Thus $\mathcal{E}(\mathfrak{A})$ is a finite subdiagonal algebra of $\mathcal{M}^\varphi$. By [3], Theorem 7, $\mathcal{E}(\mathfrak{A})$ is a maximal finite subdiagonal algebra of $\mathcal{M}^\varphi$. This completes the proof. $\blacksquare$
Finally, we remark about the question in [1], Remark 2.2.3, whether every subdiagonal algebra must necessarily be maximal subdiagonal. In [3], Exel showed that every finite subdiagonal algebra is maximal subdiagonal. By Theorem 2.4, we have following two questions.

**Question 2.6.** Is there a subdiagonal algebra which is not $\sigma_{\varphi}^{t}$-invariant for some faithful normal state $\varphi$ on $M$ such that $\varphi \circ \Phi = \varphi$?

**Question 2.7.** If $A$ is a $\sigma_{\varphi}^{t}$-invariant subdiagonal algebra of $M$ for every faithful normal state $\varphi$ on $M$ such that $\varphi \circ \Phi = \varphi$, is $A$ maximal subdiagonal?

3. SUBDIAGONAL ALGEBRAS OF $B(\mathcal{H})$

Let $\mathcal{N}$ be a nest, that is, $\mathcal{N}$ is a chain of closed subspaces of $\mathcal{H}$ containing $\{0\}$ and $\mathcal{H}$ which is closed under intersection and closed span. Then the nest algebra $\text{alg}\mathcal{N}$ is the set of all operators $T$ in $B(\mathcal{H})$ such that $TN \subseteq N$ for every $N \in \mathcal{N}$. The intersection $\mathfrak{D} = \text{alg}\mathcal{N} \cap (\text{alg}\mathcal{N})^{*}$ is the diagonal of $\text{alg}\mathcal{N}$. We recall that there exists a faithful normal expectation from $B(\mathcal{H})$ onto $\mathfrak{D}$ if and only if $\mathcal{N}$ is atomic (cf. [2], Theorem 8.6). In this case, $\text{alg}\mathcal{N}$ is a subdiagonal algebra of $B(\mathcal{H})$ with respect to the expectation. In this section, we consider the converse, that is, if $\mathfrak{A}$ is a subdiagonal algebra of $B(\mathcal{H})$ with respect to a faithful normal expectation $\Phi$, then $\mathfrak{A}$ is a nest algebra with an atomic nest.

Our main theorem in this section is the following.

**Theorem 3.1.** Let $\mathfrak{A}$ be a subdiagonal algebra of $B(\mathcal{H})$ with respect to a faithful normal expectation $\Phi$. Then there exists an atomic nest $\mathcal{N}$ such that $\mathfrak{A} = \text{alg}\mathcal{N}$.

The proof of Theorem 3.1 requires a few preliminary results.

Let $\mathfrak{A}$ be a subdiagonal algebra of $B(\mathcal{H})$ with respect to a faithful normal expectation $\Phi$. Since $\mathcal{H}$ is separable, there exists a faithful normal state $\varphi$ on $B(\mathcal{H})$ such that $\varphi \circ \Phi = \varphi$. Let $\rho$ be the canonical trace of $B(\mathcal{H})$. By [5], Lemma 9.2.19, there is a positive contraction $K$ in $B(\mathcal{H})$ such that $I - K$ is a trace-class operator and

$$\rho((I - K)A) = \varphi(KA) = \varphi(AK) \quad (A \in B(\mathcal{H})).$$

Moreover, both $K$ and $I - K$ are injective. Put $F = K^{-1}(I - K)$. Then by [5], Lemma 9.2.20, the modular automorphism group $\{\sigma_{\varphi}^{t}\}_{t \in \mathbb{R}}$ of $B(\mathcal{H})$ associated with $\varphi$ is written as

$$\sigma_{\varphi}^{t}(X) = F^{it}XF^{-it} \quad (X \in B(\mathcal{H}), \ t \in \mathbb{R}).$$
Define the holomorphic function $f$ with radius $r$. Certain structure of subdiagonal algebras...  

Let $B(\mathcal{H})^\varphi$ be the centralizer of $\varphi$ of $B(\mathcal{H})$. Since the centralizer $B(\mathcal{H})^\varphi$ is the fixed point algebra of $\{\sigma_t^\varphi\}_{t \in \mathbb{R}}$ (cf. [5], p. 697), we can show that $B(\mathcal{H})^\varphi$ is the commutant of $\{F\}$. Since $I - K$ is a trace class positive operator, we can write $I - K = \sum_{n=1}^{\infty} \lambda_n P_n$, where $\{\lambda_n\}_{n=1}^{\infty}$ are the distinct eigenvalues of $I - K$ and for every $n$, $P_n$ is the spectral projection of $I - K$ corresponding to $\lambda_n$. Since $F = K^{-1}(I - K) = \sum_{n=1}^{\infty} \lambda_n (1 - \lambda_n)^{-1} P_n$, we can decompose

$$B(\mathcal{H})^\varphi = \sum_{n=1}^{\infty} M_{k_n},$$

where $k_n$ is the dimension of $P_n^* \mathcal{H}$, and $M_{k_n}$ is the set of all $k_n \times k_n$ matrices on $P_n^* \mathcal{H}$.

As in Section 2, there exists a faithful normal expectation $E$ from $B(\mathcal{H})$ onto $B(\mathcal{H})^\varphi$. Then we have the following proposition.

**Proposition 3.2.** Keep the notation and the assumptions as above. Then there exists a family $\{Q_n\}_{n=1}^{\infty}$ of mutually orthogonal rank one projections in the diagonal $\mathcal{D}$ of $\mathfrak{A}$ such that $\sum_{n=1}^{\infty} Q_n = I$.

**Proof.** As in Section 2, let $\mathfrak{A}_m$ be the maximal subdiagonal algebra with respect to $\Phi$ of $B(H)$ containing $\mathfrak{A}$. Let $P_n$ be as above. At first, we shall show that $P_n \in \mathcal{E}(\mathfrak{D}) \subset \mathfrak{D}$. We consider an invertible operator $X = 2P_n \oplus (I - P_n)$ on $\mathcal{H} = P_n^* \mathcal{H} \oplus P_n^\perp \mathcal{H}$. By Corollary 2.5, $\mathcal{E}(\mathfrak{A}_m)$ is a finite maximal subdiagonal algebra of $B(\mathcal{H})^\varphi$ with diagonal $\mathcal{E}(\mathfrak{D})$. Hence, by [1], Theorem 4.4.1, there exist a unitary operator $U$ in $B(\mathcal{H})^\varphi$ and an invertible operator $A$ in $\mathcal{E}(\mathfrak{A}_m) \cap \mathcal{E}(\mathfrak{A}_m)^{-1}$ such that $X = UA$. Since $P_n$ is a central projection of $B(\mathcal{H})^\varphi$, we can decompose $U = U_1 \oplus U_2$ and $A = A_1 \oplus A_2$. It is clear that $A_1 = 2U_1^* \oplus A_2$ and $A_2 = U_2^* \oplus U_2$. So we have that $\sigma(A) = 2\sigma(U_1^*) \cup \sigma(U_2^*)$. It is also clear that $\sigma(U_1^*)(\subset \mathbb{T})$ is a finite subset of $\mathbb{T}$ and $\sigma(U_2^*) \subseteq \mathbb{T}$, where $\mathbb{T}$ is the unit circle. If $\sigma(U_2^*) \neq \mathbb{T}$, then we know that $\mathbb{C} \setminus \sigma(A)$ is connected. If $\sigma(U_2^*) = \mathbb{T}$, then the only bounded component of $\mathbb{C} \setminus \sigma(A)$ is the open disc $\mathbb{D}$. Thus we can choose a neighborhood $\Omega$ of $\sigma(A)$ such that

1. $\Omega = \Omega_1 \cup \Omega_2$, $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$,
2. $\sigma(A_1) \subset \Omega_1$, $\sigma(A_2) \subset \Omega_2$, and
3. the only bounded component of $\mathbb{C} \setminus \Omega$ is a closed disc $\{\lambda \in \mathbb{C} : |\lambda| \leq r\}$ with radius $r$ less than one.

Define the holomorphic function $f$ on $\Omega$ by

$$f(z) = \begin{cases} 1, & z \in \Omega_1 \\ 0, & z \in \Omega_2. \end{cases}$$
By [8], Theorem 13.9 there is a sequence \( \{R_i\} \) of rational functions with poles at 0, such that \( R_i \to f \) uniformly on compact subsets of \( \Omega \). Since both \( A \) and \( A^{-1} \) are in \( \mathcal{E}(\mathfrak{A}_m) \), we have that \( P_n(=f(A)) \in \mathcal{E}(\mathfrak{A}_m) \), and so \( P_n \in \mathcal{E}(\mathcal{D}) \). By Corollary 2.5, \( P_n \in \mathcal{D} \). Since \( P_n \mathcal{E}(\mathfrak{A}_m) P_n \) is a subdiagonal algebra of \( M_{k_n} \), by [4], Theorem 2.1, \( P_n \mathcal{E}(\mathfrak{A}_m) P_n \) is a nest subalgebra of \( M_{k_n} \) with a finite nest. So \( P_n \mathcal{E}(\mathfrak{A}_m) P_n \) contains a family of mutually orthogonal rank one projections whose sum is \( P_n \). This completes the proof. 

**Lemma 3.3.** Keep the assumptions as above and also assume that \( \mathfrak{A} \neq \mathcal{B}(\mathcal{H}) \).

Then there exists a nontrivial \( \mathfrak{A} \)-invariant subspace of \( \mathcal{H} \).

**Proof.** By Proposition 3.2, we know that \( \mathcal{D}' \) is abelian. Hence, by Theorems 6.2.1 and 6.2.2 in [1], we have \( \rho \circ \Phi = \rho \), where \( \rho \) is the canonical trace of \( \mathcal{B}(\mathcal{H}) \). By Proposition 3.2, there is a rank one projection \( e \otimes e \in \mathcal{D} \) such that \( \{\mathfrak{A}_0 e\} \neq \{0\} \), where \( e \otimes e(x) = (x,e) e, \forall x \in \mathcal{H} \). Put \( \mathfrak{M} = [\mathfrak{A}_0 e] \), then \( \mathfrak{M} \) is \( \mathfrak{A} \)-invariant. If \( T \in \mathfrak{A}_0 \), then

\[
(Te,e) = \rho(T(e \otimes e)) = \rho \circ \Phi(T(e \otimes e)) = 0.
\]

Hence, \( \mathfrak{M} \) is nontrivial. This completes the proof. 

**Proof of Theorem 3.1.** By Lemma 3.3 and Zorn’s lemma, there exists a maximal nest \( \mathcal{N} \) of \( \mathfrak{A} \)-invariant subspaces of \( \mathcal{H} \). Since \( \mathcal{N} \subset \mathcal{D}' \) and \( \mathcal{D}' \) is atomic, the nest \( \mathcal{N} \) is atomic. Let \( \{E_\lambda\}_{\lambda \in \Lambda} \) be the set of all atoms of \( \mathcal{N} \). By the maximality of \( \mathcal{N} \) and Lemma 3.3, we have \( E_\lambda \mathcal{B}(\mathcal{H}) E_\lambda \subset \mathcal{D} \). This implies that \( \mathcal{N}' = \mathcal{D} \) and so \( \Phi(T) = \sum_{\lambda \in \Lambda} E_\lambda TE_\lambda \) for every \( T \in \mathcal{B}(\mathcal{H}) \). It is clear that \( \mathfrak{A} \subset \text{alg} \mathcal{N} \).

Conversely, if \( T \in \text{alg} \mathcal{N} \), then there exist nets \( A_\lambda, B_\mu \in \mathfrak{A} \) such that \( A_\lambda + B_\mu \to T \) \( \sigma \)-weakly. For \( \lambda, \mu \in \Lambda \), there exist \( P_\lambda, P_\mu \in \mathcal{N} \) such that \( E_\lambda = P_\lambda \circ P_\lambda \) and \( E_\mu = P_\mu \circ P_\mu \) respectively. We define an order on \( \Lambda \) by

\[
\lambda < \mu \iff P_\lambda < P_\mu.
\]

If \( \lambda > \mu \), then \( E_\lambda TE_\mu = 0 \) because \( T \in \text{alg} \mathcal{N} \). If \( \lambda = \mu \), then \( E_\lambda TE_\lambda \in \mathcal{D} \subset \mathfrak{A} \).

Since \( E_\lambda B_\mu^* E_\mu = 0 \) in case that \( \lambda < \mu \), we have \( E_\lambda TE_\mu \in \mathfrak{A} \). Therefore \( T = \sum_{\lambda \leq \mu} E_\lambda TE_\mu \in \mathfrak{A} \). This completes the proof.

In [9], Theorem 2, the third author and Watatani proved that if \( \mathcal{D} \) is a subfactor of a finite dimensional factor \( \mathcal{M} \), then there exist no maximal subdiagonal algebras of \( \mathcal{M} \) with respect to the canonical conditional expectation from \( \mathcal{M} \) onto \( \mathcal{D} \) unless \( \mathcal{D} = \mathcal{M} \). The following corollary is in case that \( \mathcal{M} = \mathcal{B}(\mathcal{H}) \).
Corollary 3.4. If $\mathcal{D}$ is a subfactor of $\mathcal{B}(\mathcal{H})$, then there exist no maximal subdiagonal algebras with diagonal $\mathcal{D}$ unless $\mathcal{D} = \mathcal{B}(\mathcal{H})$.

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