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# REDUCIBLE SEMIGROUPS OF IDEMPOTENT OPERATORS

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ABSTRACT. We study the existence of common invariant subspaces for semigroups of idempotent operators. It is known that in finite dimensions every such semigroup is simultaneously triangularizable. The question of the existence of even one non-trivial invariant subspace is still open in infinite dimensions.

Working with semigroups of idempotent operators in Hilbert/Banach vector space settings, we exploit the connection between the purely algebraic structure and the operator structure to show that the answer is affirmative in a number of cases.

KEYWORDS: *Semigroups, idempotents, invariant subspaces, reducible, bands.*
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## 1. PRELIMINARIES

Much of the following is standard material in semigroup theory. A reference on the subject is [5].

An element $a$ of a set $\mathcal{S}$ is said to be *idempotent* with respect to an operation $\circ : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$, if $a \circ a = a$. A *band* is a semigroup of idempotents (the reference to some fixed operation is implicit). A *subband* of a band is a subset that is closed under the operation.

Suppose $\mathcal{S}$ is a band (with an operation '$\circ$'). Define a relation '$\sim$' on $\mathcal{S}$ by:

$$a \sim b \Longleftrightarrow \begin{cases} a \circ b \circ a = a; \\ b \circ a \circ b = b. \end{cases}$$

Then '$\sim$' is an equivalence relation.

Let $\mathcal{C}_a$ stand for the $\sim$-equivalence class of $a$. Then each $\sim$-equivalence class is a subband of $\mathcal{S}$. We shall refer to the $\sim$-equivalence classes as *components of* $\mathcal{S}$.

Define an operation '$\odot$' on the set $\mathcal{S}/\sim$ of components of $\mathcal{S}$ by

$$\mathcal{C}_a \odot \mathcal{C}_b = \mathcal{C}_{a\circ b}.$$

This operation is well defined and $\mathcal{S}/\sim$ is an abelian (!) band under $\odot$. Therefore the components of $\mathcal{S}/\sim$ are singletons. We refer to this band as *the band of components of* $\mathcal{S}$.

One simple consequence of $\mathcal{S}/\sim$ being abelian is that a free band on $n$ generators has $2^n - 1$ components. For example, if $\mathcal{S}$ is a free band with generators $a, b$ and $c$ then

$$\mathcal{S}/\sim = \{\mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_c, \mathcal{C}_{a\circ b}, \mathcal{C}_{a\circ c}, \mathcal{C}_{b\circ c}, \mathcal{C}_{a\circ b\circ c}\}.$$

The following calculation illustrates the idea of the proof:

$$\begin{aligned}\mathcal{C}_{a\circ b\circ c\circ b\circ a\circ b} &= \mathcal{C}_a \odot \mathcal{C}_b \odot \mathcal{C}_c \odot \mathcal{C}_b \odot \mathcal{C}_a \odot \mathcal{C}_b = (\mathcal{C}_a)^2 \odot (\mathcal{C}_b)^3 \odot \mathcal{C}_c \\ &= \mathcal{C}_{(a^2)} \odot \mathcal{C}_{(b^3)} \odot \mathcal{C}_c = \mathcal{C}_a \odot \mathcal{C}_b \odot \mathcal{C}_c = \mathcal{C}_{a\circ b\circ c}.\end{aligned}$$

In fact more is true about finitely generated bands: a particular case of the celebrated theorem of Green and Rees ([3]) states that every free band on $n$ generators is finite. Consequently all finitely generated bands are finite.

The next definition is implicit in standard semigroup texts (for example [5]). Define a relation '$\precsim$' on a band $\mathcal{S}$ by:

$$a \precsim b \iff a \circ b \circ a = a.$$

Then '$\precsim$' is a pre-order (i.e. is reflexive and transitive) on $\mathcal{S}$. We refer to '$\precsim$' as *the band pre-order on* $\mathcal{S}$. Clearly:

$$a \sim b \iff \begin{cases} a \precsim b; \\ b \precsim a. \end{cases}$$

It follows that '$\precsim$' is a partial order exactly when all components of $\mathcal{S}$ are singletons. Therefore the band pre-order on $\mathcal{S}/\sim$ is a partial order. We denote it by '$\preceq$' and refer to it as *the band order on* $\mathcal{S}/\sim$. It is easy to see that that

$$\mathcal{C}_a \preceq \mathcal{C}_b \iff a \precsim b.$$

Let us also agree that $\square \prec \diamond$ will stand for $\begin{cases} \square \precsim (\preceq) \diamond; \\ \square \not\sim (\neq)\ \diamond\,. \end{cases}$

With respect to the operation of $\mathcal{S}$ the band pre-order has two useful properties which we state in terms of the band order on the abelian (!) band of components of $\mathcal{S}$ (we make both $\circ$ and $\odot$ implicit notation from now on):

$$(*) \qquad \left.\begin{array}{l}\mathcal{C}_a \preceq \mathcal{C}_b \\ \mathcal{C}_a \preceq \mathcal{C}_c\end{array}\right\} \Longrightarrow \mathcal{C}_a \preceq \mathcal{C}_b\mathcal{C}_c;$$

$$(**) \qquad \mathcal{C}_a\mathcal{C}_b \preceq \mathcal{C}_a, \quad \text{for every } a, b \in \mathcal{S}.$$

One more property deserves mention. It is clear that $\mathcal{C}_a\mathcal{C}_b\mathcal{C}_a = \mathcal{C}_a$, whenever $a \precsim b$ (i.e. $aba = a$). Restated without use of components this says that $abd = ad$ whenever $a \sim d \precsim b$. Equivalently (via property $(*)$):

$$ab_1b_2 \cdots b_m d = ad \quad \text{whenever} \quad a \sim d \precsim b_1, b_2, \ldots, b_m.$$

We refer to this as *the sandwich property.*

Since property $(*)$ states that $\mathcal{S}/\!\sim$ , together with its band order, is a lower semi-lattice (with $\mathcal{C}_a\mathcal{C}_b$ being the 'meet' of $\mathcal{C}_a$ and $\mathcal{C}_b$), it is natural to ask whether every (abstract) lower semi-lattice is order isomorphic to $(\mathcal{S}/\!\sim , \preceq)$, for some band $\mathcal{S}$. The answer is affirmative. Indeed, every lower semi-lattice can be considered to be an abelian band with the operation induced by its 'meet'.

We restrict our attention to vector space and Hilbert space settings. Most of our Hilbert space results have obvious Banach space generalizations (with appropriate quotients used in place of orthogonal compliments).

All Hilbert spaces ($\mathcal{H}$) in this paper are assumed to be over the field $\mathbb{C}$ of complex numbers. All vector spaces ($\mathcal{V}$) are taken to be over a field $\mathbb{F}$ of characteristic 0, unless stated otherwise. The copy of the set of rational numbers naturally imbedded in $\mathbb{F}$ is taken to have the usual order. With this in mind, for example, a reference to 'non-negative integers' in $\mathbb{F}$ is self-explanatory. We do not deal with the trivial Hilbert/vector space $\{0\}$ explicitly.

By an *operator* we mean either a linear transformation (in vector space setting) or a bounded linear transformation (in Hilbert space setting), depending on the context. The set of all operators from a Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$ is denoted by $\mathfrak{B}(\mathcal{H}_1, \mathcal{H}_2)$, or simply by $\mathfrak{B}(\mathcal{H}_1)$, if $\mathcal{H}_1 = \mathcal{H}_2$. The notation is similar in vector space setting, except we use $\mathfrak{L}$ insted of $\mathfrak{B}$. We write $I$ and 0 for the identity and the zero operators, respectively.

In most of this paper (except for the last section) we restrict our attention to bands of operators on Hilbert/vector spaces under the operation of composition (i.e. operator multiplication). We refer to these bands as *operator bands.* It is worth observing that if $\mathcal{S}$ is an operator band containing $I$ and 0, then $0 \precsim T \precsim I$, for every $T \in \mathcal{S}$.

The rest of notation and terminology is standard.

## 2. INTRODUCTION

The goal of this paper is to study the existence of a non-trivial (closed, in Hilbert space setting) common invariant subspace for a semigroup of idempotent operators. Sets of operators possessing such a subspace are often called *reducible*.

This research is part of an ongoing program to find convenient necessary and sufficient conditions for reducibility of semigroups within various classes of operators (for example: idempotent, nilpotent, quasinilpotent, compact etc.). Some of the questions can be answered by passing to associated algebras and using the existing machinery (for example, Lomonosov's technique).

A number of results have been obtained in Hilbert space setting that relate the reducibility of semigroups to various spectral conditions satisfied by the elements of the semigroup.

In this paper we approach the problem from a more algebraic point of view: we explore the connection between the component structure of a band and its reducibility.

We will start by demonstrating that bands with finitely many components are always reducible (with special block-upper-triangular structure). Since bands in finite dimensions are shown to be of this type, this recovers and generalizes a theorem from [6] which states that bands in finite dimensions are always triangularizable.

We proceed to show that bands with finitely many components possess a uniformly bounded non-negative-integer valued 'faithful' trace which extends linearly to a trace on the algebra generated by the band.

This trace is consequently used to obtain a number of results about spans and convex hulls of general operator bands. For example: a span of an operator band contains only algebraic operators, while (under appropriate assumptions about the field) elements of a convex hull are never nilpotent and their spectrum is a subset of [0,1].

Continuing without restriction on the number of components, we prove, among other things, that if, in a Hilbert space setting, a span of an operator band contains a non-zero compact operator, then the band is reducible. This extends a theorem from [6] stating that an operator band containing a finite-rank element is reducible.

The last section of the paper deals with representing abstract bands as operator bands in finite dimensions.

## 3. OPERATOR BANDS WITH FINITELY MANY COMPONENTS

Operator bands in finite dimensions. We begin by exposing the relationship that exists between the band pre-order and the canonical trace function in finite dimensions. The reader should keep in mind that for finite-rank idempotents the trace and rank coincide (so that the trace is a non-negative integer).

The essence of our first theorem is that components of an operator band in finite dimensions are simply the maximal subbands with constant trace (rank).

Theorem 3.1. *If $\mathcal{S}$ is an operator band acting on a finite-dimensional vector space and $A \in \mathcal{S}$, then*

$$\{B \in \mathcal{S} \mid \operatorname{tr}(AB) = \operatorname{tr}(A) = \operatorname{tr}(B)\}$$

*is the component of $\mathcal{S}$ containing $A$.*

*Proof.* Rank coincides with the trace for any finite-rank idempotent.

*Claim.* For all $A, B \in \mathcal{S}$ : $\operatorname{tr}(AB) = \operatorname{tr}(A) \iff A \precsim B$. Indeed, if tr(AB) = tr(A), then

$$\operatorname{rank}(A) = \operatorname{tr}(A) = \operatorname{tr}(AB) = \operatorname{tr}(ABA) = \operatorname{rank}(ABA).$$

Since $\operatorname{ran}(ABA) \subset \operatorname{ran}(A)$ and $\ker(A) \subset \ker(ABA)$, it follows that $\operatorname{ran}(ABA) = \operatorname{ran}(A)$ and $\ker(A) = \ker(ABA)$ by the Rank-Nullity Theorem. Therefore $ABA = A$. Conversely, if $A \precsim B$ then

$$\operatorname{tr}(A) = \operatorname{tr}(ABA) = \operatorname{tr}(AB).$$

This proves the claim.

Since $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, a symmetric argument shows that

$$\operatorname{tr}(AB) = \operatorname{tr}(B) \iff B \precsim A. \quad \blacksquare$$

Theorem 3.2. *The canonical trace is constant on components of an operator band acting on a finite-dimensional vector space and is strictly monotonic with respect to the band pre-order.*

*Proof.* That canonical trace is constant on components follows from Theorem 3.1.

If $A \precsim B$ in $\mathcal{S}$, then

$$\operatorname{tr}(A) = \operatorname{tr}(ABA) = \operatorname{tr}(BA) = \operatorname{rank}(BA) \leqslant \operatorname{rank}(B) = \operatorname{tr}(B)$$

and consequently the trace is monotonic with respect to the band pre-order.

To show that it is strictly monotonic, suppose that $A \precsim B$ and tr(A)=tr(B). Then

$$\operatorname{tr}(B) = \operatorname{tr}(A) = \operatorname{tr}(ABA) = \operatorname{tr}(BA).$$

Therefore $B \precsim A$ (see Theorem 3.1), so that $A \sim B$. $\blacksquare$

A non-$\{0\}$ component $\mathcal{C}$ of a band $\mathcal{S}$ is called *minimal* when it is a minimal non-$\{0\}$ element in the commutative band of components of $\mathcal{S}$ with respect to the band order.

Theorem 3.3. *Every non-$\{0\}$ band of operators on a finite-dimensional vector space has at least one minimal non-$\{0\}$ component.*

*Proof.* Consider a component on which the trace takes its least non-zero value. This component is a minimal non-$\{0\}$ component by Theorem 3.2. ∎

Rectangular bands of operators. A simple way to pass from finite-dimensional framework to infinite-dimensional one is by first considering bands with particularly simple component structure. Presently we consider bands of operators that have only one component. Such bands are called *rectangular.*

For a band $\mathcal{S}$ of operators on a vector space $\mathcal{V}$, write:

$$\operatorname{ran}(\mathcal{S}) = \bigvee_{A\in\mathcal{S}} \operatorname{ran}(A)$$

and

$$\ker(\mathcal{S}) = \bigcap_{A\in\mathcal{S}} \ker(A).$$

If $\mathcal{H}$ is a Hilbert space write $\overline{\operatorname{ran}(\mathcal{S})}$ for the closure of $\operatorname{ran}(\mathcal{S})$ in the norm topology.

Since many of the proofs for the non-topological claims presented below can be easily adapted to the presence of topology, we shall be content with stating the topological claims without proof in such situations.

Theorem 3.4. *If $\mathcal{S}$ is a rectangular band of operators on a vector space $\mathcal{V}$, then:*

$$\left.\begin{array}{l}\operatorname{ran}(\mathcal{S}) = \mathcal{V}\\ \ker(\mathcal{S}) = \{0\}\end{array}\right\} \Longrightarrow \mathcal{S} = \{I\}.$$

*For a band $\mathcal{S} \subset \mathfrak{B}(\mathcal{H})$ replace '$\operatorname{ran}(\mathcal{S}) = \mathcal{V}$' by '$\overline{\operatorname{ran}(\mathcal{S})} = \mathcal{H}$' (and recall that for any idempotent $A$, $\operatorname{ran}(A)$ and $\ker(A)$ are closed complementary subspaces).*

*Proof.* Suppose $A \in \mathcal{S}$. We shall show that $A = I$.

Since $A$ is idempotent, $\mathcal{V} = \operatorname{ran}(A) \oplus \ker(A)$. With respect to this decomposition $A$ has matrix form

$$\begin{pmatrix} I & 0\\ 0 & 0\end{pmatrix}.$$

Every element $B \in \mathcal{S}$ satisfies:

$$\begin{cases} ABA = A \\ B = (BA)(AB) \\ B^2 = B. \end{cases}$$

Therefore every $B$ has matrix form

$$\begin{pmatrix} I & X \\ Y & YX \end{pmatrix},$$

for some $X, Y$ such that $XY = 0$.

If by way of contradiction, $A \neq I$, i.e. $\ker(A) \neq \{0\}$, then there is some $B_0$ in $\mathcal{S}$ such that $Y_0 \neq 0$ (recall that $\operatorname{ran}(\mathcal{S}) = \mathcal{V}$ is assumed). Therefore $XY_0 = 0$, for every $B \in \mathcal{S}$ (since the north-west block of $BB_0$ must be $I$). Consequently

$$\begin{pmatrix} 0 \\ y \end{pmatrix} \in \ker(\mathcal{S})$$

for any non-zero column $y$ of $Y_0$. This shows that $\ker(\mathcal{S}) \neq \{0\}$. Contradiction. ∎

THEOREM 3.5. *Suppose $\mathcal{S}$ is a non-$\{0\}$ rectangular band of operators on a vector space $\mathcal{V}$. Write:*

$$\begin{aligned} \mathcal{V}_1 &= \ker(\mathcal{S}) \cap \operatorname{ran}(\mathcal{S}), \\ \mathcal{V}_2 &= \textit{any complement of } \mathcal{V}_1 \textit{ in } \operatorname{ran}(\mathcal{S}), \\ \mathcal{V}_3 &= \textit{any (possibly } \{0\}\textit{) complement of } \operatorname{ran}(\mathcal{S}) \textit{ in } \mathcal{V}. \end{aligned}$$

*Then there exist sets $\Omega \subset \mathfrak{L}(\mathcal{V}_2, \mathcal{V}_1)$ and $\Lambda \subset \mathfrak{L}(\mathcal{V}_3, \mathcal{V}_2)$ such that $\mathcal{S}$ has matrix form*

$$\mathcal{S} = \left\{ \begin{pmatrix} 0 & A & AB \\ 0 & I & B \\ 0 & 0 & 0 \end{pmatrix} \middle| A \in \Omega,\, B \in \Lambda \right\}$$

*with respect to $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$.*

*Rectangular bands $\mathcal{S} \subset \mathfrak{B}(\mathcal{H})$ have the same operator matrix form with respect to $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, where*

$$\begin{aligned} \mathcal{H}_1 &= \ker(\mathcal{S}) \cap \overline{\operatorname{ran}(\mathcal{S})}, \\ \mathcal{H}_2 &= \mathcal{H}_1{}^{\perp} \cap \overline{\operatorname{ran}(\mathcal{S})}, \\ \mathcal{H}_3 &= \overline{\operatorname{ran}(\mathcal{S})}^{\perp}. \end{aligned}$$

*Proof.* Suppose $\mathcal{S}$ is a non-$\{0\}$ rectangular band of operators on a vector space $\mathcal{V}$. It is worth noting that $\mathcal{V}_1, \mathcal{V}_3$ cannot be $\{0\}$ at the same time, unless

$\mathcal{S} = \{I\}$ (Theorem 3.4). It is also clear that $\mathcal{V}_2 \neq \{0\}$ because $\mathcal{S} \neq \{0\}$. Every element of $\mathcal{S}$ has matrix form

$$\begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the decomposition $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$. Observe that the set of all (2,2) block-entries from these matrices is a rectangular band (call it $\mathfrak{T}$) in $\mathfrak{L}(\mathcal{V}_2)$. The fact that each element of $\mathcal{S}$ acts as an identity on its own range implies that the span of the images of $\mathcal{V}_1 \oplus \mathcal{V}_2$ $(= \operatorname{ran}(\mathcal{S}))$ under elements of $\mathcal{S}$ is $\mathcal{V}_1 \oplus \mathcal{V}_2$. Consequently, $\operatorname{ran}(\mathfrak{T}) = \mathcal{V}_2$. At the same time, $\ker(\mathfrak{T}) = \{0\}$. (Indeed: the only vectors an idempotent transformation sends into its own kernel are those already in the kernel.) Apply Theorem 3.4 to conclude that $\mathfrak{T} = \{I\}$.

The matrix

$$\begin{pmatrix} 0 & A & C \\ 0 & I & B \\ 0 & 0 & 0 \end{pmatrix}$$

is idempotent exactly when $C = AB$. Therefore

$$\mathcal{S} \subset \left\{ \begin{pmatrix} 0 & A & AB \\ 0 & I & B \\ 0 & 0 & 0 \end{pmatrix} \middle| A \text{ and } B \text{ are arbitrary} \right\}.$$

Let

$$\Omega = \left\{ A \in \mathfrak{L}(\mathcal{V}_2, \mathcal{V}_1) \middle| \begin{pmatrix} 0 & A & AB \\ 0 & I & B \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{S}, \text{ for some } B \right\}$$

and

$$\Lambda = \left\{ B \in \mathfrak{L}(\mathcal{V}_3, \mathcal{V}_2) \middle| \begin{pmatrix} 0 & A & AB \\ 0 & I & B \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{S}, \text{ for some } A \right\}.$$

It is clear that

$$\mathcal{S} \subset \left\{ \begin{pmatrix} 0 & A & AB \\ 0 & I & B \\ 0 & 0 & 0 \end{pmatrix} \middle| A \in \Omega, B \in \Lambda \right\}.$$

For the reverse inclusion, note that if $A \in \Omega$ and $B \in \Lambda$ are arbitrary, then there exist $T, Q \in \mathcal{S}$ such that

$$T = \begin{pmatrix} 0 & A & AC \\ 0 & I & C \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 0 & D & DB \\ 0 & I & B \\ 0 & 0 & 0 \end{pmatrix}.$$

Consequently:

$$QT = \begin{pmatrix} 0 & A & AB \\ 0 & I & B \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{S},$$

as needed. ∎

We say that an operator band on a vector space $\mathcal{V}$ is *maximal rectangular* if it is not properly contained in any other rectangular operator band on $\mathcal{V}$.

COROLLARY 3.6. *Suppose $\mathcal{S}$ is a maximal rectangular operator band on a vector space $\mathcal{V}$. Define $\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3$ as in Theorem* 3.5. *Then $\mathcal{S}$ has matrix form*

$$\mathcal{S} = \left\{ \begin{pmatrix} 0 & A & AB \\ 0 & I & B \\ 0 & 0 & 0 \end{pmatrix} \Big| A \text{ and } B \text{ are arbitrary} \right\}$$

*with respect to the decomposition $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$.*

*Maximal rectangular bands in $\mathfrak{B}(\mathcal{H})$ have the same operator matrix form with respect to the decomposition of the Hilbert space given in Theorem* 3.5.

*Proof.* Simply note that

$$\left\{ \begin{pmatrix} 0 & A & AB \\ 0 & I & B \\ 0 & 0 & 0 \end{pmatrix} \Big| A \text{ and } B \text{ are arbitrary} \right\}$$

is a band. The rest is by Theorem 3.5 and maximality. ∎

COROLLARY 3.7. *Every maximal rectangular band in $\mathfrak{B}(\mathcal{H})$ contains an orthogonal projection.*

OPERATOR BANDS WITH SEVERAL COMPONENTS. The first goal of this section is to demonstrate that every band of operators in finite dimensions has finitely many components. The second goal is to explore the 'converse'. What we show is that operator bands with finitely many components exhibit certain finite-dimensional behaviour. For example: such bands have special block-upper-triangular structure (with finitely many blocks) and they possess a non-negative-integer-valued 'faithful' trace that extends to a (linear) trace on the algebra generated by the band and which behaves not unlike the canonical trace (present in finite dimensions) with respect to the component structure of the band.

We say that a set of operators on a vector space is *reducible* if it has a common non-trivial invariant subspace. In the case of bounded operators on a Hilbert space 'reducibility' is defined in terms of non-trivial closed invariant subspaces. The term *irreducible* is self-explanatory.

The first result is common knowledge and is stated here as a reminder.

Lemma 3.8. *If a non-zero algebra ideal $\mathfrak{I}$ in an algebra $\mathcal{A}$ of operators on a vector space $\mathcal{V}$ is reducible, then so is $\mathcal{A}$. Moreover the spaces* $\operatorname{ran}(\mathfrak{I})$ *and* $\ker(\mathfrak{I})$ *defined by:*

$$\operatorname{ran}(\mathfrak{I}) = \bigvee_{T\in\mathfrak{I}} \operatorname{ran}(T)$$

*and*

$$\ker(\mathfrak{I}) = \bigcap_{T\in\mathfrak{I}} \ker(T),$$

*are always invariant under $\mathcal{A}$.*

*Similar results hold for bounded operators on a Hilbert space, with* $\overline{\operatorname{ran}(\mathfrak{I})}$ *replacing* $\operatorname{ran}(\mathfrak{I})$.

*Proof.* That $\ker(\mathfrak{I})$ and $\operatorname{ran}(\mathfrak{I})$ are invariant under $\mathcal{A}$ is an easy exercise. In verifying the other claim we may assume $\ker(\mathfrak{I}) = \{0\}$. In this case, if $\mathcal{W}$ is a non-trivial invariant subspace of $\mathfrak{I}$, then the span of $\{T(w) \,|\, T \in \mathfrak{I},\, w \in \mathcal{W}\}$ is a non-$\{0\}$ invariant subspace for $\mathcal{A}$. ∎

A non-empty subset $\mathfrak{J}$ of a band $\mathcal{S}$ is said to be a *band ideal* in $\mathcal{S}$ if $AT \in \mathfrak{J}$ and $TA \in \mathfrak{J}$, for all $A \in \mathcal{S}$ and $T \in \mathfrak{J}$.

Note that the span of an operator band ideal is an algebra ideal in the algebra generated by the band.

Lemma 3.9. ([6]) *Lemma* 3.8 *remains true if 'algebra ideal' is replaced by 'band ideal'.*

*Proof.* The span($\mathfrak{J}$) of $\mathfrak{J}$ is an algebra ideal in the algebra span($\mathcal{S}$). The invariant subspaces of $\mathfrak{J}$ and $\mathcal{S}$ coincide with those for span($\mathfrak{J}$) and span($\mathcal{S}$) respectively. A reference to Lemma 3.8 completes the proof. ∎

Theorem 3.10. ([2]) *A subset $\mathfrak{J}$ of a band $\mathcal{S}$ is a band ideal in $\mathcal{S}$ if and only if*

(i) *$\mathfrak{J}$ is a union of components of $\mathcal{S}$.*

(ii) *If $\mathcal{C}$ and $\mathcal{D}$ are components of $\mathcal{S}$ such that $\mathcal{C} \prec \mathcal{D}$ and $\mathcal{D} \subset \mathfrak{J}$, then $\mathcal{C} \subset \mathfrak{J}$.*

The last theorem is best considered in terms of the band of components of $\mathcal{S}$. It is also curious to note that a union of band ideals is still a band ideal.

Lemma 3.11. *If an operator band $\mathcal{S}$ on a vector space $\mathcal{V}$ has a minimal non-$\{0\}$ component then $\mathcal{S}$ is reducible. The same is true in Hilbert space setting.*

*Proof.* Without loss of generality assume $0 \in \mathcal{S}$. If $\mathcal{C}$ is a minimal non-$\{0\}$ component of $\mathcal{S}$ (for non-triviality assume $\mathcal{C} \neq \{I\}$), then $\mathcal{C} \cup \{0\}$ is a band ideal in $\mathcal{S}$, by Theorem 3.10. By Proposition 3.4, either $\operatorname{ran}(\mathcal{C})$ or $\ker(\mathcal{C})$ is non-trivial. Apply Theorem 3.9 to complete the proof. ∎

The 'length' of a chain of components of a band is the number of components in the chain.

LEMMA 3.12. *Suppose $\mathcal{S}$ is a band such that all chains of its components are bounded in length by $n$. If $\mathcal{G}$ is an image of $\mathcal{S}$ under a band homomorphism $\psi$, then all chains of components of $\mathcal{G}$ are also bounded in length by $n$.*

*Proof.* The proof is by contradiction. Suppose

$$\mathcal{C}_1 \succ \mathcal{C}_2 \succ \mathcal{C}_3 \succ \cdots \mathcal{C}_{n+1}$$

is a chain of components of $\mathcal{G}$. Then there exist $A_1, A_2, A_3, \ldots, A_{n+1}$ in $\mathcal{S}$ such that

$$\psi(A_1) \succ \psi(A_2) \succ \psi(A_3) \succ \cdots \succ \psi(A_{n+1}).$$

(Simply pick any $A_i$ such that $\psi(A_i) \in \mathcal{C}_i$.) Observe that

$$\begin{aligned}
&\psi(A_1) = \psi(A_1) \sim \psi(A_1)\\
&\psi(A_1A_2) = \psi(A_1)\psi(A_2) \sim \psi(A_2)\\
&\psi(A_1A_2A_3) = \psi(A_1)\psi(A_2)\psi(A_3) \sim \psi(A_3)\\
&\qquad\qquad\vdots\\
&\psi(A_1A_2A_3\cdots A_nA_{n+1}) = \psi(A_1)\psi(A_2)\psi(A_3)\cdots\psi(A_n)\psi(A_{n+1}) \sim \psi(A_{n+1}).
\end{aligned}$$

Consequently:

$$A_1 \not\sim A_1A_2 \not\sim A_1A_2A_3 \not\sim \cdots \not\sim (A_1A_2A_3\cdots A_{n+1}),$$

because

$$\psi(A_1) \not\sim \psi(A_1A_2) \not\sim \psi(A_1A_2A_3) \not\sim \cdots \not\sim (\psi(A_1)\psi(A_2)\psi(A_3)\cdots\psi(A_n)\psi(A_{n+1})).$$

It follows that

$$A_1 \succ A_1A_2 \succ A_1A_2A_3 \succ \cdots \succ A_1A_2A_3\cdots A_{n+1}.$$

Restated in terms of components of $\mathcal{S}$, the last statement clearly contradicts the hypothesis of the lemma. ∎

Recall that a set of operators is said to be *triangularizable* if it possesses a a chain of common invariant subspaces which is maximal among chains of subspaces.

Theorem 3.13. *If an operator band $\mathcal{S}$ on a vector space $\mathcal{V}$ is such that all chains of its components are bounded in length then $\mathcal{S}$ is triangularizable. The same is true in a Hilbert space setting.*

*Proof.* If a band $\mathcal{S}$ satisfies the hypothesis (without loss of generality $\mathcal{S} \neq \{0\}$), then it possesses a minimal non-$\{0\}$ component. Therefore $\mathcal{S}$ is reducible by Lemma 3.11.

If $\mathcal{W}$ is an invariant subspace for $\mathcal{S}$ and $\mathcal{U}$ is any complement of $\mathcal{W}$ in $\mathcal{V}$ and $P_{\mathcal{U}}$ is the projection along $\mathcal{W}$ onto $\mathcal{U}$, then $\{T|_{\mathcal{W}} \mid T \in \mathcal{S}\}$ and $\{P_{\mathcal{U}}T|_{\mathcal{U}} \mid T \in \mathcal{S}\}$ are operator bands on $\mathcal{W}$ and $\mathcal{U}$ respectively. These bands are homomorphic images of $\mathcal{S}$ and therefore the chains of components in these bands are still bounded in length (Lemma 3.12). A standard application of Zorn's Lemma completes the proof. ∎

Corollary 3.14. *An operator band which has finitely many components is reducible. This is true in both vector space and Hilbert space settings.*

We are now in a position to recover the following theorem.

Theorem 3.15. ([6]) *Every operator band on a finite-dimensional vector space is triangularizable.*

*Proof.* Every such non-$\{0\}$ band possesses a minimal non-$\{0\}$ component (Theorem 3.3) and is therefore reducible (Lemma 3.11). The rest of the proof follows the same steps as that for Theorem 3.13 ∎

Theorem 3.16. *Every operator band $\mathcal{S}$ on a finite-dimensional vector space $\mathcal{V}$ has finitely many components.*

*Proof.* By Theorem 3.15 we may think of $\mathcal{S}$ as a band of $n \times n$ upper-triangular matrices. Since the matrices are idempotent, the entries on their diagonals are either 0 or 1. For each $A \in \mathcal{S}$ we denote by $\Delta(A)$ the $n \times n$ diagonal matrix whose diagonal is equal to that of $A$.

*Claim.* $A \sim B$ in $\mathcal{S}$ if and only if $\Delta(A) = \Delta(B)$. Indeed, $A \sim B$ implies $\Delta(A) = \Delta(B)$ because the map $T \overset{\Delta}{\longmapsto} \Delta(T)$ is a homomorphism on upper-triangular matrices. Conversely if $\Delta(A) = \Delta(B)$ then

$$\Delta(AB) = \Delta(A) = \Delta(B),$$

so that

$$\operatorname{tr}(AB) = \operatorname{tr}(A) = \operatorname{tr}(B).$$

Therefore $A \sim B$ by Theorem 3.1, and the claim is proved.

Since the set of $n \times n$, 0-1 diagonal matrices is finite, so is the number of components of $\mathcal{S}$. ∎

Given a block-upper-triangular (possibly infinite) operator matrix $A$, denote by $\Delta(A)$ the block-diagonal matrix (of the same format as $A$), with the block-diagonal equal to that of $A$. We shall refer to $\Delta(A)$ as the *block-diagonal truncation* of $A$.

If $\mathcal{S}$ is a band of block-upper-triangular operator matrices, then the *block-diagonal truncation* $\Delta(\mathcal{S})$ of $\mathcal{S}$ is the band (!) defined by:

$$\Delta(\mathcal{S}) = \{\Delta(A) \mid A \in \mathcal{S}\}.$$

With this set-up, let us write $\Delta$ for the obvious band homomorphism from $\mathcal{S}$ to $\Delta(\mathcal{S})$. It is clear that $\Delta(A) \sim \Delta(B)$ whenever $A \sim B$. Therefore $\Delta$ canonically induces a band homomorphism $\Delta/\!\sim$ from $\mathcal{S}/\!\sim$ to $(\Delta(\mathcal{S}))/\!\sim$. (Think of $\Delta/\!\sim$ as sending components of $\mathcal{S}$ to the corresponding components of $\Delta(\mathcal{S})$.)

Our next result shows that the component structure of a block-upper-triangular operator band $\mathcal{S}$ (with no restriction on the number of components) is completely determined by its block-diagonal truncation $\Delta(\mathcal{S})$.

Theorem 3.17. *Suppose $\mathcal{S}$ is an operator band on a vector space $\mathcal{V}$ such that $\mathcal{S}$ has a block-upper-triangular matrix form with respect to a decomposition*

$$\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_n \oplus \mathcal{V}_{n+1} \oplus \cdots.$$

*Then $\Delta/\!\sim$ is a band isomorphism. The same is true in Hilbert space setting.*

*Proof.* What we are required to show is that $\Delta/\!\sim$ is injective. In other words: if $\Delta(A) \sim \Delta(B)$ then $A \sim B$ $(A, B \in \mathcal{S})$.

*Claim.* If $A, B \in \mathcal{S}$ satisfy $\Delta(A) = \Delta(B)$, then $A \sim B$.

Indeed, by symmetry it is enough to show that $A \precsim B$. Note that $A - ABA$ is idempotent (this is true for any $A$ and $B$ in a band) and

$$\Delta(ABA) = \Delta(A)\Delta(B)\Delta(A) = \Delta(A).$$

It follows that $\Delta(A - ABA) = 0$. In the next few lines we will show that all of this implies that $A - ABA = 0$ (i.e. $A \precsim B$). If we write $\mathcal{Z}_m = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_m$, then $\mathcal{Z}_m$ is invariant under $\mathcal{S}$. Thus $(A - ABA)|_{\mathcal{Z}_m}$ is still idempotent and is represented by an $m \times m$ block-matrix whose block-diagonal entries are all zero. This implies that the operator is both idempotent and nilpotent. The only such operator is 0. This shows that $(A - ABA)|_{\mathcal{Z}_m} = 0$, for all $m \in \mathbb{N}$. Consequently $A - ABA = 0$ as claimed.

Thus

$$\Delta(A) \sim \Delta(B) \Longrightarrow \Delta(ABA) = \Delta(A) \Longrightarrow ABA \sim A \Longrightarrow ABA = A \Longrightarrow A \precsim B$$

and

$$\Delta(A) \sim \Delta(B) \Longrightarrow B \precsim A$$

by symmetry. Therefore $A \sim B$, as required. ∎

It is worth noting that the ideas in the proof above can be easily adapted to other 'reasonably discrete' decompositions of the space.

Besides the information about the component structure of the band, the block-diagonal truncation also carries complete information about the rank, in the following sense:

THEOREM 3.18. *Suppose $T$ is an idempotent operator on a vector space $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_n$ and $T$ has block-upper-triangular matrix form with respect to this decomposition. Then*

$$T \text{ has finite rank} \Longleftrightarrow \Delta(T) \text{ has finite rank.}$$

*In this case*

$$\operatorname{rank}(T) = \operatorname{rank}(\Delta(T)).$$

*The same is true in Hilbert space setting.*

*Proof.* Indeed, if $T$ has finite rank then so do all of its compressions. Conversely, the proof is by induction on $n$. The result is trivially true for $n = 1$. The inductive step relies entirely on the following claim:

*Claim.* If $A = \begin{pmatrix} E & X \\ 0 & F \end{pmatrix}$ is idempotent and $E, F$ have finite rank then $A$ has finite rank and

$$\operatorname{rank}(A) = \operatorname{rank}\begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix}.$$

Indeed, the fact that $A$ is idempotent implies that $X = EX + XF$. Let $R$ stand for the matrix

$$\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}.$$

Then $R$ is invertible and

$$RAR = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix}.$$

The required conclusions of the claim follow. The rest of the details of the inductive proof is standard. ∎

The above theorem is also true when the space is decomposed into countably many pieces. The proof is left to the reader.

The next theorem indicates that by studying block-upper-triangular bands with special block-diagonal structure we cover a lot of ground.

Theorem 3.19. *If a band $\mathcal{S}$ of operators on a vector space $\mathcal{V}$ has finitely many components, then there exist complementary subspaces $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ of $\mathcal{V}$ (m is usually greater than the number of components), such that with respect to the decomposition $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_m$ all elements of $\mathcal{S}$ have a block-upper-triangular matrix form, with each block on the diagonal being either* $0$ *or* $I$. *Components of $\mathcal{S}$ are those subsets that consist of all elements in $\mathcal{S}$ with the same block-diagonal.*

*Similar results hold true in a Hilbert space setting.*

*Proof.* Suppose $\mathcal{S}$ is a non-$\{0\}$ operator band with finitely many components acting on a vector space $\mathcal{V}$. Without loss of generality assume $0 \in \mathcal{S}$. One of the components of $\mathcal{S}$ (say $\mathcal{C}$) must be a minimal non-$\{0\}$ component.

Decompose $\mathcal{V}$ as $\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$, via Theorem 3.5, with $\operatorname{ran}(\mathcal{C}) = \mathcal{W}_1 \oplus \mathcal{W}_2$. With respect to this decomposition $\mathcal{C}$ has matrix form

$$\mathcal{C} = \left\{ \begin{pmatrix} 0 & A & AB \\ 0 & I & B \\ 0 & 0 & 0 \end{pmatrix} \Bigg| A \in \Omega, B \in \Lambda \right\},$$

for appropriate sets $\Omega$ and $\Lambda$. The set $\mathcal{C} \cup \{0\} (= \mathcal{C}_0)$ is an ideal in $\mathcal{S}$. This implies that both $\operatorname{ran}(\mathcal{C}_0)$ and $\ker(\mathcal{C}_0)$ are invariant under $\mathcal{S}$ (Theorem 3.9), and so is their intersection. Yet: $\operatorname{ran}(\mathcal{C}_0) = \mathcal{W}_1 \oplus \mathcal{W}_2$ and $\operatorname{ran}(\mathcal{C}_0) \cap \ker(\mathcal{C}_0) = \mathcal{W}_1$. Therefore all elements of $\mathcal{S}$ have block-upper-triangular matrix form with respect to $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$.

If $D \in \mathcal{C}$ and $E \in \mathcal{S}$ then $ED \sim DE \precsim D$. Since $\mathcal{C}$ is a minimal non-$\{0\}$ component, it follows that either $ED \sim D$ or $ED = 0$, i.e. either $EDE = E$ or $EDE = 0$. This implies that every element $T$ of $\mathcal{S}$ has matrix form

$$T = \begin{pmatrix} T_{11} & * & * \\ 0 & (0 \text{ or } I) & * \\ 0 & 0 & T_{33} \end{pmatrix}.$$

The map (from $\mathcal{S}$ to $\mathfrak{L}(\mathcal{W}_1)$) that sends $T$ to $T_{11}$ is a band homomorphism whose kernel contains $\mathcal{C}$. It follows that $\{T_{11} \mid T \in \mathcal{S}\} = \mathfrak{T}$ is a band in $\mathfrak{L}(\mathcal{W}_1)$ with strictly fewer components than $\mathcal{S}$.

Apply the same steps to $\mathfrak{T}$ as were applied to $\mathcal{S}$, and so obtain a decomposition of $\mathcal{W}_1$ into (at most) three parts, such that with respect to this decomposition elements of $\mathfrak{T}$ have block-upper-triangular matrix form, with blocks on the diagonal being either 0 or $I$.

Do the same for the band $\mathfrak{R} = \{T_{33} \mid T \in \mathcal{S}\}$. This is the the first stage of the process of constructing the required decomposition of $\mathcal{V}$.

Keep passing to the (1,1) and (3,3) compressions of bands constructed as 'corners' at the previous stage ($\mathfrak{T}$ and $\mathfrak{R}$ after the first stage, for example). After each iteration the bands obtained have strictly fewer components than those at the previous stage. Consequently this process will stop after finitely many steps. Combine all intermediate decompositions in the obvious way to obtain the required decomposition of $\mathcal{V}$.

To verify the claim about the components, simply apply Theorem 3.17, keeping in mind that for 0-$I$ block-diagonal matrices $\sim$ is the same as equality. ∎

A non-zero function $\tau : \mathcal{S} \to \mathbb{R}$ on a band $\mathcal{S}$ is called a *trace* provided $\tau(ab) = \tau(ba)$, for all $a, b \in \mathcal{S}$. We shall say that a trace $\tau$ is *faithful* if it satisfies: $\tau(a) = 0 \Leftrightarrow a = 0$.

It is clear from the definition that a trace is constant on the components of $\mathcal{S}$. Consequently there is an obvious bijection between traces on $\mathcal{S}$ and those on $\Delta(\mathcal{S})$. This bijection maps faithful traces to faithful traces (in both directions).

The next theorem should be compared to Theorem 3.1.

Theorem 3.20. *If $\mathcal{S}$ is a band of operators on a vector space and $\mathcal{S}$ has finitely many components, then $\mathcal{S}$ possesses a faithful bounded non-negative-integer-valued trace $\tau$ such that*

(i) *$\tau$ extends to a linear trace $\widehat{\tau}$ on the algebra generated by $\mathcal{S}$, and $\widehat{\tau}(N) = 0$ for every nilpotent $N$ in the algebra.*

(ii) *For every $A \in \mathcal{S}$*

$$\{B \in \mathcal{S} \mid \tau(AB) = \tau(A) = \tau(B)\}.$$

*is the component of $\mathcal{S}$ containing $A$.*

*The same is true in a Hilbert space setting.*

*Proof.* Use Theorem 3.19 above to decompose the underlying space into finitely many parts (say: $m$), so that with respect to this decomposition the matrix for each element is block-upper-triangular and with each block on the diagonal being either 0 or $I$. For $A \in \mathcal{S}$ define trace $\tau(A)$ to be the number of $I$'s on the block-diagonal of $A$. That $\tau$ is a bounded non-negative-integer-valued band trace is immediate. This band trace is faithful: if $\tau(A) = 0$ then $A$ is nilpotent (since $\Delta(A) = 0$) and since $A$ is idempotent it follows that $A = 0$. It remains to be shown that $\tau$ posesses properties (i) and (ii).

(i) Each element $T$ of $\operatorname{span}(\mathcal{S})$ has matrix form

$$T = \begin{pmatrix} t_1 I & * & * & \dots & * \\ 0 & t_2 I & * & \dots & * \\ 0 & 0 & t_3 I & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & t_m I \end{pmatrix},$$

with respect to the decomposition above. Let $\xi : \operatorname{span}(\mathcal{S}) \longmapsto \mathbb{M}_m(\mathbb{F})$ be the map defined by:

$$\xi(T) = \begin{pmatrix} t_1 & 0 & 0 & \dots & 0 \\ 0 & t_2 & 0 & \dots & 0 \\ 0 & 0 & t_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & t_m \end{pmatrix}.$$

If we denote by $\rho$ the canonical trace on $\mathbb{M}_m(\mathbb{F})$, then $\rho \circ \xi$ is a linear trace extending $\tau$ to the algebra $\operatorname{span}(\mathcal{S})$. That this trace is zero on every nilpotent in $\operatorname{span}(\mathcal{S})$ is clear, since such nilpotents have only zeros on the block-diagonal.

(ii) If $A, B \in \mathcal{S}$ then

$$\begin{aligned} \tau(A) = \tau(AB) &\Longleftrightarrow \tau(A) = \tau(ABA) \Longleftrightarrow \Delta(A) = \Delta(ABA) \\ &\Longleftrightarrow \Delta(A - ABA) = 0 \Longleftrightarrow A - ABA = 0 \Longleftrightarrow A \precsim B. \end{aligned}$$

(Here we once again used the facts that $A - ABA$ is idempotent and that the only block-upper-triangular idempotent (with finitely many blocks) that has a zero block-diagonal is the zero operator.)

Since it follows by symmetry that

$$\tau(B) = \tau(AB) \Longleftrightarrow B \precsim A,$$

the proof is complete. ∎

A trace on a band $\mathcal{S}$ is said to be *a band trace on* $\mathcal{S}$ if it satisfies the condition stated as part (ii) in Theorem 3.20.

4. SPANS AND CONVEX HULLS OF OPERATOR BANDS

The existence of a well-behaved band trace for bands with finitely many components (see Corollary 3.20) turns out to be a powerful tool which can be used for arbitrary bands.

The next four results are true in both vector space and Hilbert space settings.

Theorem 4.1. *Every operator in the linear span of (i.e. in the algebra generated by) an operator band $\mathcal{S}$ is algebraic.*

*Proof.* Suppose $A_1, A_2, \dots, A_n \in \mathcal{S}$ and $B$ is a linear combination of $A_1, A_2, \dots, A_n$. If $\mathcal{S}_0$ is the subband of $\mathcal{S}$ generated by $A_1, A_2, \dots, A_n$, then $\mathcal{S}_0$ has finitely many components. Use Theorem 3.19 to decompose the underlying space into finitely many parts, so that with respect to this decomposition the matrix for each element of $\mathcal{S}_0$ is block-upper-triangular with each block on the diagonal being either 0 or $I$. Then $B$ has matrix form

$$B = \begin{pmatrix} b_1 I & * & * & \dots & * \\ 0 & b_2 I & * & \dots & * \\ 0 & 0 & b_3 I & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_m I \end{pmatrix}.$$

Apply the standard matricial inductive argument to show that $\prod_{i=1}^{m} (B - b_i I) = 0$. Thus $B$ is algebraic. ∎

Theorem 4.2. *If elements $A_1, A_2, \dots, A_n$ of an operator band $\mathcal{S}$ are such that $A_1 + A_2 + \cdots + A_n$ is idempotent, then $A_i A_j = 0$, for all $i \neq j$.*

*Proof.* If $\mathcal{S}_0$ is the subband of $\mathcal{S}$ generated by $A_1, A_2, \dots, A_n$, then $\mathcal{S}_0$ has finitely many components. Let $\tau$ be a band trace on $\mathcal{S}_0$ provided by Corollary 3.20. That $\sum\sum_{i\neq j} A_i A_j = 0$ follows from the fact that $A_1 + A_2 + \cdots + A_n$ is idempotent. Consequently

$$\widehat{\tau}\Big(\sum_{i\neq j}\sum A_i A_j\Big) = \sum_{i\neq j}\sum \tau(A_i A_j) = 0.$$

It follows that $\tau(A_i A_j) = 0$, for all $i \neq j$, because $\tau$ is non-negative-integer-valued. Since $\tau$ is faithful, it must be that $A_i A_j = 0$, for all $i \neq j$, as claimed. ∎

In the next two theorems we assume that the underlying field $\mathbb{F}_{\mathbb{Q}}$ is an ordered field extension of the field $\mathbb{Q}$ of rational numbers.

THEOREM 4.3. *If elements* $A_1, A_2, \ldots, A_n$ *of an operator band* $\mathcal{S}$ *on a vector space* $\mathcal{V}$ *over a field* $\mathbb{F}_{\mathbb{Q}}$ *are such that* $p_1A_1 + p_2A_2 + \cdots + p_nA_n$ *is nilpotent, for some positive field elements* $p_1, p_2, \ldots, p_n$*, then* $A_i = 0$*, for all* $i$*.*

*Proof.* If $\mathcal{S}_0$ is the subband of $\mathcal{S}$ generated by $A_1, A_2, \ldots, A_n$ then $\mathcal{S}_0$ has finitely many components. Let $\tau$ be a band trace on $\mathcal{S}_0$ provided by Corollary 3.20. If $p_1A_1 + p_2A_2 + \cdots + p_nA_n$ is nilpotent, then

$$0 = \widehat{\tau}\Big(\sum_{i=1}^{n} p_iA_i\Big) = \sum_{i=1}^{n} p_i\tau(A_i).$$

It follows that $\tau(A_i) = 0$, for all $i$, because $\tau$ is non-negative-integer valued. ∎

Theorem 4.3 is true in a Hilbert space setting even if 'nilpotent' is replaced with 'quasinilpotent'. Indeed, this is a mere change of wording. In such a case, if $p_1A_1 + p_2A_2 + \cdots + p_nA_n$ is quasinilpotent (for some positive constants $p_1, p_2, \ldots, p_n$) then it is nilpotent, since it must be algebraic by Theorem 4.1.

THEOREM 4.4. *The spectrum of every element in the convex hull of an operator band* $\mathcal{S}$ *on a vector space* $\mathcal{V}$ *over a field* $\mathbb{F}_{\mathbb{Q}}$ *is a subset of the interval* $[0,1]_{\mathbb{F}_{\mathbb{Q}}}$*.*

*Proof.* Suppose $A_1, A_2, \ldots, A_n \in \mathcal{S}$ and $p_1A_1 + p_2A_2 + \cdots + p_nA_n = C$ is a convex combination of $A_1, A_2, \ldots, A_n$. If $\mathcal{S}_0$ is the subband of $\mathcal{S}$ generated by $A_1, A_2, \ldots, A_n$ then $\mathcal{S}_0$ has finitely many components. Use Theorem 3.19 to decompose the underlying space into finitely many parts, so that with respect to this decomposition the matrix for each element of $\mathcal{S}_0$ is block-upper-triangular with each block on the diagonal being either 0 or $I$. The matrix for $C$ is also block-upper-triangular and each block on its diagonal is of the form $\alpha I$, for some $\alpha \in [0,1]_{\mathbb{F}_{\mathbb{Q}}}$. That the spectrum of $C$ lies in $[0,1]_{\mathbb{F}_{\mathbb{Q}}}$ is now clear. ∎

The next two technical results build towards Theorem 4.7, which states that an operator band has finitely many components exactly when all operators in its span are algebraic with degree bounded above. We begin with some simple algebraic observations.

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are elements of an infinite field $\mathbb{F}$ such that $b_i \neq 0$ whenever $a_i = 0$, then there exists $c \in \mathbb{F}$ such that $a_i \neq cb_i$, for all $i$.

Consequently, if $d_1, d_2, \ldots, d_m$ and $h_1, h_2, \ldots, h_m$ are elements of an infinite field $\mathbb{F}$ such that $h_i \neq h_j$ whenever $d_i = d_j$, then there exists $c \in \mathbb{F}$ such that $d_1 + ch_1, d_2 + ch_2, \ldots, d_m + ch_m$ are all distinct. (Indeed, this is equivalent to saying that $d_k - d_l \neq c(h_l - h_k)$ for all $k, l$, and existence of a required $c$ follows from the last paragraph).

LEMMA 4.5. *Suppose $\mathcal{M}$ is a subspace of a vector space $\mathbb{F}^n$, where $\mathbb{F}$ is an infinite field. Then either $\mathcal{M}$ contains an n-tuple whose entries are all distinct, or $\mathcal{M} \subset \{(r_1, \ldots, r_n) \in \mathbb{F}^n \mid r_i = r_j\}$, for some $i, j$.*

*Proof.* The proof is by contradiction. Suppose $\mathcal{M}$ satisfies the hypothesis but not the conclusion. Let $x$ be an element in $\mathcal{M}$ with the maximal number (say $k$; $k < n$) of distinct entries. By permuting, we may assume that $x$ is of the form

$$x = (t_1, t_1, t_3, \ldots, t_{k+1}, *, \ldots, *),$$

where $t_1, t_3, \ldots, t_{k+1}$ are all distinct.

Since $\mathcal{M} \not\subset \{(r_1, \ldots, r_n) \in \mathbb{F}^n \mid r_1 = r_2\}$, there exists some $y = (h_1, h_2, h_3, \ldots, h_n) \in \mathcal{M}$ such that $h_1 \neq h_2$. Apply the remarks preceeding this lemma to conclude that there exists some $c \in \mathbb{F}$ such that $t_1 + ch_1$, $t_1 + ch_2$, $t_3 + ch_3, \ldots, t_{k+1} + ch_{k+1}$ are all distinct. Then $x + cy$ is an element of $\mathcal{M}$ with at least $k+1$ distinct entries. This contradicts the choice of $x$. ∎

COROLLARY 4.6. *If $2^m \leqslant k$ and $m \leqslant l$ and $Z_1, \ldots, Z_k$ are distinct $l$-tuples of 0's and 1's, and $\mathbb{F}$ is an infinite field, then* $\operatorname{span}_{\mathbb{F}}\{Z_1, \ldots, Z_k\}$ *contains an element with at least $m$ distinct entries.*

*Proof.* The main part of the proof is the following:

*Claim.* After permuting the coordinates, if necessary, we may assume that the $m$-tuples $\widehat{Z}_1, \ldots, \widehat{Z}_k$ obtained by chopping off the last $l-m$ entries of $Z_1, \ldots, Z_k$ satisfy:

$$\{\widehat{Z}_1, \ldots, \widehat{Z}_k\} \not\subset \{(r_1, \ldots, r_m) \in \mathbb{F}^m \mid r_i = r_j\}, \qquad \text{for any } i, j.$$

Indeed, think of $Z_1, \ldots, Z_k$ as rows of a $k \times l$, 0-1 matrix $T$. Permute coordinates so as to collect equal columns of $T$ together:

$$T = \left( \boxed{C_1} \boxed{C_1} \cdots \boxed{C_1} \boxed{C_2} \boxed{C_2} \cdots \boxed{C_2} \cdots \boxed{C_n} \boxed{C_n} \cdots \boxed{C_n} \right),$$

with $C_i \neq C_j$, for $i \neq j$. Since there can be no more than $2^n$ distinct 0-1 $l$-tuples of the form $(c_1, \ldots, c_1, c_2, \ldots, c_2, \ldots, c_n, \ldots, c_n)$ and $Z_1, \ldots, Z_k$ are of this form and are distinct, it must be that $k \leqslant 2^n$. Yet $2^m \leqslant k$. Therefore $m \leqslant n$. By permuting again we may assume

$$T = \left( \boxed{C_1} \boxed{C_2} \cdots \boxed{C_m} \boxed{*} \cdots \boxed{*} \right).$$

Taking $\widehat{Z}_1, \ldots, \widehat{Z}_k$ to be the rows of the matrix

$$\begin{pmatrix} \boxed{C_1} & \boxed{C_2} & \cdots & \boxed{C_m} \end{pmatrix}$$

concludes the proof of the claim.

Use Lemma 4.5 to conclude that $\mathrm{span}_{\mathbb{F}}\{\widehat{Z}_1, \ldots, \widehat{Z}_k\}$ contains an $m$-tuple whose entries are all distinct. This clearly implies that $\mathrm{span}_{\mathbb{F}}\{Z_1, \ldots, Z_k\}$ contains an element with at least $m$ distinct entries. ∎

THEOREM 4.7. *For an operator band $\mathcal{S}$ the following are equivalent:*

(i) *$\mathcal{S}$ has finitely many components.*

(ii) *There exists an integer $m$ such that every element of* $\mathrm{span}(\mathcal{S})$ *is algebraic with a minimal polynomial of degree at most $m$.*

*The same is true in a Hilbert space setting.*

*Proof.* (i) $\Rightarrow$ (ii) See the proof of Theorem 4.1.

(ii) $\Rightarrow$ (i) We prove the contrapositive. Suppose $\mathcal{S}$ does not satisfy (a). Without loss of generality assume $0 \in \mathcal{S}$.

*Claim.* For every $n$ there exists some $k \geqslant n$ such that $\mathcal{S}$ has a subband with $k$ components.

Indeed, pick any elements $A_1, A_2, \ldots, A_n$ of $\mathcal{S}$, all from different components. If $\mathcal{S}_0$ is a subband of $\mathcal{S}$ generated by these elements, then $\mathcal{S}_0$ has at least $n$ (and at most $2^n - 1$) components (refer to the Preliminaries section), so the claim is proved.

Let $m$ be any natural number. Pick any $k \geqslant 2^m$ such that $\mathcal{S}$ has a subband $\mathcal{S}_0$ with $k$ components. Use Theorem 3.19 to decompose the underlying space into finitely many (say $l$) parts, so that with respect to this decomposition the matrix for each element of $\mathcal{S}_0$ is block-upper-triangular with each block on the diagonal being either 0 or $I$. Observe that since $k \geqslant 2^m$ and the components of $\mathcal{S}_0$ are distinguished by patterns of 0's and $I$'s on the block-diagonal, it must be that $l \geqslant m$.

Select elements $B_1, \ldots, B_k \in \mathcal{S}_0$, by choosing (without any restrictions) one from each of the $k$ components of $\mathcal{S}_0$. The patterns of 0's and $I$'s on their block-diagonals are all distinct. Apply the obvious modification of Corollary 4.6 to conclude that some $D \in \mathrm{span}_{\mathbb{F}}\{B_1, \ldots, B_k\}$ has matrix form

$$D = \begin{pmatrix} d_1 I & * & * & \cdots & * \\ 0 & d_2 I & * & \cdots & * \\ 0 & 0 & d_3 I & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & d_l I \end{pmatrix},$$

where at least $m$ of the $d_i$'s are distinct. Since $d_1, \dots, d_l$ must be roots of the minimal polynomial for $D$, the degree of such polynomial is at least $m$. On the other hand $D \in \operatorname{span}_{\mathbb{F}}(\mathcal{S})$ and the choice of $m$ was arbitrary. It is now that clear $\mathcal{S}$ does not satisfy (ii). ∎

## 5. REDUCIBLE OPERATOR BANDS WITH INFINITELY MANY COMPONENTS

The following lemma is common knowledge and is stated here explicitly for reader's convenience.

LEMMA 5.1. *Suppose $\mathcal{S}$ is a semigroup of operators on a vector space $\mathcal{V}$.*

(i) *If*

$$\bigvee_{T\in\mathcal{S}} \operatorname{ran}(T|_{\mathcal{Z}}) \neq \mathcal{V},$$

*for some non-$\{0\}$ subspace $\mathcal{Z}$ of $\mathcal{V}$, then $\mathcal{S}$ is reducible.*

(ii) *If $P_{\mathcal{U}}\mathcal{S}P_{\mathcal{W}} = \{0\}$, for some projections $P_{\mathcal{U}}$ and $P_{\mathcal{W}}$ onto non-$\{0\}$ subspaces $U$ and $W$ of $\mathcal{V}$, then $\mathcal{S}$ is reducible.*

*Topological analogues of these results are true in Hilbert space setting.*

*Proof.* (i) Either

$$\bigcap_{T\in\mathcal{S}} \ker(T) \neq \{0\}$$

or

$$\bigvee_{T\in\mathcal{S}} \operatorname{ran}(T|_{\mathcal{Z}})$$

is a non-trivial invariant subspace for $\mathcal{S}$. (ii) is an immediate consequence of (i). ∎

Next we present (in more generality and with a new proof) a result from [6] that we shall further extend in Theorem 5.10.

THEOREM 5.2. ([6]) *If an operator band $\mathcal{S}$ contains a non-zero finite-rank operator then $\mathcal{S}$ is reducible. This is true in both vector space and Hilbert space settings.*

*Proof.* We will show that $\mathcal{S}$ contains a minimal non-$\{0\}$ component and then simply refer to Lemma 3.11 for the required conclusion. To this end, observe that if $0 \neq A \in \mathcal{S}$ is an operator of finite rank and $B \precsim A$ (i.e. $BAB = B$) in $\mathcal{S}$, then $B$ is also of finite rank and $\operatorname{rank}(B) \leqslant \operatorname{rank}(A)$. Moreover, methods of Theorem 3.2 can be easily adapted to this situation (take two finite-rank idempotents at a time and restrict them to a large enough common invariant finite-dimensional subspace) so that we can further infer that $(E \prec B) \Longrightarrow (\mathrm{rank}(E) < \mathrm{rank}(B)\ )$, for any $E, B \precsim A$.

Pick an element $B_0, B_0 \precsim A$, of minimal non-zero rank. Then $\mathcal{C}_{B_0}$ is a minimal non-$\{0\}$ component of $\mathcal{S}$. ∎

In the original proof of the theorem above, the problem was reduced to a case when $\mathcal{S}$ contains a non-trivial finite-rank left divisor of zero and it was shown that the existence of such an operator automatically implies the reducibility of $\mathcal{S}$.

It turns out that the existence of any non-trivial (not necessarily finite-rank) zero divisors in an operator band always entails reducibility.

Theorem 5.3. *If an operator band $\mathcal{S}$ has non-trivial zero divisors then $\mathcal{S}$ is reducible. This is true in both vector space and Hilbert space settings.*

*Proof.* Suppose $P, Q \in \mathcal{S}$ are non-zero and $PQ = 0$. Then

$$QP = (QP)(PQ)(QP) = 0.$$

Therefore $PAQ = (PAQ)^2 = PAQPAQ = 0$, for all $A \in \mathcal{S}$. Similarly $QAP = 0$, for all $A \in \mathcal{S}$.

The matrix of $P$ with respect to the decomposition $\mathcal{V} = \mathrm{ran}(P) \oplus \ker(P)$ is

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

This, together with $PQ = QP = 0$, implies that the matrix of $Q$ must be

$$\begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix},$$

where $E$ is an idempotent. Therefore with respect to the decomposition $\mathcal{V} = \mathrm{ran}(P) \oplus \ker(E) \oplus \mathrm{ran}(E)$, matrices of $P$ and $Q$ are

$$\begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}$$

respectively. Since $PAQ = QAP = 0$, for every $A \in \mathcal{S}$, it follows that all elements of $\mathcal{S}$ have the matrix form

$$\begin{pmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix}$$

with respect to the decomposition above. Since $\mathrm{ran}(P)$ and $\mathrm{ran}(E)$ are non-trivial, it follows that

$$\bigvee_{T \in \mathcal{S}} \mathrm{ran}(T|_{\mathrm{ran}(E)}) \neq \mathcal{V},$$

and the proof is complete by Lemma 5.1. ∎

The following lemma paves the way for the definition immediately following it.

Lemma 5.4. *If $B \precsim A$ are two elements of an operator band $\mathcal{S}$, and the co-dimension of* $\operatorname{ran}(ABA)$ *in* $\operatorname{ran}(A)$ *is $n$ (an integer), then the co-dimension of* $\operatorname{ran}(EFE)$ *in* $\operatorname{ran}(E)$ *is also $n$, for all $F \in \mathcal{C}_B$, $E \in \mathcal{C}_A$. This is true both in vector space and Hilbert space settings.*

*Proof.* Observe that the co-dimension of $\operatorname{ran}(ABA)$ in $\operatorname{ran}(A)$ is exactly the rank of $A - ABA$. Similarly the co-dimension of $\operatorname{ran}(EFE)$ in $\operatorname{ran}(E)$ is the rank of $E - EFE$.

Let $\mathcal{S}_0 = \mathcal{C}_B \cup \mathcal{C}_A$. Then $\mathcal{S}_0$ is a subband of $\mathcal{S}$ with at most two components. Use Theorem 3.19 to decompose the underlying space into finitely many parts, so that with respect to this decomposition the matrix for each element of $\mathcal{S}_0$ is block-upper-triangular with each block on the diagonal being either 0 or $I$. It easily follows that $A - ABA$ and $E - EFE$ will also have the same matrix form. Thus (refer to the discussion preceeding Theorem 3.17 for notation)

$$\begin{aligned}\operatorname{rank}(A - ABA) &= \operatorname{rank}(\Delta(A - ABA)) = \operatorname{rank}(\Delta(A) - \Delta(A)\Delta(B)\Delta(A))\\ &= \operatorname{rank}(\Delta(E) - \Delta(E)\Delta(F)\Delta(E)) = \operatorname{rank}(\Delta(E - EFE))\\ &= \operatorname{rank}(E - EFE),\end{aligned}$$

as required. Here we made use of Theorem 3.18 and the fact that $\Delta(A) = \Delta(E)$ and $\Delta(B) = \Delta(F)$, which follows from Theorem 3.19. ∎

If $\mathcal{D} \prec \mathcal{C}$ are two components of an operator band $\mathcal{S}$ such that the co-dimension of $\operatorname{ran}(ABA)$ in $\operatorname{ran}(A)$ is non-zero and finite (say $n$), for some (and therefore for all) $B \in \mathcal{D}$, $A \in \mathcal{C}$, then we say that there is a *finite-dimensional gap (of dimension $n$)* between $\mathcal{D}$ and $\mathcal{C}$.

Corollary 5.5. *Suppose $\mathcal{C} \prec \mathcal{D} \prec \mathcal{G}$ are components of an operator band $\mathcal{S}$. Then the following are equivalent:*

(i) *There is a finite-dimensional gap (of dimension $n$) between $\mathcal{C}$ and $\mathcal{G}$.*

(ii) *There are finite-dimensional gaps between $\mathcal{C}$ and $\mathcal{D}$ (of dimension $k$) and between $\mathcal{D}$ and $\mathcal{G}$ (of dimension $m$).*

*In this case: $n = m + k$.*

*Proof.* If $E \in \mathcal{C}$, $F \in \mathcal{D}$, $G \in \mathcal{G}$ then $GFG \in \mathcal{D}$ and $(GFG)E(GFG) \in \mathcal{G}$. Clearly $\operatorname{ran}((GFG)E(GFG)) \subset \operatorname{ran}(GFG) \subset \operatorname{ran}(G)$. The rest of the proof is trivial (via Lemma 5.4). ∎

We refer to components $\mathcal{D} \prec \mathcal{C}$ of a band $\mathcal{S}$ as *consecutive*, provided no component $\mathcal{G}$ of $\mathcal{S}$ satisfies:

$$\mathcal{D} \prec \mathcal{G} \prec \mathcal{C}.$$

The next lemma sheds some light on the structure of many band ideals. The reader is encouraged to pay particular attention to what takes place in the (2,2) corners of the matrices involved.

LEMMA 5.6. *Suppose $T$ is an element of an operator band $\mathcal{S}$ acting on a vector space $\mathcal{V}$. Let $\mathcal{S}_T$ stand for the band ideal $\{Q \in \mathcal{S} \mid Q \precsim T\}$. Then:*

(i) *With respect to the decomposition $\mathcal{V} = \operatorname{ran}(T) \oplus \ker(T)$ every element of $\mathcal{S}_T$ has matrix form*

$$\begin{pmatrix} E & X \\ Y & YX \end{pmatrix},$$

*with some $X, Y$ such that $XY = 0$ (and consequently $(YX)^2 = 0$) and $X = EX$, $Y = YE$.*

(ii) *Let $(\mathcal{S}_T)_{11}$, $(\mathcal{S}_T)_{21}$, $(\mathcal{S}_T)_{12}$ and $(\mathcal{S}_T)_{22}$ stand for the sets of operators that appear as* (1,1), (2,1), (1,2) *and* (2,2) *entries (respectively) in the matrices for elements of $\mathcal{S}_T$, with respect to the decomposition $\mathcal{V} = \operatorname{ran}(T) \oplus \ker(T)$. Then*

(a) *$(\mathcal{S}_T)_{11}$ is an operator band on* ran*(T),*

(b) *if $Y_1 \in (\mathcal{S}_T)_{21}$ and $X_2 \in (\mathcal{S}_T)_{12}$, then $Y_1X_2 \in (\mathcal{S}_T)_{22}$ and consequently $(Y_1X_2)^2 = 0$.*

*The same is true in Hilbert space setting.*

*Proof.* (i) The matrix of $T$ with respect to the decomposition $\mathcal{V} = \operatorname{ran}(T) \oplus \ker(T)$ is

$$T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Suppose $Q \precsim T$ and the matrix for $Q$, with respect to the same decomposition, is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

From $Q = QTQ = (QT)(TQ)$ it follows:

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix},$$

so that $A$ is idempotent, $B = AB$, $C = CA$ and $D = CB$. Therefore

$$Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & CB \end{pmatrix},$$

which, together with the fact that $Q$ is idempotent, implies $BC = 0$.

(ii) (a) $(\mathcal{S}_T)_{11}$ is a band, because $\{TRT \mid R \in \mathcal{S}_T\}$ is a subband of $\mathcal{S}$.

(ii) (b) If $Y_1 \in (\mathcal{S}_T)_{21}$ and $X_2 \in (\mathcal{S}_T)_{12}$, then there exist $Q_1$ and $Q_2$ in $\mathcal{S}_T$ such that

$$Q_1 = \begin{pmatrix} E_1 & X_1 \\ Y_1 & Y_1X_1 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} E_2 & X_2 \\ Y_2 & Y_2X_2 \end{pmatrix}.$$

In particular, $(Q_1T)(TQ_2) \in \mathcal{S}_T$ and

$$(Q_1T)(TQ_2) = \begin{pmatrix} E_1 & 0 \\ Y_1 & 0 \end{pmatrix} \begin{pmatrix} E_2 & X_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} E_1E_2 & E_1X_2 \\ Y_1E_2 & Y_1X_2 \end{pmatrix},$$

so that $Y_1X_2 \in (\mathcal{S}_T)_{22}$. ∎

THEOREM 5.7. *Suppose $\mathcal{S}$ is an operator band on a vector space $\mathcal{V}$. If two components of $\mathcal{S} \cup \{0, I\}$ have a gap of dimension 1, 2 or 3 between them, then $\mathcal{S}$ is reducible.*

*Proof.* First of all, there is no harm in assuming that $\{0, I\} \subset \mathcal{S}$. We may also assume, again without loss of generality (see Corollary 5.5), that $\mathcal{D} \prec \mathcal{C}$ are consecutive components of $\mathcal{S}$ with a gap of dimension 1, 2 or 3 between them.

If $T$ is any element of $\mathcal{C}$ (note: $T \neq 0$) then $\{TA|_{\operatorname{ran}(T)} \mid A \precsim T\} = \mathcal{S}|_T$ is a unital operator band on $\operatorname{ran}(T)$, containing 0. Then $\{TA|_{\operatorname{ran}(T)} \,\big|\, A \in \mathcal{D}\} \prec \{I\}$, and these are consecutive components of $\mathcal{S}|_T$ with a gap of dimension 1, 2 or 3 between them (refer to Lemma 5.6). If $\mathcal{S}|_T$ is reducible, then so is the band ideal $\{A \in \mathcal{S} \mid A \precsim T\}$ of $\mathcal{S}$ (use Lemma 5.6 and Lemma 5.1), and consequently so is $\mathcal{S}$ (by Theorem 3.9).

This argument shows that we may assume from the start that $\mathcal{C} = \{I\}$. Under such assumption, if $Q$ is any element of $\mathcal{D}$ then the dimension of the gap between $\mathcal{D}$ and $\{I\}$ is the co-dimension of $\operatorname{ran}(Q)$ in $\mathcal{V}$, i.e. the dimension of $\ker(Q)$. This dimension is assumed to be 1, 2 or 3.

Fix any $Q \in \mathcal{D}$. Our strategy is to show that the band ideal $\{A \in \mathcal{S} \mid A \precsim Q\} = \mathcal{S}_Q$ of $\mathcal{S}$ is reducible, and this will yield the same conclusion about $\mathcal{S}$. By part (i) of Lemma 5.6 all elements of $\mathcal{S}_Q$ have the matrix form

$$\begin{pmatrix} E & X \\ Y & YX \end{pmatrix}$$

with respect to the decomposition $\mathcal{V} = \operatorname{ran}(Q) \oplus \ker(Q)$, with some $X$, $Y$ such that $XY = 0$. The entries in (1,1) position in these matrices form an operator band $(\mathcal{S}_Q)_{11}$ on ran(Q). The sets of operators that appear as (2,1), (1,2), (2,2) entries in these matrices shall be denoted by $(\mathcal{S}_Q)_{21}$, $(\mathcal{S}_Q)_{12}$ and $(\mathcal{S}_Q)_{22}$ respectively (just as in Lemma 5.6).

We treat the three possible dimensions of the gap as three separate cases.

*Claim* 1. $\dim(\ker(Q)) = 1$ (i.e. the gap has dimension 1).

In this case $(\mathcal{S}_Q)_{22}$ is the set of nilpotent operators (Lemma 5.6) on a one-dimensional space $\ker(Q)$, and therefore $(\mathcal{S}_Q)_{22} = \{0\}$. $\mathcal{S}_Q$ is reducible by part (ii) of Lemma 5.1.

*Claim* 2. $\dim(\ker(Q)) = 2$ (i.e. the gap has dimension 2).

*Case* (2a). Every element in $(\mathcal{S}_Q)_{21}$ has either rank 0 or 2 (i.e is either zero or is onto $\ker(Q)$).

If

$$\begin{pmatrix} E & X \\ Y & YX \end{pmatrix}$$

is an element of $\mathcal{S}_Q$ then $XY = 0$. Consequently either $Y = 0$ or $Y$ is onto $\ker(Q)$, in which case $X = 0$. In either case $YX = 0$. It follows that $(\mathcal{S}_Q)_{22} = \{0\}$ and so $\mathcal{S}_Q$ is reducible by part (ii) of Lemma 5.1.

*Case* (2b). There exists $Y_0 \in (\mathcal{S}_Q)_{21}$ with $\operatorname{rank}(Y_0) = 1$.

Note that $(Y_0X) \in (\mathcal{S}_Q)_{22}$, for each $X \in (\mathcal{S}_Q)_{12}$, and so $(Y_0X)^2 = 0$ (by part (ii) of Lemma 5.6). Choose any vector $f_1$ in $\operatorname{ran}(Y_0)$ and extend it to a basis $\{f_1, f_2\}$ of $\ker(Q)$. With respect to this basis $Y_0X$ has the matrix form

$$\begin{pmatrix} a_X & b_X \\ 0 & 0 \end{pmatrix},$$

for some $a_X, b_X \in \mathbb{F}$. Clearly $a_X$ must be zero because $(Y_0X)^2 = 0$.

We have shown that $Y_0X|_{\operatorname{span}(f_1)} = 0$, for all $X \in (\mathcal{S}_Q)_{12}$. Consequently

$$\bigvee_{X \in (\mathcal{S}_Q)_{12}} \operatorname{ran}(X|_{\operatorname{span}(f_1)}) \neq \operatorname{ran}(Q),$$

which in turn implies that

$$\bigvee_{A \in \mathcal{S}_Q} \operatorname{ran}(A|_{\operatorname{span}(f_1)}) \neq \mathcal{V}.$$

Therefore $\mathcal{S}_Q$ is reducible by part (i) of Lemma 5.1.

*Claim* 3. $\dim(\ker(Q)) = 3$ (i.e. the gap has dimension 3).

*Case* (3a). There exists $Y \in (\mathcal{S}_Q)_{21}$ with $\operatorname{rank}(Y) = 1$. (Use the same argument as in Case (2b).)

*Case* (3b). There exists $X \in (\mathcal{S}_Q)_{12}$ with $\operatorname{rank}(X) = 1$. (The argument is essentially symmetric to that of Case (2b).)

*Case* (3c). Neither $(\mathcal{S}_Q)_{12}$ nor $(\mathcal{S}_Q)_{21}$ contains any operators of rank 1.

Suppose

$$\begin{pmatrix} E & X \\ Y & YX \end{pmatrix}$$

is an element of $\mathcal{S}_Q$. Then $XY = 0$. If $\operatorname{rank}(X) = 0$ or 3, then it must be that $YX = 0$ via the same argument as in Case (2a). If $\operatorname{rank}(X) = 2$ then $\operatorname{rank}(Y) \leqslant 1$ because $XY = 0$. Since $\operatorname{rank}(Y) \neq 1$, it must be that $Y = 0$, so that again $YX = 0$. We have shown again that $(\mathcal{S}_Q)_{22} = \{0\}$, so that $\mathcal{S}_Q$ is reducible by part (ii) of Lemma 5.1. The Claim 3 is proved. The proof is complete. ∎

A much stronger result is true in Hilbert space setting. The following corollary ([4]) to a result of Aupetit ([1]) turns out to be a key ingredient.

We write 'r' for the spectral radius.

Theorem 5.8. ([4]) *For an algebra $\mathcal{A}$ of compact operators on a Banach space $\mathcal{X}$ the following are equivalent:*

(i) *$\mathcal{A}$ is triangularizable.*

(ii) $\mathrm{r}(AB) \leqslant \mathrm{r}(A)\mathrm{r}(B)$*, for all $A, B \in \mathcal{A}$.*

Lemma 5.9. *If $\mathcal{S}$ is an operator band on a vector space $\mathcal{V}$ then* $\mathrm{r}(AB) \leqslant \mathrm{r}(A)\mathrm{r}(B)$*, for all $A, B \in \operatorname{span}_{\mathbb{F}}(\mathcal{S})$. The same is true in Hilbert space setting.*

*Proof.* If $A, B \in \operatorname{span}_{\mathbb{F}}(\mathcal{S})$ then

$$A = t_1A_1 + t_2A_2 + \cdots + t_nA_n,$$

and

$$B = r_1B_1 + r_2B_2 + \cdots + r_mB_m,$$

for some $A_i, B_j \in \mathcal{S}$ and some scalars $t_i$, $r_j$.

If $\mathcal{S}_0$ is the subband of $\mathcal{S}$ generated by $A_1, A_2, \ldots, A_n, B_1, \ldots, B_m$ then $\mathcal{S}_0$ has finitely many components. Use Theorem 3.19 to decompose $\mathcal{V}$ into finitely many complementary subspaces, so that with respect to this decomposition the matrix for each element of $\mathcal{S}_0$ is block-upper-triangular with each block on the diagonal being either 0 or $I$. Matrices for $A$ and $B$ are of the form

$$A = \begin{pmatrix} a_1I & * & * & \cdots & * \\ 0 & a_2I & * & \cdots & * \\ 0 & 0 & a_3I & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_pI \end{pmatrix}, \quad B = \begin{pmatrix} b_1I & * & * & \cdots & * \\ 0 & b_2I & * & \cdots & * \\ 0 & 0 & b_3I & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_pI \end{pmatrix},$$

for some scalars $a_i, b_i$. It follows that $\sigma(A) = \{a_1, \dots, a_p\}$ and $\sigma(B) = \{b_1, \dots, b_p\}$ and $\sigma(AB) = \{a_1 b_1, \dots, a_p b_p\}$. Therefore

$$\mathrm{r}(AB) = \sup_{1 \leqslant i \leqslant p} |a_i b_i| \leqslant \Big( \sup_{1 \leqslant i \leqslant p} |a_i| \Big) \Big( \sup_{1 \leqslant i \leqslant p} |b_i| \Big) = \mathrm{r}(A)\mathrm{r}(B)$$

as required. ∎

THEOREM 5.10. *If $\mathcal{S}$ is an operator band in $\mathcal{B}(\mathcal{H})$ and the (non-closed) span of $\mathcal{S}$ contains a non-zero compact operator $K$, then $\mathcal{S}$ is reducible.*

*Proof.* First of all $\mathrm{span}_{\mathbb{F}}(\mathcal{S})$ is an algebra. We will prove that $\mathrm{span}_{\mathbb{F}}(\mathcal{S})$ is reducible. Let $\mathcal{J}_K$ stand for the algebra ideal of $\mathrm{span}_{\mathbb{F}}(\mathcal{S})$ generated by $K$. By Lemma 3.8 it is sufficient to show that $\mathcal{J}_K$ is reducible.

It is clear that $\mathcal{J}_K$ is an algebra of compact operators which (by Lemma 5.9) satisfies (ii) of Theorem 5.8. It follows that $\mathcal{J}_K$ is triangularizable. ∎

COROLLARY 5.11. *Suppose $\mathcal{S}$ is an operator band in $\mathcal{B}(\mathcal{H})$. If two distinct components of $\mathcal{S} \cup \{0, I\}$ have a finite-dimensional gap between them, then $\mathcal{S}$ is reducible.*

*Proof.* Without loss of generality assume $\{0, I\} \subset \mathcal{S}$. If the gap between components containing $A$ and $B$ has finite non-zero dimension (assume $A \prec B$), then $A - ABA$ has finite rank. ∎

COROLLARY 5.12. *If an operator band $\mathcal{S}$ in $\mathcal{B}(\mathcal{H})$ has a component $\mathcal{C}$ that is a maximal rectangular band in $\mathcal{B}(\mathcal{H})$, then $\mathcal{S}$ is reducible.*

*Proof.* Non-trivial (i.e. not $\{I\}$ or $\{0\}$) maximal rectangular bands always contain distinct idempotents whose difference has finite rank (see Corollary 3.6). ∎

An operator is said to be *essentially self-adjoint* if it is a sum of a compact and a self-adjoint operators.

LEMMA 5.13. *If $\begin{pmatrix} 0 & X \\ Y & YX \end{pmatrix}$ in $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ is essentially self-adjoint and $XY = 0$, then $X$ and $Y$ are compact.*

*Proof.* If the hypothesis holds then

$$\begin{pmatrix} 0 & X \\ Y & YX \end{pmatrix} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} + \begin{pmatrix} K & L \\ M & N \end{pmatrix},$$

for some operators $A, B, C$ and some compact operators $K, L, M, N$. Since $X = B + L$ and $Y = B^* + M$, it follows that $0 = XY = BB^* +$ compact, so that $BB^*$ is compact. Use functional calculus and polar decomposition to conclude that $B$ is compact. Thus $X$ and $Y$ are compact, as required. ∎

Corollary 5.14. *If an operator band $\mathcal{S}$ in $\mathcal{B}(\mathcal{H})$ has a component has contains two essentially self-adjoint operators, then $\mathcal{S}$ is reducible.*

*Proof.* Suppose $A \sim B$ in $\mathcal{S}$ are essentially self-adjoint operators. Then $A$ has the matrix form

$$A = \begin{pmatrix} I & Z \\ 0 & 0 \end{pmatrix},$$

with respect to decomposition $\mathcal{H} = \operatorname{ran}(A) \oplus \operatorname{ran}(A)^{\perp}$. Since $A - A^*$ is compact, it follows that $Z$ is compact, because

$$A - A^* = \begin{pmatrix} 0 & Z \\ Z^* & 0 \end{pmatrix}.$$

Note the operator $T$ represented by the matrix

$$\begin{pmatrix} I & -Z \\ 0 & I \end{pmatrix}$$

is invertible, with the inverse $\begin{pmatrix} I & Z \\ 0 & I \end{pmatrix}$, and is a compact perturbation of the identity operator. Therefore $A_1 = TAT^{-1}$ and $B_1 - TBT^{-1}$ are still essentially self-adjoint operators and

$$A_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} I & X \\ Y & YX \end{pmatrix},$$

with respect to the decomposition above, for some $Y, X$ such that $XY = 0$ (use the fact that $B_1 = (B_1A_1)(A_1B_1)$ by sandwich property).

It follows that $\begin{pmatrix} 0 & X \\ Y & YX \end{pmatrix}$ is essentially self-adjoint and consequently $X$ and $Y$ are compact by Lemma 5.13. Thus $B_1 - A_1$ is compact and so the same is true for $B - A$. Apply Theorem 5.10 to complete the proof. ∎

## 6. REPRESENTING ABSTRACT BANDS AS OPERATOR BANDS

The following result is certainly well known to those involved with semigroup theory.

Theorem 6.1. *Every abstract band is faithfully representable as an operator band on a vector space over any field.*

*Proof.* Suppose $\mathcal{S}$ is an abstract band. Without loss of generality we may assume $\mathcal{S}$ is unital with an identity element $e$. (If $\mathcal{S}$ is not unital, imbed $\mathcal{S}$ in the unital band $\mathcal{S} \cup \{e\}$, with the definition that $ea = a = ae$, for every $a \in \mathcal{S}$.)

For each $a \in \mathcal{S}$, let $\mathcal{M}_a : \mathcal{S} \to \mathcal{S}$ be the 'left multiplication by $a$'. If $\mathbb{F}$ is any field, let $\mathcal{V}_{\mathcal{S},\mathbb{F}}$ be the free vector space over $\mathbb{F}$ generated by $\mathcal{S}$. Extend each $\mathcal{M}_a$ to a linear transformation on $\mathcal{V}_{\mathcal{S},\mathbb{F}}$ and define map $\Phi : \mathcal{S} \to \mathcal{L}(\mathcal{V}_{\mathcal{S},\mathbb{F}})$ by $\Phi(a) = \mathcal{M}_a$.

It is clear that $\Phi$ is injective because $\mathcal{M}_a(e) = a$, for each $a \in \mathcal{S}$. That $\Phi$ is a homomorphism follows from the fact that $\mathcal{M}_a\mathcal{M}_b = \mathcal{M}_{ab}$ on $\mathcal{S}$ and thus also on $\mathcal{V}_{\mathcal{S},\mathbb{F}}$. ∎

Our next result is not a complete surprise. After all, we have gone to great lengths in Section 2 in exposing the reader to much of the 'finite-dimensional' behaviour of the operator bands with finitely many components. One naturally hopes that *all* bands with finitely many components are somewhat 'finite-dimensional' ...

THEOREM 6.2. *Every rectangular band can be faithfully represented as an operator band on a three-dimensional vector space over a large enough field.*

*Proof.* Suppose $\mathcal{S}$ is a rectangular band and $A \in \mathcal{S}$.

*Claim.* Every element $B \in \mathcal{S}$ admits a unique factorization $B = CD$, where $C \in \mathcal{S}A = \{TA \mid T \in \mathcal{S}\}$ and $D \in A\mathcal{S} = \{AT \mid T \in \mathcal{S}\}$.

Indeed, if $B \in \mathcal{S}$ then $B = (BA)(AB)$ by the 'sandwich property' of rectangular bands. This insures the existence of the factorization. If $B = CD$ is any other factorization (besides $B = (BA)(AB)$) then (again via the sandwich property)

$$C = CA = CDA = BA$$

and

$$D = AD = ACD = AB.$$

This gives 'uniqueness' and the claim.

Given a field $\mathbb{F}$ such that $\#\mathbb{F} \geqslant \mathrm{Max}(\#(\mathcal{S}A), \#(A\mathcal{S}))$, choose any injective maps $\mu : \mathcal{S}A \to \mathbb{F}$ and $\eta : A\mathcal{S} \to \mathbb{F}$ with the property: $\mu(A) = 0 = \eta(A)$. Define $\Psi : \mathcal{S} \to \mathbb{M}_3(\mathbb{F})$ by:

$$\Psi(B) = \begin{pmatrix} 0 & \mu(BA) & \mu(BA)\eta(AB) \\ 0 & 1 & \eta(AB) \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\Psi$ is injective (use the claim above) and

$$\begin{aligned}\Psi(B)\Psi(D) &= \begin{pmatrix} 0 & \mu(BA) & \mu(BA)\eta(AB) \\ 0 & 1 & \eta(AB) \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mu(BDA) & \mu(BDA)\eta(ABD) \\ 0 & 1 & \eta(ABD) \\ 0 & 0 & 0 \end{pmatrix} = \Psi(BD),\end{aligned}$$

for every $B, D \in \mathcal{S}$. Thus $\Psi$ is a faithful representation. ∎

Suppose that $\mathcal{S}$ is a band and $\mathcal{G} \subset \mathcal{S}$. We say that a subset $\mathcal{F}$ of $\mathcal{G}$ *separates the multiplication of $\mathcal{G}$ by $\mathcal{S}$ on the left* provided that, for every choice of $A$ and $B$ in $\mathcal{S}$, the following is satisfied:

$$(AT = BT, \text{ for all } T \in \mathcal{F}) \Rightarrow (AT = BT, \text{ for all } T \in \mathcal{G}).$$

The corresponding 'right-' concept is defined accordingly.

LEMMA 6.3. *Suppose $\mathcal{S}$ is an operator band acting on an $n$-dimensional vector space $\mathcal{V}$ and $\mathcal{G} \subset \mathcal{S}$. Then there is a subset $\mathcal{F}$ of $\mathcal{G}$, with no more than $n$ elements, such that $\mathcal{F}$ separates the multiplication of $\mathcal{G}$ by $\mathcal{S}$ on the left and on the right.*

*Proof.* Since span$(\mathcal{G})$ is $n$-dimensional, some finite subset $\mathcal{F}$ of $\mathcal{G}$ is a basis for span$(\mathcal{G})$. The conclusion follows. ∎

As we have stated above, one naturally hopes that *all* bands with finitely many components are somewhat 'finite-dimensional' ... Alas, such is not the case, as far as representing such bands as operator bands on finite-dimensional vector spaces goes.

EXAMPLE 6.4. *There exists a two-component band that cannot be faithfully represented as an operator band on any finite-dimensional vector space.*

*Proof.* Let us start by agreeing on the following notation:

(1) $\infty$ stands for $\mathbb{N}$.

(2) $e_i$ stands for the $i$-th ($i \in \mathbb{N}$) standard basis vector of $\mathbb{C}^\infty$ and is considered to be an element of $\mathbb{M}_{\infty\times 1}(\mathbb{C})$.

(3) $E_{ii}$ stands for the $ii$-th ($i \in \mathbb{N}$) standard matrix unit in $\mathbb{M}_{\infty\times\infty}(\mathbb{C})$.

(4) $F_i$ ($i \in \mathbb{N}$) stands for the matrix in $\mathbb{M}_{\infty\times\infty}(\mathbb{C})$ every column of which is $e_i$, except for the $i$-th column, which is zero.

(5) $I$ stands for the identify matrix in $\mathbb{M}_{\infty\times\infty}(\mathbb{C})$.

(6) $A_i$ $(i \in \mathbb{N})$ stands for the block-matrix

$$A_i = \begin{pmatrix} I & 0 & E_{ii} & e_i \\ 0 & I & F_i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(7) $B_i$ $(i \in \mathbb{N})$ stands for the block-matrix

$$B_i = \begin{pmatrix} I & 0 & E_{ii} & 0 \\ 0 & I & F_i & e_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(8) $C_i$ $(i \in \mathbb{N})$ stands for the block-matrix

$$C_i = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & e_i \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Observe that $E_{ii}e_i = e_i$, $E_{ii}e_j = 0$, $F_ie_i = 0$, and $F_ie_j = e_i$, for all $i \neq j$.

The set $\mathcal{S} = \{A_i \mid i \in \mathbb{N}\} \cup \{B_i \mid i \in \mathbb{N}\} \cup \{C_i \mid i \in \mathbb{N}\}$ is a band of linear transformations on the vector space $\mathbb{C}^\infty \oplus \mathbb{C}^\infty \oplus \mathbb{C}^\infty \oplus \mathbb{C}^\infty$. Its multiplication table is shown below.

| | $A_i$ | $A_j$ | $B_i$ | $B_j$ | $C_i$ | $C_j$ |
|---|---|---|---|---|---|---|
| $A_i$ | $A_i$ | $A_j$ | $B_i$ | $B_j$ | $A_i$ | $B_i$ |
| $A_j$ | $A_i$ | $A_j$ | $B_i$ | $B_j$ | $B_j$ | $A_j$ |
| $B_i$ | $A_i$ | $A_j$ | $B_i$ | $B_j$ | $A_i$ | $B_i$ |
| $B_j$ | $A_i$ | $A_j$ | $B_i$ | $B_j$ | $B_j$ | $A_j$ |
| $C_i$ | $A_i$ | $A_j$ | $B_i$ | $B_j$ | $C_i$ | $C_j$ |
| $C_j$ | $A_i$ | $A_j$ | $B_i$ | $B_j$ | $C_i$ | $C_j$ |

$(i \neq j)$.

Components of $\mathcal{S}$ are $\mathcal{C}_1 = \{C_i \mid i \in \mathbb{N}\}$ and $\mathcal{C}_2 = \{A_i \mid i \in \mathbb{N}\} \cup \{B_i \mid i \in \mathbb{N}\}$. It is easy to see that $\mathcal{C}_2 \prec \mathcal{C}_1$.

*Claim.* No finite subset of $\mathcal{C}_2$ separates multiplication of $\mathcal{C}_2$ by $\mathcal{S}$ on the right.

Indeed, if $m \in \mathbb{N}$ then

$$\begin{cases} A_iC_m = B_i = A_iC_{m+1} \\ B_iC_m = B_i = B_iC_{m+1} \end{cases}, \qquad \text{for all } 1 \leqslant i \leqslant m,$$

but

$$\begin{cases} A_mC_m = A_m \neq B_m = A_mC_{m+1} \\ B_mC_m = A_m \neq B_m = B_mC_{m+1}. \end{cases}$$

Loosely: for each finite subset of $\mathcal{C}_2$ there is a (large enough) $m$ such that the subset cannot 'tell the difference' between $C_m$ and $C_{m+1}$ multiplying $\mathcal{C}_2$ on the right. But all of $\mathcal{C}_2$ always 'knows the diference' between being multiplied by $C_m$ and $C_{m+1}$ on the right. The claim is proved.

The desired conclusion follows from Lemma 6.3. ∎

Example 6.5. *For each $n \in \mathbb{N}$ there exists a finite two-components band that cannot be faithfully represented as an operator band on any $n$-dimensional vector space.*

*Proof.* Under the setup of Example 6.4. let

$$\mathcal{G} = \{A_i \mid 1 \leqslant i \leqslant n+2\} \cup \{B_i \mid 1 \leqslant i \leqslant n+2\} \cup \{C_i \mid 1 \leqslant i \leqslant n+2\}.$$

Then $\mathcal{G}$ has two components:

$$\mathcal{D}_1 = \{C_i \mid 1 \leqslant i \leqslant n+2\}$$

and

$$\mathcal{D}_2 = \{A_i \mid 1 \leqslant i \leqslant n+2\} \cup \{B_i \mid 1 \leqslant i \leqslant n+2\}.$$

It is easy to see that $\mathcal{D}_2 \prec \mathcal{D}_1$. If $\mathcal{K}$ is a subset of $\mathcal{D}_2$ containing no more than $n$ elements, then there exist $i, j$ $(1 \leqslant i \neq j \leqslant n+2)$, such that $A_i, B_i, A_j, B_j \notin \mathcal{K}$. Observe that

$$\begin{cases} A_l C_i = B_l = A_l C_j \\ B_l C_i = B_l = B_l C_j \end{cases}, \quad \text{for all } l \neq i, j,$$

but

$$A_i C_i = A_i \neq B_i = A_i Cj.$$

Therefore $\mathcal{K}$ does not separate multiplication of $\mathcal{D}_2$ by $\mathcal{G}$ on the right. The desired conclusion follows again from Lemma 6.3. ∎

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## REFERENCES

1. B. Aupetit, *Propriétés spectrales des algèbres de Banach*, Lecture Notes in Math., vol. 735, Springer, Berlin 1979.
2. P. Fillmore, G. MacDonald, M. Radjabalipour, H. Radjavi, Towards a classification of maximal unicellular bands, *Semigroup Forum* **49**(1994), 195–215.
3. J.A. Green, D. Rees, On semigroups in which $x^r = x$, *Proc. Cambridge Philos. Soc.* **48**(1952), 35–40.
4. A. Katavolos, H. Radjavi, Simultaneous triangularization of operators on a Banach space, *J. London Math. Soc. (2)* **41**(1990), 547–554.

5. M. Petric, *Lectures in Semigroups*, Academie-Verlag, John Wiley & Sons Ltd., Berlin 1977.
6. H. Radjavi, On the reduction and triangularization of semigroups of operators, *J. Operator Theory* **13**(1985), 63-71.
7. H. Radjavi, P. Rosenthal, *Invariant Subspaces*, Springer-Verlag, New York 1973.
8. H. Radjavi, P. Rosenthal, On transitive and reductive operator algebras, *Math. Ann.* **209**(1974), 43–56.

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