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NORMS OF SOME SINGULAR INTEGRAL OPERATORS AND THEIR INVERSE OPERATORS

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Abstract. Let α and β be bounded measurable functions on the unit circle $\mathbb{T}$. Then the singular integral operator $S_{\alpha,\beta}$ is defined by $S_{\alpha,\beta}f = \alpha P_+ f + \beta P_- f$, $(f \in L^2(\mathbb{T}))$ where P_+ is an analytic projection and P_- is a co-analytic projection. In this paper, the norms of $S_{\alpha,\beta}$ and its inverse operator on the Hilbert space $L^2(\mathbb{T})$ are calculated in general, using α, β and $\alpha\bar{\beta} + H^\infty$. Moreover, the relations between these and the norms of Hankel operators are established. As an application, in some special case in which α and β are nonconstant functions, the norm of $S_{\alpha,\beta}$ is calculated in a completely explicit form. If α and β are constant functions, then it is well known that the norm of $S_{\alpha,\beta}$ on $L^2(\mathbb{T})$ is equal to $\max\{|\alpha|, |\beta|\}$. If α and β are nonzero constant functions, then it is also known that $S_{\alpha,\beta}$ on $L^2(\mathbb{T})$ has an inverse operator $S_{\alpha^{-1},\beta^{-1}}$ whose norm is equal to $\max\{|\alpha|^{-1}, |\beta|^{-1}\}$.

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1. INTRODUCTION

Let m denote the normalized Lebesgue measure on the unit circle $\mathbb{T} = \{\zeta : |\zeta| = 1\}$. That is, $\mathrm{d}m(\zeta) = \mathrm{d}\theta/2\pi$ for $\zeta = \mathrm{e}^{\mathrm{i}\theta}$. The inner product in the Hilbert space $L^2 = L^2(\mathbb{T})$ is given by

$$(f, g) = \int_{\mathbb{T}} f(\zeta)\overline{g(\zeta)}\,\mathrm{d}m(\zeta).$$

The norm in L^2 is given by $\|f\|_2 = \sqrt{(f,f)}$. By $L^\infty = L^\infty(\mathbb{T})$ we denote the space of bounded measurable functions. The norm in L^∞ is given by $\|f\|_\infty = \operatorname{ess\,sup}_{\mathbb{T}} |f|$. Let H^2 (resp. H^∞) be the Hardy space of functions $f \in L^2$ (resp. $f \in L^\infty$) whose negative Fourier coefficients are zero. Let $\overline{H_0^2}$ be the space of functions $f \in L^2$ whose nonnegative Fourier coefficients are zero. Then $L^2 = H^2 \oplus \overline{H_0^2}$. Let S be the singular integral operator defined by

$$(Sf)(\zeta) = \frac{1}{\pi \mathrm{i}} \int_{\mathbb{T}} \frac{f(\eta)}{\eta - \zeta} \, \mathrm{d}\eta \quad (\text{a.e. } \zeta \in \mathbb{T}),$$

where the integral is understood in the sense of Cauchy's principal value (cf. [4], p. 12). If f is in L^1, then $(Sf)(\zeta)$ exists for almost every ζ on $\mathbb{T}$, and Sf becomes a measurable function on $\mathbb{T}$. Let the analytic projection and the co-analytic projection be

$$P_+ = \frac{I+S}{2}, \quad \text{and} \quad P_- = \frac{I-S}{2},$$

where I denotes the identity operator. If $\alpha, \beta \in L^\infty$, then the singular integral operator $S_{\alpha,\beta}$ on L^2 is defined by

$$S_{\alpha,\beta} f = \alpha P_+ f + \beta P_- f, \quad (f \in L^2).$$

Then $S_{1,1} = I$, $S_{1,-1} = S$, $S_{1,0} = P_+$ and $S_{0,1} = P_-$. The norm of $S_{\alpha,\beta}$ is defined by

$$\|S_{\alpha,\beta}\| = \sup_{f \in L^2,\, \|f\|_2 = 1} \|S_{\alpha,\beta} f\|_2.$$

Since $\|P_+\| = \|P_-\| = 1$, we have

$$\|S_{\alpha,\beta}\| \leqslant \|\alpha\|_\infty + \|\beta\|_\infty < \infty.$$

Hence, $S_{\alpha,\beta}$ is a bounded operator on L^2. Furthermore, it is well known that

$$\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \leqslant \|S_{\alpha,\beta}\| \leqslant \left\| \sqrt{|\alpha|^2 + |\beta|^2} \right\|_\infty.$$

If α and β are constant functions, then it is well known and not difficult to establish that

$$\|S_{\alpha,\beta}\| = \max\{|\alpha|, |\beta|\}$$

(cf. [3]). If α and β are nonconstant functions, then we will show in Section 2 that the formula of $\|S_{\alpha,\beta}\|$ is more complicated.

If $\varphi \in L^\infty$, then the Toeplitz operator T_φ is defined by $T_\varphi f = P_+(\varphi f)$, $(f \in H^2)$. Its norm is equal to $\|\varphi\|_\infty$ (cf. [2], p. 179). The Hankel operator H_φ is defined by $H_\varphi f = P_-(\varphi f)$, $(f \in H^2)$. By the Nehari theorem [8] (cf. [9], p. 181), its norm is equal to $\inf\{\|\varphi - k\|_\infty : k \in H^\infty\}$. Hence,

$$\|H_\varphi\| \leqslant \|T_\varphi\| \leqslant \|S_{\varphi,1}\|.$$

Though the norm of T_φ or H_φ is known, the norm of $S_{\varphi,1}$ is not known.

In this paper, we consider the operator $S_{\alpha,\beta}$ on L^2 for functions $\alpha, \beta \in L^\infty$. In Section 2, we give the formula of the norm of $S_{\alpha,\beta}$ on L^2 for $\alpha, \beta \in L^\infty$, which involves lower bounds over the algebra H^∞. It is a little surprising that the norm of the singular integral operator $S_{\alpha,\beta}$ is related to the norm of the Hankel operator $H_{\alpha\bar\beta}$ for some special α and β. In Section 3, we also give the formula of the norm of the inverse operator of $S_{\alpha,\beta}$ on L^2 for $\alpha, \beta \in L^\infty$, which involves upper bounds over the algebra H^∞. If $S_{\alpha,\beta}$ is invertible, then $\operatorname*{ess\,inf}_{\mathbb{T}} (\min\{|\alpha|, |\beta|\}) > 0$. When $\varphi = \alpha/\beta$ and $\operatorname*{ess\,inf}_{\mathbb{T}} (\min\{|\alpha|, |\beta|\}) > 0$,

$$S_{\alpha,\beta} = \beta S_{\varphi,1} = \beta(P_+\varphi P_+ + P_-)(I + P_-\varphi P_+).$$

Then $I + P_-\varphi P_+$ is invertible and $(I + P_-\varphi P_+)^{-1} = I - P_-\varphi P_+$ (cf. [9], p. 393). Hence, $S_{\alpha,\beta}$ is invertible if and only if $\min\{\operatorname*{ess\,inf}_{\mathbb{T}} |\alpha|, \operatorname*{ess\,inf}_{\mathbb{T}} |\beta|\} > 0$ and $T_{\alpha/\beta}$ is invertible.

The first author (cf. [7]) calculated essentially the norm of the inverse operator of T_φ on H^2. We will show in Section 3 that the formula of the norm of the inverse operator of T_φ is similar to the formula of the norm of the inverse operator of $S_{\alpha,\beta}$. The second author (cf. [11]) considered the norm of $S_{\alpha,\beta}$ on the weighted space $L^2(T, W)$ with a weight function W on $\mathbb{T}$, and proved the Feldman-Krupnik-Markus theorem ([3]) using the Cotlar-Sadosky lifting theorem ([1]) when α and β are constant functions.

2. NORM OF THE OPERATOR $S_{\alpha,\beta}$

If $\alpha, \beta \in L^\infty$, then the following inequality is well known and not difficult to establish:

$$\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} \leqslant \|S_{\alpha,\beta}\| \leqslant \left\|\sqrt{|\alpha|^2 + |\beta|^2}\right\|_\infty.$$

We should mention that

$$\max\{\|\alpha\|_\infty, \|\beta\|_\infty\} = \left\|\frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{0 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}\right\|_\infty^{\frac{1}{2}},$$

and

$$\left\| \sqrt{|\alpha|^2 + |\beta|^2} \right\|_\infty = \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - 0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2} \right)^2} \right\|_\infty^{\frac{1}{2}}.$$

It is not difficult to establish that

$$\|S_{\alpha,\beta}\|^2 \leqslant \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2} \right)^2} \right\|_\infty,$$

for any $k \in H^\infty$. The following theorem gives the formula for the computation of the norm of the operator $S_{\alpha,\beta}$.

THEOREM 2.1. *Let $\alpha, \beta \in L^\infty$. Then*

$$\|S_{\alpha,\beta}\|^2 = \inf_{k \in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2} \right)^2} \right\|_\infty.$$

The infimum is attained.

Proof. For any $k \in H^\infty$, we define the quantity M_k according to

$$M_k = \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2} \right)^2} \right\|_\infty.$$

We prove that $\|S_{\alpha,\beta}\|^2 \geqslant \inf\{M_k : k \in H^\infty\}$. Let $\gamma = \|S_{\alpha,\beta}\|$. Then

$$\|S_{\alpha,\beta} f\|_2 \leqslant \gamma \|f\|_2, \quad (f \in L^2).$$

Let $W_1 = \gamma^2 - |\alpha|^2$, $W_2 = \gamma^2 - |\beta|^2$ and $W_3 = \gamma^2 - \alpha\bar{\beta}$. Then

$$(W_1 f_1, f_1) + (W_2 f_2, f_2) + 2\mathrm{Re}\,(W_3 f_1, f_2) \geqslant 0,$$

$(f_1 \in H^2, f_2 \in \overline{H_0^2})$. By the Cotlar-Sadosky lifting theorem ([1]), $W_1 \geqslant 0, W_2 \geqslant 0$, and there exists a $g \in H^\infty$ such that

$$|W_3 - g|^2 \leqslant W_1 W_2.$$

Hence, $\gamma \geqslant \max\{|\alpha|, |\beta|\}$ and

$$|\gamma^2 - \alpha\bar{\beta} - g|^2 \leqslant (\gamma^2 - |\alpha|^2)(\gamma^2 - |\beta|^2).$$

Let $k_0 = \gamma^2 - g$. Then $k_0 \in H^\infty$ and $|\gamma^2 - \alpha\bar{\beta} - g| = |\alpha\bar{\beta} - k_0|$. Hence,

$$\begin{aligned} 0 &\leqslant (\gamma^2 - |\alpha|^2)(\gamma^2 - |\beta|^2) - |\alpha\bar{\beta} - k_0|^2 \\ &= \gamma^4 - (|\alpha|^2 + |\beta|^2)\gamma^2 + |\alpha\beta|^2 - |\alpha\bar{\beta} - k_0|^2. \end{aligned}$$

Suppose

$$\gamma^2 \leqslant \frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}$$

on some measurable subset E of $\mathbb{T}$. Since

$$\gamma^2 \geqslant \max\left\{|\alpha|^2, |\beta|^2\right\} = \frac{|\alpha|^2 + |\beta|^2}{2} + \left|\frac{|\alpha|^2 - |\beta|^2}{2}\right|$$

on $\mathbb{T}$, we have

$$\left|\frac{|\alpha|^2 - |\beta|^2}{2}\right| + \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \leqslant 0$$

on E. This implies that $|\alpha| - |\beta| = |\alpha\bar{\beta} - k_0| = 0$ on E. Hence,

$$\gamma^2 \geqslant \max\left\{|\alpha|^2, |\beta|^2\right\} = \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}$$

on E. Therefore,

$$\gamma^2 \geqslant \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}$$

on $\mathbb{T}$. Hence $M_{k_0} \leqslant \gamma^2$. Since $\gamma = \|S_{\alpha,\beta}\|$, we have

$$\inf_{k \in H^\infty} M_k \leqslant M_{k_0} \leqslant \|S_{\alpha,\beta}\|^2.$$

We prove that $\|S_{\alpha,\beta}\|^2 \leqslant \inf\{M_k : k \in H^\infty\}$. This is the easy direction of the theorem. For any $k \in H^\infty$, we have

$$(kf_1, f_2) = 0, \quad (f_1 \in H^2, f_2 \in \overline{H_0^2}).$$

Since

$$\frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \leqslant M_k,$$

we have

$$(M_k - |\alpha|^2)(M_k - |\beta|^2) \geqslant \left(\sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} - \frac{|\alpha|^2 - |\beta|^2}{2} \right)$$
$$\times \left(\sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} + \frac{|\alpha|^2 - |\beta|^2}{2} \right) = |\alpha\bar{\beta} - k|^2.$$

Hence,

$$\begin{aligned} &M_k\|f_1 + f_2\|_2^2 - \|\alpha f_1 + \beta f_2\|_2^2 \\ &= \left\|\sqrt{M_k - |\alpha|^2} f_1\right\|_2^2 + \left\|\sqrt{M_k - |\beta|^2} f_2\right\|_2^2 - 2\mathrm{Re}(\alpha\bar{\beta} f_1, f_2) \\ &\geqslant 2\left\|\sqrt{M_k - |\alpha|^2} f_1\right\|_2 \left\|\sqrt{M_k - |\beta|^2} f_2\right\|_2 - 2\mathrm{Re}((\alpha\bar{\beta} - k) f_1, f_2) \\ &\geqslant 2\int_{\mathbb{T}} \left(\sqrt{M_k - |\alpha|^2}\sqrt{M_k - |\beta|^2} - |\alpha\bar{\beta} - k|\right) |f_1 f_2| \,\mathrm{d}m \geqslant 0. \end{aligned}$$

Hence, for any $k \in H^\infty$,

$$\|S_{\alpha,\beta} f\|_2^2 \leqslant M_k \|f\|_2^2, \quad (f \in L^2).$$

Therefore,

$$\inf_{k\in H^\infty} M_k \leqslant M_{k_0} \leqslant \|S_{\alpha,\beta}\|^2 \leqslant \inf_{k\in H^\infty} M_k.$$

Hence the equalities hold, and the infimum is attained by $k = k_0$. This completes the proof. ∎

REMARK 2.2. Let $\alpha, \beta \in L^\infty$, let $\varphi = \alpha\bar{\beta}$, and let $\psi = (|\alpha|^2 - |\beta|^2)/2$. Then

$$\|S_{\alpha,\beta}\|^2 = \inf_{k\in H^\infty} \left\|\sqrt{|\varphi|^2 + \psi^2} + \sqrt{|\varphi - k|^2 + \psi^2}\right\|_\infty.$$

The infimum is attained. If $|\alpha| = |\beta|$, then $\psi = 0$. Hence

$$\|S_{\alpha,\beta}\|^2 = \inf_{k\in H^\infty} \|\, |\varphi| + |\varphi - k| \,\|_\infty.$$

COROLLARY 2.3. *If $|\alpha|$ and $|\beta|$ are constant functions, then*

$$\|S_{\alpha,\beta}\|^2 = \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{\left\|H_{\alpha\bar{\beta}}\right\|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}.$$

Proof. It follows from Theorem 2.1 that

$$\|S_{\alpha,\beta}\|^2 = \inf_{k\in H^\infty} \left\| \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right\|_\infty$$
$$= \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{\left(\inf_{k\in H^\infty} \|\alpha\bar{\beta} - k\|_\infty\right)^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}.$$

By the Nehari theorem ([8]), this proves the corollary. ∎

COROLLARY 2.4. *Let $\alpha, \beta \in L^\infty$. Then*

$$\|S_{\alpha,\beta}\|^2 \leqslant \max\left\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\right\} + \|H_{\alpha\bar\beta}\|.$$

Proof. It follows from the easy direction of Theorem 2.1 that

$$\begin{aligned}
\|S_{\alpha,\beta}\|^2 &\leqslant \inf_{k\in H^\infty}\left\|\frac{|\alpha|^2+|\beta|^2}{2} + \sqrt{|\alpha\bar\beta - k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right\|_\infty \\
&\leqslant \inf_{k\in H^\infty}\left\|\frac{|\alpha|^2+|\beta|^2}{2} + |\alpha\bar\beta - k| + \left|\frac{|\alpha|^2-|\beta|^2}{2}\right|\right\|_\infty \\
&= \inf_{k\in H^\infty}\left\|\max\left\{|\alpha|^2, |\beta|^2\right\} + |\alpha\bar\beta - k|\right\|_\infty \\
&\leqslant \max\left\{\|\alpha\|_\infty^2, \|\beta\|_\infty^2\right\} + \inf_{k\in H^\infty}\|\alpha\bar\beta - k\|_\infty.
\end{aligned}$$

By the Nehari theorem ([8]), this proves the corollary. ∎

REMARK 2.5. If $\alpha\bar\beta \in H^\infty$, then the infimum in Theorem 2.1 is attained by $k = \alpha\bar\beta$. In this case,

$$\|S_{\alpha,\beta}\| = \|\max\{|\alpha|, |\beta|\}\|_\infty = \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}.$$

Hence the equality holds in Corollary 2.4 because $\|H_{\alpha\bar\beta}\| = 0$. By Corollary 2.3, if $|\alpha| = |\beta| =$ constant, then the equality holds in Corollary 2.4.

By Corollary 2.4, if $\varphi \in L^\infty$ and $\|\varphi\|_\infty \leqslant 1$, then $\|S_{\varphi,1}\|^2 \leqslant 1 + \|H_\varphi\|$. By Corollary 2.3, if $|\varphi| = 1$, then $\|S_{\varphi,1}\|^2 = 1 + \|H_\varphi\|$. If φ is a real function whose range consists of only two points, then the norm of $S_{\varphi,1}$ will be calculated in a completely explicit form in Corollary 2.7. For example, by Corollary 2.7, if E is a measurable subset of the unit circle $\mathbb{T}$ satisfying $0 < m(E) < 1$, then $\|S_{\chi_E,1}\| = 2/\sqrt{3}$. It is well known that $\|H_{\chi_E}\| = 1/2$. In this case, $\|S_{\chi_E,1}\|^2 < 1 + \|H_{\chi_E}\|$. Hence the equality does not hold, in general, in Corollary 2.4. For the proof of Corollary 2.7 we need Lemma 2.6.

LEMMA 2.6. *Let a and b be real numbers satisfying $a \neq b$. Then the equality*

$$\frac{a^2+1}{2} + \sqrt{(a-x)^2 + \left(\frac{a^2-1}{2}\right)^2} = \frac{b^2+1}{2} + \sqrt{(b-x)^2 + \left(\frac{b^2-1}{2}\right)^2}$$

holds for some real number x if and only if $|a+b| < 2$. Then x is unique and it is given by

$$x = x_0 = \frac{2(a+b)(1-ab)}{4-(a+b)^2}.$$

Proof. Suppose the equality holds for $x = 0$. Then $a^2 = b^2$. Since $a \neq b$, we have $a = -b$. Hence $|a + b| = 0 < 2$. Suppose the equality holds for some real number x satisfying $x \neq 0$. Then

$$\left\{ \frac{b^2 - a^2}{2} + \sqrt{(b - x)^2 + \left(\frac{b^2 - 1}{2} \right)^2} \right\}^2 = (a - x)^2 + \left(\frac{a^2 - 1}{2} \right)^2 .$$

It follows by direct computation that

$$2(b^2 - a^2)\sqrt{(x - b)^2 + \left(\frac{b^2 - 1}{2} \right)^2} = (b - a) \left\{ 4x - (a + b)(b^2 + 1) \right\} .$$

Since $a \neq b$, we have

$$2(a + b)\sqrt{(x - b)^2 + \left(\frac{b^2 - 1}{2} \right)^2} = 4x - (a + b)(b^2 + 1).$$

Hence,

$$\left\{ 4 - (a + b)^2 \right\} x^2 = 2(a + b)(1 - ab)x.$$

Since $x \neq 0$, we have

$$\left\{ 4 - (a + b)^2 \right\} x = 2(a + b)(1 - ab).$$

If $(a + b)^2 = 4$, then $ab = 1$. Hence $a^2 = b^2 = 1$. Since $a \neq b$, we have $a + b = 0$. This contradiction implies that $(a + b)^2 \neq 4$. Hence,

$$x = \frac{2(a + b)(1 - ab)}{4 - (a + b)^2}.$$

It follows by direct computation that

$$4x - (a + b)(b^2 + 1) = \frac{(a + b) \left\{ (a - b)^2 + (b^2 + ab - 2)^2 \right\}}{4 - (a + b)^2}.$$

Since $a \neq b$, we have

$$2\sqrt{(x - b)^2 + \left(\frac{b^2 - 1}{2} \right)^2} = \frac{(a - b)^2 + (b^2 + ab - 2)^2}{4 - (a + b)^2} > 0$$

because $(a - b)^2 > 0$. Hence, $|a + b| < 2$. This proof is reversible. ∎

Corollary 2.7. *Let E be a measurable subset of $\mathbb{T}$ satisfying $0 < m(E) < 1$, let a and b be real numbers, and let $\varphi = a\chi_E + b\chi_{E^c}$. Then*

(i) *If $\max\{a^2, b^2\} \geqslant 2 - ab$, then*

$$\|S_{\varphi,1}\| = \max\{|a|, |b|\}.$$

(ii) *If $\max\{a^2, b^2\} < 2 - ab$, then*

$$\|S_{\varphi,1}\|^2 = \frac{4(1-ab)}{4-(a+b)^2}.$$

Proof. We define the functions $f(x)$ and $g(x)$ according to

$$f(x) = \frac{a^2+1}{2} + \sqrt{(a-x)^2 + \left(\frac{a^2-1}{2}\right)^2},$$

and

$$g(x) = \frac{b^2+1}{2} + \sqrt{(b-x)^2 + \left(\frac{b^2-1}{2}\right)^2}.$$

For the real number x_0 defined in Lemma 2.6, it follows that

$$\begin{aligned} f(x_0) = g(x_0) &= \frac{a^2+1}{2} + \sqrt{\left(\frac{(a-b)(a^2+ab-2)}{4-(a+b)^2}\right)^2 + \left(\frac{a^2-1}{2}\right)^2} \\ &= \frac{b^2+1}{2} + \sqrt{\left(\frac{(a-b)(b^2+ab-2)}{4-(a+b)^2}\right)^2 + \left(\frac{b^2-1}{2}\right)^2} = \frac{4(1-ab)}{4-(a+b)^2}. \end{aligned}$$

We prove (i). Since $\max\left\{a^2, b^2\right\} \geqslant ab$ and $\max\left\{a^2, b^2\right\} \geqslant 2 - ab$, we have $\max\left\{a^2, b^2\right\} \geqslant 1$. It follows that $\max\left\{a^2, b^2, 1\right\} = \max\left\{a^2, b^2\right\}$. Since $\|\varphi\|_\infty = \max\{|a|, |b|\}$ and $\max\{\|\varphi\|_\infty, 1\} \leqslant \|S_{\varphi,1}\|$, we have

$$\max\left\{a^2, b^2\right\} = \max\left\{a^2, b^2, 1\right\} \leqslant \|S_{\varphi,1}\|^2.$$

Suppose $a = b$. Then φ becomes a constant function. Hence, by Remark 2.5,

$$\|S_{\varphi,1}\| = \max\{|\varphi|, 1\} = \max\{|a|, |b|, 1\} = \max\{|a|, |b|\}.$$

Suppose $a \neq b$. Since $0 < m(E) < 1$, we have

$$\begin{aligned} \|S_{\varphi,1}\|^2 &\leqslant \inf_{x\in\mathbb{R}} \left\| \frac{|\varphi|^2+1}{2} + \sqrt{(\varphi-x)^2 + \left(\frac{\varphi^2-1}{2}\right)^2} \right\|_\infty \\ &= \inf_{x\in\mathbb{R}} \|f(x)\chi_E + g(x)\chi_{E^c}\|_\infty = \inf_{x\in\mathbb{R}} \left(\max\{f(x), g(x)\}\right). \end{aligned}$$

By Lemma 2.6, if $|a+b| \geqslant 2$, then the equality $f(x) = g(x)$ does not hold for any real number x. Hence, $f(x) < g(x)$ or $f(x) > g(x)$. Hence,

$$\inf_{x\in\mathbb{R}} (\max\{f(x), g(x)\}) = \max\{f(a), g(b)\}.$$

Since $f(a) = \max\left\{a^2, 1\right\}$ and $g(b) = \max\left\{b^2, 1\right\}$, we have

$$\max\left\{a^2, b^2\right\} \leqslant \|S_{\varphi,1}\|^2 \leqslant \max\{f(a), g(b)\} = \max\left\{a^2, b^2, 1\right\} = \max\left\{a^2, b^2\right\}.$$

By Lemma 2.6, if $|a+b| < 2$, then the equation $f(x) = g(x)$ has a unique solution $x = x_0$ which is given in Lemma 2.6. Hence, $f(x_0) = g(x_0)$. Without loss of generality, we assume $a < b$. If $a \leqslant x_0 \leqslant b$, then $\max\left\{a^2, b^2\right\} \leqslant 2 - ab$. Since $\max\left\{a^2, b^2\right\} \geqslant 2 - ab$, we have $\max\left\{a^2, b^2\right\} = 2 - ab$. By this equality,

$$\begin{aligned}\max\left\{a^2, b^2\right\} &\leqslant \|S_{\varphi,1}\|^2 \leqslant \inf_{x\in\mathbb{R}} (\max\{f(x), g(x)\}) \\ &= f(x_0) = g(x_0) = \frac{4(1-ab)}{4-(a+b)^2} = \max\left\{a^2, b^2\right\}.\end{aligned}$$

If $x_0 \leqslant a \leqslant b$ or $a \leqslant b \leqslant x_0$, then

$$\begin{aligned}\max\left\{a^2, b^2\right\} &\leqslant \|S_{\varphi,1}\|^2 \leqslant \inf_{x\in\mathbb{R}} (\max\{f(x), g(x)\}) \\ &= \max\{f(a), g(b)\} = \max\left\{a^2, b^2, 1\right\} = \max\left\{a^2, b^2\right\}.\end{aligned}$$

We prove (ii). Suppose $a = b$. Then φ becomes a constant function and $\varphi = a = b$. Since $\max\left\{a^2, b^2\right\} < 2 - ab$, we have $a^2 = b^2 < 1$. By Remark 2.5,

$$\|S_{\varphi,1}\|^2 = \max\{\varphi^2, 1\} = \max\{a^2, 1\} = \max\{b^2, 1\} = 1.$$

Suppose $a \neq b$. It is sufficient to prove that $\|S_{\varphi,1}\|^2 = f(x_0) = g(x_0)$. Without loss of generality, we assume $a < b$. Since $\max\left\{a^2, b^2\right\} < 2 - ab$, we have $a^2 + b^2 < 2(2-ab)$. Hence $|a+b| < 2$. Let $f(x)$ and $g(x)$ be functions defined in the proof of (i). By Lemma 2.6, the equation $f(x) = g(x)$ has a unique solution $x = x_0$ which is given in Lemma 2.6. Hence, $f(x_0) = g(x_0)$. It follows by direct computation that

$$a - x_0 = \frac{(b-a)(a^2+ab-2)}{4-(a+b)^2} < 0,$$

$$b - x_0 = \frac{(a-b)(b^2+ab-2)}{4-(a+b)^2} > 0.$$

Hence, $a < x_0 < b$. By Theorem 2.1, there exists a $k_0 \in H^\infty$ such that

$$\begin{aligned}\|S_{\varphi,1}\|^2 &= \left\| \frac{|\varphi|^2+1}{2} + \sqrt{|\varphi-k_0|^2 + \left(\frac{|\varphi|^2-1}{2}\right)^2} \right\|_\infty \\ &= \inf_{k\in H^\infty} \left\| \frac{|\varphi|^2+1}{2} + \sqrt{|\varphi-k|^2 + \left(\frac{|\varphi|^2-1}{2}\right)^2} \right\|_\infty \\ &\leqslant \inf_{x\in\mathbb{R}} \left\| \frac{|\varphi|^2+1}{2} + \sqrt{|\varphi-x|^2 + \left(\frac{|\varphi|^2-1}{2}\right)^2} \right\|_\infty \\ &= \inf_{x\in\mathbb{R}} \|f(x)\chi_E + g(x)\chi_{E^c})\|_\infty \leqslant \inf_{x\in\mathbb{R}} \left(\max\{f(x), g(x)\}\right).\end{aligned}$$

Since $a < x_0 < b$, we have

$$\inf_{x\in\mathbb{R}} \left(\max\{f(x), g(x)\}\right) = f(x_0) = g(x_0).$$

Hence,

$$\|S_{\varphi,1}\|^2 \leqslant f(x_0) = g(x_0).$$

Then

$$f(x_0) = \frac{a^2+1}{2} + \sqrt{(a-x_0)^2 + \left(\frac{a^2-1}{2}\right)^2},$$

$$g(x_0) = \frac{b^2+1}{2} + \sqrt{(b-x_0)^2 + \left(\frac{b^2-1}{2}\right)^2}.$$

Suppose there exists an $\varepsilon > 0$ such that $\|S_{\varphi,1}\|^2 \leqslant f(x_0) - \varepsilon$. Then

$$\left\| \frac{|\varphi|^2+1}{2} + \sqrt{|\varphi-k_0|^2 + \left(\frac{|\varphi|^2-1}{2}\right)^2} \right\|_\infty \leqslant \frac{a^2+1}{2} + \sqrt{(a-x_0)^2 + \left(\frac{a^2-1}{2}\right)^2} - \varepsilon.$$

Since $f(x_0) = g(x_0)$, we have $\|S_{\varphi,1}\|^2 \leqslant g(x_0) - \varepsilon$. Hence,

$$\left\| \frac{|\varphi|^2+1}{2} + \sqrt{|\varphi-k_0|^2 + \left(\frac{|\varphi|^2-1}{2}\right)^2} \right\|_\infty \leqslant \frac{b^2+1}{2} + \sqrt{(b-x_0)^2 + \left(\frac{b^2-1}{2}\right)^2} - \varepsilon.$$

Hence, there exists an $\varepsilon' > 0$ such that

$$|a-k_0| \leqslant |a-x_0| - \varepsilon' \quad \text{on } E,$$

$$|b-k_0| \leqslant |b-x_0| - \varepsilon' \quad \text{on } E^{\mathrm{c}}.$$

Since $a < x_0 < b$, we have $|a - x_0| + |b - x_0| = b - a$. If $|a - x_0| \geqslant |b - x_0|$, then

$$\begin{aligned}|2|a - x_0|\chi_{E^c} + a - k_0| &= |a\chi_E + (b + |a - x_0| - |b - x_0|)\chi_{E^c} - k_0| \\ &\leqslant |a - k_0|\chi_E + |b - k_0|\chi_{E^c} + (|a - x_0| - |b - x_0|)\chi_{E^c} \\ &\leqslant |a - x_0| - \varepsilon'.\end{aligned}$$

Hence, $\inf\{\|2\chi_{E^c} - k\|_\infty : k \in H^\infty\} < 1$. This is a contradiction (cf. [5], p. 198). If $|a - x_0| \leqslant |b - x_0|$, then

$$\begin{aligned}|-2|b - x_0|\chi_E + b - k_0| &= |(a + |a - x_0| - |b - x_0|)\chi_E + b\chi_{E^c} - k_0| \\ &\leqslant |a - k_0|\chi_E + |b - k_0|\chi_{E^c} + (|b - x_0| - |a - x_0|)\chi_E \\ &\leqslant |b - x_0| - \varepsilon'.\end{aligned}$$

Hence, $\inf\{\|2\chi_E - k\|_\infty : k \in H^\infty\} < 1$. This is a contradiction (cf. [5], p. 198). These two contradictions imply that ε' must be zero. This contradiction implies that ε must be zero. Hence,

$$\|S_{\varphi,1}\|^2 = f(x_0) = g(x_0).$$

This completes the proof. ∎

When a and b are complex numbers, we give Corollary 2.9. When $|\varphi|$ is not constant, Corollary 2.9 does not contain the completely explicit form of the norm of $S_{\varphi,1}$. For the proof of Corollary 2.9 we need Lemma 2.8.

LEMMA 2.8. *Let a and b be complex numbers, and let θ be a real number satisfying* $\mathrm{Re}\big(\mathrm{e}^{\mathrm{i}\theta}(a - b)\big) = 0$. *If the equality*

$$\frac{|a|^2 + 1}{2} + \sqrt{|\mathrm{e}^{\mathrm{i}\theta}a - x|^2 + \left(\frac{|a|^2 - 1}{2}\right)^2} = \frac{|b|^2 + 1}{2} + \sqrt{|\mathrm{e}^{\mathrm{i}\theta}b - x|^2 + \left(\frac{|b|^2 - 1}{2}\right)^2}$$

holds for some real number x, then $|a| = |b|$ and the equality holds for any real numbers x.

Proof. Suppose the equality holds for some real number x. Then

$$\left\{\frac{|a|^2 - |b|^2}{2} + \sqrt{|\mathrm{e}^{\mathrm{i}\theta}a - x|^2 + \left(\frac{|a|^2 - 1}{2}\right)^2}\right\}^2 = |\mathrm{e}^{\mathrm{i}\theta}b - x|^2 + \left(\frac{|b|^2 - 1}{2}\right)^2.$$

Since $\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\theta}(a-b)\right)=0$, we have

$$\begin{aligned}(|a|^2-|b|^2)&\sqrt{|\mathrm{e}^{\mathrm{i}\theta}a-x|^2+\left(\frac{|a|^2-1}{2}\right)^2}\\&=2\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\theta}(a-b)\right)x+|b|^2-|a|^2+\frac{(|b|^2-|a|^2)(|a|^2-1)}{2}\\&=\frac{(|b|^2-|a|^2)(|a|^2+1)}{2}.\end{aligned}$$

Hence, $|a|=|b|$ and the equality

$$|\mathrm{e}^{\mathrm{i}\theta}a-x|^2=|\mathrm{e}^{\mathrm{i}\theta}b-x|^2$$

holds for any x because $\mathrm{Re}\big(\mathrm{e}^{\mathrm{i}\theta}(a-b)\big)=0$. ∎

COROLLARY 2.9. *Let E be a measurable subset of $\mathbb{T}$ satisfying $0<m(E)<1$, let a and b be complex numbers, and let $\varphi=a\chi_E+b\chi_{E^c}$. Then*

(i) *If θ is a real number satisfying $\mathrm{Re}\big(\mathrm{e}^{\mathrm{i}\theta}(a-b)\big)=0$, then*

$$\|S_{\varphi,1}\|^2\leqslant\max_{z=a,b}\left\{\frac{|z|^2+1}{2}+\sqrt{\big(\mathrm{Im}(\mathrm{e}^{\mathrm{i}\theta}z)\big)^2+\left(\frac{|z|^2-1}{2}\right)^2}\right\}.$$

The equality does not hold in general.

(ii) *If $|a|=|b|$, then the equality holds for some θ in* (i), *and*

$$\|S_{\varphi,1}\|^2=\frac{|a|^2+1}{2}+\sqrt{\left(\frac{|a-b|}{2}\right)^2+\left(\frac{|a|^2-1}{2}\right)^2}.$$

Proof. We prove (ii). Suppose $a=b$. Then φ becomes a constant function. Hence, by Remark 2.5,

$$\|S_{\varphi,1}\|^2=\max\left\{|\varphi|^2,1\right\}=\max\left\{|a|^2,1\right\}=\frac{|a|^2+1}{2}+\sqrt{0+\left(\frac{|a|^2-1}{2}\right)^2}.$$

Suppose $a\neq b$. Since $|\varphi|=|a|=|b|$, it follows from Corollary 2.3 that

$$\|S_{\varphi,1}\|^2=\frac{|a|^2+1}{2}+\sqrt{\|H_\varphi\|^2+\left(\frac{|a|^2-1}{2}\right)^2}.$$

Let

$$\psi=\frac{2\varphi-(a+b)}{a-b}.$$

Then

$$\psi = \begin{cases} 1 & \text{on } E; \\ -1 & \text{on } E^{c}. \end{cases}$$

Hence, $\inf\{\|\psi - k\|_\infty : k \in H^\infty\} = 1$ (cf. [5], p. 198). By the Nehari theorem ([8]),

$$\|H_\varphi\| = \inf_{k \in H^\infty} \|\varphi - k\|_\infty = \frac{|a-b|}{2}.$$

Hence,

$$\|S_{\varphi,1}\|^2 = \frac{|a|^2+1}{2} + \sqrt{\left(\frac{|a-b|}{2}\right)^2 + \left(\frac{|a|^2-1}{2}\right)^2}.$$

Since $a \neq b$, $|a| = |b|$ and $\mathrm{Re}\big(\mathrm{e}^{\mathrm{i}\theta}(a-b)\big) = 0$, it follows that there exist real numbers u and v such that

$$\mathrm{e}^{\mathrm{i}\theta} a = u + \mathrm{i}v \quad \text{and} \quad \mathrm{e}^{\mathrm{i}\theta} b = u - \mathrm{i}v.$$

Since $|a| = |b|$, we have

$$\begin{aligned}\left(\frac{|a-b|}{2}\right)^2 &= \frac{|a|^2 - \mathrm{Re}(a\bar{b})}{2} = \frac{u^2+v^2-(u^2-v^2)}{2} \\ &= v^2 = \big(\mathrm{Im}(\mathrm{e}^{\mathrm{i}\theta}a)\big)^2 = \big(\mathrm{Im}(\mathrm{e}^{\mathrm{i}\theta}b)\big)^2.\end{aligned}$$

We prove (i). By (ii), it is sufficient to prove (i) when $|a| \neq |b|$. We define the functions $f(x)$ and $g(x)$ according to

$$f(x) = \frac{|a|^2+1}{2} + \sqrt{|\mathrm{e}^{\mathrm{i}\theta}a - x|^2 + \left(\frac{|a|^2-1}{2}\right)^2},$$

and

$$g(x) = \frac{|b|^2+1}{2} + \sqrt{|\mathrm{e}^{\mathrm{i}\theta}b - x|^2 + \left(\frac{|b|^2-1}{2}\right)^2}.$$

Since $|a| \neq |b|$, it follows from Lemma 2.8 that the equality $f(x) = g(x)$ does not hold for any real number x. If $|a| < |b|$, then $f(x) < g(x)$ because $\mathrm{Re}\big(\mathrm{e}^{\mathrm{i}\theta}(a-b)\big) = 0$. Hence, by Theorem 2.1,

$$\begin{aligned}\|S_{\varphi,1}\|^2 &= \inf_{k \in H^\infty} \left\| \frac{|\varphi|^2+1}{2} + \sqrt{|\varphi - k|^2 + \left(\frac{|\varphi|^2-1}{2}\right)^2} \right\|_\infty \\ &\leqslant \inf_{x \in \mathbb{R}} \left\| \frac{|\varphi|^2+1}{2} + \sqrt{|\varphi - \mathrm{e}^{-\mathrm{i}\theta}x|^2 + \left(\frac{|\varphi|^2-1}{2}\right)^2} \right\|_\infty \\ &= \inf_{x \in \mathbb{R}} \|f(x)\chi_E + g(x)\chi_{E^{c}}\|_\infty \leqslant \inf_{x \in \mathbb{R}} g(x) = g\left(\mathrm{Re}\big(\mathrm{e}^{\mathrm{i}\theta}b\big)\right) \\ &= \frac{|b|^2+1}{2} + \sqrt{\big(\mathrm{Im}(\mathrm{e}^{\mathrm{i}\theta}b)\big)^2 + \left(\frac{|b|^2-1}{2}\right)^2}.\end{aligned}$$

Similarly, if $|a| > |b|$, then

$$\|S_{\varphi,1}\|^2 \leqslant \frac{|a|^2+1}{2} + \sqrt{\left(\mathrm{Im}\left(\mathrm{e}^{\mathrm{i}\theta}a\right)\right)^2 + \left(\frac{|a|^2-1}{2}\right)^2}.$$

This completes the proof. ∎

3. NORM OF THE INVERSE OPERATOR OF $S_{\alpha,\beta}$

The first author essentially gave Theorem 3.1 (cf. [7], Corollary 3). It is not difficult to establish that

$$\inf_{f\in H^2,\,\|f\|_2=1} \|T_\varphi f\|_2^2 \geqslant \operatorname*{ess\,inf}_{\mathbb{T}} \left(|\varphi|^2 - |\varphi - k|^2\right),$$

for any $k \in H^\infty$. The following theorem gives the formula for the computation of the norm of the inverse operator of T_φ.

Theorem 3.1. *Let $\varphi \in L^\infty$. Then*

$$\inf_{f\in H^2,\,\|f\|_2=1} \|T_\varphi f\|_2^2 = \sup_{k\in H^\infty} \left(\operatorname*{ess\,inf}_{\mathbb{T}} \left(|\varphi|^2 - |\varphi - k|^2\right)\right).$$

The supremum is attained. If T_φ is invertible, then the supremum is equal to $\|T_\varphi^{-1}\|^{-2}$.

Proof. For any $k \in H^\infty$, we define the quantity J_k according to

$$J_k = \operatorname*{ess\,inf}_{\mathbb{T}} \left(|\varphi|^2 - |\varphi - k|^2\right).$$

We prove that $\inf\{\|T_\varphi f\|_2^2 : f \in H^2, \|f\|_2 = 1\} \leqslant \sup\{J_k : k \in H^\infty\}$. Let $\varepsilon = \inf\{\|T_\varphi f\|_2 : f \in H^2, \|f\|_2 = 1\}$. Then

$$\varepsilon\|f_1\|_2 \leqslant \|T_\varphi f_1\|_2, \quad (f_1 \in H^2).$$

Since $\|T_\varphi f_1\|_2^2 + \|H_\varphi f_1\|_2^2 = \|\varphi f_1\|_2^2$, we have

$$\|H_\varphi f_1\|_2^2 \leqslant \left(\left(|\varphi|^2 - \varepsilon^2\right) f_1, f_1\right).$$

Let $W_1 = |\varphi|^2 - \varepsilon^2$, $W_2 = 1$, $W_3 = \varphi$. Since $(\varphi f_1, f_2) = (H_\varphi f_1, f_2)$, for any $f_1 \in H^2$, $f_2 \in \overline{H_0^2}$, we have

$$\begin{aligned}(W_1f_1, f_1) + (W_2f_2, f_2) + 2\mathrm{Re}\big(W_3f_1, f_2\big) &\geqslant \|H_\varphi f_1\|_2^2 + \|f_2\|_2^2 + 2\mathrm{Re}\big(H_\varphi f_1, f_2\big)\\ &= \|H_\varphi f_1 + f_2\|_2^2 \geqslant 0,\end{aligned}$$

($f_1 \in H^2, f_2 \in \overline{H_0^2}$). By the Cotlar-Sadosky lifting theorem ([1]), $W_1 \geqslant 0, W_2 \geqslant 0$, and there exists a $k_0 \in H^\infty$ such that

$$|W_3 - k_0|^2 \leqslant W_1 W_2.$$

Hence,

$$|\varphi - k_0|^2 \leqslant |\varphi|^2 - \varepsilon^2.$$

Since $\varepsilon = \inf\{\|T_\varphi f\|_2 : f \in H^2, \|f\|_2 = 1\}$, we have

$$\inf_{f\in H^2,\,\|f\|_2=1} \|T_\varphi f\|_2^2 \leqslant J_{k_0} \leqslant \sup_{k\in H^\infty} J_k.$$

We prove that $\inf\{\|T_\varphi f\|_2^2 : f \in H^2, \|f\|_2 = 1\} \geqslant \sup\{J_k : k \in H^\infty\}$. This is the easy direction of the theorem. For any $k \in H^\infty$, $|\varphi|^2 - J_k \geqslant |\varphi - k|^2$ and $H_{\varphi-k} = H_\varphi$. Hence, for any $f_1 \in H^2$,

$$\begin{aligned}
\|T_\varphi f_1\|_2^2 - J_k\|f_1\|_2^2 &= \|\varphi f_1\|_2^2 - \|H_\varphi f_1\|_2^2 - J_k\|\varphi f_1\|_2^2 \\
&= \big((|\varphi|^2 - J_k)f_1, f_1\big) - \|H_\varphi f_1\|_2^2 \\
&\geqslant \big(|\varphi - k|^2 f_1, f_1\big) - \|H_{\varphi-k} f_1\|_2^2 \\
&= \|(\varphi - k)f_1\|_2^2 - \|H_{\varphi-k} f_1\|_2^2 = \|T_{\varphi-k} f_1\|_2^2 \geqslant 0.
\end{aligned}$$

Hence, for any $k \in H^\infty$,

$$\|T_\varphi f_1\|_2^2 \geqslant J_k\|f_1\|_2^2, \quad (f_1 \in H^2).$$

Therefore,

$$\sup_{k\in H^\infty} J_k \leqslant \inf_{f\in H^2,\,\|f\|_2=1} \|T_\varphi f\|_2^2 \leqslant J_{k_0} \leqslant \sup_{k\in H^\infty} J_k.$$

Hence the equalities hold, and the infimum is attained by $k = k_0$. This completes the proof. ∎

COROLLARY 3.2. *If $|\varphi|$ is a constant function, then*

$$\inf_{f\in H^2,\,\|f\|_2=1} \|T_\varphi f\|_2^2 = |\varphi|^2 - \|H_\varphi\|^2.$$

Proof. It follows from Theorem 3.1 that

$$\begin{aligned}
\inf_{f\in H^2,\,\|f\|_2=1} \|T_\varphi f\|_2^2 &= \sup_{k\in H^\infty} \Big(\operatorname*{ess\,inf}_{\mathbb{T}} \big(|\varphi|^2 - |\varphi - k|^2\big)\Big) \\
&= |\varphi|^2 + \sup_{k\in H^\infty} \Big(\operatorname*{ess\,inf}_{\mathbb{T}} \big(-|\varphi - k|^2\big)\Big) \\
&= |\varphi|^2 - \inf_{k\in H^\infty} \|\varphi - k\|_\infty^2.
\end{aligned}$$

By the Nehari theorem ([8]), this proves the corollary. ∎

COROLLARY 3.3. *Let $\varphi \in L^\infty$. Then*

$$\inf_{f\in H^2,\, \|f\|_2=1} \|T_\varphi f\|_2^2 \geqslant \operatorname*{ess\,inf}_{\mathbb{T}} |\varphi|^2 - \|H_\varphi\|^2.$$

Proof. It follows from the easy direction of Theorem 3.1 that

$$\begin{aligned}\inf_{f\in H^2,\, \|f\|_2=1} \|T_\varphi f\|_2^2 &\geqslant \sup_{k\in H^\infty} \Big(\operatorname*{ess\,inf}_{\mathbb{T}} \big(|\varphi|^2 - |\varphi-k|^2\big)\Big)\\ &\geqslant \sup_{k\in H^\infty} \Big(\operatorname*{ess\,inf}_{\mathbb{T}} |\varphi|^2 + \operatorname*{ess\,inf}_{\mathbb{T}} \big(-|\varphi-k|^2\big)\Big)\\ &= \operatorname*{ess\,inf}_{\mathbb{T}} |\varphi|^2 + \sup_{k\in H^\infty} \Big(\operatorname*{ess\,inf}_{\mathbb{T}} \big(-|\varphi-k|^2\big)\Big)\\ &= \operatorname*{ess\,inf}_{\mathbb{T}} |\varphi|^2 - \inf_{k\in H^\infty} \|\varphi-k\|_\infty^2.\end{aligned}$$

By the Nehari theorem ([8]), this proves the corollary. ∎

If $\alpha, \beta \in L^\infty$, then the following inequality is well known and not difficult to establish.

$$\begin{aligned}\inf_{f\in L^2,\, \|f\|_2=1} \|S_{\alpha,\beta} f\|_2^2 &\leqslant \operatorname*{ess\,inf}_{\mathbb{T}} \big(\min\big\{|\alpha|^2, |\beta|^2\big\}\big)\\ &= \operatorname*{ess\,inf}_{\mathbb{T}} \left(\frac{|\alpha|^2+|\beta|^2}{2} - \sqrt{0 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right).\end{aligned}$$

It is not difficult to establish that

$$\inf_{f\in L^2,\, \|f\|_2=1} \|S_{\alpha,\beta} f\|_2^2 \geqslant \operatorname*{ess\,inf}_{\mathbb{T}} \left(\frac{|\alpha|^2+|\beta|^2}{2} - \sqrt{|\alpha\bar\beta - k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right),$$

for any $k \in H^\infty$. The following theorem gives the formula for the computation of the norm of the inverse operator of $S_{\alpha,\beta}$. The proof of the following theorem is essentially the same as the proof of Theorem 2.1.

THEOREM 3.4. *Let $\alpha, \beta \in L^\infty$. Then*

$$\begin{aligned}&\inf_{f\in L^2, \|f\|_2=1} \|S_{\alpha,\beta} f\|_2^2\\ &\quad = \sup_{k\in H^\infty} \left(\operatorname*{ess\,inf}_{\mathbb{T}} \left(\frac{|\alpha|^2+|\beta|^2}{2} - \sqrt{|\alpha\bar\beta - k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right)\right).\end{aligned}$$

The supremum is attained. If $S_{\alpha,\beta}$ is invertible, then the supremum is equal to $\|S_{\alpha,\beta}^{-1}\|^{-2}$.

Proof. For any $k \in H^\infty$, we define the quantity N_k according to

$$N_k = \operatorname*{ess\,inf}_{\mathbb{T}} \left(\frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \right).$$

We prove that $\inf\{\|S_{\alpha,\beta} f\|_2^2 : f \in L^2, \|f\|_2 = 1\} \leqslant \sup\{N_k : k \in H^\infty\}$. Let $\varepsilon = \inf\{\|S_{\alpha,\beta} f\|_2 : f \in L^2, \|f\|_2 = 1\}$. Then

$$\varepsilon \|f\|_2 \leqslant \|S_{\alpha,\beta} f\|_2, \quad (f \in L^2).$$

Let $W_1 = |\alpha|^2 - \varepsilon^2$, $W_2 = |\beta|^2 - \varepsilon^2$ and $W_3 = \alpha\bar{\beta} - \varepsilon^2$. Then

$$\big(W_1 f_1, f_1\big) + \big(W_2 f_2, f_2\big) + 2\mathrm{Re}\big(W_3 f_1, f_2\big) \geqslant 0,$$

$(f_1 \in H^2, f_2 \in \overline{H_0^2})$. By the Cotlar-Sadosky lifting theorem ([1]), $W_1 \geqslant 0, W_2 \geqslant 0$, and there exists a $g \in H^\infty$ such that

$$|W_3 - g|^2 \leqslant W_1 W_2.$$

Hence, $\varepsilon \leqslant \min\{|\alpha|, |\beta|\}$ and there exists a $g \in H^\infty$ such that

$$|\alpha\bar{\beta} - \varepsilon^2 - g|^2 \leqslant \left(|\alpha|^2 - \varepsilon^2\right)\left(|\beta|^2 - \varepsilon^2\right).$$

Let $k_0 = \varepsilon^2 + g$. Then $k_0 \in H^\infty$ and $|\alpha\bar{\beta} - \varepsilon^2 - g| = |\alpha\bar{\beta} - k_0|$. Hence,

$$0 \leqslant \left(|\alpha|^2 - \varepsilon^2\right)\left(|\beta|^2 - \varepsilon^2\right) - |\alpha\bar{\beta} - k_0|^2 = \varepsilon^4 - \left(|\alpha|^2 + |\beta|^2\right)\varepsilon^2 + |\alpha\beta|^2 - |\alpha\bar{\beta} - k|^2.$$

Suppose

$$\varepsilon^2 \geqslant \frac{|\alpha|^2 + |\beta|^2}{2} + \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}$$

on some measurable subset E of $\mathbb{T}$. Since

$$\varepsilon^2 \leqslant \min\left\{|\alpha|^2, |\beta|^2\right\} = \frac{|\alpha|^2 + |\beta|^2}{2} - \left|\frac{|\alpha|^2 - |\beta|^2}{2}\right|$$

on $\mathbb{T}$, we have

$$\left|\frac{|\alpha|^2 - |\beta|^2}{2}\right| + \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2} \leqslant 0$$

on E. This implies that $|\alpha| - |\beta| = |\alpha\bar{\beta} - k_0| = 0$ on E. Hence,

$$\varepsilon^2 \leqslant \min\left\{|\alpha|^2, |\beta|^2\right\} = \frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{|\alpha\bar{\beta} - k_0|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2}\right)^2}$$

on E. Therefore,

$$\varepsilon^2 \leqslant \frac{|\alpha|^2+|\beta|^2}{2} - \sqrt{|\alpha\bar{\beta}-k_0|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}$$

on $\mathbb{T}$. Hence $\varepsilon^2 \leqslant N_{k_0}$. Since $\varepsilon = \inf\{\|S_{\alpha,\beta}f\|_2 : f \in L^2, \|f\|_2 = 1\}$, we have

$$\inf_{f\in L^2,\, \|f\|_2=1} \|S_{\alpha,\beta}f\|_2^2 \leqslant N_{k_0} \leqslant \sup_{k\in H^\infty} N_k.$$

We prove that $\inf\{\|S_{\alpha,\beta}f\|_2^2 : f \in L^2, \|f\|_2 = 1\} \geqslant \sup\{N_k : k \in H^\infty\}$. This is the easy direction of the theorem. For any $k \in H^\infty$, we have

$$(kf_1, f_2) = 0, \quad (f_1 \in H^2, f_2 \in \overline{H_0^2}).$$

Since

$$\frac{|\alpha|^2+|\beta|^2}{2} - \sqrt{|\alpha\bar{\beta}-k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} \geqslant N_k,$$

we have

$$\begin{aligned}(|\alpha|^2-N_k)(|\beta|^2-N_k) &\geqslant \left(\sqrt{|\alpha\bar{\beta}-k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} + \frac{|\alpha|^2-|\beta|^2}{2}\right)\\ &\qquad\times\left(\sqrt{|\alpha\bar{\beta}-k|^2 + \left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2} - \frac{|\alpha|^2-|\beta|^2}{2}\right)\\ &= |\alpha\bar{\beta}-k|^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\|\alpha f_1+\beta f_2\|_2^2 - N_k\|f_1+f_2\|_2^2\\ &\quad= \left\|\sqrt{|\alpha|^2-N_k}f_1\right\|_2^2 + \left\|\sqrt{|\beta|^2-N_k}f_2\right\|_2^2 + 2\mathrm{Re}(\alpha\bar{\beta}f_1,f_2)\\ &\quad\geqslant 2\left\|\sqrt{|\alpha|^2-N_k}f_1\right\|_2\left\|\sqrt{|\beta|^2-N_k}f_2\right\|_2 + 2\mathrm{Re}((\alpha\bar{\beta}-k)f_1,f_2)\\ &\quad\geqslant 2\int_{\mathbb{T}}\left(\sqrt{|\alpha|^2-N_k}\sqrt{|\beta|^2-N_k} - |\alpha\bar{\beta}-k|\right)|f_1f_2|\,\mathrm{d}m \geqslant 0.\end{aligned}$$

Hence, for any $k \in H^\infty$,

$$\|S_{\alpha,\beta}f\|_2^2 \geqslant N_k\|f\|_2^2, \quad (f \in L^2).$$

Therefore,

$$\sup_{k\in H^\infty} N_k \leqslant \inf_{f\in L^2,\, \|f\|_2=1} \|S_{\alpha,\beta}f\|_2^2 \leqslant N_{k_0} \leqslant \sup_{k\in H^\infty} N_k.$$

Hence the equalities hold, and the infimum is attained by $k = k_0$. This completes the proof. ∎

REMARK 3.5. Let $\alpha, \beta \in L^\infty$, let $\varphi = \alpha\bar{\beta}$, and let $\psi = (|\alpha|^2 - |\beta|^2)/2$. Then

$$\inf_{f \in L^2,\, \|f\|_2 = 1} \|S_{\alpha,\beta} f\|_2^2 = \sup_{k \in H^\infty} \left(\operatorname*{ess\,inf}_{\mathbb{T}} \left(\sqrt{|\varphi|^2 + \psi^2} - \sqrt{|\varphi - k|^2 + \psi^2} \right) \right).$$

The supremum is attained. If $|\alpha| = |\beta|$, then $\psi = 0$. Hence

$$\inf_{f \in L^2,\, \|f\|_2 = 1} \|S_{\alpha,\beta} f\|_2^2 = \sup_{k \in H^\infty} \left(\operatorname*{ess\,inf}_{\mathbb{T}} \left(|\varphi| - |\varphi - k| \right) \right).$$

COROLLARY 3.6. *If $|\alpha|$ and $|\beta|$ are constant functions, then*

$$\inf_{f \in L^2,\, \|f\|_2 = 1} \|S_{\alpha,\beta} f\|_2^2 = \frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{\|H_{\alpha\bar{\beta}}\|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2} \right)^2}.$$

Proof. It follows from Theorem 3.4 that

$$\begin{aligned}
&\inf_{f \in L^2,\, \|f\|_2 = 1} \|S_{\alpha,\beta} f\|_2^2 \\
&\quad = \sup_{k \in H^\infty} \left(\operatorname*{ess\,inf}_{\mathbb{T}} \left(\frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2} \right)^2} \right) \right) \\
&\quad = \frac{|\alpha|^2 + |\beta|^2}{2} + \sup_{k \in H^\infty} \left(\operatorname*{ess\,inf}_{\mathbb{T}} \left(- \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2} \right)^2} \right) \right) \\
&\quad = \frac{|\alpha|^2 + |\beta|^2}{2} + \sup_{k \in H^\infty} \left(- \left\| \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2} \right)^2} \right\|_\infty \right) \\
&\quad = \frac{|\alpha|^2 + |\beta|^2}{2} - \inf_{k \in H^\infty} \left\| \sqrt{|\alpha\bar{\beta} - k|^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2} \right)^2} \right\|_\infty \\
&\quad = \frac{|\alpha|^2 + |\beta|^2}{2} - \sqrt{\inf_{k \in H^\infty} \|\alpha\bar{\beta} - k\|_\infty^2 + \left(\frac{|\alpha|^2 - |\beta|^2}{2} \right)^2}.
\end{aligned}$$

By the Nehari theorem ([8]), this proves the corollary. ∎

COROLLARY 3.7. *Let $\alpha, \beta \in L^\infty$. Then*

$$\inf_{f \in L^2,\, \|f\|_2 = 1} \|S_{\alpha,\beta} f\|_2^2 \geqslant \operatorname*{ess\,inf}_{\mathbb{T}} \left(\min\{|\alpha|^2, |\beta|^2\} \right) - \|H_{\alpha\bar{\beta}}\|.$$

Proof. It follows from the easy direction of Theorem 3.4 that

$$\begin{aligned}
&\inf_{f\in L^2,\,\|f\|_2=1}\|S_{\alpha,\beta}f\|_2^2\\
&\quad\geqslant \sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_{\mathbb{T}}\left(\frac{|\alpha|^2+|\beta|^2}{2}-\sqrt{|\alpha\bar\beta-k|^2+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right)\right)\\
&\quad\geqslant \sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_{\mathbb{T}}\left(\frac{|\alpha|^2+|\beta|^2}{2}-|\alpha\bar\beta-k|-\left|\frac{|\alpha|^2-|\beta|^2}{2}\right|\right)\right)\\
&\quad= \sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_{\mathbb{T}}\left(\min\left\{|\alpha|^2,|\beta|^2\right\}-|\alpha\bar\beta-k|\right)\right)\\
&\quad\geqslant \sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_{\mathbb{T}}\left(\min\left\{|\alpha|^2,|\beta|^2\right\}\right)+\operatorname*{ess\,inf}_{\mathbb{T}}\left(-|\alpha\bar\beta-k|\right)\right)\\
&\quad= \operatorname*{ess\,inf}_{\mathbb{T}}\left(\min\left\{|\alpha|^2,|\beta|^2\right\}\right)+\sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_{\mathbb{T}}\left(-|\alpha\bar\beta-k|\right)\right)\\
&\quad= \operatorname*{ess\,inf}_{\mathbb{T}}\left(\min\left\{|\alpha|^2,|\beta|^2\right\}\right)+\sup_{k\in H^\infty}\left(-\|\alpha\bar\beta-k\|_\infty\right)\\
&\quad= \operatorname*{ess\,inf}_{\mathbb{T}}\left(\min\left\{|\alpha|^2,|\beta|^2\right\}\right)-\inf_{k\in H^\infty}\|\alpha\bar\beta-k\|_\infty.
\end{aligned}$$

By the Nehari theorem ([8]), this proves the corollary. ∎

REMARK 3.8. If $\alpha\bar\beta\in H^\infty$, then the supremum in Theorem 3.4 is attained by $k=\alpha\bar\beta$. In this case,

$$\inf_{f\in L^2,\,\|f\|_2=1}\|S_{\alpha,\beta}f\|_2=\operatorname*{ess\,inf}_{\mathbb{T}}\left(\min\left\{|\alpha|,|\beta|\right\}\right).$$

For functions $\alpha,\beta\in L^\infty$, $S_{\alpha,\beta}$ is left invertible if and only if

$$\operatorname*{ess\,inf}_{\mathbb{T}}\left(\frac{|\alpha|^2+|\beta|^2}{2}-\sqrt{|\alpha\bar\beta-k|^2+\left(\frac{|\alpha|^2-|\beta|^2}{2}\right)^2}\right)>0,$$

for some $k\in H^\infty$. By Corollary 3.2, if $|\varphi|=1$, then T_φ is left invertible if and only if $S_{\varphi,1}$ is left invertible if and only if $\|H_\varphi\|<1$ (cf. [9], p. 203).

COROLLARY 3.9. *Let $\varphi\in L^\infty, |\varphi|=1$. Then*

$$\inf_{f\in H^2,\,\|f\|_2=1}\|T_\varphi f\|_2=\|S_{\varphi,1}\|\left(\inf_{f\in L^2,\,\|f\|_2=1}\|S_{\varphi,1}f\|_2\right).$$

Proof. It follows from Corollary 3.2 that

$$\inf_{f\in H^2,\,\|f\|_2=1}\|T_\varphi f\|_2^2=1-\|H_\varphi\|^2.$$

It follows from Corollary 2.3 that

$$\|S_{\varphi,1}\|^2 = 1 + \|H_\varphi\|.$$

It follows from Corollary 3.6 that

$$\inf_{f\in L^2,\, \|f\|_2=1} \|S_{\varphi,1}f\|_2^2 = 1 - \|H_\varphi\|.$$

These equalities prove the corollary. ∎

COROLLARY 3.10. *If a and b are invertible functions in H^∞, then*

$$\|S_{a,\bar b}L_{1/\bar b}\|^{-2} = \sup_{k\in H^\infty}\left(\operatorname*{ess\,inf}_{\mathbb{T}}\left(\frac{|a|^2+|b|^2}{2|a|^2} - \sqrt{\left|\frac{\bar b}{a}-k\right|^2 + \left(\frac{|a|^2-|b|^2}{2|a|^2}\right)^2}\right)\right),$$

where $L_{1/\bar b}$ denotes the Laurent operator on L^2. The supremum is attained.

Proof. Since a and b are invertible functions H^∞, it follows that $S_{\bar b/a,1}$ is invertible, and

$$S_{\bar b/a,1}^{-1} = S_{a,\bar b}L_{1/\bar b}$$

(cf. [6], p. 88 and [10], Theorem 3). Hence,

$$\|S_{a,\bar b}L_{1/\bar b}\|^{-2} = \|S_{\bar b/a,1}^{-1}\|^{-2} = \inf_{f\in L^2, \|f\|_2=1}\|S_{\bar b/a,1}f\|_2^2.$$

We apply Theorem 3.4 with $\alpha = \bar b/a$ and $\beta = 1$. This completes the proof. ∎

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