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THE COMPACT QUANTUM GROUP $\mathrm{SO}(3)_q$

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Abstract. We study the various definitions that have been suggested for the compact quantum group $\mathrm{SO}(3)_q$, and obtain a new presentation of it which has all the properties that one would hope for.

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INTRODUCTION

The C^*-algebraic theory of compact quantum groups is by now quite well understood. There is a very satisfactory axiomatic definition ([17], [20]) from which the existence of the Haar state can be deduced, and which provides a secure basis for the development of a general representation theory. Constructions have been given for quantised versions of the standard series of compact Lie groups ([19], [12]), and there are powerful general methods for constructing other classes of examples ([16]). But the only nonclassical compact quantum group for which a detailed, concrete description has been given is $\mathrm{SU}(2)_q$, the seminal example introduced by Woronowicz ([18]) and Drinfel′d ([4]). The aim of this paper is to double this stock of examples by providing a similar concrete, spatial analysis of $\mathrm{SO}(3)_q$ to that given by Woronowicz for $\mathrm{SU}(2)_q$. (See also [10], which gives a presentation, but not a Hilbert space representation, of $C\big(\mathrm{Sp}(2)_q\big)$). We give an explicit presentation of the C^*-algebra of $\mathrm{SO}(3)_q$ and obtain a faithful Hilbert space representation of it. The C^*-algebra, like that of $\mathrm{SU}(2)_q$, is an extension of $\mathcal{K}\otimes C(\mathbb{T})$ by $C(\mathbb{T})$; but it is a different extension, and the two C^*-algebras are not isomorphic.

As in the classical case, where SU(2) is a double covering of SO(3), there is an embedding of C^*-bialgebras $C\big(\mathrm{SO}(3)_q\big) \hookrightarrow C\big(\mathrm{SU}(2)_{q^{1/2}}\big)$. (The change in the quantum parameter, q becoming $q^{1/2}$, is just an artefact of the way in which the parameters were chosen; but the existence of the embedding is important, and in fact it would be very natural to *define* $\mathrm{SO}(3)_q$ as the subalgebra of $\mathrm{SU}(2)_{q^{1/2}}$ given by the range of this mapping.) Using this embedding, we can give a neat description of the irreducible representations of $\mathrm{SO}(3)_q$. In the case $q = 1$, this yields a description of the irreducible representations of SO(3) in which the coefficients are Jacobi polynomials, which seems to be considerably more manageable than the standard construction [15] using ultraspherical polynomials.

1. DEFINITION BY R-MATRIX

In this Section we reformulate the construction of $\mathrm{SO}(3)_q$ given in [12]. The general setting is that one considers a unital C^*-algebra A generated (as a C^*-algebra) by n^2 elements $u_{i,j}$ subject to various relations. Firstly, the element $U = (u_{i,j})$ should be unitary in the C^*-algebra $M_n(A)$ of $n \times n$ matrices over A. That is,

$$\sum_{k=1}^{n} u_{i,k}u_{j,k}^* = \sum_{k=1}^{n} u_{k,i}^* u_{k,j} = \delta_{i,j}1 \quad (i, j = 1, \dots, n).$$

Secondly, the generators $u_{i,j}$ satisfy some commutation relations specified in the following way. We can regard U as an adjointable operator on the Hilbert C^*-module A^n (see [7]). In fact, since A^n is unitarily equivalent to $\mathbb{C}^n \otimes A$, we can regard U as acting on $\mathbb{C}^n \otimes A$. Now let R be an $n^2 \times n^2$ matrix over $\mathbb{C}$, and regard R as an operator on $\mathbb{C}^n \otimes \mathbb{C}^n$. Use leg-numbering notation ([7], p. 80) to write $U_{1\,3}$, $U_{2\,3}$, $R_{1\,2}$, for the operators U, R acting on the appropriate 'legs' of the triple tensor product C^*-module $\mathbb{C}^n \otimes \mathbb{C}^n \otimes A$. The R-matrix equation is

$$R_{1\,2}U_{1\,3}U_{2\,3} = U_{2\,3}U_{1\,3}R_{1\,2}. \tag{1.1}$$

If we regard operators on $\mathbb{C}^n \otimes \mathbb{C}^n \otimes A$ as $n^2 \times n^2$ matrices over A then the matrix entries of (1.1) give n^4 relations among the generators of A. For certain very special choices of R these relations are realised by a nontrivial algebra A; and the mapping $\delta\colon A \to A \otimes A$ given on the generators by

$$\delta(u_{i,j}) = \sum_{k=1}^{n} u_{i,k} \otimes u_{k,j} \tag{1.2}$$

defines a comultiplication making A into a C^*-bialgebra.

Thirdly, there may be some selfadjointness relations among the elements of A, corresponding (in the classical case) to the fact that some Lie groups are represented by real rather than complex matrices.

Turning now to the particular case of $\mathrm{SO}(3)_q$, we observe first that the classical Lie group $\mathrm{SO}(3)$ consists of all 3×3 unitary matrices with determinant 1 in which each element is real. But on conjugating such a matrix by the unitary matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & \mathrm{i} \\ 0 & 1+\mathrm{i} & 0 \\ \mathrm{i} & 0 & 1 \end{pmatrix},$$

we obtain a 3×3 unitary matrix of the form

$$\begin{pmatrix} a & z^* & y^* \\ x & h & x^* \\ y & z & a^* \end{pmatrix}, \tag{1.3}$$

where $a, x, y, z \in \mathbb{C}$, $h \in \mathbb{R}$ and an asterisk denotes complex conjugation; and $\mathrm{SO}(3)$ could be defined as the set of all such matrices that have determinant 1. We shall see in Section 4 that for some purposes this description of $\mathrm{SO}(3)$ is more convenient than the usual one.

In [12], Reshetikhin, Takhtajan and Faddeev define the C^*-algebra of $\mathrm{SO}(3)_q$ to be a unital C^*-algebra A with five generators a, h, x, y, z such that $h^* = h$,

$$U = \begin{pmatrix} a & q^{\frac{1}{2}} z^* & q y^* \\ x & h & q^{\frac{1}{2}} x^* \\ y & z & a^* \end{pmatrix} \tag{1.4}$$

is unitary in $M_3(A)$, and (1.1) is satisfied for a certain 9×9 R-matrix which is specified in [12]. Here, q is a fixed real parameter with $0 < q \leqslant 1$. Writing out these relations in full, we see from the fact that U is unitary that

$$\begin{cases} aa^* + qz^*z + q^2y^*y = 1 & a^*a + x^*x + y^*y = 1 \\ xx^* + h^2 + qx^*x = 1 & qzz^* + h^2 + z^*z = 1 \\ yy^* + zz^* + a^*a = 1 & q^2yy^* + qxx^* + aa^* = 1 \\ ax^* + q^{\frac{1}{2}}z^*h + q^{\frac{3}{2}}y^*x = 0 & q^{\frac{1}{2}}za + hx + z^*y = 0 \\ ay^* + q^{\frac{1}{2}}z^{*2} + qy^*a = 0 & qya + q^{\frac{1}{2}}x^2 + ay = 0 \\ xy^* + hz^* + q^{\frac{1}{2}}x^*a = 0 & q^{\frac{3}{2}}yz^* + q^{\frac{1}{2}}xh + az = 0. \end{cases} \tag{1.5}$$

In addition, the 81 R-matrix equations lead (by routine but tedious calculations which we omit) to the following commutation relations:

$$\text{(1.6)}\qquad \begin{cases} xz^* = z^*x & ax = qxa & xh - hx = (1-q^2)yz^* \\ yh = hy & az^* = qz^*a & z^*h - hz^* = (1-q^2)y^*x \\ yy^* = y^*y & xy = qyx & q^{\frac{1}{2}}(ha - ah) = (1-q^2)xz^* \\ ay = q^2ya & xy^* = qy^*x & q(zx - xz) = (1-q^2)yh \\ ay^* = q^2y^*a & z^*y = qyz^* & xx^* - qx^*x = (1-q^2)y^*y. \\ & z^*y^* = qy^*z^* & \end{cases}$$

One obtains the C^*-algebra A by taking the $*$-algebra A_0 given by the above generators and relations and completing with respect to its greatest C^*-seminorm (which exists, because the relations (1.5) easily imply that all the generators lie in the unit ball of A_0, for any seminorm). In the case $q = 1$, the C^*-algebra so obtained is commutative. It is not the algebra $C\big(\mathrm{SO}(3)\big)$, however, but $C\big(\mathrm{O}(3)\big)$, since it is clear that in this case we have not imposed any condition that would require the matrix in (1.3) to have determinant 1. This was observed by Takeuchi ([13]), who pointed out that the definition of Reshetikhin, Takhtajan and Faddeev should be modified by the introduction of a quantum determinant, as we shall now explain.

For an $n \times n$ matrix $U = (u_{i,j})$ with entries in an algebra A, let

$$\text{(1.7)}\qquad \mathrm{Det}_{\mathbf{A}}(U) = \sum_{\sigma \in S_n} (-q)^{\mathrm{inv}(\sigma)} u_{1,\sigma(1)} u_{2,\sigma(2)} \cdots u_{n,\sigma(n)},$$

where $\mathrm{inv}(\sigma)$ is the number of inversions in σ. (The subscript A in the notation $\mathrm{Det}_{\mathbf{A}}$ refers not to the algebra A but to the fact that $\mathrm{Det}_{\mathbf{A}}$ is the form of quantum determinant appropriate for dealing with quantisations of Lie groups of type A. We are about to see that quantisations of Lie groups of type B require a different quantum determinant $\mathrm{Det}_{\mathbf{B}}$. For further information on quantum determinants, see [8].) As is well known (see [19] for example), this is the definition of quantum determinant appropriate for constructing $\mathrm{SU}(n)_q$. But it was observed by Takeuchi that a different formula for the quantum determinant is needed in the case of $\mathrm{SO}(n)_q$; and for $n = 3$ this formula was computed explicitly by Fiore ([5]) in the form

$$\text{(1.8)}\qquad \begin{aligned} \mathrm{Det}_{\mathbf{B}}(U) = u_{1,1}u_{2,2}u_{3,3} - qu_{1,1}u_{2,3}u_{3,2} - qu_{1,2}u_{2,1}u_{3,3} + qu_{1,2}u_{2,3}u_{3,1} \\ + qu_{1,3}u_{2,1}u_{3,2} - q^2u_{1,3}u_{2,2}u_{3,1} + q^{\frac{1}{2}}(1-q)u_{1,2}u_{2,2}u_{3,2} \end{aligned}$$

(note the bizarre seventh term at the end of this expression). If we set $w = \mathrm{Det}_{\mathbf{B}}(U)$ for the U in (1.4), it can be verified that w is a central involution in A. Takeuchi imposes the additional relation $w = 1$ on A to obtain his version of the quantum SO(3) algebra. So to our existing relations (1.5) and (1.6), we add the extra relation $w = 1$, or more explicitly

$$aha^* - q^{\frac{3}{2}}ax^*z - q^{\frac{5}{2}}z^*xa^* + q^2z^*x^*y + q^2y^*xz - q^3y^*hy + q(1-q)z^*hz = 1. \tag{1.9}$$

From (1.5), (1.6) and (1.9) one can derive by laborious algebraic calculations the equation

$$xzy^* = (1 + qh)yy^*.$$

However, in order to show that the Hilbert space representation that we shall construct in Section 3 is faithful, we need the stronger condition

$$xz = (1 + qh)y,$$

which does not appear to follow from the above relations. We shall also need some other simple relations among the generators that do not appear to be derivable from those considered so far. For this reason, we shall now redefine $C\big(\mathrm{SO}(3)_q\big)$ using a tensorial approach to the quantum determinant similar that of [5] and [19].

2. DEFINITION BY QUANTUM DETERMINANT

We start by recalling the definition of $C(\mathrm{SU}(N)_q)$ given in [19]. For an algebra A, a matrix $X \in M_N(A)$ and an element

$$\eta = (\eta_1, \dots, \eta_N) \in \{1, \dots, N\}^N,$$

let $X^{(\eta)}$ be the matrix in $M_N(A)$ in which row j is equal to row η_j of X, for $1 \leqslant j \leqslant N$. Then $C(\mathrm{SU}(N)_q)$ is defined to be the C^*-algebra with N^2 generators $u_{i,j}$ $(1 \leqslant i, j \leqslant N)$ satisfying relations given by the requirements that $U = (u_{i,j})$ is unitary and

$$\mathrm{Det}_{\mathbf{A}}(U^{(\eta)}) = \mathrm{Det}_{\mathbf{A}}(I^{(\eta)}) \quad (\eta \in \{1, \dots, N\}^N), \tag{2.1}$$

where I is the identity $N \times N$ matrix and $\mathrm{Det}_{\mathbf{A}}$ is as in (1.7). The N^N equations (2.1) are hard to handle; but for small values of N they can be simplified by a technique described in Appendix 1 of [17] (for the case $N = 2$) and used in [3] (for $N = 3$). For simplicity, we shall outline this technique just for the case $N = 3$.

For $\eta \in \{1, 2, 3\}^3$, let

$$q^{(\eta)} = \begin{cases} (-q)^{\mathrm{inv}(\eta)} & \text{if } \eta \text{ is a permutation,} \\ 0 & \text{otherwise.} \end{cases}$$

In the space $\mathbb{C}^{27} \cong \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, let ξ be the vector whose components (indexed by the elements of $\{1, 2, 3\}^3$) are $\xi_\eta = q^{(\eta)}$. As at the start of Section 1, we regard the 3×3 matrix U as acting on the Hilbert C^*-module $\mathbb{C}^3 \otimes A$, and we use leg-numbering notation on the module $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes A$. Then it is easy to see that (2.1) is equivalent to the condition

$$U_{14}U_{24}U_{34}(\xi \otimes 1) = \xi \otimes 1.$$

But since U is unitary, we can write this equation in either of the equivalent forms

$$U_{24}U_{34}(\xi \otimes 1) = U_{14}^*(\xi \otimes 1), \quad U_{34}(\xi \otimes 1) = U_{24}^*U_{14}^*(\xi \otimes 1).$$

When written out in component form, this results in equations involving products of at most two of the generators $u_{i,j}$ and their adjoints. In this way, we obtain a much more tractable set of relations than those in (2.1), which involve triple products.

Our presentation of $C(\mathrm{SO}(3)_q)$ will be modelled very closely on the above. We define it to be the C^*-algebra (formed by completion with respect to the maximal C^*-norm) of the $*$-algebra A_0 with five generators a, h, x, y, z such that h is selfadjoint, the matrix U in (1.4) is unitary, and

$$\mathrm{Det}_{\mathbf{B}}(U^{(\eta)}) = \mathrm{Det}_{\mathbf{B}}(I^{(\eta)}) \quad (\eta \in \{1, 2, 3\}^3), \tag{2.2}$$

where I is the identity 3×3 matrix and $\mathrm{Det}_{\mathbf{B}}$ is as in (1.8).

Let ζ be the vector in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ whose only nonzero components are given by

$$\zeta_{111} = 1, \quad \zeta_{132} = \zeta_{213} = -q, \quad \zeta_{231} = \zeta_{312} = q, \quad \zeta_{321} = -q^2, \quad \zeta_{222} = q^{\frac{1}{2}}(1-q).$$

Then (2.2) is equivalent to the condition

$$U_{14}U_{24}U_{34}(\zeta \otimes 1) = \zeta \otimes 1$$

in the module $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes A$. Since U is unitary, we can rewrite these equations as

$$U_{24}U_{34}(\zeta \otimes 1) = U_{14}^*(\zeta \otimes 1), \quad U_{34}(\zeta \otimes 1) = U_{24}^* U_{14}^*(\zeta \otimes 1).$$

When written out in component form, this set of 54 equations yields (apart from the relations (1.5) that we already know to hold from the fact that U is unitary) the following relations:

$$(2.3)\quad \begin{cases} xz^* = z^*x & ax = qxa & x(1-h) = q(1-h)x = (1+q)z^*y \\ yh = hy & az^* = qz^*a & z^*(1-h) = q(1-h)z^* = (1+q)xy^* \\ yy^* = y^*y & xy = qyx & a(1-h) = q^2(1-h)a = -q^{\frac{3}{2}}z^*x \\ ay = q^2ya & xy^* = qy^*x & x^*x - xx^* = q^{-1}(1-q)h(1-h) \\ ay^* = q^2y^*a & z^*y = qyz^* & xz = (1+qh)y \\ y^*y = \frac{1}{(1+q)^2}(1-h)^2 & z^*y^* = qy^*z^* & zx = (1+q^{-1}h)y. \end{cases}$$

It is easily seen that the equations in the right-hand column of (1.6) follow from those in (1.5) and (2.3), and so does (1.9).

From the equation $y^*y = (1+q)^{-2}(1-h)^2$ it would follow in a C^*-algebra that $|y| = (1+q)^{-1}(1-h)$. We have not yet exhibited a C^*-norm on A, but we can use the right-hand side of this equation (which is clearly in A) to define $|y|$. From the equation $x(1-h) = (1+q)z^*y$ one then sees that $x|y| = z^*y$, and similarly $xy^* = z^*|y|$.

Let

$$(2.4)\qquad B = \{a^k x^\varepsilon y^\alpha |y|^l : k, l \in \mathbb{N},\ \alpha \in \mathbb{Z},\ \varepsilon \in \{0,1\}\} \cup \{a^k z^* y^{*l} : k, l \in \mathbb{N}\},$$

where y^α is interpreted to mean $(y^*)^{-\alpha}$ when $\alpha < 0$. The set $B \cup B^*$ linearly spans the algebra A: for it certainly contains all the generators and their adjoints, and one can painstakingly check (using (1.5) and (2.3)) that any product of elements in $B \cup B^*$ can be expressed as a linear combination of elements in the set. (Alternatively, one could presumably use the Diamond Lemma ([1]) for a more stylish proof.) Note that $B \cap B^* = \{y^\alpha |y|^l : \alpha \in \mathbb{Z},\ l \in \mathbb{N}\}$. We shall show in the next section that $B \cup B^*$ (with the elements of $B \cap B^*$ counted only once) is a linear basis for A.

3. FUNDAMENTAL REPRESENTATION OF THE C^*-ALGEBRA

In this section we shall assume that $0 < q < 1$, since the representation that we are about to construct is not defined for $q = 1$. Let $\{\varepsilon_{k,n} : k \in \mathbb{N},\, n \in \mathbb{Z}\}$ be the standard orthonormal basis of the Hilbert space $\ell^2(\mathbb{N} \times \mathbb{Z})$ and let H be the closed subspace spanned by $\{\varepsilon_{k,n} : k + n \text{ is even}\}$. We define operators $\pi(a)$, $\pi(h)$, $\pi(x)$, $\pi(y)$, $\pi(z)$ in $B(H)$ by

$$\tag{3.1} \begin{cases} \pi(a)\varepsilon_{k,n} = (1-q^k)^{\frac12}(1-q^{k-1})^{\frac12}\varepsilon_{k-2,n} \\ \pi(h)\varepsilon_{k,n} = (1-q^k-q^{k+1})\varepsilon_{k,n} \\ \pi(x)\varepsilon_{k,n} = \mathrm{i}[(1+q)q^{k-1}(1-q^k)]^{\frac12}\varepsilon_{k-1,n-1} \qquad (k \in \mathbb{N},\ n \in \mathbb{Z}). \\ \pi(y)\varepsilon_{k,n} = q^k\varepsilon_{k,n-2} \\ \pi(z)\varepsilon_{k,n} = \mathrm{i}[(1+q)q^k(1-q^{k+1})]^{\frac12}\varepsilon_{k+1,n-1} \end{cases}$$

It is routine to verify that these operators satisfy all the relations (1.5) and (2.3), and thus we have constructed a Hilbert space $*$-representation π of the $*$-algebra A defined by (2.2).

By considering the effect of the elements of $B \cup B^*$ on the basis elements $\varepsilon_{k,n}$ (where B is as in (2.4)), it is easy to see that $B \cup B^*$ is linearly independent. This shows both that $B \cup B^*$ is a linear basis for A and also that π is a faithful representation of A. Thus the operator norm in $B(H)$ gives a C^*-norm on A, and we define $B_q = C(\mathrm{SO}(3)_q)$ to be the completion of A with respect to this norm. In fact, this norm is the greatest C^*-norm on A, and the C^*-algebra B_q is independent of q. If we denote by B_0 the "crystal limit" algebra, namely the C^*-algebra generated by the operators in (3.1) when $q = 0$, then $B_q \cong B_0$. The proofs of all these assertions are very similar to those of the corresponding results for $C(\mathrm{SO}(3)_q)$ as outlined in Appendix 2 of [17], and we omit them. We call π the *fundamental representation* of $C(\mathrm{SO}(3)_q)$.

Putting $q = 0$ in (3.1), we see that the generators of A_0 are the operators given by

$$\widehat{a}\varepsilon_{k,n} = \varepsilon_{k-2,n} \quad \widehat{h}\varepsilon_{k,n} = (1-\delta_{k,0})\varepsilon_{k,n}$$

$$\widehat{x}\varepsilon_{k,n} = i\delta_{k,1}\varepsilon_{k-1,n-1} \quad \widehat{y}\varepsilon_{k,n} = \delta_{k,0}\varepsilon_{k,n-2} \quad \widehat{z}\varepsilon_{k,n} = i\delta_{k,0}\varepsilon_{k+1,n-1}$$

(where $\delta_{i,j}$ is a Kronecker delta symbol, and $\varepsilon_{k,n}$ is interpreted to mean 0 when $k < 0$). Identify $\ell^2(\mathbb{N} \times \mathbb{Z})$ with $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{Z})$ and write w for the bilateral shift $\varepsilon_n \mapsto \varepsilon_{n-2}$ of multiplicity two on $\ell^2(\mathbb{Z})$; and use e_0 to denote the projection onto the first basis vector ε_0 in the algebra $\mathcal{K}$ of compact operators on $\ell^2(\mathbb{N})$. Then $\widehat{y} = e_0 \otimes w$, so the C^*-subalgebra of A_0 generated by $\widehat{y}$ can be identified with

$e_0 \otimes C(\mathbb{T})$. From this, one sees (using the technique of Lemma 1.1 in [11]) that the C^*-subalgebra J of A_0 generated by $1-\widehat{h}$, $\widehat{x}$, $\widehat{y}$ and $\widehat{z}$ is an ideal, isomorphic to $\mathcal{K} \otimes C(\mathbb{T})$; and that the quotient algebra (generated by the image under the quotient mapping of the unilateral shift $\widehat{a}$ of multiplicity two) is isomorphic to $C(\mathbb{T})$.

Thus $C(\mathrm{SO}(3)_q)$ is an extension of $\mathcal{K} \otimes C(\mathbb{T})$ by $C(\mathbb{T})$, as is $C(\mathrm{SU}(2)_q)$ (see [18]). In order to distinguish between these two extensions, we look first at the simpler case of extensions of $\mathcal{K}$ by $C(\mathbb{T})$. Any such extension is determined by its Busby invariant τ, which is a homomorphism from $C(\mathbb{T})$ into the Calkin algebra $\mathcal{Q}$ (see Chapter 15 of [2]).

For the extension

$$0 \longrightarrow \mathcal{K} \longrightarrow E \xrightarrow{\varphi} C(\mathbb{T}) \longrightarrow 0,$$

write $E = E_n$ if $\operatorname{index}(\tau(\zeta)) = n$, where ζ is the identity function in $C(\mathbb{T})$. We aim to show that the C^*-algebras E_n $(n \in \mathbb{N})$ are pairwise nonisomorphic. Note that E_n is generated by $\mathcal{K}$ and an element z such that $\varphi(z) = \zeta$; and that, for $n > 0$, E_n has a faithful representation ρ on $\ell^2(\mathbb{N})$ given by

$$\rho = \text{identity on } \mathcal{K}, \quad \rho(z) = s^n, \tag{3.2}$$

where s is the unilateral shift. If $t \in E_n$ and $\varphi(t)$ is invertible then $\operatorname{index}(\tau(\varphi(t)))$ is just the Fredholm index of $\rho(t)$.

We shall say that a unital C^*-algebra has property $\operatorname{Proj}(m)$ if it contains an isometry u such that $1 - uu^*$ is the sum of m (necessarily pairwise orthogonal) minimal projections. If v is an isometry in E_n (with $n > 0$) then $\varphi(v)$ is unitary in $C(\mathbb{T})$, and hence is a $\mathbb{T}$-valued function on $\mathbb{T}$, with winding number r say; and $\varphi(v)$ is connected by a path of unitaries to the function ζ^r. Thus $\operatorname{index}(\tau(\varphi(v))) = \operatorname{index}(\tau(\zeta^r))$. So the Fredholm index of $\rho(v)$ is the same as that of s^{rn}, namely rn. Therefore $\rho(1 - vv^*)$ has rank rn and (since $\mathcal{K} \subseteq \rho(E_n)$) $1 - vv^*$ is the sum of rn minimal projections in E_n.

Conversely, if, given $r \in \mathbb{N}$, we choose $v = z^r \in E_n$ then $\rho(1 - vv^*)$ has rank rn. It follows that, for $n > 0$, E_n has property $\operatorname{Proj}(m)$ precisely when m is a multiple of n. Finally, if $n = 0$ then E_0 has no nonunitary isometries and so does not have property $\operatorname{Proj}(m)$ for any $m \neq 0$. Thus all the algebras E_n are nonisomorphic. (We could also construct extensions E_n with negative index n. A faithful representation of such an extension would be given by (3.2) with s^n replaced by $(s^*)^{-n}$. This makes it clear that $E_n \cong E_{-n}$, so there is in fact no need for us to consider the case of negative index. This applies also to the extensions F_n considered below.)

Next, we claim that a very similar analysis applies to extensions of the form

$$0 \longrightarrow \mathcal{K} \otimes C(\mathbb{T}) \longrightarrow F \xrightarrow{\varphi} C(\mathbb{T}) \longrightarrow 0.$$

In this case, the Busby invariant τ maps $C(\mathbb{T})$ into the outer multiplier algebra of $\mathcal{K} \otimes C(\mathbb{T})$. But this is just $\mathcal{Q}$ again, so the extension is still specified up to strong equivalence ([2], Chapter 15) by the nonnegative integer index$(\tau(\zeta))$. We write $F = F_n$ if this index is n. If t is an isometry in F_n then $p = \rho(1 - tt^*)$ is a projection in $\mathcal{K} \otimes C(\mathbb{T})$. Using the natural isomorphism between $\mathcal{K} \otimes C(\mathbb{T})$ and $C(\mathbb{T}, \mathcal{K})$, the algebra of continuous functions from $\mathbb{T}$ to $\mathcal{K}$, we can identify p with a projection-valued function $\lambda \mapsto p(\lambda)$ from $\mathbb{T}$ to $\mathcal{K}$. The continuity of this function implies that rank$(p(\lambda))$ is constant on $\mathbb{T}$, and it is this constant value, which is equal to index$(\tau(\varphi(t)))$, that we call the rank of p. With this interpretation of rank, the minimal projections in $\mathcal{K} \otimes C(\mathbb{T})$ are just those of rank one. For $n > 0$, F_n has a faithful representation on $\ell^2(\mathbb{N}) \otimes L^2(\mathbb{T})$ given by

$$\rho = \text{identity on } \mathcal{K} \otimes C(\mathbb{T}), \quad \rho(z) = s^n \otimes 1$$

(where $C(\mathbb{T})$ acts by multiplication on $L^2(\mathbb{T})$). The argument of the previous two paragraphs now goes through just as before to show that the C^*-algebras F_n $(n \geqslant 0)$ are all nonisomorphic.

It is evident from the representation of $C(\mathrm{SU}(2)_q)$ constructed in [17] that this algebra is isomorphic to F_1. From the third paragraph of this section we see that $C(\mathrm{SO}(3)_q) \cong F_2$. Therefore the C^*-algebras $C(\mathrm{SU}(2)_q)$ and $C(\mathrm{SO}(3)_q)$ are nonisomorphic.

4. Haar State and Irreducible Representations

So far, we have considered only the C^*-algebraic structure of $C(\mathrm{SO}(3)_q)$. But the interest of this algebra lies in the fact that it has a comultiplication (given by (1.2)) that makes it into a C^*-bialgebra. In this final Section, we shall consider the additional structure associated with the comultiplication. Our analysis will be based on the observation of Takeuchi ([13]) that there is an embedding $C\big(\mathrm{SO}(3)_q\big) \hookrightarrow C\big(\mathrm{SU}(2)_{q^{1/2}}\big)$. This can be described as follows.

For convenience, we write A_q, B_q for the C^*-algebras $C\big(\mathrm{SU}(2)_q\big)$ and $C\big(\mathrm{SO}(3)_q\big)$ respectively. Let α, γ be the generators of A_q, so that A_q by definition is the C^*-algebra with these generators together with relations that make

$$\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

unitary in $M_2(A_q)$. The embedding $\theta : B_q \hookrightarrow A_{q^{1/2}}$ is given on the generators of B_q by

$$\begin{gathered} \theta(a) = \alpha^2 \quad \theta(h) = 1 - (1+q)\gamma^*\gamma \\ \theta(x) = \mathrm{i}(1+q)^{\frac{1}{2}}\gamma\alpha \quad \theta(y) = \gamma^2 \quad \theta(z) = -\mathrm{i}(1+q)^{\frac{1}{2}}\alpha^*\gamma. \end{gathered} \tag{4.1}$$

The easiest way to see that this gives a $*$-homomorphism is to observe that if $\pi : A_{q^{1/2}} \to B\big(\ell^2(\mathbb{N}\times\mathbb{Z})\big)$ is the fundamental representation of $A_{q^{1/2}}$ ([17], Theorem 1.1), given by

$$\pi(\alpha)\varepsilon_{k,n} = (1-q^k)^{\frac{1}{2}}\varepsilon_{k-1,n} \quad \pi(\gamma)\varepsilon_{k,n} = q^{\frac{k}{2}}\varepsilon_{k,n-1},$$

then $\pi\theta$ is just the fundamental representation of B_q given in (3.1).

Let W be the unitary

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{i} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in $M_3(\mathbb{C})$. With $U \in \mathcal{L}(\mathbb{C}^3 \otimes B_q)$ as in (1.4), $W\big((1\otimes\theta)U\big)W^*$ is just the three-dimensional irreducible unitary representation of $\mathrm{SU}(2)_{q^{1/2}}$ as described in [6], [9], [14], [17], namely

$$\begin{pmatrix} \alpha^2 & -q^{\frac{1}{2}}(1+q)^{\frac{1}{2}}\gamma^*\alpha & q\gamma^{*2} \\ (1+q)^{\frac{1}{2}}\gamma\alpha & 1-(1+q)\gamma^*\gamma & -q^{\frac{1}{2}}(1+q)^{\frac{1}{2}}\alpha^*\gamma^* \\ \gamma^2 & (1+q)^{\frac{1}{2}}\alpha^*\gamma & \alpha^{*2} \end{pmatrix}.$$

[Note: by a representation of a quantum group we mean a corepresentation of its C^*-algebra.] The fact that this is a representation of $\mathrm{SU}(2)_{q^{1/2}}$ tells us at once that $\delta : B_q \to B_q \otimes B_q$, defined by $(1\otimes\delta)U = U_{12}U_{13}$ is a well-defined comultiplication on B_q.

The close connection between B_q and $A_{q^{1/2}}$ enables us to read off some properties of B_q without any further work. For example, the Haar state η of B_q is given by $\eta = (1-q)\sum\limits_{k\geqslant 0} q^k\omega_k$, where ω_k is the vector state induced by the vector $\varepsilon_{k,0}$ in the fundamental representation. Also, the irreducible representations of $\mathrm{SO}(3)_q$ can be quickly obtained from those of $\mathrm{SU}(2)_{q^{1/2}}$ as follows.

Denote by $\mathcal{A}$ the dense $*$-subalgebra of A_q generated algebraically by α and γ. By Theorem 1.2 of [17], $\mathcal{A}$ has a linear basis consisting of $A \cup A^*$, where

$$A = \{\alpha^l\gamma^m\gamma^{*n} : l,m,n \in \mathbb{N}\}$$

(and elements of $A \cap A^* = \{\gamma^m\gamma^{*n} : m,n \in \mathbb{N}\}$ are counted only once). Similarly, denote by $\mathcal{B}$ the linear span of $B \cup B^*$ in B_q, where B is as in (2.4). From (1.4),

$\theta(\mathcal{B})$ is spanned by the words of even length in $A \cup A^*$: for every such word can be split into "syllables" of length two (such as $\alpha\gamma^*$, for example), and each such syllable is the image under θ of one of the generators a, x, y, z, $|y|$, of B_q, or their adjoints.

A basic result in representation theory (Proposition 4.7 in [18]) states that the coefficients in the irreducible unitary representations of $\mathrm{SU}(2)_{q^{1/2}}$ form a basis for $\mathcal{A}$. The formulas for these coefficients given in [6], [9] and [14] show that they are linear combinations of words of even length in $A \cup A^*$ if the representation is odd-dimensional, and vice versa. Hence every odd-dimensional irreducible unitary representation of $\mathrm{SU}(2)_{q^{1/2}}$ is the image (under $1 \otimes \theta$) of an irreducible unitary representation of $\mathrm{SO}(3)_q$; and every such representation arises in this way (because the coefficients of these representations form a basis for $\mathcal{B}$). In this way, the analysis of the representations of $\mathrm{SU}(2)_q$ given in [6], [9] and [14], in which the coefficients are described in terms of little q-Jacobi polynomials, can be transferred directly to give a similar analysis for $\mathrm{SO}(3)_q$. As mentioned in the Introduction, this analysis works also in the case $q = 1$, and enables the representation theory of SO(3), which is usually considered to be much harder than that of SU(2), to be deduced directly from it.

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