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THE NEHARI PROBLEM FOR THE HARDY SPACE ON THE TORUS

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Abstract. We explicitly construct functions in $H^2(\mathbb{T}^2)^\perp$ which determine bounded (big) Hankel operators on $H^2(\mathbb{T}^2)$ but are not of the form $P_\perp \psi$ for any $\psi \in L^\infty(\mathbb{T}^2)$. We use this construction to show that the norm of a Hankel operator with bounded symbol is not, in general, comparable to the distance the symbol is from $H^\infty(\mathbb{T}^2)$. We also characterize the vector space quotient of symbols of bounded Hankel operators modulo those which lift to $L^\infty(\mathbb{T}^2)$ in terms of a Toeplitz completion problem on vector-valued Hardy space in one-variable.

Keywords: *Hankel operator, Toeplitz operator, Nehari theorem, restricted BMO.*

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1. INTRODUCTION

Let $L^2(\mathbb{T}^2)$ denote the space of Lebesgue measurable, square-integrable functions on the torus, $\mathbb{T}^2$. The Hardy space, $H^2(\mathbb{T}^2)$, is the closed subspace of $L^2(\mathbb{T}^2)$ consisting of those functions whose Fourier coefficients vanish off $\mathbb{Z}^+ \times \mathbb{Z}^+$. For $\phi \in H^2(\mathbb{T}^2)^\perp$, the *big* Hankel operator with symbol ϕ is densely defined by $\Gamma_\phi f = P_\perp(\phi f)$ where $f \in H^2(\mathbb{T}^2)$ is a polynomial and $P_\perp : L^2(\mathbb{T}^2) \to H^2(\mathbb{T}^2)^\perp$ is the orthogonal projection onto $H^2(\mathbb{T}^2)^\perp$. Note that $\Gamma_\phi = \Gamma_{P_\perp \phi}$ on polynomials and the correspondence between the operator Γ_ϕ and the function $P_\perp \phi$ is one-to-one. Let $\mathrm{Hank}(\mathbb{T}^2)$ denote the space of functions $\phi \in H^2(\mathbb{T}^2)^\perp$ for which Γ_ϕ extends to a bounded operator from $H^2(\mathbb{T}^2)$ into $L^2(\mathbb{T}^2)$ equipped with the operator norm, $\|\phi\|_{\mathrm{Hank}(\mathbb{T}^2)} = \|\Gamma_\phi\|$.

If $\psi \in L^\infty(\mathbb{T}^2)$ then $P_\perp\psi \in \mathrm{Hank}(\mathbb{T}^2)$ and $\|P_\perp\psi\|_{\mathrm{Hank}(\mathbb{T}^2)} \leqslant \|\psi\|_\infty$. It follows that $\|P_\perp\psi\|_{\mathrm{Hank}(\mathbb{T}^2)} \leqslant \|\psi + h\|_\infty$ for all $h \in H^\infty(\mathbb{T}^2)$ and thus $\|P_\perp\psi\|_{\mathrm{Hank}(\mathbb{T}^2)} \leqslant \mathrm{dist}_\infty(\psi, H^\infty(\mathbb{T}^2))$. Identifying $P_\perp(L^\infty(\mathbb{T}^2))$ with the quotient $L^\infty(\mathbb{T}^2)/H^\infty(\mathbb{T}^2)$, we have that $P_\perp(L^\infty(\mathbb{T}^2))$ is contained contractively in $\mathrm{Hank}(\mathbb{T}^2)$. This, of course, holds in the one variable case and Nehari's Theorem says that $P_\perp(L^\infty(\mathbb{T})) = \mathrm{Hank}(\mathbb{T})$ and the norms are the same. However in [6], Cotlar and Sadosky proved that the analogue of Nehari's Theorem does not hold for $\mathrm{Hank}(\mathbb{T}^2)$ by showing that $P_\perp(L^\infty(\mathbb{T}^2))$ was strictly smaller than $\mathrm{Hank}(\mathbb{T}^2)$. Using duality and a representation theorem for functions in $\mathrm{Hank}(\mathbb{T}^2)$, the authors proved only the existence of a function in $\mathrm{Hank}(\mathbb{T}^2)$ which does not lift to $L^\infty(\mathbb{T}^2)$.

In this note we exhibit such functions and characterize the vector space quotient $\mathrm{Hank}(\mathbb{T}^2)/P_\perp(L^\infty(\mathbb{T}^2))$ in terms of a certain Toeplitz completion problem. The paper is divided into two parts. In Section 2 we study the subspace of $\mathrm{Hank}(\mathbb{T}^2)$ consisting of those functions which lie in the second *quadrant* of $L^2(\mathbb{T}^2)$. This allows us to transfer the Nehari problem for this subclass to a Toeplitz completion problem on vector-valued Hardy space. The Toeplitz completion problem is then used to produce functions in this class which are not in $P_\perp(L^\infty(\mathbb{T}^2))$. The following result from Section 2 implies, in particular, that the function $\phi(\zeta_1,\zeta_2) = \log(1-\bar{\zeta}_1\zeta_2)$ is in $\mathrm{Hank}(\mathbb{T}^2)$ but $\phi \neq P_\perp\psi$ for any $\psi \in L^\infty(\mathbb{T}^2)$.

THEOREM. *Let $f \in H^2(\mathbb{T})$, $f(0) = 0$, and let $\phi(\zeta_1,\zeta_2) = f(\bar{\zeta}_1\zeta_2) \in H^2(\mathbb{T})^\perp \otimes H^2(\mathbb{T})$. Then $\phi \in \mathrm{Hank}(\mathbb{T}^2)$ if and only if $\bar{f} \in \mathrm{Hank}(\mathbb{T})$ and in this case, $\|\phi\|_{\mathrm{Hank}(\mathbb{T}^2)} = \|\bar{f}\|_{\mathrm{Hank}(\mathbb{T})}$. Furthermore, $\phi \in P_\perp(L^\infty(\mathbb{T}^2))$ if and only if $f \in H^\infty(\mathbb{T})$ and in this case $\frac{1}{2}\|f\|_\infty \leqslant \mathrm{dist}_\infty(\phi, H^\infty(\mathbb{T}^2)) \leqslant \|f\|_\infty$.*

It will follow easily from the theorem above that the quotient norm and the Hankel norm on the space $P_\perp(L^\infty(\mathbb{T}^2))$ are not equivalent, Corollary 2.4. This question was posed in [6]. In Section 3 we prove the following lifting theorem which says that as a vector space the quotient $\mathrm{Hank}(\mathbb{T}^2)/P_\perp(L^\infty(\mathbb{T}^2))$ can be identified with the Hankel operators whose symbols lie in the second quadrant of $L^2(\mathbb{T}^2)$ and are not of the form $P_\perp\psi$ for any $\psi \in L^\infty(\mathbb{T}^2)$. In other words, the Nehari problem for $H^2(\mathbb{T}^2)$ is equivalent to the Toeplitz completion problem discussed in Section 2.

THEOREM. *Let $\phi \in \mathrm{Hank}(\mathbb{T}^2)$. Then there exists $\phi_0 \in H^2(\mathbb{T})^\perp \otimes H^2(\mathbb{T})$ such that $\phi - \phi_0 \in P_\perp(L^\infty(\mathbb{T}^2))$.*

DEFINITIONS AND NOTATION. Recall that the space $L^2(\mathbb{T}^2)$ is isomorphic to $L^2(\mathbb{T}) \otimes L^2(\mathbb{T})$. If $e_n(z) = z^n$, $|z| = 1$, denotes the standard orthonormal basis for $L^2(\mathbb{T})$ then via the Fourier transform every function $f \in L^2(\mathbb{T}^2)$ corresponds to an element of the form

$$f \sim \sum_{m,n=-\infty}^{\infty} a_{mn} e_m \otimes e_n \quad \text{where} \quad \|f\|^2 = \sum_{m,n=-\infty}^{\infty} |a_{mn}|^2.$$

The first sum can be split into four direct summands according to the identification $\mathbb{Z} \times \mathbb{Z} = (\mathbb{Z}^+ \times \mathbb{Z}^+) \oplus (\mathbb{Z}^- \times \mathbb{Z}^+) \oplus (\mathbb{Z}^- \times \mathbb{Z}^-) \oplus (\mathbb{Z}^+ \times \mathbb{Z}^-)$. In this way, $L^2(\mathbb{T}^2)$ can be identified as a four quadrant space where the quadrants are labeled as follows.

$$\begin{array}{c|c}
 & \uparrow \zeta_2 \\
H^2(\mathbb{T})^{\perp} \otimes H^2(\mathbb{T}) & H^2(\mathbb{T}) \otimes H^2(\mathbb{T}) \\
\hline
H^2(\mathbb{T})^{\perp} \otimes H^2(\mathbb{T})^{\perp} & H^2(\mathbb{T}) \otimes H^2(\mathbb{T})^{\perp}
\end{array} \longrightarrow \zeta_1$$

In particular, $H^2(\mathbb{T}^2) \cong H^2(\mathbb{T}) \otimes H^2(\mathbb{T})$ and every function $f \in H^2(\mathbb{T}^2)$ corresponds to a sum of the form $\sum_{n=0}^{\infty} e_n \otimes f_n$ where $f_n \in H^2(\mathbb{T})$ and $\|f\|^2 = \sum_{n=0}^{\infty} \|f_n\|^2$. Similarly, every $\phi \in L^2(\mathbb{T}^2)$ which lies in the quadrant $H^2(\mathbb{T})^{\perp} \otimes H^2(\mathbb{T})$ can be identified with an element of the form $\sum_{n=1}^{\infty} e_{-n} \otimes a_n$ where $a_n \in H^2(\mathbb{T})$ and $\|\phi\|^2 = \sum_{n=1}^{\infty} \|a_n\|^2$.

For $\psi \in L^{\infty}(\mathbb{T})$ let T_{ψ} denote the Toeplitz operator on $H^2(\mathbb{T})$ defined by $T_{\psi} f = P_+(\psi f)$ where $P_+ : L^2(\mathbb{T}) \to H^2(\mathbb{T})$ is the orthogonal projection onto $H^2(\mathbb{T})$. If $a \in H^{\infty}(\mathbb{T})$ then T_a is just multiplication by a and referred to as the *analytic* Toeplitz operator with symbol a. The notation $\mathcal{P}(\mathbb{D}^2)$ denotes the space of polymonials in two variables equipped with the sup norm, $\|p\|_{\infty} = \sup_{|\zeta_1|=1,\, |\zeta_2|=1} \|p(\zeta_1, \zeta_2)\|$.

2. FUNCTIONS IN $\mathrm{Hank}(\mathbb{T}^2) \setminus P_\perp(L^\infty(\mathbb{T}^2))$

In this section we focus on the subclass of $\mathrm{Hank}(\mathbb{T}^2)$ consisting of those symbols in $H^2(\mathbb{T})^\perp \otimes H^2(\mathbb{T})$. Recall that the operator-valued version of Nehari's Theorem states that the Hankel matrix with operator entries, (A_{i+j+1}), is bounded on the infinite direct sum $\bigoplus_1^\infty H^2(\mathbb{T})$ if and only if there exist operators $X_n \in \mathcal{B}(\ell^2)$ such that

$$\sup_{|\zeta|=1} \Big\| \sum_{n=0}^\infty \zeta^n X_n + \sum_{n=1}^\infty \bar{\zeta}^n A_n \Big\|_{\mathcal{B}(\ell^2)} < \infty.$$

Furthermore, in this case, there exists (see [9]) a sequence X_n for which

$$\|(A_{i+j+1})\| = \sup_{|\zeta|=1} \Big\| \sum_{n=0}^\infty \zeta^n X_n + \sum_{n=1}^\infty \bar{\zeta}^n A_n \Big\|_{\mathcal{B}(\ell^2)}.$$

If we take the operators A_n to be analytic Toeplitz operators, say $A_n = T_{a_n}$, then there is no reason to assume that we can find a solution of the form $X_n = T_{b_n}$, where $b_n \in H^\infty(\mathbb{T})$. However, by using the projection onto the Toeplitz subspace $\{T_\psi \mid \psi \in L^\infty(\mathbb{T})\}$, see [1], we do have that

$$\|(T_{a_{i+j+1}})\| = \inf_{\{\psi_n\} \subseteq L^\infty(\mathbb{T})} \Big\| \sum_{n=0}^\infty \zeta^n T_{\psi_n} + \sum_{n=1}^\infty \bar{\zeta}^n T_{a_n} \Big\|_\infty.$$

The next result shows that we can find analytic functions b_n such that

$$\|(T_{a_{i+j+1}})\| = \Big\| \sum_{n=0}^\infty \zeta^n T_{b_n} + \sum_{n=1}^\infty \bar{\zeta}^n T_{a_n} \Big\|_\infty$$

if and only if $\phi \sim \sum_{n=1}^\infty e_{-n} \otimes a_n$ is in $P_\perp(L^\infty(\mathbb{T}^2))$.

PROPOSITION 2.1. *Let $\phi \sim \sum_{n=1}^\infty e_{-n} \otimes a_n$ be in $H^2(\mathbb{T})^\perp \otimes H^2(\mathbb{T})$. Then $\|\Gamma_\phi\| = \sup_{|\lambda|=1} \|(a_{i+j+1}(\lambda))\|_{\mathcal{B}(\ell^2)}$. The function ϕ is in $P_\perp(L^\infty(\mathbb{T}^2))$ if and only if there exist functions $b_n \in H^\infty(\mathbb{T})$ such that*

$$\sup_{|\lambda|=1} \left\| \begin{pmatrix} b_0(\lambda) & a_1(\lambda) & a_2(\lambda) & \dots \\ b_1(\lambda) & b_0(\lambda) & a_1(\lambda) & \dots \\ b_2(\lambda) & b_1(\lambda) & b_0(\lambda) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right\|_{\mathcal{B}(\ell^2)} < \infty.$$

Proof. Let $f \sim \sum_{n=0}^{\infty} e_n \otimes f_n$ be in $H^2(\mathbb{T}^2)$. Then

$$\Gamma_\phi f \sim \sum_{m=1}^{\infty} e_{-m} \otimes \sum_{n=0}^{\infty} a_{m+n} f_n. \tag{2.1}$$

Hence $\|\Gamma_\phi f\|^2 = \sum_{k=1}^{\infty} \Big\| \sum_{n=0}^{\infty} T_{a_{n+k}} f_n \Big\|^2$. It follows that $\|\Gamma_\phi\| = \|(T_{a_{i+j+1}})\|$. Identifying $\bigoplus_1^\infty H^2(\mathbb{T}^2)$ with the Hardy space of ℓ^2-valued functions on $\mathbb{T}$, the operator $(T_{a_{i+j+1}})$ corresponds to the analytic multiplier $z \mapsto (a_{i+j+1}(z))$ and thus $\|(T_{a_{i+j+1}})\| = \sup_{|\lambda|=1} \|(a_{i+j+1}(\lambda))\|_{\mathcal{B}(\ell^2)}$. The function ϕ is in $P_\perp(L^\infty(\mathbb{T}^2))$ if and only if there exists a function $b \in H^2(\mathbb{T}^2)$ such that $\phi + b \in L^\infty(\mathbb{T}^2)$. If $b \sim \sum_{n=0}^{\infty} e_n \otimes b_n$ then

$$\|\phi+b\|_\infty = \sup_{|\lambda|=1} \|\phi(\,\cdot\,,\lambda)+b(\,\cdot\,,\lambda)\|_\infty = \sup_{|\lambda|=1} \left\| \begin{pmatrix} b_0(\lambda) & a_1(\lambda) & a_2(\lambda) & \dots \\ b_1(\lambda) & b_0(\lambda) & a_1(\lambda) & \dots \\ b_2(\lambda) & b_1(\lambda) & b_0(\lambda) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right\|_{\mathcal{B}(\ell^2)}.$$

The proof is now complete. ∎

Now if $a_n, b_n \in H^\infty(\mathbb{T}^2)$ then

$$\sup_{|\lambda|=1} \left\| \begin{pmatrix} b_0(\lambda) & a_1(\lambda) & a_2(\lambda) & \dots \\ b_1(\lambda) & b_0(\lambda) & a_1(\lambda) & \dots \\ b_2(\lambda) & b_1(\lambda) & b_0(\lambda) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right\|_{\mathcal{B}(\ell^2)} = \left\| \begin{pmatrix} T_{b_0} & T_{a_1} & T_{a_2} & \dots \\ T_{b_1} & T_{b_0} & T_{a_1} & \dots \\ T_{b_2} & T_{b_1} & T_{b_0} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right\|.$$

Identifying the Toeplitz operator matrix on the right with the operator-valued function defined on the circle by $\Phi(\zeta) = \sum_{n=0}^{\infty} \zeta^n T_{b_n} + \sum_{n=1}^{\infty} \bar{\zeta}^n T_{a_n}$, it follows that

$$\left\| \begin{pmatrix} T_{b_0} & T_{a_1} & T_{a_2} & \dots \\ T_{b_1} & T_{b_0} & T_{a_1} & \dots \\ T_{b_2} & T_{b_1} & T_{b_0} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right\| = \sup_{|\zeta|=1} \Big\| \sum_{n=0}^{\infty} \zeta^n T_{b_n} + \sum_{n=1}^{\infty} \bar{\zeta}^n T_{a_n} \Big\|.$$

Thus, by Proposition 2.1, a function $\phi \sim \sum_{n=1}^{\infty} e_{-n} \otimes a_n \in \mathrm{Hank}(\mathbb{T}^2)$ lifts to $L^\infty(\mathbb{T}^2)$ if and only if there is an analytic solution to the Toeplitz completion problem for the operators $A_n = T_{a_n}$.

Therefore, to find a function $\phi \in \mathrm{Hank}(\mathbb{T}^2)$ which lies in the second quadrant and which is not in $P_\perp(L^\infty(\mathbb{T}^2))$ we need to construct a sequence $a_n \in H^\infty(\mathbb{T})$ such that

$$\sup_{|\lambda|=1} \left\| \begin{pmatrix} a_1(\lambda) & a_2(\lambda) & a_3(\lambda) & \dots \\ a_2(\lambda) & a_3(\lambda) & a_4(\lambda) & \dots \\ a_3(\lambda) & a_4(\lambda) & a_5(\lambda) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right\|_{\mathcal{B}(\ell^2)} < \infty$$

but

$$\sup_{|\lambda|=1} \left\| \begin{pmatrix} b_0(\lambda) & a_1(\lambda) & a_2(\lambda) & \dots \\ b_1(\lambda) & b_0(\lambda) & a_1(\lambda) & \dots \\ b_2(\lambda) & b_1(\lambda) & b_0(\lambda) & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right\|_{\mathcal{B}(\ell^2)} = \infty$$

for all sequences $\{b_n\} \subset H^\infty(\mathbb{T})$. The following lemma will be used in the proof of Theorem 2.3.

LEMMA 2.2. (i) *If $\{\alpha_n\} \in \ell^2$ then for all $|\lambda| = 1$, $\|(\alpha_{i+j}\lambda^{i+j})\|_{\mathcal{B}(\ell^2)} = \|(\alpha_{i+j})\|_{\mathcal{B}(\ell^2)}$.*

(ii) *If $\{\alpha_n\} \in \ell^2(\mathbb{Z})$ then for all $|\lambda| = 1$,*

$$\left\| \begin{pmatrix} \alpha_0 & \alpha_{-1}\bar{\lambda} & \alpha_{-2}\bar{\lambda}^2 & \dots \\ \alpha_1\lambda & \alpha_0 & \alpha_{-1}\bar{\lambda} & \dots \\ \alpha_2\lambda^2 & \alpha_1\lambda & \alpha_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right\|_{\mathcal{B}(\ell^2)} = \left\| \begin{pmatrix} \alpha_0 & \alpha_{-1} & \alpha_{-2} & \dots \\ \alpha_1 & \alpha_0 & \alpha_{-1} & \dots \\ \alpha_2 & \alpha_1 & \alpha_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right\|_{\mathcal{B}(\ell^2)}.$$

Proof. To prove (i) let $H_\lambda = (\alpha_{i+j}\lambda^{i+j})$, $|\lambda| = 1$. Then H_λ is the Schur product of the Hankel matrices (λ^{i+j}) and $H_1 = (\alpha_{i+j})$. Since $|\alpha| = 1$, it is easy to see that the Schur multiplier (λ^{i+j}) acts isometrically on $\mathcal{B}(\ell^2)$. Hence $\|H_\lambda\| = \|H_1\|$.

To prove (ii) let $\psi \sim \sum_{n=-\infty}^{\infty} \alpha_n e_n \in L^2(\mathbb{T})$. For $|\lambda| = 1$ let $\psi_\lambda(\zeta) = \psi(\lambda\zeta)$. Then

$$\|\psi_\lambda\|_\infty = \sup_{|\zeta|=1} |\psi(\lambda\zeta)| = \|\psi\|_\infty \quad \text{for all } |\lambda| = 1.$$

Since $\psi_\lambda \sim \sum_{n=-\infty}^{\infty} \alpha_n \lambda^n e_n$, (ii) now follows. ∎

THEOREM 2.3. *Let $f \in H^2(\mathbb{T})$, $f(0) = 0$, and let $\phi(\zeta_1, \zeta_2) = f(\bar{\zeta}_1\zeta_2) \in H^2(\mathbb{T})^\perp \otimes H^2(\mathbb{T})$. Then $\phi \in \mathrm{Hank}(\mathbb{T}^2)$ if and only if $\bar{f} \in \mathrm{Hank}(\mathbb{T})$ and in this case, $\|\phi\|_{\mathrm{Hank}(\mathbb{T}^2)} = \|\bar{f}\|_{\mathrm{Hank}(\mathbb{T})}$. Furthermore, $\phi \in P_\perp(L^\infty(\mathbb{T}^2))$ if and only if $f \in H^\infty(\mathbb{T})$ and in this case $\frac{1}{2}\|f\|_\infty \leqslant \mathrm{dist}_\infty(\phi, H^\infty(\mathbb{T}^2)) \leqslant \|f\|_\infty$.*

Proof. Let $f \sim \sum_{n=1}^{\infty} \alpha_n e_n$. Then $\phi \sim \sum_{n=1}^{\infty} e_{-n} \otimes a_n$ where $a_n(\lambda) = \alpha_n \lambda^n \in H^\infty(\mathbb{T})$. By Proposition 2.1, $\|\Gamma_\phi\| = \sup_{|\lambda|=1} \|(\alpha_{i+j+1}\lambda^{i+j+1})\|$. Thus by Lemma 2.2 and Nehari's theorem, $\|\Gamma_\phi\| = \|(\alpha_{i+j+1})\| = \|(\overline{\alpha_{i+j+1}})\| = \|\bar{f}\|_{\mathrm{Hank}(\mathbb{T})}$. Note that if $f \in H^\infty(\mathbb{T})$ then $\phi \in L^\infty(\mathbb{T}^2)$ and $\|\phi\|_\infty = \|f\|_\infty$. Since $\phi = P_\perp \phi$ we have that

$$f \in H^\infty(\mathbb{T}) \Rightarrow \phi \in L^\infty(\mathbb{T}^2) \quad \text{and} \quad \mathrm{dist}_\infty(\phi, H^\infty(\mathbb{T}^2)) \leqslant \|f\|_\infty. \tag{2.2}$$

Now suppose that $\phi \in P_\perp(L^\infty(\mathbb{T}^2))$ and let $b \sim \sum_{n=0}^{\infty} e_n \otimes b_n$ be a function in $H^2(\mathbb{T}^2)$ such that $\phi + b \in L^\infty(\mathbb{T}^2)$. Then

$$\|\phi + b\|_\infty = \sup_{|\lambda|=1} \left\| \begin{pmatrix} b_0(\lambda) & \alpha_1\lambda & \alpha_2\lambda^2 & \cdots \\ b_1(\lambda) & b_0(\lambda) & \alpha_1\lambda & \cdots \\ b_2(\lambda) & b_1(\lambda) & b_0(\lambda) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right\|_{\mathcal{B}(\ell^2)} < \infty.$$

By Lemma 2.2,

$$\sup_{|\lambda|=1} \left\| \begin{pmatrix} b_0(\lambda) & \alpha_1\lambda & \alpha_2\lambda^2 & \cdots \\ b_1(\lambda) & b_0(\lambda) & \alpha_1\lambda & \cdots \\ b_2(\lambda) & b_1(\lambda) & b_0(\lambda) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right\|_{\mathcal{B}(\ell^2)} = \sup_{|\lambda|=1} \left\| \begin{pmatrix} b_0(\lambda) & \alpha_1 & \alpha_2 & \cdots \\ \lambda b_1(\lambda) & b_0(\lambda) & \alpha_1 & \cdots \\ \lambda^2 b_2(\lambda) & \lambda b_1(\lambda) & b_0(\lambda) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \right\|_{\mathcal{B}(\ell^2)}.$$

Now the function

$$B(z) = \begin{pmatrix} b_0(z) & \alpha_1 & \alpha_2 & \cdots \\ z b_1(z) & b_0(z) & \alpha_1 & \cdots \\ z^2 b_2(z) & z b_1(z) & b_0(z) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is an analytic $\mathcal{B}(\ell^2)$-valued function on the unit disc with constant term equal to

$$B(0) = \begin{pmatrix} b(0,0) & \alpha_1 & \alpha_2 & \cdots \\ 0 & b(0,0) & \alpha_1 & \cdots \\ 0 & 0 & b(0,0) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hence $\|B(0)\| = \|b(0,0) + f\|_\infty \leqslant \sup_{|\lambda|<1} \|B(\lambda)\| = \|\phi + b\|_\infty$. Thus $f \in H^\infty(\mathbb{T})$ and $\|\phi + b\|_\infty \geqslant \inf\{\|w + f\|_\infty \mid w \in \mathbb{C}\} = \mathrm{dist}_\infty(f, \mathbb{C})$. Since $f(0) = 0$, $\mathrm{dist}_\infty(f, \mathbb{C}) \geqslant \frac{1}{2}\|f\|_\infty$. The result now follows from (2.2) above. ∎

Corollary 2.4. *For any $\varepsilon > 0$ there exists $\phi \in L^\infty(\mathbb{T}^2)$ which lies in the quadrant $H^2(\mathbb{T})^\perp \otimes H^2(\mathbb{T})$ and such that $\|\Gamma_\phi\| < \varepsilon$ but $\operatorname{dist}_\infty(\phi, H^\infty(\mathbb{T}^2)) \geqslant \frac{1}{2}$.*

Proof. Let $f \in H^\infty(\mathbb{T})$, $f(0) = 0$, such that $\|f\|_\infty = 1$ but $\|\bar{f}\|_{\mathrm{Hank}(\mathbb{T})} < \varepsilon$. We know that such a function exists since the Hank($\mathbb{T}$) norm and the sup norm are not equivalent for bounded functions. Let $\phi(\zeta, \lambda) = f(\bar{\zeta}\lambda)$. Then $\phi \in L^\infty(\mathbb{T}^2)$ and by Theorem 2.3, $\|\Gamma_\phi\| = \|\bar{f}\|_{\mathrm{Hank}(\mathbb{T})} < \varepsilon$ and $\operatorname{dist}_\infty(\phi, H^\infty(\mathbb{T}^2)) \geqslant \frac{1}{2}\|f\|_\infty = \frac{1}{2}$. ∎

It follows by Corollary 2.4 that there does *not* exist a uniform constant $C > 0$ such that $\operatorname{dist}_\infty(\phi, H^\infty(\mathbb{T}^2)) \leqslant C\|\Gamma_\phi\|$ for every $\phi \in L^\infty(\mathbb{T}^2)$.

3. THE QUOTIENT SPACE $\mathrm{Hank}(\mathbb{T}^2)/P_\perp(L^\infty(\mathbb{T}^2))$

In [5], Cotlar and Sadosky characterized the symbols of bounded Hankel operators as those functions in the so called *restricted* BMO class and prove that the restricted BMO norm and the Hankel norm are equivalent. In this section we prove an alternative lifting result which turns out to be equivalent to the lifting *part* of Cotlar and Sadosky's result. In other words, a proof of Theorem 3.1 below can be based on the result in [5]. On the other hand, with the exception of the statement on norms, Theorem 3.1 can in turn be used to prove the lifting part of the result in [5]. In this sense, the two lifting results are equivalent.

Theorem 3.1. *Let $\phi \in \mathrm{Hank}(\mathbb{T}^2)$. Then there exists $\phi_0 \in H^2(\mathbb{T})^\perp \otimes H^2(\mathbb{T})$ such that $\phi - \phi_0 \in P_\perp(L^\infty(\mathbb{T}^2))$.*

The proof given here is algebraic and motivated by known results on first order Ext groups over the disk algebra. Several facts of independent interest will be used in the proof and presented first. Throughout the operators $U_i \in \mathcal{B}(L^2(\mathbb{T}^2))$ denote the bilateral shifts defined by $U_i\varphi(\zeta_1, \zeta_2) = \zeta_i\varphi(\zeta_1, \zeta_2)$, $i = 1, 2$. The space $H^2(\mathbb{T}^2)$ is invariant under both U_1 and U_2 and the restriction operators $S_i = U_i|H^2(\mathbb{T}^2)$, $i = 1, 2$, are the unilateral shifts on $H^2(\mathbb{T}^2)$. Note that Hankel operators are completely characterized by the intertwining relations $\Gamma_\psi S_i = P_\perp U_i \Gamma_\psi$, $i = 1, 2$. Furthermore, if $\psi \in L^\infty(\mathbb{T}^2)$ then the Toeplitz operator $T_\psi = P_+ M_\psi | H^2(\mathbb{T}^2)$ is bounded on $H^2(\mathbb{T}^2)$ and satisfies $P_+ U_i \Gamma_\psi = T_\psi S_i - S_i T_\psi$, $i = 1, 2$. The next lemma gives the converse of this and so we have an algebraic characterization of the functions in $P_\perp(L^\infty(\mathbb{T}^2))$.

LEMMA 3.2. *Let $\phi \in \mathrm{Hank}(\mathbb{T}^2)$. Then $\phi \in P_\perp(L^\infty(\mathbb{T}^2))$ if and only if there exists an operator $Y \in \mathcal{B}(H^2(\mathbb{T}^2))$ such that $P_+U_i\Gamma_\phi = S_iY - YS_i$, $i = 1, 2$.*

Proof. Suppose that $P_+U_i\Gamma_\phi = S_iY - YS_i$, $i = 1, 2$, for some operator $Y \in \mathcal{B}(H^2(\mathbb{T}^2))$. Define $A : H^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ by $Af = \Gamma_\phi f + Yf$. Then A satisfies $AS_i = U_iA$, $i = 1, 2$. Let $\psi = A(1) \in L^2(\mathbb{T}^2)$ where 1 denotes the constant function 1. Then for any polynomial $p \in \mathcal{P}(\mathbb{D}^2)$, $Ap = Ap(S_1, S_2)1 = p(U_1, U_2)A(1) = p\psi$. Since $\mathcal{P}(\mathbb{D}^2)$ is dense in $H^2(\mathbb{T}^2)$, $Af = \psi f$ for all $f \in H^2(\mathbb{T}^2)$. One way to see that $\psi \in L^\infty(\mathbb{T}^2)$ is by using the result in [3] that says that $L^2(\mathbb{T}^2)$ is injective as a module over the bidisc algebra. This implies that the operator $A = M_\varphi | H^2(\mathbb{T}^2)$ where $M_\varphi \in \mathcal{B}(L^2(\mathbb{T}^2))$ is multiplication by the function $\varphi \in L^\infty(\mathbb{T}^2)$. Hence $\psi = A(1) = M_\varphi(1) = \varphi$ and so $\psi \in L^\infty(\mathbb{T}^2)$. By definition, $\Gamma_\phi = P_\perp A$ and thus $\phi = \Gamma_\phi(1) = P_\perp A(1) = P_\perp \psi$. ∎

A proof of the following theorem appears in [7] and so will be ommitted here. This theorem cast in a homological context is essentially a one-variable lifting result. We are actually interested in the two-variable case in determining when a Hankel operator Γ_ϕ *lifts* to $L^2(\mathbb{T}^2)$. It is Theorem 3.3 that we will use to find a function ϕ_0 in the second quadrant of $L^2(\mathbb{T}^2)$ for which $\phi - \phi_0$ is in $P_\perp(L^\infty(\mathbb{T}^2))$.

THEOREM 3.3. *Let S and T be isometries on Hilbert spaces H and K, respectively. Let $X \in \mathcal{B}(K, H)$. If the operator $\begin{pmatrix} S & X \\ 0 & T \end{pmatrix}$ is polynomially bounded on $H \oplus K$ then there exists $Y \in \mathcal{B}(K, H)$ such that $X = SY - YT$.*

Let T_1 and T_2 be a pair of commuting isometries on a Hilbert space H and let $A \in \mathcal{B}(H)$ satisfying $AT_2 = T_2A$. If the operator $\begin{pmatrix} T_1 & A \\ 0 & T_1 \end{pmatrix}$ is polynomially bounded on $H \oplus H$ then, by Theorem 3.3, there exists an operator $B \in \mathcal{B}(H)$ such that $A = T_1B - BT_1$. One question is when can we find a solution $B \in \{T_2\}'$? The answer is not always and in fact this problem for the operators $T_i = S_i$ is essentially the Nehari problem for $H^2(\mathbb{T}^2)$. In fact, it will follow from the proof of Theorem 3.1 together with Theorem 2.3, that there exists an operator $A \in \{S_2\}'$ such that the operators $\begin{pmatrix} S_1 & A \\ 0 & S_1 \end{pmatrix}$ and $\begin{pmatrix} S_2 & 0 \\ 0 & S_2 \end{pmatrix}$ are *jointly* polynomially bounded but $A \neq S_1B - BS_1$ for any $B \in \{S_2\}'$.

The next lemma which is a slight refinement of a result found in [8] gives us a significant reduction in the two-variable problem.

LEMMA 3.4. *Let T_1 and T_2 be a pair of doubly commuting isometries on a Hilbert space H and let $A \in \mathcal{B}(H)$ which commutes with T_2. If the operator $\begin{pmatrix} T_1 & A \\ 0 & T_1 \end{pmatrix}$ is polynomially bounded on $H \oplus H$ then there exist operators $A_0, B \in \{T_2\}'$ such that*

(i) $A = T_1B - BT_1 + A_0$ *and*

(ii) $T_1^*A_0 = 0$.

Proof. If the operator $\begin{pmatrix} T_1 & A \\ 0 & T_1 \end{pmatrix}$ is polynomially bounded then the operators $A_N = \sum_{j=0}^{N-1} T_1^{N-1-j}AT_1^j$, $N \geqslant 1$ are uniformly bounded since

$$\begin{pmatrix} T_1 & A \\ 0 & T_1 \end{pmatrix}^N = \begin{pmatrix} T_1{}^N & A_N \\ 0 & T_1{}^N \end{pmatrix}$$

for all $N \geqslant 1$. As in the proof of Lemma 1 in [8], see also [4], we define an operator $B \in \mathcal{B}(H)$ weakly by the formula, $\langle Bh, k\rangle = \text{glim}\{\langle T_1^{*N} A_N h, k\rangle\}_{N=1}^{\infty}$ where glim is a fixed generalized Banach limit on ℓ^∞. A direct computation using the translation invariance of glim shows that $A = T_1B - BT_1 - (I - T_1T_1^*)T_1^*B$. Furthermore, since T_1 and T_2 commute and $T_1^*T_2 = T_2T_1^*$, we have that $BT_2 = T_2B$. The result now follows with $A_0 = -(I - T_1T_1^*)T_1^*B$. ∎

Proof. (Theorem 3.1) Let $\phi \in \text{Hank}(\mathbb{T}^2)$. By Lemma 3.2, we need to find a $\phi_0 \in H^2(\mathbb{T})^{\perp} \otimes H^2(\mathbb{T})$ such that Γ_{ϕ_0} is bounded and $P_+U_i\Gamma_{\phi-\phi_0} = S_iY - YS_i$ $i = 1, 2$ for some operator $Y \in \mathcal{B}(H^2(\mathbb{T}^2))$. Let $X_i = P_+U_i\Gamma_\phi \in \mathcal{B}(H^2(\mathbb{T}^2))$, $i = 1, 2$. We claim that the operators

$$R_i = \begin{pmatrix} S_i & X_i \\ 0 & S_i \end{pmatrix}, \quad i = 1, 2$$

commute and are jointly polynomially bounded. To see this let $A_\phi : H^2(\mathbb{T}^2) \oplus H^2(\mathbb{T}^2) \to L^2(\mathbb{T}^2)$ be the operator defined by $A_\phi(f, g) = f + \Gamma_\phi g$. Then

$$A_\phi R_i = U_i A_\phi, \quad i = 1, 2. \tag{3.1}$$

Since $U_1U_2 = U_2U_1$, $A_\phi R_1R_2 = A_\phi R_2R_1$. It follows by comparing the (1,2) corners that $S_1X_2 + X_1S_2 = S_2X_1 + X_2S_1$ which is the condition we need for R_1 and R_2 to commute.

To show that R_1 and R_2 are jointly polynomially bounded, note that for $p \in \mathcal{P}(\mathbb{D}^2)$,

$$p(R_1, R_2) = \begin{pmatrix} p(S_1, S_2) & \delta_\phi(p) \\ 0 & p(S_1, S_2) \end{pmatrix}$$

where $\delta_\phi : \mathcal{P}(\mathbb{D}^2) \to \mathcal{B}(H^2(\mathbb{T}^2))$ is a linear map (in fact, a derivation). By (3.1), $A_\phi p(R_1, R_2) = p(U_1, U_2)A_\phi$ for all polynomials $p \in \mathcal{P}(\mathbb{D}^2)$. It follows by comparing the (1,2) corners, that $\delta_\phi(p) = P_+ p(U_1, U_2)\Gamma_\phi$ and thus $\|\delta_\phi(p)\| \leqslant \|\Gamma_\phi\|\,\|p\|_\infty$ for all $p \in \mathcal{P}(\mathbb{D}^2)$. Hence, $\|p(R_1, R_2)\| \leqslant C\|p\|_\infty$ where $C = \sqrt{2}\max\{1, \|\Gamma_\phi\|\}$.

By Theorem 3.3, there exists an operator $Y \in \mathcal{B}(H^2(\mathbb{T}^2))$ such that $X_2 = S_2Y - YS_2$. Substituting this into the equation $S_1X_2 + X_1S_2 = S_2X_1 + X_2S_1$ yields $X_1 - (S_1Y - YS_1) \in \{S_2\}'$. Thus, $X_1 = S_1Y - YS_1 + A$ where A is a bounded operator which commutes with S_2. It follows that the operator $\begin{pmatrix} S_1 & A \\ 0 & S_1 \end{pmatrix}$ is polynomially bounded on $H^2(\mathbb{T}^2) \oplus H^2(\mathbb{T}^2)$. In fact,

$$\begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}\begin{pmatrix} S_1 & X_1 \\ 0 & S_1 \end{pmatrix} = \begin{pmatrix} S_1 & A \\ 0 & S_1 \end{pmatrix}\begin{pmatrix} I & Y \\ 0 & I \end{pmatrix}$$

and hence, the two operators are similar. By Lemma 3.4, there exists operators $A_0, B \in \{S_2\}'$ satisfying $A = S_1B - BS_1 + A_0$ and $S_1^*A_0 = 0$. Therefore,

$$X_1 = S_1(Y+B) - (Y+B)S_1 + A_0, \quad X_2 = S_2(Y+B) - (Y+B)S_2. \tag{3.2}$$

Now $S_1^*A_0 = 0$ and $A_0S_2 = S_2A_0$ implies that there exists functions $a_n \in H^\infty(\mathbb{T})$, $n \geqslant 1$ such that $A_0 f = e_0 \otimes \sum\limits_{n=0}^{\infty} a_{n+1} f_n$ for all $f \sim \sum\limits_{n=0}^{\infty} e_n \otimes f_n$ in $H^2(\mathbb{T}^2)$. In other words, A_0 can be identified with the operator

$$\begin{pmatrix} T_{a_1} & T_{a_2} & T_{a_3} & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

acting on $\bigoplus\limits_1^\infty H^2(\mathbb{T})$. By (3.2), the operator $\begin{pmatrix} S_1 & A_0 \\ 0 & S_1 \end{pmatrix}$ is polynomially bounded on $H^2(\mathbb{T}^2) \oplus H^2(\mathbb{T}^2)$ and thus by Theorem 3.3, $A_0 = S_1L - LS_1$ for some bounded operator L. It follows that $L = S_1^*LS_1$ so that L is unitarily equivalent to an operator on $\bigoplus\limits_1^\infty H^2(\mathbb{T})$ of the form

$$\begin{pmatrix} L_0 & T_{a_1} & T_{a_2} & \dots \\ L_1 & L_0 & T_{a_1} & \dots \\ L_2 & L_1 & L_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

By Page's theorem ([9]), the operator $(T_{a_{i+j+1}})$ is bounded on $\bigoplus\limits_1^\infty H^2(\mathbb{T})$. Now let $\phi_0 \sim \sum\limits_{n=1}^{\infty} e_{-n} \otimes a_n$. Then Γ_{ϕ_0} is bounded by Proposition 2.1 and since $P_+U_1\Gamma_{\phi_0} = A_0$ and $P_+U_2\Gamma_{\phi_0} = 0$, we have by (3.2), $P_+U_1\Gamma_{\phi-\phi_0} = S_1Y_0 - Y_0S_1$ and $P_+U_2\Gamma_{\phi-\phi_0} = S_2Y_0 - Y_0S_2$ where $Y_0 = Y + B$. ∎

By Theorem 3.1, the quotient vector space $\mathrm{Hank}(\mathbb{T}^2)/P_\perp(L^\infty(\mathbb{T}^2))$ can be identified with the space of all $\phi \in \mathrm{Hank}(\mathbb{T}^2)$ which are analytic in the second variable modulo those which lift to $L^\infty(\mathbb{T}^2)$. The fact that this quotient is nontrivial means that a certain map is not onto. More precisely, let $L^\infty(\mathbb{T})\otimes H^\infty(\mathbb{T})$ denote the subspace of $L^\infty(\mathbb{T}^2)$ consisting of those functions which are analytic in the second variable. The projection $P_\perp \otimes I$ maps $L^\infty(\mathbb{T})\otimes H^\infty(\mathbb{T})$ into the spatial tensor product $P_\perp(L^\infty(\mathbb{T}))\otimes_{\min} H^\infty(\mathbb{T})$ and has kernel equal to $H^\infty(\mathbb{T})\otimes H^\infty(\mathbb{T}) = H^\infty(\mathbb{T}^2)$. By Theorem 3.1 and Theorem 2.3, this map is not onto $P_\perp(L^\infty(\mathbb{T}))\otimes_{\min} H^\infty(\mathbb{T})$. To see this, note that every element in $P_\perp(L^\infty(\mathbb{T}))\otimes_{\min} H^\infty(\mathbb{T})$ can be identified with a function in $H^2(\mathbb{T})^\perp \otimes H^2(\mathbb{T})$ of the form $\phi(\zeta,\lambda) = \sum_{n=1}^{\infty} \bar{\zeta}^n a_n(\lambda)$, where $\|\phi\|_{\min} = \sup_{|\lambda|=1} \|\phi(\,\cdot\,,\lambda)\|_{\mathrm{Hank}(\mathbb{T})} = \sup_{|\lambda|=1} \|(a_{i+j+1}(\lambda))\|_{\mathcal{B}(\ell^2)}$. Hence, by Proposition 2.1, $\|\phi\|_{\min} = \|\Gamma_\phi\|$. Moreover, the function ϕ lifts to $L^\infty(\mathbb{T}^2)$ if and only if there exist functions $b_n \in H^\infty(\mathbb{T})$ such that the function $\psi(\zeta,\lambda) = \sum_{n=0}^{\infty} \zeta^n b_n(\lambda) + \sum_{n=1}^{\infty} \overline{\zeta^n} a_n(\lambda)$ is in $L^\infty(\mathbb{T})\otimes H^\infty(\mathbb{T})$. In other words, if and only if there is a $\psi \in L^\infty(\mathbb{T})\otimes H^\infty(\mathbb{T})$ such that $\phi = (P_\perp \otimes I)\psi$.

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Note added in proof. Recently and independently, M. Bakonyi and D. Timotin ([2]), gave an alternative proof of Corollary 2.4 by showing that the quotient norms of the functions $\sum_{n=1}^{N} \frac{1}{n}\bar{\zeta}^n\lambda^n$ grow like $\mathrm{Log}(N)$. The authors then deduced that the function $\phi(\zeta,\lambda) = \log(1-\bar{\zeta}\lambda)$ is in $\mathrm{Hank}(\mathbb{T}^2)$ but does not lift to $L^\infty(\mathbb{T}^2)$. It should be pointed out that both the $\mathrm{Log}(N)$ estimate and the non-lifting of ϕ follow from our results by applying Theorem 2.3 to the polynomials $\sum_{n=1}^{N} \frac{1}{n} z^n$, as well as, the function $f(z) = \log(1-z)$.

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