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NONCOMMUTATIVE H^p SPACES

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Abstract. Let $\mathcal{M}$ be a von Neumann algebra equipped with a finite, normalised, normal faithful trace τ and let H^∞ be a finite maximal subdiagonal subalgebra of $\mathcal{M}$. For $1 \leqslant p < \infty$ let H^p be the closure of H^∞ in the noncommutative Lebesgue space $L^p(\mathcal{M})$. Then H^p is shown to possess many of the properties of the classical Hardy space $H^p(\mathbb{T})$ of the circle, such as various factorisation results including a Riesz factorisation theorem, a Riesz-Bochner theorem on the existence and boundedness of harmonic conjugates, direct sum decompositions, and duality.

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1. INTRODUCTION

The classical Hardy spaces $H^p(\mathbb{D})$, $1 \leqslant p < \infty$, are Banach spaces of analytic functions on the unit disk which satisfy the growth condition

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{\mathrm{i}\theta})|^p \,\mathrm{d}\theta < \infty.$$

The Banach algebra H^∞ consists of the bounded analytic functions on the unit disk. By taking radial limits, $H^p(\mathbb{D})$ can be identified with $H^p(\mathbb{T})$, the space of functions on the unit circle which are in $L^p(\mathbb{T})$ with respect to Lebesgue measure and whose negative Fourier coefficients vanish. These spaces have played an important role in modern analysis and prediction theory. One of the central results in the functional analytic approach to Hardy spaces is Szegö's Theorem ([38]), which is a formula for the weighted $L^2(\mathbb{T})$ distance from 1 to the analytic polynomials which vanish at the origin. Kolmogorov ([22]) and Wiener ([39]) independently adapted Szegö's work to solve the prediction problem for certain stochastic processes. Beurling ([2]), unaware of the connection with Szegö's work, used Hardy space theory to characterize the invariant subspaces of the unilateral shift, i.e. multiplication by z acting on $H^2(\mathbb{D})$. Beurling's work was generalized to shifts of higher multiplicity by Halmos ([16]) and Lax ([25]).

In the fifties, the theory of Hardy spaces was generalized in two directions. Masani and Wiener ([28], [29]) extended Szegö's theorem to the theory of multivariate stochastic processes by studying matrix valued functions. Concurrently, Helson and Lowdenslager ([17]) introduced techniques from functional analysis to extend the theory to the setting of a compact group with ordered dual, thus laying the foundation for the theory of function algebras. This eventually led to the definition of a weak* Dirichlet algebra of functions by Srinivasan and Wang ([37]). In this setting Srinivasan and Wang were able to prove a version of Beurling's theorem, Szegö's theorem, and several other important results in the theory of function algebras. In seemingly unrelated work, Kadison and Singer ([20]) introduced the notion of a triangular operator algebra, thus inaugurating the systematic study of nonselfadjoint operator algebras.

In 1967 Arveson ([1]) introduced the concept of a subdiagonal algebra in order to unify the perspectives of [17], [20] and [28]. Roughly, a subdiagonal algebra $\mathcal{A}$ is a subalgebra of a von Neumann algebra $\mathcal{M}$ which has many of the structural properties of the Hardy space $H^\infty(\mathbb{T})$. In effect, subdiagonal algebras are the noncommutative analogue of weak* Dirichlet algebras. Subsequently, several authors studied the invariant subspaces of $\mathcal{A}$ acting on the noncommutative Lebesgue space $L^p(\mathcal{M})$ ([21], [26], [30], [33]). There has also been considerable investigation of analytic crossed products, which are a type of subdiagonal algebra introduced by McAsey, Muhly and Saito, including their invariant subspace structure ([30], [33]), maximality among weak* closed subalgebras of $\mathcal{M}$ ([30]), associated Toeplitz operators ([34], [35]) and Hankel operators ([18]). We shall study H^p, the closure of $\mathcal{A}$ in the noncommutative Lebesgue space $L^p(\mathcal{M})$, as an analogue of the classical Hardy space $H^p(\mathbb{T})$, and so obtain generalizations of several classical results including a Riesz factorization theorem for H^1, a Riesz-Bochner theorem on the existence and L^p boundedness of harmonic conjugates, a projection from L^p to H^p, and the duality of H^p and H^q.

2. PRELIMINARIES

Throughout $\mathcal{M}$ is a von Neumann algebra equipped with a finite, normalised, normal faithful trace τ. The set of orthogonal projections in $\mathcal{M}$ is denoted by $\mathcal{M}^{\mathrm{p}}$. A closed densely defined operator x is said to be affiliated to $\mathcal{M}$ if $u^*xu = x$ for every unitary operator u belonging to the commutant $\mathcal{M}'$ of $\mathcal{M}$. The set of all closed densely defined operators affiliated with $\mathcal{M}$ will be denoted $\widetilde{\mathcal{M}}$, this is a $*$-algebra of operators on the underlying Hilbert space where the sum and product operation is the closure of the ordinary sum and product. For $x \in \widetilde{\mathcal{M}}$ the generalised singular function

$$\mu_t(x) = \inf\{\|xp\| : p \in \mathcal{M}^{\mathrm{p}},\ \tau(1-p) \leqslant t\} \tag{2.1}$$

is finite valued and decreasing on $(0,1]$. For further information on the generalised singular function we refer the reader to [13]. For $1 \leqslant p \leqslant \infty$ we define the set $L^p(\mathcal{M})$ to be all those $x \in \widetilde{\mathcal{M}}$ for which $\mu_t(x) \in L^p(0,1]$, and set $\|x\|_p = \|\mu_t(x)\|_p$. In fact $\|x\|_p = \sqrt[p]{\tau(|x|^p)}$ for $1 \leqslant p < \infty$ and $\|x\|_\infty = \|x\|$ and so this formulation coincides with other traditional formulations of the L^p spaces, as seen in [36] and [32], for example.

It is shown in [6] and [9] that $L^p(\mathcal{M})$ is a Banach space under $\|\cdot\|_p$ satisfying hoped for properties. In particular, for $1 \leqslant p < \infty$, the dual of $L^p(\mathcal{M})$ is $L^q(\mathcal{M})$ (where $\frac{1}{p} + \frac{1}{q} = 1$) under the pairing $\langle x, y\rangle = \tau(xy)$. Furthermore, the ultraweak topology on $\mathcal{M}$ is just the weak* topology on $L^\infty(\mathcal{M})$ when it is regarded as the dual of $L^1(\mathcal{M})$.

One of the key ideas to emerge in the recent theory, initiated in [6], is that of submajorisation of operators. This is a generalisation of the notion of submajorisation of functions introduced by Hardy, Littlewood and Polya (see [23] for example). If f and g are measurable positive decreasing functions on $(0,\infty)$, then we say that g is submajorised by f and write $g \prec\prec f$ if

$$\int_0^\theta g(t)\,\mathrm{d}t \leqslant \int_0^\theta f(t)\,\mathrm{d}t$$

for all $\theta > 0$. Given y and x in $\widetilde{\mathcal{M}}$, we say that y is submajorised by x and write $y \prec\prec x$ if $\mu(y) \prec\prec \mu(x)$. The importance of this notion of submajorisation in $\widetilde{\mathcal{M}}$ lies in that it serves as a tool for establishing norm inequalities in a wide variety of settings. Firstly, it has been established that $\mu(x+y) \prec\prec \mu(x) + \mu(y)$ and $\mu(xy) \prec\prec \mu(x)\mu(y)$ for all $x, y \in \widetilde{\mathcal{M}}$. Secondly, a large class of rearrangement invariant (fully) symmetric Banach operator spaces have been identified. A normed space $E \subset \widetilde{\mathcal{M}}$ is rearrangement invariant fully symmetric if whenever $y \in \widetilde{\mathcal{M}}$ and $x \in E$ with $y \prec\prec x$ then $y \in E$ and $\|y\|_E \leqslant \|x\|_E$. The spaces $L^p(\mathcal{M})$, for $1 \leqslant p \leqslant \infty$, are such. For more details the reader is referred to [5].

For a comprehensive survey on the history and theory of non-commutative Banach function spaces, the reader may consult [5]. The general setting is that of a semifinite von Neumann algebra equipped with a faithful semifinite normal trace, in that case, the generalised singular function is defined on $(0, \infty)$.

The following lemma about submajorisation will provide the bridge from the classical to the noncommutative Riesz-Bochner Theorem.

LEMMA 2.1. *Suppose* $x_1, x_2, \ldots, x_n \in \widetilde{\mathcal{M}}$. *Then*

$$\mu(x_1 x_2 \cdots x_n) \prec\prec \mu(x_1)\mu(x_2) \cdots \mu(x_n).$$

Proof. The case $n = 2$ has become well known with different proofs appearing in [13], [3] and [31]. The proof of the general case follows from a simple inductive argument based on a Hardy-type inequality: if $0 \leqslant f, g, h$ are decreasing measurable functions on $(0, \infty)$, and $f \prec\prec g$, then $fh \prec\prec gh$. This Hardy inequality is easily deduced from [23], II (2.36). ∎

For $t \in \mathcal{M}, x \in L^2(\mathcal{M})$, let $L_t(x) = tx$ and $R_t(x) = xt$. Then $\mathcal{L} = \{L_t : t \in \mathcal{M}\}$ and $\mathcal{R} = \{R_t : t \in \mathcal{M}\}$ are von Neumann algebras on the Hilbert space $L^2(\mathcal{M})$ which are each other's commutants. Furthermore, the map $t \to L_t$ (respectively $t \to R_t$) is a normal, $*$-isomorphism (respectively $*$-anti-isomorphism) of $\mathcal{M}$ onto $\mathcal{L}$ (respectively $\mathcal{R}$), and the identity 1 is a cyclic and separating vector for $\mathcal{L}$ and $\mathcal{R}$. The map $x \to x^*$ on $\mathcal{M}$ extends to a conjugate linear isometry on $L^p(\mathcal{M})$ for $1 \leqslant p < \infty$.

$\mathcal{M}_1$ denotes the unit ball of $\mathcal{M}$. Given $S \subset \widetilde{\mathcal{M}}$, S_1 denotes the intersection of S with $\mathcal{M}_1$, S^+ denotes the positive members of S, and S^{sa} the self adjoint members of S.

3. EXPECTATIONS, DEFINITIONS, AND EXAMPLES

Given a von Neumann algebra $\mathcal{M}$ and a von Neumann subalgebra $\mathcal{N}$, an expectation $\Phi : \mathcal{M} \to \mathcal{N}$ is defined to be a positive linear map which preserves the identity and satisfies $\Phi(xy) = x\Phi(y)$ for all $x \in \mathcal{N}$ and $y \in \mathcal{M}$.

Note that since Φ is positive, it is hermitian i.e. $\Phi(x)^* = \Phi(x^*)$ for all $x \in \mathcal{M}$.

As a consequence, $\Phi(yx) = \Phi(y)x$ for all $x \in \mathcal{N}$ and $y \in \mathcal{M}$. For a comprehensive survey on expectations, see [1], Section 6. In the following proposition we summarize the properties of Φ that we will need in our work.

Proposition 3.1. *Suppose $\Phi : \mathcal{M} \to \mathcal{N}$ is an expectation.*

(i) *$\mathcal{N}$ is the set of fixed points of Φ;*

(ii) *$\Phi \circ \Phi = \Phi$;*

(iii) *$\Phi(x)^*\Phi(x) \leqslant \Phi(x^*x)$ for all $x \in \mathcal{M}$ (generalized Schwarz inequality);*

(iv) *$\|\Phi(x)\|_\infty \leqslant \|x\|_\infty$ for all $x \in \mathcal{M}$.*

Proof. (i) and (ii) are clear. (iii) follows by expanding $\big(x - \Phi(x)\big)^*\big(x - \Phi(x)\big) \geqslant 0$ and then applying Φ throughout, as seen in [1], Proposition 6.1.1. (iv) then follows from (iii) as follows:

$$\|\Phi(x)\|^2 = \|\Phi(x)^*\Phi(x)\| \leqslant \|\Phi(x^*x)\| \leqslant \big\|\Phi\big(\|x^*x\|\big)\big\| = \|x^*x\| = \|x\|^2;$$

or from [20], Lemma 8.2.2. ∎

We now introduce the noncommutative analogue of $H^\infty(\mathbb{T})$ (cf. [1], [30]).

Definition 3.2. Let $\mathcal{A}$ be a weak* closed unital subalgebra of $\mathcal{M}$, and let Φ be a faithful, normal expectation from $\mathcal{M}$ onto the diagonal von Neumann algebra $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$. Then $\mathcal{A}$ is a *finite, maximal subdiagonal subalgebra of $\mathcal{M}$ with respect to Φ* if:

(i) $\mathcal{A} + \mathcal{A}^*$ is weak* dense in $\mathcal{M}$;

(ii) $\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in \mathcal{A}$;

(iii) $\tau \circ \Phi = \tau$.

It has been typical in the recent literature to require in addition that $\mathcal{A}$ be maximal among those subalgebras satisfying (i) and (ii). In fact Arveson did not require in [1] that subdiagonal algebras be weak* closed, and he asked ([1], Remark 2.2.3) if the weak* closure was the largest subdiagonal algebra containing a given one. R. Exel ([12]) has shown that this is the case when the subdiagonal algebra is finite. As a consequence the requirement that $\mathcal{A}$ be maximal among those subalgebras satisfying (i) and (ii) is redundant.

We give four examples of finite maximal subdiagonal algebras in order to indicate the scope of the definition. Additional illuminating examples are in [1], [26] and [30].

Example 3.3. Let (X, μ) be a probability space. Let $\mathcal{A}$ be a subalgebra of $L^\infty(\mu)$ such that $1 \in \mathcal{A}$, $\mathcal{A} + \overline{\mathcal{A}}$ is weak* dense in $L^\infty(\mu)$, and $\int ab \,\mathrm{d}\mu = (\int a \,\mathrm{d}\mu)(\int b \,\mathrm{d}\mu)$ for all $a, b \in \mathcal{A}$. Let $\Phi(a) = (\int a \,\mathrm{d}\mu)1$ and $\tau(a) = (\int a \,\mathrm{d}\mu)$. Then $\mathcal{A}$ is a finite subdiagonal algebra in $\mathcal{M} = L^\infty(\mu)$, and $\mathcal{A}$ is maximal if it is weak* closed. In particular, $H^\infty(\mathbb{T})$ is a subdiagonal algebra. These examples are the weak* Dirichlet algebras of Srinivasan and Wang.

EXAMPLE 3.4. Let $\mathcal{M} = M_n(L^\infty(\mathbb{T}))$ be the algebra of $n \times n$ matrices with entries from $L^\infty(\mathbb{T})$ and $\mathcal{A} = M_n(H^\infty(\mathbb{T}))$ be the algebra of $n \times n$ matrices with entries from $H^\infty(\mathbb{T})$. For $x \in \mathcal{M}$ with entries $x_{i,j}$, define $\Phi(x)$ to be the matrix with entries $\int x_{i,j}\,\mathrm{d}m$ and $\tau(x) = \frac{1}{n}\sum_{i=1}^{n}\int x_{i,i}\,\mathrm{d}m$. Then $\mathcal{A}$ is a finite maximal subdiagonal algebra. These examples provide the setting for the work of Masani and Wiener.

EXAMPLE 3.5. Let G be a countable, discrete, ordered group. Let $\ell^2(G) = \left\{f : G \to \mathbb{C} : \sum_{g\in G}|f(g)|^2 < \infty\right\}$. For each $g \in G$, define $U_g : \ell^2(G) \to \ell^2(G)$ by $(U_g f)(h) = f(g^{-1}h), h \in G, f \in \ell^2(G)$. The map $g \to U_g$ is a unitary representation of G on $\ell^2(G)$. Let $\mathcal{M}$ be the von Neumann algebra generated by $\{U_g\}_{g\in G}$. Each $x \in \mathcal{M}$ has a matrix $(x_{k,h})$ relative to the standard basis for $\ell^2(G)$. Let $\mathcal{A} = \{x \in \mathcal{M} : x_{h,k} = 0 \text{ for } k < h\}$. Let $\tau(x) = x_{e,e}$, where e is the identity of G, and $\Phi(x) = \tau(x)1$. Then $\mathcal{A}$ is a finite maximal subdiagonal algebra. These examples encompass much of the work of Helson and Lowdenslager.

EXAMPLE 3.6. Let $\mathcal{M} = M_n(\mathbb{C})$ be the algebra of $n \times n$ matrices with complex entries equipped with the trace $\tau(x) = \frac{1}{n}\sum_{i=1}^{n} x_{i,i}$. Let $\mathcal{A}$ be the subalgebra of upper triangular matrices, then $\mathcal{D}$ is the diagonal matrices and Φ is the natural projection onto the diagonal. Then $\mathcal{A}$ is a finite maximal subdiagonal algebra.

EXAMPLE 3.7. As a generalisation of the above example, let $\mathcal{M}$ be a finite von Neumann algebra with trace τ and fix $\mathcal{P}$, a totally ordered set of projections in $\mathcal{M}$ which contains 0 and 1. Let $\mathcal{A} = \{a \in \mathcal{M} : ap = pap \text{ for all } p \in \mathcal{P}\}$. By a theorem of Dye ([11]) there is a trace preserving expectation Φ onto $\mathcal{D} = \{x \in \mathcal{M} : xp = px \text{ for all } p \in \mathcal{P}\}$. Then $\mathcal{A}$ is a finite maximal subdiagonal algebra called the nest subalgebra of $\mathcal{M}$ relative to the nest $\mathcal{P}$. These examples encompass some of the work of Kadison and Singer. But not all triangular algebras are subdiagonal algebras.

Having established the analogue of finite maximal subdiagonal algebras with the classical Hardy spaces, we will henceforth denote a given such algebra by H^∞. Also let $H_0^\infty = \{x \in H^\infty : \Phi(x) = 0\}$. From the multiplicative condition on Φ we get that H_0^∞ is an ideal in H^∞.

For $S \subset L^p(\mathcal{M})$, $1 \leqslant p < \infty$, let $[S]_p$ denote the closure of S in $L^p(\mathcal{M})$. Let $H^p = [H^\infty]_p$ and $H_0^p = [H_0^\infty]_p$. As shown in [30], $L^2(\mathcal{M})$ may be decomposed into the orthogonal direct sum $H^2 \oplus (H_0^2)^* = H_0^2 \oplus L^2(\mathcal{D}) \oplus (H_0^2)^*$. In [33], Section 3 it is shown that for $1 \leqslant p \leqslant \infty$,

$$H^p = H^1 \cap L^p(\mathcal{M}) = \{x \in L^p(\mathcal{M}) : \tau(xy) = 0 \text{ for all } y \in H_0^\infty\} \tag{3.1}$$

$$H_0^p = H_0^1 \cap L^p(\mathcal{M}) = \{x \in L^p(\mathcal{M}) : \tau(xy) = 0 \text{ for all } y \in H^\infty\}\,. \tag{3.2}$$

Proposition 3.8. *$\tau\big(\Phi(x)y\big) = \tau\big(x\Phi(y)\big)$ for all $x, y \in \mathcal{M}$.*

Proof.

$$\tau\big(\Phi(x)y\big) = \tau\big(\Phi(\Phi(x)y)\big) = \tau\big(\Phi(x)\Phi(y)\big) = \tau\big(\Phi(x\Phi(y))\big) = \tau\big(x\Phi(y)\big). \quad ∎$$

Proposition 3.9. *$\Phi(x) \prec\prec x$ for all $x \in L^1(\mathcal{M})$. Furthermore, Φ extends to a continuous linear operator on $L^p(\mathcal{M})$ for $1 \leqslant p \leqslant \infty$, with $\big\|\Phi(x)\big\|_p \leqslant \|x\|_p$ for all $x \in L^p(\mathcal{M})$.*

Proof. The case $p = \infty$ is Proposition 3.1 (iv). For the case $p = 1$, we use the same proposition and also Proposition 3.8 to calculate:

$$\big\|\Phi(x)\big\|_1 = \sup_{y \in \mathcal{M}_1} \big|\tau\big(\Phi(x)y\big)\big| = \sup_{y \in \mathcal{M}_1} \big|\tau\big(x\Phi(y)\big)\big| \leqslant \sup_{y \in \mathcal{M}_1} \big|\tau(xy)\big| = \|x\|_1.$$

As a consequence, Φ extends uniquely to a norm decreasing map from $L^1(\mathcal{M})$ to $L^1(\mathcal{D})$, which we will continue to denote by Φ. Hence Φ can be considered to be a continuous map from $\mathcal{M}$ to $\mathcal{D}$ and from $L^1(\mathcal{M})$ to $L^1(\mathcal{D})$; in the terminology of [23], Chapter 1, Φ is a continuous linear operator from the Banach couple $\big(L^1(\mathcal{M}), \mathcal{M}\big)$ to the Banach couple $\big(L^1(\mathcal{D}), \mathcal{D}\big)$, and Φ has norm 1. It now follows from [9], Proposition 4.1, that $\Phi(x) \prec\prec x$ for all $x \in L^1(\mathcal{M})$. It follows, via the fact that $L^p(0, \infty)$ is a rearrangement invariant fully symmetric Banach space, that Φ extends to a continuous linear operator on $L^p(\mathcal{M})$ for $1 \leqslant p \leqslant \infty$, with $\big\|\Phi(x)\big\|_p \leqslant \|x\|_p$ for all $x \in L^p(\mathcal{M})$. ∎

Proposition 3.10. *The adjoint of the map $\Phi : L^p(\mathcal{M}) \to L^p(\mathcal{M})$ ($1 \leqslant p \leqslant \infty$) is the map $\Phi : L^q(\mathcal{M}) \to L^q(\mathcal{M})$.*

Proof. This is an immediate consequence of the fact that $\tau\big(\Phi(x)y\big) = \tau\big(x\Phi(y)\big)$ for all $x, y \in \mathcal{M}$ and so by continuity (of Φ and of τ) for all $x \in L^p(\mathcal{M}), y \in L^q(\mathcal{M})$. ∎

4. FACTORISATION THEOREMS

LEMMA 4.1. *Suppose* $1 \leqslant r,\ p, q \leqslant \infty$ *with* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. *If* $x \in H^p$ *and* $y \in H^q$ *then* $xy \in H^r$ *and* $\|xy\|_r \leqslant \|x\|_p\|y\|_q$. *If in addition* $x \in H_0^p$ *or* $y \in H_0^q$, *then* $xy \in H_0^r$.

Proof. First suppose $x \in H^p$ and $y \in H^q$. That $xy \in L^r(\mathcal{M})$, and the norm estimate, is a consequence of the Hölder inequality, [13], Theorem 4.2. It suffices to show that $xy \in H^r$, and for this it suffices by (3.1) to show that $\tau(xyz) = 0$ for all $z \in H_0^\infty$. Suppose then $z \in H_0^\infty$.

Since $x \in H^p$, we can choose a sequence $(x_n) \subset H^\infty$ such that $x_n \xrightarrow{\|\cdot\|_p} x$. Via the Hölder inequality we get that $x_n y \xrightarrow{\|\cdot\|_r} xy$, and hence $x_n y \xrightarrow{\|\cdot\|_1} xy$. Thus $x_n yz \xrightarrow{\|\cdot\|_1} xyz$; and $\tau(xyz) = \lim\limits_n \tau(x_n yz) = \lim\limits_n \tau(yzx_n)$.

For all $n \in \mathbb{N}$ we have $zx_n \in H_0^\infty$ since H_0^∞ is an ideal in H^∞. Hence $\tau(yzx_n) = 0$ for all $n \in \mathbb{N}$ by (3.1). Thus $\tau(xyz) = 0$, and $xy \in H^r$.

A similar argument shows that if $x \in H_0^p$ or $y \in H_0^q$ then $xy \in H_0^r$. ∎

THEOREM 4.2. [Left and right factorisations] *Suppose* $1 \leqslant p \leqslant \infty$. *For all* $\varepsilon > 0$ *and* $z \in L^p(\mathcal{M})$ *there exist* $h_i \in H^p$ *and* $v_i \in \mathcal{M} (i = 1, 2)$ *such that*

(i) $z = h_1 v_1 = v_2 h_2$;
(ii) $\|v_i\|_\infty \leqslant 1$;
(iii) $\|h_i\|_p < (1 + \varepsilon)\|z\|_p$;
(iv) h_i *is invertible, and* $h_i^{-1} \in H^\infty$.

Proof. Suppose $\varepsilon > 0$ and $z \in L^p(\mathcal{M})$ are given. Choose $0 < \delta < \sqrt{\varepsilon}\|z\|_p$.

First suppose $2 \leqslant p \leqslant \infty$. Since τ is finite we have $z \in L^2(\mathcal{M})$. We now consider the positive weak* continuous linear form $\omega_{\delta I} + \omega_z$ on the von Neumann algebra $\mathcal{R}$. Now $I \in L^2(\mathcal{M})$ is a separating vector for $\mathcal{R}$, so by [19], Theorem 7.2.3 there exists $y \in L^2(\mathcal{M})$ such that $\omega_y = \omega_{\delta I} + \omega_z$ on $\mathcal{R}$.

Thus $\tau(y^*yx) = \tau(\delta^2 x) + \tau(z^*zx)$ for all $x \in \mathcal{M}$, or $\tau\big((y^*y - \delta^2 I - z^*z)x\big) = 0$ for all $x \in \mathcal{M}$. Hence $y^*y - \delta^2 I - z^*z = 0$ from the tracial duality, that is, $y^*y = \delta^2 I + z^*z$.

Now $z \in L^p(\mathcal{M})$ implies that $z^*z \in L^{p/2}(\mathcal{M})$. Hence $y^*y = \delta^2 I + z^*z \in L^{p/2}(\mathcal{M})$, and so $y \in L^p(\mathcal{M})$. Furthermore $\|y\|_p^2 = \|y^*y\|_{p/2} = \|\delta^2 I + z^*z\|_{p/2} \leqslant \|\delta^2 I\|_{p/2} + \|z^*z\|_{p/2} = \delta^2 + \|z\|_p^2$.

Now since $\omega_y \geqslant \omega_{\delta I}$ we can find $r \in \mathcal{M}_1$ such that $\delta I = ry$. Likewise since $\omega_y \geqslant \omega_z$ we can find $s \in \mathcal{M}_1$ such that $z = sy$.

Since $\frac{r}{\delta}$ is right invertible in $\widetilde{\mathcal{M}}$, with right inverse y, it is in fact invertible (because of the finiteness of $\mathcal{M}$) and y is that inverse. Thus $\left(\frac{r}{\delta}\right)^{-1} \in L^2(\mathcal{M})$, and so by [33], Proposition 1 there are unitary operators $u_i \in \mathcal{M}$ and invertible operators $a_i \in H^\infty$ such that $\frac{r}{\delta} = u_1 a_1 = a_2 u_2$ and $a_i^{-1} \in H^2$.

It follows that $I = u_1 a_1 y = a_2 u_2 y$, and so $y = a_1^{-1} u_1^* = u_2^* a_2^{-1}$. In particular a_i^{-1} has the same singular function as y, and so $a_i^{-1} \in L^p(\mathcal{M})$. By applying (3.1) we get that $a_i^{-1} \in H^p$.

Thus $z = sy = (su_2^*)a_2^{-1} := v_2 h_2$ where $v_2 \in \mathcal{M}_1, h_2 \in H^p$ and $h_2^{-1} \in H^\infty$ as required. Finally

$$\|h_2\|_p = \|y\|_p \leqslant \sqrt{\delta^2 + \|z\|_p^2} < \sqrt{1+\varepsilon}\|z\|_p < (1+\varepsilon)\|z\|_p.$$

Now suppose $1 \leqslant p < 2$. Consider the polar decomposition $z = v|z| = v|z|^{1/2}|z|^{1/2}$. Then $v|z|^{1/2}$ and $|z|^{1/2}$ both belong to $L^{2p}(\mathcal{M})$.

Since $2p \geqslant 2$ we have from what has already been proved that $|z|^{1/2} = w_2 g_2$ where $w_2 \in \mathcal{M}_1$, $g_2 \in H^{2p}$, $g_2^{-1} \in H^\infty$, and

$$\|g_2\|_{2p} < \sqrt{1+\varepsilon}\big\||z|^{1/2}\big\|_{2p} = \sqrt{1+\varepsilon}\sqrt{\|z\|_p}.$$

Now we have $z = v|z|^{1/2} w_2 g_2$. Again $v|z|^{1/2} w_2 \in L^{2p}(\mathcal{M})$, and so again $v|z|^{1/2} w_2 = w_1 g_1$ where $w_1 \in \mathcal{M}_1, g_1 \in H^{2p}, g_1^{-1} \in H^\infty$ and

$$\|g_1\|_{2p} < \sqrt{1+\varepsilon}\big\|v|z|^{1/2} w_2\big\|_{2p} \leqslant \sqrt{1+\varepsilon}\big\||z|^{1/2}\big\|_{2p} = \sqrt{1+\varepsilon}\sqrt{\|z\|_p}.$$

Thus $z = w_1(g_1 g_2) := v_2 h_2$. Since $g_i \in H^{2p}$ we have from Lemma 4.1 that $h_2 \in H^p$. Furthermore, $h_2^{-1} = g_2^{-1} g_1^{-1} \in H^\infty$. Finally

$$\|h_2\|_p \leqslant \|g_1\|_{2p}\|g_2\|_{2p} < (1+\varepsilon)\|z\|_p.$$

Thus we have completed the proof for $1 \leqslant p \leqslant \infty$ of the one decomposition $z = v_2 h_2$. By considering the von Neumann algebra $\mathcal{L}$ instead of $\mathcal{R}$ one likewise derives the other decomposition $z = h_1 v_1$. ∎

Subsequently we shall refer to the factorisation $z = h_1 v_1$ of the above theorem as the left factorisation, and $z = v_2 h_2$ as the right factorisation.

THEOREM 4.3. [Riesz Factorisation Theorem] *Suppose $f \in H^1, 1 \leqslant p \leqslant \infty$, and $\varepsilon > 0$. Then there exist $g \in H^p$ and $h \in H^q$ such that $f = gh$ and $\|f\|_1 \leqslant \|g\|_p \|h\|_q < (1+\varepsilon)\|f\|_1$. If $f \in H_0^1$ then we can arrange that $g \in H_0^p$. If $f \in H_0^1$ and $1 < p \leqslant \infty$ then we can arrange that either $g \in H_0^p$ or $h \in H_0^q$.*

Proof. If $p = 1$ then of course we put $g = f$, $h = I$. So we suppose that $1 < p < \infty$. Since $f \in H^1 \subset L^1(\mathcal{M})$, we can easily write $f = xy$ where $x \in L^p(\mathcal{M})$, $y \in L^q(\mathcal{M})$ and $\|f\|_1 = \|x\|_p \|y\|_q$.

By the right factorisation of Theorem 4.2 we can write $y = vh$ where $v \in \mathcal{M}_1$, $h \in H^q$, $h^{-1} \in H^\infty$ and $\|h\|_q < (1+\varepsilon)\|y\|_q$. Then $f = xy = (xv)h := gh$. Certainly $g \in L^p(\mathcal{M})$ and $\|g\|_p \leqslant \|x\|_p$. Thus $\|f\|_1 \leqslant \|g\|_p\|h\|_q \leqslant \|x\|_p \cdot (1+\varepsilon)\|y\|_q = (1+\varepsilon)\|f\|_1$.

To complete the proof of the first statement it suffices to show that $g \in H^p$. Since $f \in H^1$ we have that $\tau(gha) = \tau(fa) = 0$ for all $a \in H_0^\infty$. However, by virtue of the fact that $h^{-1} \in H^\infty$, we get that $\tau(ga) = \tau\big(gh(h^{-1}a)\big) = 0$ for all $a \in H_0^\infty$. Here we use the fact that H_0^∞ is an ideal in H^∞. Hence $g \in H^p$.

Now suppose $f \in H_0^1$. Then as above $\tau(ga) = \tau(ghh^{-1}a) = \tau\big(f(h^{-1}a)\big) = 0$ for all $a \in H^\infty$. Thus $g \in H_0^p$.

If instead we implement the left factorisation (applied to x) in the argument above we will be able to deduce that if $f \in H_0^1$ then we can arrange that $h \in H_0^q$. The arguments parallel those already given, and so are omitted. ∎

5. THE CONJUGATION AND HERGLOTZ MAPS

Given an operator x, we let $\operatorname{Re} x = \frac{x+x^*}{2}$ and $\operatorname{Im} x = \frac{x-x^*}{2\mathrm{i}}$. Furthermore, given a set S of operators, $\operatorname{Re} S$ and $\operatorname{Im} S$ are defined in the obvious manner.

In this section we will concern ourselves with the question of recovering H^∞ from $\operatorname{Re} H^\infty$. The first author has shown in [27], Section 4, that if $x \in \operatorname{Re} H^\infty$ then there exists a uniquely determined $\widetilde{x} \in \operatorname{Re} H^\infty$ such that $x + \mathrm{i}\,\widetilde{x} \in H^\infty$ and $\Phi(\widetilde{x}) = 0$. The map $\sim : \operatorname{Re} H^\infty \to \operatorname{Re} H^\infty : x \to \widetilde{x}$ is called the conjugation map, while $h : \operatorname{Re} H^\infty \to H^\infty : x \to x + \mathrm{i}\,\widetilde{x}$ is called the Herglotz transform. Both $\sim$ and h are linear maps over the real scalar field.

The main aim of this section is an exact generalization of the Riesz theorem which guarantees that the conjugation map is bounded in the $L^p(\mathcal{M})$-norm for $1 < p < \infty$. The basic procedure is to prove the result for even numbers and then use interpolation for the interval $[2, \infty)$ and then adjoints for the interval $(1, 2)$. This technique is similar to that of Gohberg and Kreĭn for the classic Schatten-p-class case — see [15], Theorem III 6.2 and the interesting discussion there and is motivated by the 'Bootstrap' method due to Cotlar ([4]).

After the work on this paper was completed we learned that Narcisse Randrianantoanina has independently obtained this result using similar methods.

LEMMA 5.1. *Suppose* $x, y \in \operatorname{Re} H^\infty$.

(i) *If* $x \in \operatorname{Re} H_0^\infty$ *then* $x + \mathrm{i}\,\widetilde{x} \in H_0^\infty$;

(ii) $x - \Phi(x) \in \operatorname{Re} H_0^\infty$, *and* $\widetilde{x} = \big(x - \Phi(x)\big)^\sim$;

(iii) $\widetilde{\widetilde{x}} = \Phi(x) - x$;

(iv) $\tau(\widetilde{x}y) = -\tau(x\widetilde{y})$.

Proof. (i) Immediate from the fact that $\Phi(\operatorname{Re} a) = \operatorname{Re}(\Phi(a))$.

(ii) Clearly $x + \mathrm{i}\,\widetilde{x} - \Phi(x) \in H_0^\infty$, and therefore

$$x - \Phi(x) = \frac{\big[x + \mathrm{i}\,\widetilde{x} - \Phi(x)\big] + \big[x + \mathrm{i}\,\widetilde{x} - \Phi(x)\big]^*}{2} \in \operatorname{Re} H_0^\infty.$$

Thus $\widetilde{x} = \big(x - \Phi(x)\big)^\sim$ by uniqueness of conjugates.

(iii) Since $x + \mathrm{i}\,\widetilde{x} \in H^\infty$, multiplying by $-\mathrm{i}$ we get $\widetilde{x} - \mathrm{i}x \in H^\infty$. Therefore $\widetilde{x} + \mathrm{i}\big(\Phi(x) - x\big) \in H^\infty$, and certainly $\Phi\big(\Phi(x) - x\big) = 0$. Once again the desired result follows by means of the uniqueness of conjugates.

(iv) First note that if $a, b \in \mathcal{M}^+$ then

$$\tau(ab) = \tau(a^{\frac{1}{2}}a^{\frac{1}{2}}b) = \tau(a^{\frac{1}{2}}ba^{\frac{1}{2}}) \geqslant 0.$$

It is now immediate that if $a, b \in \mathcal{M}^{\mathrm{sa}}$ then $\tau(ab) \in \mathbf{R}$. Now

$$\begin{aligned}\tau(\Phi(x)\Phi(y)) &= \tau(\Phi(h(x))\Phi(h(y))) = \tau(\Phi(h(x)h(y))) = \tau(h(x)h(y)) \\ &= \tau((x + \mathrm{i}\,\widetilde{x})(y + \mathrm{i}\,\widetilde{y})) = \tau(xy) - \tau(\widetilde{x}\widetilde{y}) + \mathrm{i}[\tau(\widetilde{x}y) + \tau(x\widetilde{y})].\end{aligned}$$

We have that x, y, $\widetilde{x}$, $\widetilde{y}$, $\Phi(x)$, $\Phi(y) \in \mathcal{M}^{\mathrm{sa}}$. Via our initial remarks we deduce that $\tau(\widetilde{x}y) + \tau(x\widetilde{y}) = 0$. ∎

What follows is the larger part of the argument for a non-commutative generalization of the Riesz theorem. The argument which we are able to imitate closely is that of Bochner. This appears in [14], Chapter IV, Theorem 1.3, although we have repeated most of the details for the convenience of the reader.

The bridging of the gap between the commutative and non-commutative setting is provided by Lemma 2.1.

THEOREM 5.2. [Generalised Riesz theorem] *Suppose p is even. Then $\|\widetilde{x}\|_p \leqslant \frac{2p}{\ln 2}\|x\|_p$ for all $x \in \operatorname{Re} H^\infty$.*

Proof. For convenience we denote $\widetilde{x}$ by y.

Firstly we suppose that $x \in \operatorname{Re} H_0^\infty$. Then $x + \mathrm{i}\,y \in H_0^\infty$ and hence $(x + \mathrm{i}y)^p \in H_0^\infty$ since H_0^∞ is an ideal. Thus

$$0 = \tau\big((x + \mathrm{i}y)^p\big) = \sum_{k=0}^{p} \mathrm{i}^{p-k} \sum_{u \in \mathcal{Q}(k,p)} \tau(u)$$

where $\mathcal{Q}(k,p)$ is the set

$$\Big\{x^{\alpha_1}y^{\beta_1}x^{\alpha_2}y^{\beta_2}\cdots x^{\alpha_n}y^{\beta_n} : n \in \mathbb{N}, \alpha_i, \beta_i \geqslant 0, \sum \alpha_i = k, \sum \beta_i = p - k\Big\}.$$

Now $y^p = \mathcal{Q}(0,p)$ and thus

$$-\mathrm{i}^p \tau(y^p) = \sum_{k=1}^{p} \mathrm{i}^{p-k} \sum_{u\in\mathcal{Q}(k,p)} \tau(u)$$

and so

$$\|y\|_p^p = \tau(y^p) \leqslant \sum_{k=1}^{p} \sum_{u\in\mathcal{Q}(k,p)} \bigl|\tau(u)\bigr|.$$

If u is a typical member of $\mathcal{Q}(k,p)$, then via Lemma 2.1 we have that

$$\begin{aligned}\bigl|\tau(u)\bigr| \leqslant \|u\|_1 &= \int_0^1 \mu_t\bigl(x^{\alpha_1}y^{\beta_1}x^{\alpha_2}y^{\beta_2}\cdots x^{\alpha_n}y^{\beta_n}\bigr)\mathrm{d}t \\ &\leqslant \int_0^1 \mu_t(x)^{\alpha_1}\mu_t(y)^{\beta_1}\cdots\mu_t(x)^{\alpha_n}\mu_t(y)^{\beta_n}\,\mathrm{d}t = \int_0^1 \mu_t(x)^k\mu_t(y)^{p-k}\mathrm{d}t \\ &\leqslant \left[\int_0^1 \mu_t(x)^p\,\mathrm{d}t\right]^{\frac{k}{p}} \left[\int_0^1 \mu_t(y)^p\,\mathrm{d}t\right]^{\frac{p-k}{p}} = \|x\|_p^k\,\|y\|_p^{p-k}.\end{aligned}$$

Here we use Hölder's inequality, with the indices $\frac{p}{k}$ and $\frac{p}{p-k}$. Since $\mathcal{Q}(k,p)$ has $\binom{p}{k}$ elements, we thus have $\|y\|_p^p \leqslant \sum\limits_{k=1}^{p}\binom{p}{k}\,\|x\|_p^k\,\|y\|_p^{p-k}$. Let $\xi = \frac{\|y\|_p}{\|x\|_p}$. Then $\xi^p \leqslant \sum\limits_{k=1}^{p}\binom{p}{k}\xi^{p-k} = (1+\xi)^p - \xi^p$. Hence $2\xi^p \leqslant (1+\xi)^p$ and so $\xi \leqslant \frac{1}{2^{1/p}-1} \leqslant \frac{p}{\ln 2}$. Thus $\|\widetilde{x}\|_p \leqslant \frac{p}{\ln 2}\|x\|_p$.

Now suppose $x \in \operatorname{Re} H^\infty$. Then by making use of Lemma 5.1 (ii), and what has already been shown, we see that

$$\|\widetilde{x}\|_p \leqslant \frac{p}{\ln 2}\bigl\|x-\Phi(x)\bigr\|_p \leqslant \frac{p}{\ln 2}\bigl[\|x\|_p + \bigl\|\Phi(x)\bigr\|_p\bigr] \leqslant \frac{2p}{\ln 2}\|x\|_p$$

since $\bigl\|\Phi(x)\bigr\|_p \leqslant \|x\|_p$. ∎

It follows that for every even natural number p, $\sim$ extends to a real linear map from $[\operatorname{Re} H^\infty]_p$ to itself which has norm of order p. We now make a more convenient identification of $[\operatorname{Re} H^\infty]_p$.

PROPOSITION 5.3. $[\operatorname{Re} H^\infty]_p = [\mathcal{M}^{\mathrm{sa}}]_p = L^p(\mathcal{M})^{\mathrm{sa}}$ *for all* $1 \leqslant p < \infty$.

Proof. The second equality is apparent, we concern ourselves only with the first. Since $\operatorname{Re} H^\infty \subset \mathcal{M}^{\mathrm{sa}}$, we have $[\operatorname{Re} H^\infty]_p \subset [\mathcal{M}^{\mathrm{sa}}]_p$.

Conversely, suppose $x \in \mathcal{M}^{\mathrm{sa}}$. Since $H^\infty + H^{\infty^*}$ is weak* dense in $\mathcal{M}$, we can find nets $(x_\alpha), (y_\alpha) \subset H^\infty$ such that $x_\alpha + y_\alpha^* \xrightarrow{\mathrm{w}*} x$. Then $\operatorname{Re} H^\infty \supset z_\alpha := \frac{x_\alpha + x_\alpha^*}{2} + \frac{y_\alpha^* + y_\alpha}{2} \xrightarrow{\mathrm{w}*} x$. Here we make use of the weak* continuity of adjunction.

Recall that the weak* topology is just the $\sigma(L^\infty(\mathcal{M}), L^1(\mathcal{M}))$ topology. Since $L^q(\mathcal{M}) \subset L^1(\mathcal{M})$, we thus get that the convergence also takes place in the $\sigma(L^p(\mathcal{M}), L^q(\mathcal{M}))$ topology, so x belongs to the $\sigma(L^p(\mathcal{M}), L^q(\mathcal{M}))$-closure of $\operatorname{Re} H^\infty$. Since $\operatorname{Re} H^\infty$ is convex, we may use Mazur's theorem to conclude that x belongs to $[\operatorname{Re} H^\infty]_p$. ∎

Combining Theorem 5.2 with the above proposition we conclude that for an even natural number p the map $\sim$: $L^p(\mathcal{M})^{\mathrm{sa}} \to L^p(\mathcal{M})^{\mathrm{sa}}$ is real linear and has norm of order p. Now consider the standard complexification $\approx : L^p(\mathcal{M}) \to L^p(\mathcal{M})$, then given $x, y \in L^p(\mathcal{M})^{\mathrm{sa}}$ we have

$$\left\| \approx(x + \mathrm{i}\, y) \right\|_p = \|\widetilde{x} + \mathrm{i}\, \widetilde{y}\|_p \leqslant \|\widetilde{x}\|_p + \|\widetilde{y}\|_p \leqslant \frac{2p}{\ln 2} [\|x\|_p + \|y\|_p] \leqslant \frac{4p}{\ln 2} \|x + \mathrm{i}\, y\|_p.$$

Here we use the fact that $x, y \prec\prec x + \mathrm{i}y$. [This follows from the statements that $\frac{a+a^*}{2} \prec\prec a$, $\frac{a-a^*}{2\mathrm{i}} \prec\prec a$ which are immediate consequences of the triangle 'inequality' $\mu(a+b) \prec\prec \mu(a) + \mu(b)$.]

We conclude that $\approx$ has norm of order p.

Having constructed linear (as opposed to merely real linear) operators, we can now use the noncommutative Riesz-Thorin interpolation theorem first proved by Kunze ([24], [7], Theorem 3.1, [8]), to conclude that $\approx : L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is defined and has norm of order p for all $p \in [2, \infty)$.

From Lemma 5.1 (iv) we deduce that for $p \in [2, \infty)$ the adjoint of $\approx : L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is $-\approx : L^q(\mathcal{M}) \to L^q(\mathcal{M})$. Therefore $\approx : L^p(\mathcal{M}) \to L^p(\mathcal{M})$ is defined for $p \in (1, 2)$ and has norm of order $\frac{1}{p-1}$.

By restriction we get that $\sim : L^p(\mathcal{M})^{\mathrm{sa}} \to L^p(\mathcal{M})^{\mathrm{sa}}$ is defined and has norm of the appropriate orders for $p \in (1, \infty)$.

Summarizing the above discussion, we have the most general version of the Riesz-Bochner theorem.

THEOREM 5.4. [Generalised Riesz theorem] *Suppose* $1 < p < \infty$. *The real linear maps*

$$\sim : \operatorname{Re} H^\infty \to \operatorname{Re} H^\infty, \quad h : \operatorname{Re} H^\infty \to H^\infty$$

extend to real linear maps

$$\sim : L^p(\mathcal{M})^{\mathrm{sa}} \to L^p(\mathcal{M})^{\mathrm{sa}}, \quad h : L^p(\mathcal{M})^{\mathrm{sa}} \to H^p.$$

If $x \in L^p(\mathcal{M})^{\mathrm{sa}}$ then $h(x) = x + \mathrm{i}\,\widetilde{x} \in H^p$, and $\Phi(\widetilde{x}) = 0$. Both $\sim$ and h have norms of order p for $p \in [2, \infty)$ and order $\frac{1}{p-1}$ for $p \in (1, 2)$.

COROLLARY 5.5. $\operatorname{Re} H^p = [\operatorname{Re} H^\infty]_p = L^p(\mathcal{M})^{\mathrm{sa}}$ *for all* $1 < p < \infty$.

Proof. The second equality has already been established.

If $f \in H^p$ then we can choose $(f_n) \subset H^\infty$ such that $f_n \xrightarrow{\|\cdot\|_p} f$. Then $\operatorname{Re} f_n \xrightarrow{\|\cdot\|_p} \operatorname{Re} f$, showing that $\operatorname{Re} H^p \subset [\operatorname{Re} H^\infty]_p$.

For the converse, it suffices to show that $\operatorname{Re} H^p$ is closed. So suppose $(f_n) \subset H^p$ and $\operatorname{Re} f_n \xrightarrow{\|\cdot\|_p} z$. By continuity of adjunction we have that $z \in L^p(\mathcal{M})^{\mathrm{sa}}$. Then by Theorem 5.4, $H^p \supset h(\operatorname{Re} f_n) \xrightarrow{\|\cdot\|_p} h(z) \in H^p$. Now $z = \operatorname{Re}(z + \mathrm{i}\,\tilde{z}) = \operatorname{Re} h(z) \in \operatorname{Re} H^p$. ∎

We should remark that the existence of an abstract Riesz projection in a noncommutative setting (namely, relative to a flow on the von Neumann algebra) was first established by Zsidó ([40], Theorem 4.2). Recently an extension of this result has been made to spaces having non-trivial Boyd indices. The interested reader is referred to [10], in particular Section 4, and the further references there.

6. DIRECT SUM DECOMPOSITIONS

As previously noted, it has been shown in [30] that $L^2(\mathcal{M})$ admits the direct sum decomposition $H_0^2 \oplus L^2(\mathcal{D}) \oplus (H_0^2)^*$. In this section we aim at a similar result for the $L^p(\mathcal{M})$ case ($1 < p < \infty$).

Given $1 < p < \infty$, $\sim$ denotes the map $L^p(\mathcal{M})^{\mathrm{sa}} \to L^p(\mathcal{M})^{\mathrm{sa}}$ from the Riesz-Bochner theorem of the previous section; $\approx$ denotes its complexification.

LEMMA 6.1. *Suppose $x \in H^p$.*

(i) $\operatorname{Im} x = \Phi(\operatorname{Im} x) + \widetilde{\operatorname{Re} x}$;

(ii) $\operatorname{Re} x = \Phi(\operatorname{Re} x) - \widetilde{\operatorname{Im} x}$;

(iii) $x = \Phi(x) + \mathrm{i}\,\overline{\widetilde{x}}$.

Proof. (i) and (ii) follow in a manner similar to Lemma 5.1 (ii) and (iii), exploiting the uniqueness property of $\sim$, and so the proofs are omitted. (iii) follows from combining (i) and (ii). ∎

THEOREM 6.2. *Suppose $1 < p < \infty$. Then $L^p(\mathcal{M}) = H_0^p \oplus L^p(\mathcal{D}) \oplus (H_0^p)^*$. The relevant projections are $x \to \frac{1}{2}\big[x + \mathrm{i}\,\overline{\widetilde{x}} - \Phi(x)\big]$ which has norm of the same order as $\sim$; $x \to \Phi(x)$ which is of norm 1; and $x \to \frac{1}{2}\big[x - \mathrm{i}\,\overline{\widetilde{x}} - \Phi(x)\big]$ which has norm of the same order as $\sim$.*

Proof. First observe that $H^p = H_0^p \oplus L^p(\mathcal{D})$; the relevant projections are $x \to x - \Phi(x)$, $x \to \Phi(x)$. Also, if we assume for the moment that we have $\frac{1}{2}\big[x + \mathrm{i}\,\widetilde{\overline{x}} + \Phi(x)\big] \in H^p$, then it is easy to verify using the basic properties of Φ that the above decomposition splits this into $\frac{1}{2}\big[x + \mathrm{i}\,\widetilde{\overline{x}} - \Phi(x)\big] \in H_0^p$ and $\Phi(x) \in L^p(\mathcal{D})$. To complete the proof it thus suffices to show that $Q : L^p(\mathcal{M}) \to L^p(\mathcal{M}) : x \to \frac{1}{2}\big[x + \mathrm{i}\,\widetilde{\overline{x}} + \Phi(x)\big]$ is a projection along $(H_0^p)^*$ onto H^p.

Suppose $x \in L^p(\mathcal{M})$. Then $\mathrm{Re}\,x + \mathrm{i}\,\widetilde{\mathrm{Re}\,x}, \mathrm{Im}\,x + \mathrm{i}\,\widetilde{\mathrm{Im}\,x} \in H^p$ and hence $x + \mathrm{i}\,\widetilde{\overline{x}} + \Phi(x) = \mathrm{Re}\,x + \mathrm{i}\,\widetilde{\mathrm{Re}\,x} + \mathrm{i}\big[\mathrm{Im}\,x + \mathrm{i}\,\widetilde{\mathrm{Im}\,x}\big] + \Phi(x) \in H^p$.

Taking into account the previous lemma we see that Q is a projection onto H^p with norm of the appropriate order.

To finish we need to show that Q has null space $(H_0^p)^*$. So suppose $x \in L^p(\mathcal{M})$ and $Qx = 0$. Thus $x + \mathrm{i}\,\widetilde{\overline{x}} + \Phi(x) = 0$; applying Φ throughout we see that $\Phi(x) = 0$. Hence $x + \mathrm{i}\,\widetilde{\overline{x}} = 0$. Comparing real and imaginary parts we see that $\mathrm{Re}\,x = \widetilde{\mathrm{Im}\,x}$ and $\mathrm{Im}\,x = -\widetilde{\mathrm{Re}\,x}$. Hence $x = \mathrm{Re}\,x + \mathrm{i}\,\mathrm{Im}\,x = \mathrm{Re}\,x - \mathrm{i}\,\widetilde{\mathrm{Re}\,x} = (\mathrm{Re}\,x + \mathrm{i}\,\widetilde{\mathrm{Re}\,x})^* \in (H_0^p)^*$.

Conversely, if $x^* \in H_0^p$ then $\Phi(x^*) = 0 = \Phi(x)$ and hence $\Phi(\mathrm{Re}\,x) = \Phi(\mathrm{Im}\,x) = 0$. It is then immediate that $\widetilde{\mathrm{Re}\,x} = -\mathrm{Im}\,x$ and $\widetilde{\mathrm{Im}\,x} = \mathrm{Re}\,x$. Thus $x = -\mathrm{i}\,\widetilde{\overline{x}}$, and so $Qx = 0$. ∎

COROLLARY 6.3. *Let $1 < p < \infty$. The dual space of H^p is conjugate isomorphic to H^q via the canonical pairing.*

Proof. This follows easily from the decompostion of Theorem 6.2 and general duality theory. More specifically: since $L^q(\mathcal{M}) = H_0^q \oplus (H^q)^*$, we have an isomorphism between $(H^q)^*$ and $L^q(\mathcal{M})/H_0^q$. But the latter is isometrically isomorphic to the dual of H^p, because H_0^q is the annihilator of H^p in $L^q(\mathcal{M})$. ∎

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