J. OPERATOR THEORY
40(1998), 357–372

CHARACTERISTIC MATRIX OF THE TENSOR PRODUCT OF OPERATORS

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Communicated by William B. Arveson

Abstract. The characteristic matrix of the tensor product of two Hilbert space operators is analyzed. The case where operators are not necessarily closable is also considered, and we determine (i) the closure of the graph of the algebraic tensor product $T_1 \otimes_{\mathrm{alg}} T_2$, (ii) the maximal closable part of $T_1 \otimes_{\mathrm{alg}} T_2$ in the sense of P. Jørgensen ([1]).

Keywords: *Characteristic matrix, maximal closable part, tensor product of unbounded operators.*

MSC (2000): 47A05.

1. INTRODUCTION

For a densely defined closed operator T on a Hilbert space $\mathcal{H}$, let P be the orthogonal projection from the direct sum $\mathcal{H} \oplus \mathcal{H}$ onto the closed subspace

$$\Gamma(T) = \{(\xi, T\xi) : \xi \in \mathcal{D}(T), \text{ the domain of } T\},$$

the graph of T. Note

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

which is a 2×2-matrix with operator entries, and it is known as the characteristic matrix of T. The notion of a characteristic matrix was introduced by M. Stone and J. von Neumann, and it is a standard tool to deal with unbounded operators.

The purpose of the present article is to determine the characteristic matrix of the tensor product (see Subsection 2.1) of two operators (Theorem 4.1). We will also investigate the case when involved operators are not necessarily closable. Let T_1, T_2 be densely defined operators (not necessarily closable) on $\mathcal{H}$. The intersection of $\overline{\Gamma(T_i)}$ and the "y-axis" $0 \oplus \mathcal{H}$ could be large, but nevertheless the algebraic tensor product $T_1 \otimes_{\rm alg} T_2$ makes sense on the algebraic tensor product $\mathcal{D}(T_1) \otimes_{\rm alg} \mathcal{D}(T_2)$. By making use of characteristic matrices, we will be able to describe the closure of the graph $\Gamma(T_1 \otimes_{\rm alg} T_2)$ (Theorem 4.7). Of course $T_1 \otimes_{\rm alg} T_2$ could be far from being closable, but by removing the intersection of $\overline{\Gamma(T_1 \otimes_{\rm alg} T_2)}$ and the "y-axis" (i.e., the obstruction for closability) one obtains a certain maximal closable part. In fact, the general theory on such a procedure was thoroughly studied by P. Jørgensen ([1]) as a certain Lebesgue type decomposition for an operator (see Subsection 2.4 for a quick introduction). Our graph analysis will enable us to determine the maximal closable part of $T_1 \otimes_{\rm alg} T_2$ in the sense of [1], and we will indeed show the multiplicativity for the maximal closable part under the algebraic tensor product (Theorem 4.8).

2. CHARACTERISTIC MATRIX

Here we summarize basic facts on characteristic matrices, and full details can be found for example in [1], [2], [4].

2.1. TENSOR PRODUCT. (see Section 8.10 in [3] for details) Assume that T_1, T_2 are densely defined operators on a Hilbert space $\mathcal{H}$. Let $\mathcal{D}(T_1) \otimes_{\rm alg} \mathcal{D}(T_2)$ be the algebraic tensor product of the two domains. The algebraic tensor product $T_1 \otimes_{\rm alg} T_2$ of operators is the one with the domain $\mathcal{D}(T_1 \otimes_{\rm alg} T_2) = \mathcal{D}(T_1) \otimes_{\rm alg} \mathcal{D}(T_2)$ defined by

$$(T_1 \otimes_{\rm alg} T_2)\Big(\sum_{i=1}^n \xi_i \otimes \eta_i\Big) = \sum_{i=1}^n T_1\xi_i \otimes T_2\eta_i \quad (\xi_i \in \mathcal{D}(T_1), \eta_i \in \mathcal{D}(T_2)),$$

which is well defined. It is plain to see $T_1^* \otimes_{\rm alg} T_2^* \subseteq (T_1 \otimes_{\rm alg} T_2)^*$. When T_1 and T_2 are closable, both of $\mathcal{D}(T_1^*)$ and $\mathcal{D}(T_2^*)$ (hence their algebraic tensor product) are dense. Thus, so is $\mathcal{D}((T_1 \otimes_{\rm alg} T_2)^*)$, that is, the algebraic tensor product $T_1 \otimes_{\rm alg} T_2$ is closable. The closure $\overline{T_1 \otimes_{\rm alg} T_2}$ is defined as the tensor product $T_1 \otimes T_2$ in this case.

2.2. CHARACTERISTIC MATRIX. Let T_1 be a densely defined closed operator on a Hilbert space $\mathcal{H}$ with the characteristic matrix

$$P_1 = \begin{pmatrix} p_{11} & p_{21}^* \\ p_{21} & p_{22} \end{pmatrix}.$$

Therefore, the operator T_1 is described by

$$p_{11}\xi_1 + p_{21}^*\xi_2 \to p_{21}\xi_1 + p_{22}\xi_2.$$

We can explicitly write down p_{ij}'s as follows (see [4]):

$$P_1 = \begin{pmatrix} (1+T_1^*T_1)^{-1} & (1+T_1^*T_1)^{-1}T_1^* \\ T_1(1+T_1^*T_1)^{-1} & T_1T_1^*(1+T_1T_1^*)^{-1} \end{pmatrix}.$$

However, we will not use this expression, and we cannot actually use it because a more general situation is to be considered. Since P_1 is a projection, we have

$$\begin{aligned} &0 \leqslant p_{11} \leqslant 1,\ 0 \leqslant p_{22} \leqslant 1, \\ &p_{11}^2 + p_{21}^* p_{21} = p_{11}, \\ &p_{21}p_{21}^* + p_{22}^2 = p_{22}, \\ &p_{21}p_{11} + p_{22}p_{21} = p_{21}. \end{aligned}$$

Furthermore, we have

$$\ker p_{11} = 0 \quad \text{and} \quad \ker(1-p_{22}) = 0,$$

and these two conditions correspond to the density of $\mathcal{D}(T_1)$ and the closability of T_1 (i.e., the density of $\mathcal{D}(T_1^*)$) respectively. Conversely, if a projection P satisfies these conditions, then P is a characteristic matrix and the corresponding operator T_1 is given by $T_1 = (1-p_{22})^{-1}p_{21}$.

2.3. Adjoint operator. It is well-known that the graph of the adjoint operator T_1^* is given by $\Gamma(T_1^*) = U\Gamma(T_1)^\perp$ with the unitary U defined by $U\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}-\xi_2\\ \xi_1\end{pmatrix}$. The characteristic matrix of T_1^* is thus given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left(1-\begin{pmatrix} p_{11} & p_{21}^* \\ p_{21} & p_{22}\end{pmatrix}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1-p_{22} & p_{21} \\ p_{21}^* & 1-p_{11}\end{pmatrix}.$$

2.4. Maximal closable part. (see [1] for details) We assume that a densely defined operator T_1 is not necessarily closable, and let P_1 (as in Subsection 2.2) be the projection onto the closure $\overline{\Gamma(T_1)}$. By the assumption, the closure of the graph might meet the "y-axis" in a non-trivial way, i.e., $P_1(\mathcal{H}\oplus\mathcal{H})\cap(0\oplus\mathcal{H})$ could contain non-zero vectors. A vector $\begin{pmatrix}0\\ \xi\end{pmatrix}$ is in this intersection, i.e., $P_1\begin{pmatrix}0\\ \xi\end{pmatrix} = \begin{pmatrix}0\\ \xi\end{pmatrix}$ if and only if $p_{21}^*\xi = 0$ and $p_{22}\xi = \xi$. Actually, the first requirement automatically follows from the second (see (3.1) in the next section). Let q be the projection onto $\ker(1-p_{22})$. Then, the discussion so far means that $\begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$ is the projection (commuting with P_1) onto the above intersection. We set

$$P_{1\mathrm{op}} = \left(1 - \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}\right) \begin{pmatrix} p_{11} & p_{21}^* \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} p_{11} & p_{21}^* \\ p_{21} & (1-q)p_{22} \end{pmatrix}.$$

(Note that we have $(1-q)p_{21} = p_{21}$ by (3.1) again.) Obviously, the above matrix is a projection, and from the construction we have $\ker(1-(1-q)p_{22}) = 0$. Therefore, $P_{1\mathrm{op}}$ is the characteristic matrix of an operator, and $P_{1\mathrm{op}}$ is called the operator part of P_1. We consider the operator $T_{1\mathrm{c}} = (1-q)T_1$ with the domain $\mathcal{D}(T_{1\mathrm{c}}) = \mathcal{D}(T_1)$. For each $\xi \in \mathcal{D}(T_1)$ we have

$$\begin{pmatrix} \xi \\ T_1\xi \end{pmatrix} = \begin{pmatrix} p_{11}\xi_1 + p_{21}^*\xi_2 \\ p_{21}\xi_1 + p_{22}\xi_2 \end{pmatrix}$$

for some vectors ξ_1, ξ_2 so that we have

$$\begin{pmatrix} \xi \\ T_{1\mathrm{c}}\xi \end{pmatrix} = \begin{pmatrix} p_{11}\xi_1 + p_{21}^*\xi_2 \\ (1-q)p_{21}\xi_1 + (1-q)p_{22}\xi_2 \end{pmatrix} = \begin{pmatrix} p_{11}\xi_1 + p_{21}^*\xi_2 \\ p_{21}\xi_1 + (1-q)p_{22}\xi_2 \end{pmatrix}.$$

Therefore, $\Gamma(T_{1\mathrm{c}})$ is in the range of $P_{1\mathrm{c}}$, and hence $T_{1\mathrm{c}}$ is closable. The decomposition $T_1 = T_{1\mathrm{c}} + T_{1\mathrm{s}}$ (with the singular part $T_{1\mathrm{s}} = qT_1$) was investigated in [1] as a Lebesgue type decomposition, and one can show that

(i) $T_{1\mathrm{c}}$ satisfies a certain maximality condition and
(ii) the characteristic matrix of the closure $\overline{T_{1\mathrm{c}}}$ is exactly $P_{1\mathrm{op}}$.

We will refer to the operator $T_{1\mathrm{c}}$ as the maximal closable part of T_1.

3. TECHNICAL LEMMAS

Let

$$P = \begin{pmatrix} p_{11} & p_{21}^* \\ p_{21} & p_{22} \end{pmatrix}$$

be a projection. We do not assume $\ker p_{11} = \ker(1-p_{22}) = 0$ here so that $P(\mathcal{H} \oplus \mathcal{H})$ may not be the graph of a densely defined closed operator. Let $p_{11} = \int_{[0,1]} \lambda \, de_\lambda^p$ be the spectral decomposition, and $p_{21} = u|p_{21}|$ be the polar decomposition. Since

$$p_{21}^* p_{21} = p_{11} - p_{11}^2 = p_{11}(1-p_{11}),$$

we observe $|p_{21}| = (p_{11}(1-p_{11}))^{1/2}$ and the support projection of $|p_{21}|$ is $u^*u = e^p_{(0,1)}$. On the other hand, notice that

$$p_{21}p_{21}^* = p_{22} - p_{22}^2 = p_{22}(1-p_{22}).$$

Let $p_{22} = \int_{[0,1]} \lambda \, \mathrm{d}f^p_\lambda$ be the spectral decomposition, and the above computation shows that $|p_{21}^*| = (p_{22}(1-p_{22}))^{1/2}$ and the support projection of $|p_{21}^*|$ is $uu^* = f^p_{(0,1)}$. Since $u|p_{21}|u^* = |p_{21}^*|$, we conclude $u\,(p_{11}(1-p_{11}))^{1/2}\,u^* = (p_{22}(1-p_{22}))^{1/2}$ and

$$p_{21} = u\,(p_{11}(1-p_{11}))^{\frac{1}{2}} = (p_{22}(1-p_{22}))^{\frac{1}{2}}\,u, \tag{3.1}$$

which will be repeatedly used. Also the last relation among p_{ij}'s (see Subsection 2.2) means

$$(1-p_{22})p_{21} = p_{21}p_{11} \quad \text{and} \quad p_{22}p_{21} = p_{21}(1-p_{11}). \tag{3.2}$$

We point out the next result although we will not use it (see Remark 3.1 in [1]).

Lemma 3.1. *The phase part u of p_{21} satisfies $u(1-p_{11}) = p_{22}u$.*

Proof. The last relation between p_{ij}'s (see Subsection 2.2) and (3.1) show

$$u\,(p_{11}(1-p_{11}))^{\frac{1}{2}}\,p_{11} + p_{22}u\,(p_{11}(1-p_{11}))^{\frac{1}{2}} = u\,(p_{11}(1-p_{11}))^{\frac{1}{2}}\,,$$

that is,

$$u(1-p_{11})\,(p_{11}(1-p_{11}))^{\frac{1}{2}} = p_{22}u\,(p_{11}(1-p_{11}))^{\frac{1}{2}}\,.$$

For $\xi \in e^p_{(0,1)}\mathcal{H}$ (the support of $(p_{11}(1-p_{11}))^{\frac{1}{2}}$), we have $u(1-p_{11})\xi = p_{22}u\xi$. For $\xi \in e^p_{\{0\}}$ we have

$$u(1-p_{11})\xi = u\xi = 0 \quad \text{and} \quad p_{22}u\xi = p_{22}0 = 0$$

while for $\xi \in e^p_{\{1\}}$ we have

$$u(1-p_{11})\xi = u0 = 0 \quad \text{and} \quad p_{22}u\xi = p_{22}0 = 0.$$

Thus, we have $u(1-p_{11})\xi = p_{22}u\xi$ for every vector ξ. ∎

Let

$$P_2 = \begin{pmatrix} q_{11} & q_{21}^* \\ q_{21} & q_{22} \end{pmatrix}$$

be another projection. Of course the same relations ((3.1), (3.2)) are valid for q_{ij}'s. We set

$$D_1 = (1 - p_{11}) \otimes (1 - q_{11}) + p_{11} \otimes q_{11},$$
$$D_2 = (1 - p_{22}) \otimes (1 - q_{22}) + p_{22} \otimes q_{22}.$$

Since $\ker D_1 = \ker(p_{11} \otimes q_{11}) \cap \ker((1 - p_{11}) \otimes (1 - q_{11}))$ and $\ker D_2 = \ker(p_{22} \otimes q_{22}) \cap \ker((1 - p_{22}) \otimes (1 - q_{22}))$, we have

$$\begin{cases} \ker D_1 = e^p_{\{0\}}\mathcal{H} \otimes e^q_{\{1\}}\mathcal{H} + e^p_{\{1\}}\mathcal{H} \otimes e^q_{\{0\}}\mathcal{H}, \\ \ker D_2 = f^p_{\{0\}}\mathcal{H} \otimes f^q_{\{1\}}\mathcal{H} + f^p_{\{1\}}\mathcal{H} \otimes f^q_{\{0\}}\mathcal{H}, \end{cases}$$

where e^q, f^q denote the spectral projections of q_{11}, q_{22} respectively.

REMARKS 3.2. (i) If P_1, P_2 are characteristic matrices of densely defined closed operators T_1, T_2, then $e^p_{\{0\}} = f^p_{\{1\}} = e^q_{\{0\}} = f^q_{\{1\}} = 0$ (see the last part of Subsection 2.2) so that both of D_1, D_2 are injective and D_1^{-1}, D_2^{-1} are defined as (generally unbounded) non-singular positive operators.

(ii) Furthermore, if T_1, T_2 are bounded, we have

$$p_{11} \geqslant \varepsilon_1 = \frac{1}{1 + \|T_1\|^2}, \quad q_{11} \geqslant \varepsilon_2 = \frac{1}{1 + \|T_2\|^2}.$$

(see the explicit expression of a characteristic matrix in Subsection 2.2 or Proposition 4 (1) in [2]). Since $\|T_1^*\| = \|T_1\|$ and $\|T_2^*\| = \|T_2\|$, we also have $1 - p_{22} \geqslant \varepsilon_1, 1 - q_{22} \geqslant \varepsilon_2$. We can certainly choose a positive ε such that $\varepsilon \leqslant (1 - \alpha)(1 - \beta) + \alpha\beta \leqslant 1$ for scalars satisfying $\varepsilon_1 \leqslant \alpha \leqslant 1$, $\varepsilon_2 \leqslant \beta \leqslant 1$. Therefore, in this case the positive operators D_1, D_2 are invertible. For example, when T_1, T_2 are contractions, it is straightforward to see $1/2 \leqslant D_1,\ D_2 \leqslant 1$.

Due to (3.2) we observe the intertwining property

$$\begin{aligned} D_2(p_{21} \otimes q_{21}) &= (1 - p_{22})p_{21} \otimes (1 - q_{22})q_{21} + p_{22}p_{21} \otimes q_{22}q_{21} \\ &= p_{21}p_{11} \otimes q_{21}q_{11} + p_{21}(1 - p_{11}) \otimes q_{21}(1 - q_{11}) = (p_{21} \otimes q_{21})D_1. \end{aligned}$$

When D_1, D_2 are invertible, this means $D_2^{-1}(p_{21} \otimes q_{21}) = (p_{21} \otimes q_{21})\, D_1^{-1}$. To deal with the general case, by a slight abuse of notation we set for $i = 1, 2$

$$D_i^{-1} = \begin{cases} (D_i \,|_{\text{the support of } D_i})^{-1} & \text{on the support of } D_i, \\ 0 & \text{on the kernel of } D_i. \end{cases}$$

LEMMA 3.3. (i) *The ranges of* $p_{11} \otimes q_{11}$ *and* $p_{21}^* \otimes q_{21}^*$ *are in the support of* D_1.

(ii) *The ranges of* $p_{22} \otimes q_{22}$ *and* $p_{21} \otimes q_{21}$ *are in the support of* D_2.

Proof. The range of $p_{11} \otimes q_{11}$ is in the range of $e^p_{(0,1]} \otimes e^q_{(0,1]}$ while that of $p_{21}^* \otimes q_{21}^*$ is also in the range of $e^p_{(0,1)} \otimes e^q_{(0,1)}$ thanks to (3.1). These projections are obviously smaller than the support projection of D_1

$$1 - e^p_{\{0\}} \otimes e^q_{\{1\}} - e^p_{\{1\}} \otimes e^q_{\{0\}},$$

and we have (i).

We also get (ii) by using the spectral projections f^p, f^q instead. ∎

In the rest of the article we have to keep these facts in mind since D_1^{-1}, D_2^{-1} were defined in the extended sense, and at first we show that the same identity as before remains valid.

LEMMA 3.4. *We have* $D_2^{-1}(p_{21} \otimes q_{21}) = (p_{21} \otimes q_{21})D_1^{-1}$, *and this product is a bounded operator.*

Proof. On $\ker D_1 = e^p_{\{0\}}\mathcal{H} \otimes e^q_{\{1\}}\mathcal{H} + e^p_{\{1\}}\mathcal{H} \otimes e^q_{\{0\}}\mathcal{H}$ both operators vanish since $p_{21} \otimes q_{21} = 0$ on $\ker D_1$ thanks to (3.1) and $D_1^{-1} = 0$ on $\ker D_1$ from the definition. Let us assume that a vector ζ is in the support of D_1 and in $\mathcal{D}(D_1^{-1})$. Then, we have

$$(p_{21} \otimes q_{21})\zeta = (p_{21} \otimes q_{21})D_1 D_1^{-1}\zeta = D_2(p_{21} \otimes q_{21})D_1^{-1}\zeta$$

by the preceding intertwining property. Note that the left side is

$$(p_{22}(1-p_{22}) \otimes q_{22}(1-q_{22}))^{\frac{1}{2}}\,\zeta'$$

with $\zeta' = (u \otimes v)\zeta$ (see (3.1)). The vector ζ' is in $f^p_{(0,1)}\mathcal{H} \otimes f^q_{(0,1)}\mathcal{H}$ (the final space of $u \otimes v$), and hence the computation

$$\begin{aligned} D_2^{-1}\,(p_{22}(1-p_{22}) \otimes q_{22}(1-q_{22}))^{\frac{1}{2}}\,\zeta' &= \left(\frac{(p_{22}(1-p_{22}) \otimes q_{22}(1-q_{22}))^{\frac{1}{2}}}{(1-p_{22}) \otimes (1-q_{22}) + p_{22} \otimes q_{22}} \right)\zeta' \\ &= \left(\frac{\sqrt{\frac{p_{22}}{1-p_{22}} \otimes \frac{q_{22}}{1-q_{22}}}}{1 + \frac{p_{22}}{1-p_{22}} \otimes \frac{q_{22}}{1-q_{22}}} \right)\zeta' \end{aligned}$$

is legitimate. Since $\frac{\sqrt{\alpha}}{1+\alpha} \leqslant 1/2$ for $\alpha \in \mathbb{R}_+$, this computation shows that $(p_{21} \otimes q_{21})\zeta$ is certainly in $\mathcal{D}(D_2^{-1})$. Since the range of $p_{21} \otimes q_{21}$ is in the support of D_2 (Lemma 3.3), by hitting D_2^{-1} to the equation at the beginning we get

$$D_2^{-1}(p_{21} \otimes q_{21})\zeta = (p_{21} \otimes q_{21})D_1^{-1}\zeta.$$

Our computation also shows the boundedness of the product. ∎

The above proof shows that $D_2^{-1}(p_{21}\otimes q_{21})$ is defined everywhere and bounded. On the other hand, the domain of the right side is $\mathcal{D}(D_1^{-1})$. Thus, the precise meaning of the lemma is that the closure of the right side is the left side which is a bounded operator.

DEFINITION 3.5. Keeping Lemma 3.4 and the obvious commutativity $[D_1, p_{11}\otimes q_{11}] = [D_2, p_{22}\otimes q_{22}] = 0$ in our mind, we define the projection $P = P(P_1, P_2)$ (see Lemma 3.6 below) by

$$\begin{aligned} P &= \begin{pmatrix} D_1^{-1} & 0 \\ 0 & D_2^{-1} \end{pmatrix} \begin{pmatrix} p_{11}\otimes q_{11} & p_{21}^*\otimes q_{21}^* \\ p_{21}\otimes q_{21} & p_{22}\otimes q_{22} \end{pmatrix} \\ &= \begin{pmatrix} p_{11}\otimes q_{11} & p_{21}^*\otimes q_{21}^* \\ p_{21}\otimes q_{21} & p_{22}\otimes q_{22} \end{pmatrix} \begin{pmatrix} D_1^{-1} & 0 \\ 0 & D_2^{-1} \end{pmatrix} \end{aligned}$$

with the "inverses" D_1^{-1}, D_2^{-1} explained right before Lemma 3.3.

Note that the components of P are

$$\begin{aligned} P_{11} &= D_1^{-1}(p_{11}\otimes q_{11}) = (p_{11}\otimes q_{11})D_1^{-1}, \\ P_{22} &= D_2^{-1}(p_{22}\otimes q_{22}) = (p_{22}\otimes q_{22})D_2^{-1}, \\ P_{21} &= D_2^{-1}(p_{21}\otimes q_{21}) = (p_{21}\otimes q_{21})D_1^{-1}. \end{aligned}$$

For example, in the first equation both sides vanish on $\ker D_1$ because one of p_{11} and q_{11} always vanish here. Obviously, both of P_{11}, P_{22} are positive contractions. From the proof of Lemma 3.5 we observe that the following alternative expression is also possible:

$$P_{21} = \frac{\sqrt{AB}}{A+B}\,(u\otimes v)$$

with the commuting contractions

$$A = \left(p_{22}f^p_{(0,1)}\right)\otimes\left(q_{22}f^q_{(0,1)}\right), \quad B = \left((1-p_{22})f^p_{(0,1)}\right)\otimes\left((1-q_{22})f^q_{(0,1)}\right).$$

LEMMA 3.6. *The above P is indeed a projection.*

Proof. At first we show $P_{11}^2 + P_{21}^*P_{21} = P_{11}$. Note that both sides vanish on $\ker D_1$ due to the presence of D_1^{-1} in P_{11} and P_{21}. If a vector ζ is in the support of D_1 and in $\mathcal{D}(D_1^{-1})$, we have

$$(P_{11}^2 + P_{11}^*P_{21})D_1\zeta = D_1^{-1}\left((p_{11}\otimes q_{11})^2 + (p_{21}^*\otimes q_{21}^*)(p_{21}\otimes q_{21})\right)\zeta.$$

The above second term being equal to

$$\begin{aligned} p_{21}^* p_{21} \otimes q_{21}^* q_{21} &= (p_{11} - p_{11}^2) \otimes (q_{11} - q_{11}^2) = p_{11}(1 - p_{11}) \otimes q_{11}(1 - q_{11}) \\ &= (p_{11} \otimes p_{11}) \left((1 - p_{11}) \otimes (1 - q_{11}) \right), \end{aligned}$$

we conclude

$$(P_{11}^2 + P_{21}^* P_{21}) D_1 \zeta = D_1^{-1} (p_{11} \otimes q_{11}) D_1 \zeta = P_{11} D_1 \zeta$$

and we see $P_{11} + P_{21}^* P_{21} = P_{11}$ (since both sides are bounded). By analogous computations (by using D_2^{-1} instead), we can also show $P_{21} P_{21}^* + P_{22}^2 = P_{22}$.

Next we show $P_{21} P_{11} + P_{22} P_{21} = P_{21}$. Note that both sides vanish on $\ker D_1$. If a vector ζ is in the support of D_1 and in $\mathcal{D}(D_1^{-1})$, we compute

$$\begin{aligned} &(P_{21} P_{11} + P_{22} P_{21}) D_1 \zeta \\ &\quad = D_2^{-1} (p_{21} \otimes q_{21})(p_{11} \otimes q_{11}) \zeta + (p_{22} \otimes q_{22}) D_2^{-1} (p_{21} \otimes q_{21}) \zeta \\ &\quad = D_2^{-1} ((1 - p_{22}) \otimes (1 - q_{22}))(p_{21} \otimes q_{21}) \zeta \\ &\qquad + (p_{22} \otimes q_{22}) D_2^{-1} (p_{21} \otimes q_{21}) \zeta \qquad \text{(by (3.2))} \\ &\quad = \left(D_2^{-1} ((1 - p_{22}) \otimes (1 - q_{22})) + (p_{22} \otimes q_{22}) D_2^{-1} \right) (p_{21} \otimes q_{21}) \zeta. \end{aligned}$$

Since

$$\begin{aligned} &D_2^{-1} ((1 - p_{22}) \otimes (1 - q_{22})) + (p_{22} \otimes q_{22}) D_2^{-1} \\ &\qquad = D_2^{-1} ((1 - p_{22}) \otimes (1 - q_{22}) + p_{22} \otimes q_{22}) = D_2^{-1} D_2 \end{aligned}$$

and $(p_{21} \otimes q_{21}) \zeta$ is in the support of D_2 (Lemma 3.3), we conclude

$$(P_{21} P_{11} + P_{22} P_{21}) D_1 \zeta = (p_{21} \otimes q_{21}) \zeta = P_{21} D_1 \zeta$$

as desired. ∎

LEMMA 3.7. *Any vector in the direct sum* $(\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H})$ *of the form*

$$((p_{11} \xi_1 + p_{21}^* \xi_2) \otimes (q_{11} \eta_1 + q_{21}^* \eta_2), (p_{21} \xi_1 + p_{22} \xi_2) \otimes (q_{21} \eta_1 + q_{22} \eta_2))$$

$(\xi_i, \eta_i \in \mathcal{H})$ *is in the range of the projection* P.

Proof. It suffices to show that the above vector is fixed under P. We set

$$\begin{aligned} \zeta_1 &= p_{11}(p_{11} \xi_1 + p_{21}^* \xi_2) \otimes q_{11}(q_{11} \eta_1 + q_{21}^* \eta_2) \\ &\qquad + p_{21}^* (p_{21} \xi_1 + p_{22} \xi_2) \otimes q_{21}^* (q_{21} \eta_1 + q_{22} \eta_2), \\ \zeta_2 &= p_{21}(p_{11} \xi_1 + p_{21}^* \xi_2) \otimes q_{21}(q_{11} \eta_1 + q_{21}^* \eta_2) \\ &\qquad + p_{22}(p_{21} \xi_1 + p_{22} \xi_2) \otimes q_{22}(q_{21} \eta_1 + q_{22} \eta_2) \end{aligned}$$

for convenience, and we have to show

$$\begin{pmatrix} D_1^{-1} & 0 \\ 0 & D_2^{-1} \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} (p_{11}\xi_1 + p_{21}^*\xi_2) \otimes (q_{11}\eta_1 + q_{21}^*\eta_2) \\ (p_{21}\xi_1 + p_{22}\xi_2) \otimes (q_{21}\eta_1 + q_{22}\eta_2) \end{pmatrix}.$$

It is plain to see that ζ_1 and the first component in the right side (respectively ζ_2 and the second component in the right side) are in the support of D_1 (respectively D_2) thanks to (3.1) and Lemma 3.3. Therefore, it suffices to see

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} (p_{11}\xi_1 + p_{21}^*\xi_2) \otimes (q_{11}\eta_1 + q_{21}^*\eta_2) \\ (p_{21}\xi_1 + p_{22}\xi_2) \otimes (q_{21}\eta_1 + q_{22}\eta_2) \end{pmatrix}.$$

The two components in the right side are

$$\begin{aligned} &(p_{11}p_{11}\xi_1 + p_{11}p_{21}^*\xi_2) \otimes (q_{11}q_{11}\eta_1 + q_{11}q_{21}^*\eta_2) \\ &\qquad + ((1-p_{11})p_{11}\xi_1 + (1-p_{11})p_{21}^*\xi_2) \otimes ((1-q_{11})q_{11}\eta_1 + (1-q_{11})q_{21}^*\eta_2), \\ &(p_{22}p_{21}\xi_1 + p_{22}^2\xi_2) \otimes (q_{22}q_{21}\eta_1 + q_{22}^2\eta_2) \\ &\qquad + ((1-p_{22})p_{21}\xi_1 + (1-p_{22})p_{22}\xi_2) \otimes ((1-q_{22})q_{21}\eta_1 + (1-q_{22})q_{22}\eta_2) \end{aligned}$$

respectively. Thanks to $(1-p_{11})p_{21}^* = p_{21}^*p_{22}, (1-p_{22})p_{21} = p_{21}p_{11}, (1-p_{22})p_{22} = p_{21}p_{21}^*$ and the same relations for q's (see (3.2), (3.1)), it is straightforward to see that the above vectors are ζ_1, ζ_2. ∎

LEMMA 3.8. *The range of the projection P is exactly the closed subspace* (*in* $\mathcal{H} \otimes \mathcal{H} \oplus \mathcal{H} \otimes \mathcal{H}$) *generated by the vectors in the previous lemma.*

Proof. Assume that a vector $\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \in \mathcal{H} \otimes \mathcal{H} \oplus \mathcal{H} \otimes \mathcal{H}$ satisfies $P\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$ and is perpendicular to the vectors in the previous lemma. The first condition means

$$\begin{aligned} P_{11}\zeta_1 + P_{21}^*\zeta_2 &= D_1^{-1}(p_{11} \otimes q_{11})\zeta_1 + D_1^{-1}(p_{21}^* \otimes q_{21}^*)\zeta_2 = \zeta_1, \\ P_{21}\zeta_1 + P_{22}\zeta_2 &= D_2^{-1}(p_{21} \otimes q_{21})\zeta_1 + D_2^{-1}(p_{22} \otimes q_{22})\zeta_2 = \zeta_2. \end{aligned}$$

Since $(p_{11} \otimes q_{11})\zeta_1, (p_{21}^* \otimes q_{21}^*)\zeta_2$ and $(p_{21} \otimes q_{21})\zeta_1, (p_{22} \otimes q_{22})\zeta_2$ are in the support of D_1, D_2 respectively (Lemma 3.3), by hitting D_1 and D_2 from the left we see that the above equations imply

$$(p_{21}^* \otimes q_{21}^*)\zeta_2 = ((1-p_{11}) \otimes (1-q_{11}))\zeta_1, \tag{3.3}$$

$$(p_{21} \otimes q_{21})\zeta_1 = ((1-p_{22}) \otimes (1-q_{22}))\zeta_2 \tag{3.4}$$

(recall the definition of D_i). The second condition says that in particular $\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} p_{11}\xi_1 \otimes q_{11}\eta_1 \\ p_{21}\xi_1 \otimes q_{21}\eta_1 \end{pmatrix}$ and $\begin{pmatrix} p_{21}^*\xi_2 \otimes q_{21}^*\eta_2 \\ p_{22}\xi_2 \otimes q_{22}\eta_2 \end{pmatrix}$ (which is actually an equivalent requirement thanks to the "polarization"). Since

$$\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \perp \begin{pmatrix} p_{11}\xi_1 \otimes q_{11}\eta_1 \\ p_{21}\xi_1 \otimes q_{21}\eta_1 \end{pmatrix},$$

we have

$$\begin{aligned} 0 &= \left(\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}, \begin{pmatrix} p_{11}\xi_1 \otimes q_{11}\eta_1 \\ p_{21}\xi_1 \otimes q_{21}\eta_1 \end{pmatrix} \right) = (\zeta_1, p_{11}\xi_1 \otimes q_{11}\eta_1) + (\zeta_2, p_{21}\xi_1 \otimes q_{21}\eta_1) \\ &= ((p_{11} \otimes q_{11})\zeta_1 + (p_{21}^* \otimes q_{21}^*)\zeta_2, \xi_1 \otimes \eta_1). \end{aligned}$$

Since $\xi_1, \eta_1 \in \mathcal{H}$ are arbitrary vectors, we have

$$(p_{11} \otimes q_{11})\zeta_1 = -(p_{21}^* \otimes q_{21}^*)\zeta_2. \tag{3.5}$$

From $\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \perp \begin{pmatrix} p_{21}^*\xi_2 \otimes q_{21}^*\eta_2 \\ p_{22}\xi_2 \otimes q_{22}\eta_2 \end{pmatrix}$, we similarly get

$$(p_{21} \otimes q_{21})\zeta_1 = -(p_{22} \otimes q_{22})\zeta_2. \tag{3.6}$$

From (3.3) and (3.5) we get

$$(p_{11} \otimes q_{11})\zeta_1 = -((1 - p_{11}) \otimes (1 - q_{11}))\zeta_1.$$

Notice

$$\begin{aligned} 0 \leqslant \|(p_{11} \otimes q_{11})\zeta_1\|^2 &= ((p_{11} \otimes p_{11})\zeta_1, (p_{11} \otimes p_{11})\zeta_1) \\ &= -((p_{11} \otimes q_{11})\zeta_1, ((1 - p_{11}) \otimes (1 - q_{11}))\zeta_1) \\ &= -((p_{11}(1 - p_{11}) \otimes q_{11}(1 - q_{11}))\zeta_1, \zeta_1) \\ &= -\|(p_{11}(1 - p_{11}) \otimes q_{11}(1 - q_{11}))^{\frac{1}{2}}\zeta_1\|^2 \leqslant 0. \end{aligned}$$

Here, the above computation is valid since $p_{11}(1 - p_{11}) \otimes q_{11}(1 - q_{11})$ is positive. Therefore, we conclude

$$(p_{11} \otimes q_{11})\zeta_1 = (p_{11}(1 - p_{11}) \otimes q_{11}(1 - q_{11}))^{\frac{1}{2}}\zeta_1 = 0.$$

Analogously, from (3.4) and (3.6) we get $(p_{22} \otimes q_{22})\zeta_2 = -((1 - p_{22}) \otimes (1 - q_{22}))\zeta_2$ and

$$(p_{22} \otimes q_{22})\zeta_2 = (p_{22}(1 - p_{22}) \otimes q_{22}(1 - q_{22}))^{\frac{1}{2}}\zeta_2 = 0.$$

Recalling (3.1) and the definition of P_{ij}'s, we have

$$P_{11}\zeta_1 = P_{21}^*\zeta_2 = P_{21}\zeta_1 = P_{22}\zeta_2 = 0.$$

Thus, from the two equations at the very beginning of the proof we conclude $\zeta_1 = \zeta_2 = 0$ as desired. ∎

4. MAIN RESULTS

In this section we prove the results mentioned in Section 1. In the first half of the section we assume that operators T_1, T_2 are densely defined and closed, and in the second half we will deal with the case when operators are not necessarily closable.

Let us assume that P_1, P_2 are the characteristic matrices of densely defined closed operators T_1, T_2 respectively. Lemma 3.8 means that the range of the projection $P = P(P_1, P_2)$ is the closed subspace generated by $\begin{pmatrix} \varphi \otimes \psi \\ T_1\varphi \otimes T_2\psi \end{pmatrix}$ ($\varphi \in \mathcal{D}(T_1), \psi \in \mathcal{D}(T_2)$). Hence, the range of the projection P is

$$\overline{\Gamma(T_1 \otimes_{\mathrm{alg}} T_2)} = \Gamma(\overline{T_1 \otimes_{\mathrm{alg}} T_2}),$$

and since $T_1 \otimes T_2 = \overline{T_1 \otimes_{\mathrm{alg}} T_2}$ (see Subsection 2.1) we have

THEOREM 4.1. *The characteristic matrix of the tensor product $T_1 \otimes T_2$ of densely defined closed operators T_1, T_2 is the projection $P = P(P_1, P_2)$ constructed from the characteristic matrices P_1, P_2 of T_1, T_2.*

It is also possible to prove the theorem by looking at $(1-P_{22})^{-1}P_{21}$ (without having Lemma 3.8); details are left to the reader as an exercise.

In the present case D_1, D_2 are non-singular as was pointed out in Remark 3.2, and let us recall (see Subsection 2.3) that the characteristic matrices of the adjoint operators T_1^*, T_2^* are

$$P_1' = \begin{pmatrix} 1-p_{22} & p_{21} \\ p_{21}^* & 1-p_{11} \end{pmatrix} \quad \text{and} \quad P_2' = \begin{pmatrix} 1-q_{22} & q_{21} \\ q_{21}^* & 1-q_{11} \end{pmatrix}$$

respectively. From these we construct the matrix $P' = P(P_1', P_2')$ as before, i.e.,

$$P' = \begin{pmatrix} D_2^{-1}((1-p_{22}) \otimes (1-q_{22})) & D_2^{-1}(p_{21} \otimes q_{21}) \\ D_1^{-1}(p_{21}^* \otimes q_{21}^*) & D_1^{-1}((1-p_{11}) \otimes (1-q_{11})) \end{pmatrix}.$$

Note that the roles of D_1 and D_2 have been switched here. By the above theorem, P' is the characteristic matrix of $T_1^* \otimes T_2^*$, and hence the characteristic matrix of $(T_1^* \otimes T_2^*)^*$ is

$$\begin{pmatrix} 1 - D_1^{-1}((1-p_{11}) \otimes (1-q_{11})) & D_1^{-1}(p_{21}^* \otimes q_{21}^*) \\ D_2^{-1}(p_{21} \otimes q_{21}) & 1 - D_2^{-1}((1-p_{22}) \otimes (1-q_{22})) \end{pmatrix}.$$

We observe that this matrix is equal to

$$\begin{pmatrix} D_1^{-1}(p_{11} \otimes q_{11}) & D_1^{-1}(p_{21}^* \otimes q_{21}^*) \\ D_2^{-1}(p_{21} \otimes q_{21}) & D_2^{-1}(p_{22} \otimes q_{22}) \end{pmatrix} = P$$

because of $D_1^{-1}D_1 = D_2^{-1}D_2 = 1$. Therefore, we have shown $(T_1^* \otimes T_2^*)^* = T_1 \otimes T_2$, and hence

COROLLARY 4.2. *For densely defined closed operators T_1, T_2 we always have $T_1^* \otimes T_2^* = (T_1 \otimes T_2)^*$.*

This is a standard result and of course not new. However, compare our proof with that in p. 231 in [5].

In the rest we will deal with the case when T_1, T_2 are not necessarily closable. Let P_1, P_2 be the projections onto the closures $\overline{\Gamma(T_1)}, \overline{\Gamma(T_2)}$, and P be the projection constructed from these P_1, P_2 as in the previous section. For simplicity, let us assume that T_1, T_2 have dense domains. Since p_{11}, q_{11} are injective by the assumption (see the last part of Subsection 2.2), D_1^{-1} is non-singular and P_{11} is also injective. Notice that P may not be a characteristic matrix of an operator since $P((\mathcal{H}\otimes\mathcal{H}) \oplus (\mathcal{H}\otimes\mathcal{H})) \cap (0 \oplus (\mathcal{H}\otimes\mathcal{H}))$ could contain non-zero vectors. As in Subsection 2.4, let Q be the projection onto $\ker(1 - P_{22})$, and let

$$P_{\mathrm{op}} = \left(1 - \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix}\right) \begin{pmatrix} p_{11} & p_{21}^* \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} P_{11} & P_{21}^* \\ P_{21} & (1-Q)P_{22} \end{pmatrix}$$

be the operator part of P.

When T_i's are closable, D_i's are injective (see Remark 3.2) and hence we have

$$1 - P_{22} = D_2^{-1} D_2 - D_2^{-1}(p_{22} \otimes q_{22}) = D_2^{-1}((1-p_{22}) \otimes (1-q_{22})).$$

Since $\ker(1-p_{22}) = \ker(1-q_{22}) = 0$ in the present case (see Subsection 2.2), we observe that $1 - P_{22}$ is injective. Therefore, $Q = 0$ and $P_{\mathrm{op}} = P$ as expected, which of course corresponds to the closability of $T_1 \otimes_{\mathrm{alg}} T_2$ (see Subsection 2.1).

To see what P_{op} is in the general case, we need to write down the projection $1 - Q$ explicitly.

LEMMA 4.3. *We have $1 - Q = f^p_{[0,1)} \otimes f^q_{[0,1)} + f^p_{\{0\}} \otimes f^q_{\{1\}} + f^p_{\{1\}} \otimes f^q_{\{0\}}$.*

Proof. We claim that $\zeta \in \ker(1 - P_{22})$ if and only if $\zeta \in f^p_{(0,1]}\mathcal{H} \otimes f^q_{(0,1]}\mathcal{H}$ and $\zeta \in \ker((1-p_{22}) \otimes (1-q_{22}))$. In fact, if a vector ζ belongs to $\ker(1-P_{22})$, i.e., $D_2^{-1}(p_{22} \otimes q_{22})\zeta = \zeta$, then at first ζ must be in $f^p_{(0,1]}\mathcal{H} \otimes f^q_{(0,1]}\mathcal{H}$ because $D_2^{-1}(p_{22} \otimes q_{22})$ kills the orthogonal complement. Secondly, since the range of $p_{22} \otimes q_{22}$ is in the support of D_2 (Lemma 3.3), we have $(p_{22} \otimes q_{22})\zeta = D_2\zeta$, i.e., $((1-p_{22}) \otimes (1-q_{22}))\zeta = 0$. Conversely, if $\zeta \in f^p_{(0,1]}\mathcal{H} \otimes f^q_{(0,1]}\mathcal{H}$ and $((1-p_{22}) \otimes (1-q_{22}))\zeta = 0$, then we have $(p_{22} \otimes q_{22})\zeta = D_2\zeta$ at first and then $D_2^{-1}(p_{22} \otimes q_{22})\zeta = \zeta$ because $\zeta \in f^p_{(0,1]}\mathcal{H} \otimes f^q_{(0,1]}\mathcal{H}$ is in the support of D_2.

So far we have shown

$$Q = (f^p_{(0,1]} \otimes f^q_{(0,1]}) \wedge (\text{the projection onto } \ker((1-p_{22}) \otimes (1-q_{22}))) \,.$$

Therefore, by passing to the orthogonal complement, we have

$$\begin{aligned} 1-Q &= (1-f^p_{(0,1]}\otimes f^q_{(0,1]}) \vee (\text{the support projection of } (1-p_{22})\otimes(1-q_{22})) \\ &= (1-f^p_{(0,1]}\otimes f^q_{(0,1]}) \vee (f^p_{[0,1)}\otimes f^q_{[0,1)}), \end{aligned}$$

which is obviously the projection in the lemma. ∎

This lemma shows the $(2,2)$-component of P_{op} is

$$\begin{aligned} (1-Q)D_2^{-1}(p_{22}\otimes q_{22}) &= (f^p_{[0,1)}\otimes f^q_{[0,1)} + f^p_{\{0\}}\otimes f^q_{\{1\}} + f^p_{\{1\}}\otimes f^q_{\{0\}})D_2^{-1}(p_{22}\otimes q_{22}) \\ &\quad (= (f^p_{[0,1)}\otimes f^q_{[0,1)})D_2^{-1}(p_{22}\otimes q_{22})). \end{aligned}$$

We set

$$p'_{22} = f^p_{[0,1)}p_{22},\ q'_{22} = f^q_{[0,1)}q_{22}.$$

Then, the corresponding $D'_2 = p'_{22}\otimes q'_{22} + (1-p'_{22})\otimes(1-q'_{22})$ is injective (since $\ker(1-p'_{22}) = \ker(1-q'_{22}) = 0$ from the construction), and we observe

LEMMA 4.4. *We have*

$$\begin{aligned} (1-Q)D_2^{-1}(p_{22}\otimes q_{22}) &= D'^{-1}_2(p'_{22}\otimes q'_{22}), \\ D'^{-1}_2(p_{21}\otimes q_{21}) &= D_2^{-1}(p_{21}\otimes q_{21}). \end{aligned}$$

Proof. We observe from (3.1) that the range of $p_{21}\otimes q_{21}$ is in $f^p_{(0,1)}\mathcal{H}\otimes f^q_{(0,1)}\mathcal{H}$ on which D'_2 and D_2 are the same. Therefore, we have the second equation.

It remains to show the first. For a vector in the range of the projection $f^p_{\{1\}}\otimes f^q_{(0,1)} + f^p_{(0,1)}\otimes f^q_{\{1\}} + f^p_{\{1\}}\otimes f^q_{\{1\}}$, both sides give us 0 since we have $1-Q$ on the left side (see Lemma 4.3) and $p'_{22}f^p_{\{1\}} = q'_{22}f^q_{\{1\}} = 0$ from the definition. For a vector in the range of $f^p_{\{0\}}\otimes f^q_{(0,1]} + f^p_{(0,1]}\otimes f^q_{\{0\}} + f^p_{\{0\}}\otimes f^q_{\{0\}}$, the same thing happens again since the vector is killed by $p_{22}\otimes q_{22}$ and $p'_{22}\otimes q'_{22}$ ($p_{22}f^p_{\{0\}} = p'_{22}f^p_{\{0\}} = q_{22}f^q_{\{0\}} = q'_{22}f^q_{\{0\}} = 0$). On the other hand, the two operators are obviously the same against a vector in the range of $f^p_{(0,1)}\otimes f^q_{(0,1)}$, and hence we are done. ∎

It follows from Lemma 4.3 and Lemma 4.4 that

$$\begin{aligned} P_{\mathrm{op}} &= \begin{pmatrix} D_1^{-1} & 0 \\ 0 & D'^{-1}_2 \end{pmatrix}\begin{pmatrix} p_{11}\otimes q_{11} & p^*_{21}\otimes q^*_{21} \\ p_{21}\otimes q_{21} & p'_{22}\otimes q'_{22} \end{pmatrix} \\ &= P\left(\begin{pmatrix} p_{11} & p^*_{21} \\ p_{21} & p'_{22} \end{pmatrix}, \begin{pmatrix} q_{11} & q^*_{21} \\ q_{21} & q'_{22} \end{pmatrix}\right). \end{aligned}$$

Note that $1-f^p_{[0,1)} = f^p_{\{1\}}$ (respectively $1-f^q_{[0,1)} = f^q_{\{1\}}$) is the projection onto $\ker(1-p_{22})$ (respectively $\ker(1-q_{22})$). Therefore, the above two matrices $\begin{pmatrix} p_{11} & p^*_{21} \\ p_{21} & p'_{22} \end{pmatrix}$, $\begin{pmatrix} q_{11} & q^*_{21} \\ q_{21} & q'_{22} \end{pmatrix}$ are the characteristic matrices of the closures $\overline{T_{1c}}$, $\overline{T_{2c}}$ of the maximal closable parts respectively (see Subsection 2.4), and hence Theorem 4.1 implies

THEOREM 4.5. *Let T_1, T_2 be densely defined operators, and P_1, P_2 be the projections from $\mathcal{H} \oplus \mathcal{H}$ onto $\overline{\Gamma(T_1)}, \overline{\Gamma(T_2)}$ respectively. The operator part P_{op} of the projection $P = P(P_1, P_2)$ constructed from P_1 and P_2 is the characteristic matrix of the tensor product $\overline{T_{1c}} \otimes \overline{T_{2c}}$.*

COROLLARY 4.6. *With the same notation as in the above theorem, the algebraic tensor product $T_1 \otimes_{\mathrm{alg}} T_2$ is closable if and only if either both of T_1, T_2 are closable or one of them is identically zero.*

Proof. The algebraic tensor product is closable if and only if

$$(Q =) f^p_{(0,1)} \otimes f^q_{\{1\}} + f^p_{\{1\}} \otimes f^q_{(0,1)} + f^p_{\{1\}} \otimes f^q_{\{1\}} = 0$$

(see Lemma 4.3). When both of T_1, T_2 are closable, of course so is $T_1 \otimes_{\mathrm{alg}} T_2$. When none of them is closable, then $f^p_{\{1\}} \neq 0$ and $f^q_{\{1\}} \neq 0$ so that $f^p_{\{1\}} \otimes f^q_{\{1\}} \neq 0, Q \neq 0$ and hence $T_1 \otimes_{\mathrm{alg}} T_2$ is not closable.

For example, let us assume that T_1 is closable while T_2 is not, i.e., $f^p_{\{1\}} = 0$ and $f^q_{\{1\}} \neq 0$. Since $Q = f^p_{(0,1)} \otimes f^q_{\{1\}}$ in this case, we conclude that $Q = 0$ if and only if $f^p_{(0,1)} = 0$, that is, $p_{22} = 0$. Recalling (3.1), we easily observe that $p_{22} = 0$ if and only if T_1 is the zero operator. ∎

The following theorem clarifies the meaning of the projection $P = P(P_1, P_2)$:

THEOREM 4.7. *Let T_1, T_2 be densely defined operators, and P_1, P_2 be the projections from $\mathcal{H} \oplus \mathcal{H}$ onto $\overline{\Gamma(T_1)}, \overline{\Gamma(T_2)}$ respectively. Then, $P = P(P_1, P_2)$ is the projection onto the closure $\overline{\Gamma(T_1 \otimes_{\mathrm{alg}} T_2)}$.*

Proof. The range of P is obvious larger than $\Gamma(T_1 \otimes_{\mathrm{alg}} T_2)$ and hence than its closure (Lemma 3.7). Let us assume that a vector $\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \in \mathcal{H} \otimes \mathcal{H} \oplus \mathcal{H} \otimes \mathcal{H}$ is in the range of P (i.e., satisfying $P\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$) and perpendicular to $\Gamma(T_1 \otimes_{\mathrm{alg}} T_2)$. Choose and fix $\begin{pmatrix} \varphi'_1 \\ \varphi'_2 \end{pmatrix}, \begin{pmatrix} \psi'_1 \\ \psi'_2 \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H}$ from the ranges of P_1, P_2 respectively. Then, we can choose sequences $\left\{\begin{pmatrix} \varphi_i \\ T_1\varphi_i \end{pmatrix}\right\}_{i=1,2,\cdots}, \left\{\begin{pmatrix} \psi_j \\ T_2\psi_j \end{pmatrix}\right\}_{j=i,2,\cdots}$ (from $\Gamma(T_1)$ and $\Gamma(T_2)$ respectively) converging to $\begin{pmatrix} \varphi'_1 \\ \varphi'_2 \end{pmatrix}, \begin{pmatrix} \psi'_1 \\ \psi'_2 \end{pmatrix}$ respectively, and we have $\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} \perp \begin{pmatrix} \varphi_i \otimes \psi_j \\ T_1\varphi_i \otimes T_2\psi_j \end{pmatrix}$ by the assumption. For each fixed j the sequence $\left\{\begin{pmatrix} \varphi_i \otimes \psi_j \\ T_1\varphi_i \otimes T_2\psi_j \end{pmatrix}\right\}_{i=1,2,\cdots}$ in $\Gamma(T_1 \otimes_{\mathrm{alg}} T_2)$ converges to $\begin{pmatrix} \varphi'_1 \otimes \psi_j \\ \varphi'_2 \otimes T_2\psi_j \end{pmatrix}$ as $i \to \infty$ so this limit is perpendicular to $\begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$. By letting $j \to \infty$ next, we see that $\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}$ is also perpendicular to $\begin{pmatrix}\varphi'_1\otimes\psi'_1\\ \varphi'_2\otimes\psi'_2\end{pmatrix}$. Therefore, for vectors $\begin{pmatrix}\varphi'_1\\ \varphi'_2\end{pmatrix}=\begin{pmatrix}p_{11}\xi_1+p^*_{21}\xi_2\\ p_{21}\xi_1+p_{22}\xi_2\end{pmatrix}$, $\begin{pmatrix}\psi'_1\\ \psi'_2\end{pmatrix}=\begin{pmatrix}q_{11}\eta_1+q^*_{21}\eta_2\\ q_{21}\eta_1+q_{22}\eta_2\end{pmatrix}$ in the ranges of P_1, P_2, we can repeat the computations in the proof of Lemma 3.8 to conclude $\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$. ∎

This theorem means that $P_{\rm op}$ is the characteristic matrix of $\overline{(T_1\otimes_{\rm alg}T_2)_{\rm c}}$ (see Subsection 2.4 and the discussion after Corollary 4.2), and hence Theorem 4.5 guarantees

$$\overline{(T_1\otimes_{\rm alg}T_2)_{\rm c}}=\overline{T_{1\rm c}}\otimes\overline{T_{2\rm c}}.$$

Recall (see Subsection 2.1 and Subsection 2.4) that

$$\mathcal{D}((T_1\otimes_{\rm alg}T_2)_{\rm c})=\mathcal{D}(T_1\otimes_{\rm alg}T_2)=\mathcal{D}(T_1)\otimes_{\rm alg}\mathcal{D}(T_2),$$

$$\mathcal{D}(T_{1\rm c}\otimes_{\rm alg}T_{2\rm c})=\mathcal{D}(T_{1\rm c})\otimes_{\rm alg}\mathcal{D}(T_{2\rm c})=\mathcal{D}(T_1)\otimes_{\rm alg}\mathcal{D}(T_2).$$

Thus, by restricting the two operators to this common domain we have

THEOREM 4.8. *For densely defined operators T_1, T_2 the maximal closable part of the algebraic tensor product $T_1\otimes_{\rm alg}T_2$ is given by $(T_1\otimes_{\rm alg}T_2)_{\rm c}=T_{1\rm c}\otimes_{\rm alg}T_{2\rm c}$, and we have $\overline{(T_1\otimes_{\rm alg}T_2)_{\rm c}}=\overline{T_{1\rm c}}\otimes\overline{T_{2\rm c}}$.*

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Received April 1, 1997.