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INVARIANT SUBSPACES FOR DOUBLY COMMUTING CONTRACTIONS WITH RICH TAYLOR SPECTRUM

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Abstract. In the present paper we prove the existence of nontrivial common invariant subspaces for N-tuples of doubly commuting contractions with dominating Taylor spectrum for the H^∞ algebra of the polydisc. We also obtain the reflexivity of the WOT-closed subalgebra generated by finite systems of completely non-unitary doubly commuting contractions with dominating essential Taylor spectrum.

Keywords: *Invariant subspaces, reflexive operators, N-tuple of contractions, Taylor spectrum.*

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1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

One of the big steps in the invariant subspace problem for a single operator was the paper [4], where the existence of nontrivial invariant subspaces was proved for contractions having the dominating spectrum for the H^∞ algebra. The corresponding reflexivity result was shown in [2] and [14], where this property for completely non-unitary (c.n.u.) contractions with dominancy of left essential and essential spectrum, respectively, were obtained.

It is natural to ask if the above results are also valid for pairs or N-tuples of contractions. One of the approaches was made in [12], where reflexivity, thus also existence of nontrivial common invariant subspaces for N-tuples of doubly commuting c.n.u. contractions with dominancy of joint left essential spectrum was proved. In [11] a different result for this spectrum for pairs of contractions was shown. See also [6] and [8] for further results. However, the natural notion of a spectrum for N-tuples is the Taylor spectrum ([18], [19]). Some progress in that direction was made in [1]. In this note we assume double commutativity for the contractions and obtain a natural extension of the result from [4]. The definitions of the notation in the following statements will be given below.

THEOREM 1.1. *Let $T = (T_1, \dots, T_N)$ be an N-tuple of doubly commuting contractions. If the intersection of the Taylor spectrum with the open polydisc $\sigma(T) \cap \mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then $T = (T_1, \dots, T_N)$ has a common nontrivial invariant subspace.*

As we shall see, because of the double commutativity, it will be sufficient in the proof to consider the case where the Taylor spectrum of T coincides with the essential Taylor spectrum. This allows us to reduce the proof of Theorem 1.1 to the following:

THEOREM 1.2. *Let $T = (T_1, \dots, T_N)$ be an N-tuple of doubly commuting completely non-unitary contractions. If the intersection of the Taylor essential spectrum with the open polydisc $\sigma_{\mathrm{e}}(T) \cap \mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then $T = (T_1, \dots, T_N)$ is reflexive.*

Of course, the reflexivity is a much stronger property than having a nontrivial common invariant subspace. Theorem 1.2 is a generalization of the reflexivity result for a single contraction case from [2] and [14]. It also improves the result of [12], Theorem 5.4, with respect to the type of spectrum considered. Recently, the existence of a nontrivial common invariant subspace for spherical contracting instead of N-tuples of contractions is also considered, see [7].

Throughout this paper let $\mathcal{H}$ be a separable infinite-dimensional complex Hilbert space. If $\mathcal{R}$ is a subset of the algebra $L(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$, we denote by $\mathcal{W}(\mathcal{R})$, respectively $\mathcal{A}(\mathcal{R})$, the WOT (= weak operator topology)-closed, respectively the weak-star closed subalgebras of $L(\mathcal{H})$ generated by $\mathcal{R}$ and the identity Id. $\operatorname{Lat}\mathcal{R}$ will be the lattice of all (closed) invariant subspaces for $\mathcal{R}$, and $\operatorname{Alg}\operatorname{Lat}\mathcal{R}$ is as usual the algebra of all $T \in L(\mathcal{H})$ such that $T\mathcal{L} \subset \mathcal{L}$ for all $\mathcal{L} \in \operatorname{Lat}\mathcal{R}$. $\mathcal{R}$ is said to be *reflexive* if $\mathcal{W}(\mathcal{R}) = \operatorname{Alg}\operatorname{Lat}\mathcal{R}$. We say that a commutative set $\mathcal{R} \subset L(\mathcal{H})$ is *doubly commuting* (respectively *almost doubly commuting*) if $ST^* - T^*S$ is a zero (respectively compact) operator for all $S, T \in \mathcal{R}$ with $S \neq T$. In particular sets consisting of a single operator are doubly commuting.

Recall, that a set E contained in the open polydisc $\mathbb{D}^N$ is dominating for the algebra $H^\infty(\mathbb{D}^N)$ of all bounded analytic functions on $\mathbb{D}^N$, if for all $h \in H^\infty(\mathbb{D}^N)$ we have $\|h\|_\infty = \sup\limits_{z \in E} |h(z)|$. $A(\mathbb{D}^N)$ denotes the closed subalgebra of all those $h \in H^\infty(\mathbb{D}^N)$ having continuous extensions to the closed polydisc.

2. PRELIMINARIES FROM SEVERAL VARIABLES SPECTRAL THEORY

Recall from [18], [19] that the Koszul (cochain) complex $K(T,\mathcal{H})$ for a N-tuple $T=(T_1,\dots,T_N)$ of commuting operators in $L(\mathcal{H})$ with respect to $\mathcal{H}$ is given by

$$0 \longrightarrow \Lambda^0(\mathcal{H}) \xrightarrow{\delta^0(T)} \Lambda^1(\mathcal{H}) \xrightarrow{\delta^1(T)} \cdots \xrightarrow{\delta^{N-1}(T)} \Lambda^N(\mathcal{H}) \longrightarrow 0,$$

where $\Lambda^p(\mathcal{H})$ denotes the set of all p-forms with coefficients in $\mathcal{H}$ and the cochain mapping $\delta^p(T):\Lambda^p(\mathcal{H})\to\Lambda^{p+1}(\mathcal{H})$ is defined by

$$\delta^p(T)\sum_{|I|=p}{}' x_I s_I := \sum_{j=1}^{N}\ \sum_{|I|=p}{}' T_j x_I\, s_j\wedge s_I,$$

where $s_1,\dots,s_N$ is a fixed basis of $\Lambda^1(\mathbb{C})$ and $\sum\limits_{|I|=p}{}'$ denotes that the sum is taken over all $I=(i_1,\dots,i_p)\in\mathbb{N}^p$ with $1\leqslant i_1<\cdots<i_p\leqslant N$, $s_I:=s_{i_1}\wedge\cdots\wedge s_{i_p}$. Let us notice that $\Lambda^p(\mathcal{H})$ can be endowed with the natural scalar product

$$\left(\sum_{|I|=p}{}' x_I s_I, \sum_{|I|=p}{}' y_I s_I\right) := \sum_{|I|=p}{}'(x_I,y_I),$$

which gives us a canonical isomorphism with a direct sum of $\binom{N}{p}$ copies of $\mathcal{H}$. Following [18], [19], λ belongs to the Taylor spectrum $\sigma(T)\subset\mathbb{C}^N$ if, by definition, the complex $K(\lambda-T,\mathcal{H})$ is not exact, and λ belongs to the Taylor essential spectrum $\sigma_{\mathrm{e}}(T)\subset\mathbb{C}^N$ if, by definition, at least one of the cohomology groups $H^p(\lambda-T):=\ker\delta^p(\lambda-T)/\operatorname{ran}\delta^{p-1}(\lambda-T)$ has infinite dimension. Following [1], we can decompose $\sigma_{\mathrm{e}}(T)=\bigcup\limits_{p=0}^{N}\sigma_{\mathrm{e}}^p(T)$, where $\sigma_{\mathrm{e}}^p(T)$ is the set of all $\lambda\in\mathbb{C}^N$ such that the induced mapping

$$\hat{\delta}^p(\lambda-T):\Lambda^p(\mathcal{H})/\overline{\operatorname{ran}\delta^{p-1}(\lambda-T)}\to\Lambda^{p+1}(\mathcal{H})$$

has non-closed range or infinite dimensional kernel.

The points of $\sigma_{\mathrm{e}}^p(T)$ have the following property:

LEMMA 2.1. *Let $T=(T_1,\dots,T_N)$ be an N-tuple of commuting operators. If λ is a point in $\sigma_{\mathrm{e}}^p(T)$, there exists an orthonormal sequence $\{\eta_n\}_{n=1}^{\infty}$ in $\Lambda^p(\mathcal{H})$ such that*

$$\delta^{p-1}(\lambda-T)\delta^{p-1}(\lambda-T)^*\eta_n+\delta^p(\lambda-T)^*\delta^p(\lambda-T)\eta_n\to 0 \tag{2.1}$$

for $n\to\infty$.

Proof. Indeed, by the definition of $\sigma_{\mathrm{e}}^p(T)$ and a standard argument we can find an orthonormal sequence $\{\eta_n\}_{n=1}^{\infty}$ in $\Lambda^p(\mathcal{H})\ominus\operatorname{ran}\delta^{p-1}(\lambda-T)$ such that $\delta^p(\lambda-T)\eta_n\to 0$. Obviously, this sequence satisfies (2.1). ∎

We will need the following fact (see [5], Corollary 3.7).

LEMMA 2.2. *Let $T = (T_1, \ldots, T_N)$ be a N-tuple of almost doubly commuting operators. Then there is a compact operator K on $\Lambda^p(\mathcal{H})$ such that*

$$\delta^{p-1}(T)\delta^{p-1}(T)^* + \delta^p(T)^*\delta^p(T) = K + \bigoplus_F \sum_{j=1}^N T_F(j),$$

where the orthogonal sum is taken over all functions $F : \{1, \ldots, N\} \to \{0, 1\}$, $\operatorname{card}\{j : F(j) = 0\} = p$ and $T_F(j) = T_j T_j^*$ for $F(j) = 0$ and $T_F(j) = T_j^* T_j$ for $F(j) = 1$. Moreover, if the N-tuple doubly commutes, then $K = 0$.

We apply this lemma in the following form.

LEMMA 2.3. *Let $T = (T_1, \ldots, T_N)$ be a N-tuple of doubly commuting operators in $L(\mathcal{H})$. If, for some $\lambda \in \mathbb{C}^N$ and some $p \in \{1, \ldots, N\}$, we have*

$$\ker \delta^p(\lambda - T) \cap \operatorname{ran} \delta^{p-1}(\lambda - T)^{\perp} \neq \{0\},$$

then there is $i_0 \in \{1, \ldots, N\}$ such that the operator T_{i_0} has nontrivial hyperinvariant subspaces or the tuple consists of scalar operators.

Proof. By our assumptions, there exists some $0 \neq \omega \in \Lambda^p(\mathcal{H})$ with $\delta^p(\lambda - T)\omega = 0 = \delta^{p-1}(\lambda - T)^* \omega$.

Hence

$$\delta^{p-1}(\lambda - T)\, \delta^{p-1}(\lambda - T)^* \omega + \delta^p(\lambda - T)^* \delta^p(\lambda - T)\, \omega = 0.$$

By Lemma 2.2 there are disjoint sets $\mathcal{S}, \mathcal{T}$ with $\mathcal{S} \cup \mathcal{T} = \{1, \ldots, N\}$ and a vector $0 \neq x \in \mathcal{H}$ such that

$$\sum_{i \in \mathcal{S}} (\lambda_i - T_i)^*(\lambda_i - T_i)x + \sum_{k \in \mathcal{T}} (\lambda_k - T_k)(\lambda_k - T_k)^* x = 0.$$

Hence $x \in \ker(\lambda_i - T_i)$ for $i \in \mathcal{S}$ and $x \in \ker(\lambda_k - T_k)^*$ for $k \in \mathcal{T}$. From this we obtain a nontrivial hyperinvariant subspace or $T_j = \lambda_j$ for all $j \in \{1, \ldots, N\}$. ∎

Since the space $\mathcal{H}$ is infinite dimensional, we have the following.

COROLLARY 2.4. *Let $T = (T_1, \ldots, T_N)$ be a N-tuple of doubly commuting operators in $L(\mathcal{H})$. If $\sigma(T) \setminus \sigma_{\mathrm{e}}(T) \neq \emptyset$, then T_{i_0} has nontrivial hyperinvariant subspace for some $i_0 \in \{1, \ldots, N\}$.*

It seems to be a difficult problem to remove the double commutativity assumptions from the preceding statement.

3. REDUCTION TO THE APPROXIMATION PROBLEM

We start with *Reduction from Theorem* 1.1 *to Theorem* 1.2. If one of the contractions $T_1, \dots, T_N$, say T_{i_0}, has a nontrivial unitary part, then from the formula for the subspace $\mathcal{H}_{\mathrm{u}\, i_0}$ on which the contraction T_{i_0} is unitary (cf. [17], Theorem I.3.2), we see that $\mathcal{H}_{\mathrm{u}\, i_0}$ is invariant for all operators S doubly commuting with T_{i_0} and $\mathrm{Lat}(T)$ will be nontrivial. Hence we may assume that all $T_1, \dots, T_N$ are c.n.u.. Also, because of Corollary 2.4, we may assume that $\sigma(T) = \sigma_{\mathrm{e}}(T)$ and we are in the situation of Theorem 1.2. ∎

For further reductions we need some notation. Recall that a representation (a unital algebra homomorphism) $\Phi : A(\mathbb{D}^N) \to L(\mathcal{H})$ is called *contractive* if $\|\Phi(f)\| \leqslant \|f\|_\infty$ for $f \in A(\mathbb{D}^N)$. Standard techniques show that for every $x, y \in \mathcal{H}$ there is a complex, Borel, regular measure $\mu_{x,y}$ on $\overline{\mathbb{D}}^N$, called the *elementary measure,* satisfying $(\Phi(f)x, y) = \int f \,\mathrm{d}\mu_{x,y}$ for all $f \in A(\mathbb{D}^N)$. The representation Φ is called *absolutely continuous (a.c.)* if it has a system of elementary measures $\{\mu_{x,y}\}_{x,y\in\mathcal{H}}$ such that each measure $\mu_{x,y}$ is absolutely continuous with respect to some positive representing measure ν_z for some $z \in \mathbb{D}^N$.

We shall also need the language of dual algebras. Recall that $L(\mathcal{H}) = \mathcal{C}_1(\mathcal{H})^*$, where $\mathcal{C}_1(\mathcal{H})$ is the ideal of trace class operators and the duality is given by the form $\langle T, S\rangle := \mathrm{tr}(TS)$ for $T \in L(\mathcal{H})$, $S \in \mathcal{C}_1(\mathcal{H})$. Hence, every ultraweakly closed subalgebra $\mathcal{A}$ of $L(\mathcal{H})$ is a dual Banach space with predual space $\mathcal{Q}_{\mathcal{A}} \cong \mathcal{C}_1(\mathcal{H})/{}^{\perp}\mathcal{A}$ via $\langle T, [S]\rangle := \mathrm{tr}(TS)$ for $T \in \mathcal{A}$, $[S] \in \mathcal{C}_1(\mathcal{H})/{}^{\perp}\mathcal{A}$. Thus, for a rank one operator $x \otimes y$ $(z \mapsto (z, y)x)$, we have $\langle T, [x \otimes y]\rangle = (T\,x, y)$.

Now we can start the

Proof of Theorem 1.2. Under the assumptions of the theorem we can construct in a standard way, using a dilation ([17], Proposition I.9.2), a contractive representation $\Phi : A(\mathbb{D}^N) \to L(\mathcal{H})$ with $\Phi(z_i) = T_i$ for $i = 1, \dots, N$. Since $T_1, \dots, T_N$ are c.n.u., this representation is a.c. (see [10], Theorem 1), in which case we say that the N-tuple $T = (T_1, \dots, T_N)$ is a.c. By [12], Proposition 4.1 the functional calculus can be extended to $H^\infty(\mathbb{D}^N)$ as a homeomorphism for the weak-star topologies as $\Phi : H^\infty(\mathbb{D}^N) \to L(\mathcal{H})$. We will write $h(T)$ instead of $\Phi(h)$ for $h \in H^\infty(\mathbb{D}^N)$. Since the representation is contractive, to have an isometric functional calculus, we need the following known fact.

Lemma 3.1. *Suppose that the assumptions of Theorem* 1.2 *are satisfied. If* $f \in H^\infty(\mathbb{D}^N)$ *then* $\|f\|_\infty \leqslant \|f(T)\|$.

For the sake of completeness we include the elementary

Proof. Let $\lambda \in \sigma_{\mathrm{e}}(T) \cap \mathbb{D}^N \subset \sigma(T) \cap \mathbb{D}^N$. By the Gleason property of the polydisc $f(T) - f(\lambda) = \sum\limits_{i=1}^{N} (\lambda_i - T_i) u_i(T)$ for some $u_i \in H^\infty(\mathbb{D}^N)$. Then the operator $\sum\limits_{i=1}^{N} (\lambda_i - T_i) u_i(T)$ can not be invertible. Hence $f(\lambda) \in \sigma(f(T))$ and $\|f\|_\infty \leqslant r(f(T)) \leqslant \|f(T)\|$, which completes the proof of the lemma. ∎

Because of the isometry and the weak-star continuity of the functional calculus, the predual of $H^\infty(\mathbb{D}^N)$ and that of $\mathcal{A}(T)$ (which will be denoted by $\mathcal{Q}$) are isometrically isomorphic. Denote by $[C_\lambda]$ the element of $\mathcal{Q}$ corresponding to the point evaluation at $\lambda \in \mathbb{D}^N$. Since $\sigma_{\mathrm{e}}(T) \cap \mathbb{D}^N$ is dominating, standard techniques (see [4]) show that the set $\overline{\mathrm{aco}}\{[C_\lambda] : \lambda \in \sigma_{\mathrm{e}}(T) \cap \mathbb{D}^N\}$ contains the closed unit ball in $\mathcal{Q}$.

We shall need the well known fact from [3] that every dual algebra with property $X_{0,1}$ is reflexive. Recall that $\mathcal{A}$ has property $X_{0,1}$ if the unit ball of $\mathcal{Q}_{\mathcal{A}}$ is contained in $\mathcal{X}_{0,1}$ — the set of all those $[L] \in \mathcal{Q}_{\mathcal{A}}$ such that there exist sequences $\{x_n\}_{n=1}^\infty$, $\{y_n\}_{n=1}^\infty$, $\subset \mathcal{H}$ with $\|x_n\| \leqslant 1, \|y_n\| \leqslant 1$ for all n, such that

$$\lim_{n\to\infty} \|[x_n \otimes y_n] - [L]\|_{\mathcal{Q}} = 0, \tag{3.1}$$

$$\lim_{n\to\infty} \|[x_n \otimes w]\|_{\mathcal{Q}} = 0 \qquad \text{for all } w \in \mathcal{H}, \tag{3.2}$$

$$\lim_{n\to\infty} \|[w \otimes y_n]\|_{\mathcal{Q}} = 0 \qquad \text{for all } w \in \mathcal{H}. \tag{3.3}$$

Since $\mathcal{X}_{0,1}$ is absolutely convex and closed ([3]), it suffices to show that $\mathcal{X}_{0,1}$ contains all $[C_\lambda]$, $\lambda \in \sigma_{\mathrm{e}}(T) \cap \mathbb{D}^N$.

Hence, let us take $\lambda \in \sigma_{\mathrm{e}}(T) \cap \mathbb{D}^N$. Then, by Lemma 2.1 there is a number $p \in \{1, \ldots, N\}$ and an orthonormal sequence $\{\eta_n\}_{n=1}^\infty$ in $\Lambda^p(\mathcal{H})$ such that (2.1) holds. Passing, if necessary, to some subsequence, we may assume that for some $I = (i_1, \ldots, i_p) \in \mathbb{N}^p$ the coefficients x_n of $s_{i_1} \wedge \cdots \wedge s_{i_p}$ in η_n satisfy $\|x_n\| \geqslant \alpha$ for all $n \in \mathbb{N}$ and some $\alpha > 0$. By Lemma 2.2 there are disjoint sets $\mathcal{S}, \mathcal{T}$ with $\mathcal{S} \cup \mathcal{T} = \{1, \ldots, N\}$ such that

$$\sum_{i \in \mathcal{S}} (\lambda_i - T_i)^* (\lambda_i - T_i) x_n + \sum_{k \in \mathcal{T}} (\lambda_k - T_k)(\lambda_k - T_k)^* x_n \to 0.$$

Taking the scalar product with x_n we get $\|(T_i - \lambda_i) x_n\| \to 0$ for all $i \in \mathcal{S}$ and $\|(T_k^* - \overline{\lambda}_k) x_n\| \to 0$ for all $k \in \mathcal{T}$. The sequence $\{\eta_n\}_{n=1}^\infty$ being orthonormal we have $x_n \to 0$ weakly. Moreover, since the numbers $\|x_n\|$ are bounded below, we can assume, without loss of generality, that $\|x_n\| = 1$. The approximation lemmas in the next section will now prove that $[C_\lambda] \in \mathcal{X}_{0,1}$ for $\lambda \in \sigma_{\mathrm{e}}(T) \cap \mathbb{D}^N$. This will show the following structure result.

Proposition 3.2. *Let $T = (T_1, \ldots, T_N)$ be an N-tuple of doubly commuting completely non-unitary contractions. If the intersection of the Taylor essential spectrum with the open polydisc $\sigma_{\mathrm{e}}(T) \cap \mathbb{D}^N$ is dominating for $H^\infty(\mathbb{D}^N)$, then the algebra $\mathcal{A}(T_1, \ldots, T_N)$ has property $X_{0,1}$.*

The proposition implies the reflexivity of $\mathcal{A}(T_1, \ldots, T_N)$, hence also of $T = (T_1, \ldots, T_N)$.

4. THE APPROXIMATION LEMMAS

We will start with the approximation of the point evaluation.

Lemma 4.1. *Let $T = (T_1, \ldots, T_N)$ be an a.c. N-tuple of commuting contractions, $\mathcal{S}, \mathcal{T}$ be disjoint sets such that $\mathcal{S} \cup \mathcal{T} = \{1, \ldots, N\}$, and $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{D}^N$. Assume that $x_n \to 0$ weakly, $\|x_n\| = 1$ for all n. If $\|(T_i - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{S}$ and $\|(T_i^* - \overline{\lambda}_i)x_n\| \to 0$ for all $i \in \mathcal{T}$ then $\lim\limits_{n\to\infty} \|[x_n \otimes x_n] - [C_\lambda]\|_{\mathcal{Q}} = 0$.*

Proof. By the Hahn-Banach theorem, for each n, there exist some $f_n \in H^\infty(\mathbb{D}^N)$ such that $\|f_n(T)\| = \|f_n\| = 1$ and $\|[x_n \otimes x_n] - [C_\lambda]\|_{\mathcal{Q}} = |\langle f_n(T), [x_n \otimes x_n] - [C_\lambda]\rangle|$. The polydisc has the Gleason property, thus there are $g_i^n \in H^\infty(\mathbb{D}^N)$ satisfying $\|g_i^n\| \leqslant M_\lambda$ for $i = K+1, \ldots, N$ and $f_n(z) = f_n(\lambda) - \sum\limits_{i=1}^{N} (z_i - \lambda_i) g_i^n(z)$, where $z = (z_1, \ldots, z_N) \in \mathbb{D}^N$. Hence

$$\begin{aligned}
&\|[x_n \otimes x_n] - [C_\lambda]\|_{\mathcal{Q}} \\
&\quad = \Big|\Big\langle f_n(\lambda) + \sum_{i=1}^{N} (T_i - \lambda_i) g_i^n(T),\, [x_n \otimes x_n] - [C_\lambda]\Big\rangle\Big| \\
&\quad = \Big|\Big(\sum_{i=1}^{N} (T_i - \lambda_i) g_i^n(T) x_n, x_n\Big)\Big| \\
&\quad \leqslant \sum_{i\in\mathcal{S}} |(g_i^n(T)(T_i - \lambda_i)x_n, x_n)| + \sum_{i\in\mathcal{T}} |(g_i^n(T)x_n, (T_i^* - \overline{\lambda}_i)x_n)| \\
&\quad \leqslant \sum_{i\in\mathcal{S}} \|g_i^n(T)\|\, \|(T_i - \lambda_i)x_n\| + \sum_{i\in\mathcal{T}} \|g_i^n(T)\|\, \|(T_i^* - \overline{\lambda}_i)x_n\| \\
&\quad \leqslant M_\lambda \Big(\sum_{i\in\mathcal{S}} \|(T_i - \lambda_i)x_n\| + \sum_{i\in\mathcal{T}} \|(T_i^* - \overline{\lambda}_i)x_n\|\Big).
\end{aligned}$$

Thus, the proof of the lemma is finished since $\|(T_i - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{S}$ and $\|(T_i^* - \overline{\lambda}_i)x_n\| \to 0$ for all $i \in \mathcal{T}$. ∎

Next, the approximate orthogonality (3.2) will be shown.

LEMMA 4.2. *Let $T = (T_1, \dots, T_N)$ be an a.c. N-tuple of commuting contractions. Let $\mathcal{S}, \mathcal{T}$ be disjoint sets such that $\mathcal{S} \cup \mathcal{T} = \{1, \dots, N\}$. Assume that $\{T_i : i \in \mathcal{S}\}$ is doubly commuting. Let $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{D}^N$ and $x_n \to 0$ weakly, $\|x_n\| = 1$ for all n. If $\|(T_i - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{S}$ and $\|(T_i^* - \overline{\lambda}_i)x_n\| \to 0$ for all $i \in \mathcal{T}$ then $\lim\limits_{n\to\infty} \|[y \otimes x_n]\|_{\mathcal{Q}} = 0$ for all $y \in \mathcal{H}$.*

Proof. Without loss of generality we can assume that $\mathcal{S} = \{1, \dots, K\}$. There can be found some functions $f_n \in H^\infty(\mathbb{D}^N)$ such that $\|f_n(T)\| = \|f_n\| = 1$ and $\|[y \otimes x_n]\|_{\mathcal{Q}} = |(f_n(T)y, x_n)|$. Notice that

$$\|[y \otimes x_n]\|_{\mathcal{Q}} \leqslant |(f_n(T_1, \dots, T_K, \lambda_{K+1}, \dots, \lambda_N)y, x_n)| + |((f_n(T_1, \dots, T_N) \\ - f_n(T_1, \dots, T_K, \lambda_{K+1}, \dots, \lambda_N))y, x_n)|.$$

By an obvious modification of the Gleason property for polydomains, there are functions $g_{K+1}^n, \dots, g_N^n \in H^\infty(\mathbb{D}^N)$ such that $\|g_i^n\| \leqslant M_\lambda$ for $i = K+1, \dots, N$ and

$$f_n(z) - f_n(z_1, \dots, z_K, \lambda_{K+1}, \dots, \lambda_N) = \sum_{i=K+1}^{N} (\lambda_i - z_i) g_i^n(z),$$

where $z = (z_1, \dots, z_N)$. The point λ being fixed, let us denote by $h_n \in H^\infty(\mathbb{D}^K)$ the function $h_n(z_1, \dots, z_K) = f_n(z_1, \dots, z_K, \lambda_{K+1}, \dots, \lambda_N)$ and write $\widetilde{T} = (T_1, \dots, T_K)$. Obviously, we have a natural functional calculus for $\widetilde{T} = (T_1, \dots, T_K)$. Hence, for any $\varepsilon > 0$, we have

$$\begin{aligned} \|[y \otimes x_n]\|_{\mathcal{Q}} &\leqslant |(f_n(T_1, \dots, T_K, \lambda_{K+1}, \dots, \lambda_N)y, x_n)| \\ &\qquad + \Big|\Big(\sum_{i=K+1}^{N} (\lambda_i - T_i) g_i^n(T)y, x_n \Big)\Big| \\ &\leqslant |(h_n(\widetilde{T})y, x_n)| + \sum_{i=K+1}^{N} \|g_i^n(T)\| \, \|y\| \, \|(\overline{\lambda}_i - T_i^*)x_n\| \\ &\leqslant |(h_n(\widetilde{T})y, x_n)| + \varepsilon, \end{aligned}$$

for n sufficiently large, since $\|(T_i^* - \overline{\lambda}_i)x_n\| \to 0$ for $i = K+1, \dots, N$.

By the same construction as in [15], p. 1234 (see also [13], Lemma 1), we can construct a doubly commuting K-tuple of isometries $V = (V_1, \dots, V_K) \subset L(\mathcal{K})$, which is a minimal isometric dilation of the K-tuple $\widetilde{T}^* = (T_1^*, \dots, T_K^*)$. Then $V^* = (V_1^*, \dots, V_K^*)$ is an extension of $\widetilde{T} = (T_1, \dots, T_K)$. By the minimality, as in [11] using [10], Section 3 (see also [9], [12]) the K-tuple $V = (V_1, \dots, V_K)$ is a.c. and so is V^*, see [11], Lemma 2.1. Moreover, by [12], Proposition 4.1 we can construct a functional calculus for each of them.

For any contraction A and any $\mu \in \mathbb{D}$, we will denote by A^{μ} the operator $(A-\mu)(\mathrm{Id}-\overline{\mu}A)^{-1}$. Let $\mathcal{K} = \mathcal{K}_{\mathrm{s}}^{i} \oplus \mathcal{K}_{\mathrm{u}}^{i}$ be the decomposition of V_i in a shift and a unitary part, respectively. The decomposition coincides with the decomposition of $V_i^{\lambda_i}$ (see [17], Proposition I.4.3 and its proof). Moreover $(V^{\lambda})^* = ((V_1{}^{\lambda_1})^*, \dots, (V_K{}^{\lambda_K})^*)$ is an extension of $\widetilde{T}^{\lambda} = (T_1{}^{\lambda_1}, \dots, T_K{}^{\lambda_K})$. Since $\|(T_i - \lambda_i)x_n\| \to 0$ for $i = 1, \dots, K$, we have $\|T_i^{\lambda_i} x_n\| \to 0$ for $i = 1, \dots, K$.

By the double commutativity of $V = (V_1, \dots, V_K)$, using [16], Theorem 3, we obtain an orthogonal decomposition $\mathcal{K} = \mathcal{K}_{\mathrm{s}} \oplus \mathcal{K}_{\mathrm{r}} = \mathcal{K}_{\mathrm{s}} \oplus \mathcal{K}_{\mathrm{r}_1} \oplus \cdots \oplus \mathcal{K}_{\mathrm{r}_K}$, into reducing invariant subspaces for V, such that $V_i^{\mathrm{s}} = V_i|_{\mathcal{K}_{\mathrm{s}}}$ is a shift operator and $V_i|_{\mathcal{K}_{\mathrm{r}_i}}$ is a unitary operator for all $i = 1, \dots, K$. We will write $V_i^{\mathrm{r}} = V_i|_{\mathcal{K}_{\mathrm{r}}}$. Form $x_n = x_n^{\mathrm{s}} \oplus x_n^{\mathrm{r}}$, $y = y^{\mathrm{s}} \oplus y^{\mathrm{r}}$ with respect to this orthogonal decomposition. Let P_i denote the projection onto $\ker(V_i^{\lambda_i})^*$. By the double commutativity one can see that P_i, P_j commute for $i, j = 1, \dots, K$. We have also $P_1 \cdots P_K \mathcal{H} \subset \mathcal{K}_{\mathrm{s}}$. Thus

$$
\begin{aligned}
\|x_n^{\mathrm{r}}\| &\leqslant \|x_n - P_1 \cdots P_K x_n\| \leqslant \sum_{i=1}^{K} \|P_1 \cdots P_{i-1}(x_n - P_i x_n)\| \\
&\leqslant \sum_{i=1}^{K} \|x_n - P_i x_n\| = \sum_{i=1}^{K} \|V_i^{\lambda_i}(V_i^{\lambda_i})^* x_n\| \\
&= \sum_{i=1}^{K} \|(V_i^{\lambda_i})^* x_n\| = \sum_{i=1}^{K} \|T_i^{\lambda_i} x_n\| \to 0.
\end{aligned}
$$

There is also a natural functional calculus for $V_{\mathrm{s}}^* = (V_1^{\mathrm{s}*}, \dots, V_K^{\mathrm{s}*})$ and for $V_{\mathrm{r}}^* = (V_1^{\mathrm{r}*}, \dots, V_K^{\mathrm{r}*})$, since V_{s}^* and V_{r}^* are the restrictions of V^*. Hence, because of $\|x_n^{\mathrm{r}}\| \to 0$, we have

$$
\begin{aligned}
|(h_n(\widetilde{T})y,\, x_n)| = |(h_n(V^*)y,\, x_n)| &\leqslant |(h_n(V_{\mathrm{s}}^*)y^{\mathrm{s}},\, x_n^{\mathrm{s}})| + |(h_n(V_{\mathrm{r}}^*)y^{\mathrm{r}},\, x_n^{\mathrm{r}})| \\
&\leqslant |(h_n(V_{\mathrm{s}}^*)y^{\mathrm{s}},\, x_n^{\mathrm{s}})| + \|h_n\|\, \|y^{\mathrm{r}}\|\, \|x_n^{\mathrm{r}}\| \leqslant |(h_n(V_{\mathrm{s}}^*)y^{\mathrm{s}},\, x_n^{\mathrm{s}})| + \varepsilon,
\end{aligned}
$$

for n sufficiently large. Since $\|x_n^{\mathrm{r}}\| \to 0$, we have $\|x_n^{\mathrm{s}}\| \to 1$, hence we may assume that $\|x_n^{\mathrm{s}}\| = 1$.

Now we know that $V_i^{\mathrm{s}*}$ is a shift, for $i = 1, \dots, K$. For $2 < M \in \mathbb{N}$ let R_M be the orthogonal projection onto the space $\bigvee_{i=1}^{K} \operatorname{ran} V_i^{\mathrm{s}M}$. Using the obvious extension of [16], Theorem 1, from pairs to K-tuples of doubly commuting shifts, there is M (sufficiently large) such that $\|R_M\, y^{\mathrm{s}}\| \leqslant \varepsilon/2$. Let $y_1 = (\mathrm{Id} - R_M)y^{\mathrm{s}}$ and $y_2 = R_M\, y^{\mathrm{s}}$. Then $V_i^{\mathrm{s}*M}\, y_1 = 0$ for $i = 1, \dots, K$. We can write

$$
h_n(z) = \sum_{|I|=0}^{M-1} a_I^n z^I + \sum_{i=1}^{K} z_i^M q_i^n(z),
$$

where $z = (z_1, \dots, z_K)$, $a_I^n \in \mathbb{C}$, $q_i^n \in H^\infty(\mathbb{D}^K)$. Moreover, since a_I^n is a Fourier coefficient of h_n we can estimate that $|a_I^n| \leqslant 1$.

Since $x_n^{\mathrm{s}} \to 0$ weakly, we have for n sufficiently large $|(V_I^{*I} y_1, x_n^{\mathrm{s}})| \leqslant \varepsilon/2M^K$, for all I such that $|I| \leqslant M-1$. Hence

$$\begin{aligned} |(h_n(V_{\mathrm{s}}^*)y^{\mathrm{s}},\, x_n^{\mathrm{s}})| &\leqslant |(h_n(V_{\mathrm{s}}^*)y_2,\, x_n^{\mathrm{s}})| + \sum_{|I|=0}^{M-1} |a_I^n|\, |(V_{\mathrm{s}}^{*I} y_1, x_n^{\mathrm{s}})| \\ &\quad + \sum_{i=1}^{K} |(q_i^n(V_{\mathrm{s}}^*) V_i^{\mathrm{s}*M}\, y_1, x_n^{\mathrm{s}})| \\ &\leqslant \|h_n\|\, \|y_2\|\, \|x_n^{\mathrm{s}}\| + \sum_{i=1}^{K} \frac{\varepsilon}{2M^K} + 0 \leqslant 2\varepsilon. \end{aligned}$$

Thus, for n sufficiently large $\|[y \otimes x_n]\|_{\mathcal{Q}} \leqslant 4\varepsilon$. ∎

The second orthogonality condition (3.3) turns out to be symmetric to the previous one.

LEMMA 4.3. *Let $T = (T_1, \dots, T_N)$ be an a.c. N-tuple of commuting contractions. Let $\mathcal{S}', \mathcal{T}'$ be disjoint sets such that $\mathcal{S}' \cup \mathcal{T}' = \{1, \dots, N\}$. Assume that $\{T_i : i \in \mathcal{T}'\}$ is doubly commuting. Let $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{D}^N$ and $x_n \to 0$ weakly, $\|x_n\| = 1$ for all n. If $\|(T_i - \lambda_i)x_n\| \to 0$ for all $i \in \mathcal{S}'$ and $\|(T_i^* - \overline{\lambda}_i)x_n\| \to 0$ for all $i \in \mathcal{T}'$ then $\lim\limits_{n\to\infty} \|[x_n \otimes y]\|_{\mathcal{Q}} = 0$ for all $y \in \mathcal{H}$.*

Proof. The set $\{T_i^* : i \in \mathcal{T}'\}$ is also doubly commuting and we can apply Lemma 4.2 to $\mathcal{S} = \mathcal{T}'$ and $\mathcal{T} = \mathcal{S}'$. Hence we get $\|[x_n \otimes y]\|_{\mathcal{Q}} = |[y \otimes x_n]\|_{\mathcal{Q}(\mathcal{A}(\mathcal{T}^*))} \to 0$. ∎

5. FINAL REMARKS

REMARK 5.1. Notice that (with the same proof) the c.n.u. assumptions for contractions in Theorem 1.2 can be replaced by the requirement that the N-tuple is a.c..

REMARK 5.2. Let $T = (T_1, \dots, T_N)$ be a N-tuple fulfilling the assumptions of Theorem 1.1 or Theorem 1.2. Let $T^a = (T_1^a, \dots, T_N^a)$ be an N-tuple such that $T_i^a = T_i$ or $T_i^a = T_i^*$ for $i = 1, \dots, N$. Then Lemma 2.2 and easy calculations show that the dominating property still holds for T^a. Thus, Theorem 1.1 or Theorem 1.2, respectively, holds for T^a.

Some of our methods work even under the weaker conditions that the contractions are only almost doubly commuting.

PROPOSITION 5.3. *Let $T = (T_1, T_2)$ be an a.c. pair of almost doubly commuting contractions. If the middle part $\sigma_e^1(T)$ of the essential Taylor spectrum is dominating for $H^\infty(\mathbb{D}^2)$, then $T = (T_1, T_2)$ is reflexive.*

Proof. We only need to make some remarks, which cover the difference to the proof of Theorem 1.2. First, for any pair of contractions there is a unitary dilation ([17], Theorem I.6.4), so we can construct the functional calculus and it will have the same property as before. Next, since compact operators convert weak convergence of sequences of vectors to strong convergence, using Lemma 2.2, one can see that a compact perturbation does not disturb us in choosing the sequence x_n in Section 3. Finally, notice that we consider the spectrum $\sigma_e^1(T_1, T_2)$ and that $\mathcal{S}$ and $\mathcal{T}$ have exactly one element and all the approximation Lemmas 4.1, 4.2, 4.3 remain true. ∎

REMARK 5.4. Let us observe that the statement of the preceding proposition is still valid if the dominancy assumption is replaced by one of the following conditions:

(i) T_1 is of class $C_{0\cdot}$, T_2 is of class $C_{\cdot 0}$ and $\sigma_e(T)$ is dominating for $H^\infty(\mathbb{D}^2)$;

(ii) one of the operators is of class $C_{0\cdot}$ and $\sigma_e^0(T) \cup \sigma_e^1(T)$ is dominating for $H^\infty(\mathbb{D}^2)$;

(iii) one of the operators is of class $C_{\cdot 0}$ and $\sigma_e^1(T) \cup \sigma_e^2(T)$ is dominating for $H^\infty(\mathbb{D}^2)$;

(iv) T is diagonally extendable (for definition see [11]) and $\sigma_e^0(T) \cup \sigma_e^1(T)$ is dominating for $H^\infty(\mathbb{D}^2)$.

For the proof of the approximation lemmas for $\lambda \in \sigma_e^1(T)$ we proceed as in the proof of Proposition 5.3. For the left case (i.e. $\lambda \in \sigma_e^0(T)$) one uses the methods of [11]. Finally, the right exterior case (i.e. $\lambda \in \sigma_e^2(T)$) follows by duality.

Note added in proof. B. Chevreau announced during the conference "Deuxiemes Journées Lilloises de Théorie des Opérateurs", March 1997, that he removed from Theorem 1.2 the doubly commutativity assumption keeping the existence of a unitary dilation.

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