J. OPERATOR THEORY
40(1998), 409–417

A NEW LOMONOSOV LEMMA

BERNARD CHEVREAU, WING SUET LI and CARL PEARCY

Communicated by Florian-Horia Vasilescu

ABSTRACT. In 1973 V. Lomonosov obtained a beautiful result on the existence of nontrivial invariant subspaces for compact-related operators on Banach space. The key idea has become known as "Lomonosov's lemma". Since then, generalizations have been obtained by various authors. In this note we give more elementary proofs for the main theorems from [8] (and [4]), including a "new" Lomonosov-type lemma. This allows us to generalize earlier work in this area.

KEYWORDS: *Invariant subspaces, transitive algebras, Lomonosov lemma.*

MSC (2000): Primary 47A15; Secondary 47B07.

Throughout this note $\mathcal{X}$ will be an arbitrary complex, infinite dimensional, Banach space, and $\mathcal{L}(\mathcal{X})$ will denote the algebra of all bounded linear operators on $\mathcal{X}$. In what follows we write $\mathcal{K}(\mathcal{X})$ for the closed ideal of compact operators in $\mathcal{L}(\mathcal{X})$ and π for the quotient map of $\mathcal{L}(\mathcal{X})$ onto $\mathcal{L}(\mathcal{X})/\mathcal{K}(\mathcal{X})$. The spectrum of an operator T in $\mathcal{L}(\mathcal{X})$ will be denoted as usual by $\sigma(T)$, the point spectrum (i.e. the set of eigenvalues) of T by $\sigma_{\mathrm{p}}(T)$, and the essential spectrum of T (i.e., the spectrum of $\pi(T)$) by $\sigma_{\mathrm{e}}(T)$. The essential norm of T (i.e., the norm of $\pi(T)$) will be denoted by $\|T\|_{\mathrm{e}}$, and the convex hull of a set S in a linear space by $\mathrm{conh}(S)$.

Recall that a subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{X})$ is said to be transitive if the only invariant subspaces of $\mathcal{A}$ are (0) and $\mathcal{X}$. In 1973, V. Lomonosov ([7]) proved an elegant and powerful result via the Schauder-Tychonoff fixed point theorem that had as a consequence some spectacular progress on the existence of nontrivial invariant and hyperinvariant subspaces for compact-related operators in $\mathcal{L}(\mathcal{X})$. (The reader may consult [10] or [11] for a survey of these results.) This key theorem has come to be called "Lomonosov's lemma", and goes as follows.

THEOREM 1.1. ([7]) *Let $\mathcal{A}$ be a transitive subalgebra of $\mathcal{L}(\mathcal{X})$ and let K be a nonzero operator in $\mathcal{K}(\mathcal{X})$. Then there exists an operator $A \in \mathcal{A}$ such that $1 \in \sigma_{\mathrm{p}}(AK)$.*

More recently Lomonosov ([8]) obtained some stronger results than those in [7] using a variety of different and sophisticated techniques, and Scott Brown ([4]) proved another "Lomonosov-type lemma" in the case when $\mathcal{X}$ is a Hilbert space.

In this paper, we first prove a "new" theorem (Theorem 1.2), which is a modification and distillation of what was proved in [7] and [4]. Then, on the basis of this, we prove a new "Lomonosov-type lemma" (Theorem 1.4), which is stronger than Theorem 1.1 above. On the basis of these two results, we give proofs of the main theorems from [8] (and [4]) which are somewhat more elementary than those in [8]. In addition to Theorems 1.2 and 1.4, we also establish two additional results (Theorems 1.9 and 1.10) which generalize earlier work in this area (cf. [3]).

Our first result is the following.

THEOREM 1.2. *Let $\mathcal{C} \subset \mathcal{L}(\mathcal{X})$ be a set and suppose there exist $y_0 \in \mathcal{X}^*$ and $\rho_0 > \|y_0\| = 2$ such that for every y in $\mathcal{X}^*$ satisfying $\|y - y_0\| \leqslant 1$, y_0 belongs to the norm closure of the set $\{C^*y : C \in \mathrm{conh}(\mathcal{C}), \|C\|_{\mathrm{e}} \leqslant 1/\rho_0\}$. Then there exists a convex combination C_0 of elements of $\mathcal{C}$ such that $1 \in \sigma_{\mathrm{p}}(C_0)$ and $\|C_0\|_{\mathrm{e}} \leqslant 1/\rho_0$. Consequently, $\|C_0^*\|_{\mathrm{e}} \leqslant 1/\rho_0$, 1 is an isolated point in both $\sigma(C_0)$ and $\sigma(C_0^*)$, the operators $C_0 - 1$ and $C_0^* - 1$ are Fredholm operators of index zero, and the root spaces in $\mathcal{X}$ and $\mathcal{X}^*$ corresponding to the eigenvalue 1 of C_0 and C_0^* are finite dimensional.*

Proof. Let $\mathcal{D}$ denote the closed ball of radius one centered at y_0 in $\mathcal{X}^*$, and observe that $\mathcal{D}$ is weak* compact. Fix $\rho, \varepsilon > 0$ such that $2 < \rho < \rho_0$ and $2/\rho + 2\varepsilon < 1$. For each $y \in \mathcal{D}$, let $C_y \in \mathrm{conh}(\mathcal{C})$ satisfy $\|C_y^* y - y_0\| < \varepsilon$ and $\|C_y\|_{\mathrm{e}} \leqslant 1/\rho_0$. Write $C_y = T_y + K_y$ with K_y compact such that $\|T_y\| < 1/\rho$. The operator $K_y^* : (\mathcal{D}, w^*) \to (\mathcal{X}^*, \|\cdot\|)$ is continuous, so the set $\mathcal{U}_y = \{z \in \mathcal{D} : \|K_y^* z - K_y^* y\| < \varepsilon\}$ (being the inverse image of an open set in the metric topology on $\mathcal{X}^*$) is weak* open in $\mathcal{D}$, and $y \in \mathcal{U}_y$. Thus $\{\mathcal{U}_y\}_{y \in \mathcal{D}}$ is a weak* open cover of $\mathcal{D}$, so has a finite subcover $\{\mathcal{U}_{y_1}, \ldots, \mathcal{U}_{y_n}\}$. We observe next that $C_y^*(\mathcal{U}_y) \subset \mathcal{D}$ for every $y \in \mathcal{D}$, since for every such y and for every $z \in \mathcal{U}_y$,

$$\begin{aligned}\|C_y^* z - y_0\| &\leqslant \|C_y^* z - C_y^* y\| + \|C_y^* y - y_0\| \leqslant \|C_y^* z - C_y^* y\| + \varepsilon \\ &\leqslant \|T_y^*(z - y)\| + \|K_y^*(z - y)\| + \varepsilon \leqslant 2/\rho + \varepsilon + \varepsilon < 1.\end{aligned}$$

Now choose $f_1,\dots,f_n$ weak* continuous functions from $\mathcal{D}$ to $[0,1]$ such that the support of f_i is contained in $\mathcal{U}_{y_i}$, $i=1,\dots,n$, and $\sum_{i=1}^n f_i = 1$ on $\mathcal{D}$. Next we define $g:\mathcal{D}\to\mathcal{X}^*$ by $g(w)=\sum_{i=1}^n f_i(w)C_{y_i}^*(w)$, $w\in\mathcal{D}$. Since adjoint operators acting on $\mathcal{X}^*$ are $(\mathcal{X}^*,w^*)\to(\mathcal{X}^*,w^*)$ continuous, $g:(\mathcal{D},w^*)\to(\mathcal{X}^*,w^*)$ is continuous. We show that $g(\mathcal{D})\subset\mathcal{D}$. To this end, let $w\in\mathcal{D}$. Then

$$\|g(w)-y_0\| = \Big\|\sum_{i=1}^n f_i(w)(C_{y_i}^*(w)-y_0)\Big\| \leqslant \sum_{i=1}^n |f_i(w)|\,\|C_{y_i}^*(w)-y_0\|.$$

But w belongs to some $\mathcal{U}_{y_i}$ (perhaps more than one), and for each such, $\|C_{y_i}^*(w)-y_0\|\leqslant 1$, whereas for the others, $f_i(w)=0$, so $\|g(w)-y_0\|\leqslant 1$. Thus g maps $\mathcal{D}$ into itself, so by the Schauder-Tychonoff fixed point theorem for locally convex spaces ([6], p. 456), g has a fixed point $w_0\in\mathcal{D}$. Define $C_0=\sum_{i=1}^n f_i(w_0)C_{y_i}$. Then C_0 is a convex combination of (convex combinations of) elements of $\mathcal{C}$, $\|C_0\|_{\mathrm{e}}\leqslant 1/\rho_0 < 1/2$, and $C_0^*(w_0)=w_0$, which proves that $1\in\sigma_{\mathrm{p}}(C_0^*)$. Since the map $T\mapsto T^*$ is known to decrease essential norms [1], $\|C_0^*\|_{\mathrm{e}}\leqslant 1/\rho_0<1/2$ also. Thus the essential spectral radii of C_0 and C_0^* are both less then $1/2$, and consequently 1 belongs to neither essential spectrum. Since $1\in\sigma(C_0)=\sigma(C_0^*)$, the other statements of the theorem now are immediate consequences of the Fredholm theory. ∎

The following lemma, while elementary, makes the proofs to follow somewhat simpler.

Lemma 1.3. *Suppose $\mathcal{C}\subset\mathcal{L}(\mathcal{X})$ [respectively, $\mathcal{C}\subset\mathcal{L}(\mathcal{X}^*)$, $\mathcal{C}\subset\mathcal{L}(\mathcal{X}^{**})$] is a convex set and there exists a net $\{C_\lambda\}$ in $\mathcal{C}$ converging to an operator C_0 in the weak operator topology and satisfying $\|C_\lambda\|_{\mathrm{e}}\to 0$. Then there exists another net $\{C_\mu\}$ in $\mathcal{C}$ converging to C_0 in the strong operator topology and satisfying $\|C_\mu\|_{\mathrm{e}}\to 0$.*

Proof. Let Γ be the directed set of strong operator topology open neighborhoods of C_0 in $\mathcal{L}(\mathcal{X})$ [resp. $\mathcal{L}(\mathcal{X}^*)$, $\mathcal{L}(\mathcal{X}^{**})$], and consider the directed set $M=\Gamma\times\mathbb{R}_+$ where $\mu=(\gamma,r)\geq\mu'=(\gamma',r')$ in M if and only if $\gamma\subset\gamma'$ and $(0<)r\leqslant r'$. Fix an element (γ_0,r_0) of M. To prove the lemma, it suffices to exhibit C_{μ_0} in $\mathcal{C}$ such that $C_{\mu_0}\in\gamma_0$ and $\|C_{\mu_0}\|_{\mathrm{e}}<r_0$. Choose $\lambda_0\in\Lambda$ such that for all $\lambda\geq\lambda_0$, $\|C_\lambda\|_{\mathrm{e}}<r_0$. Consider the convex set $\mathcal{C}'\subset\mathcal{C}$ consisting of all convex combinations of elements in the set $\{C_\lambda:\lambda\geq\lambda_0\}$. Clearly C_0 belongs to the closure in the weak operator topology of $\mathcal{C}'$, and since $\mathcal{C}'$ is convex, its closure in the weak operator topology coincides with its closure in the strong operator topology ([6], p. 477), so there exists $C_{\mu_0}\in\mathcal{C}'$ such that $C_{\mu_0}\in\gamma_0$. Furthermore,

C_{μ_0} is a convex combination of operators each having essential norm less than r_0, so $\|C_{\mu_0}\|_{\mathrm{e}} < r_0$, and the lemma is proved. ∎

THEOREM 1.4. (New "Lomonosov Lemma") *Suppose $\mathcal{A}$ is a transitive subalgebra of $\mathcal{L}(\mathcal{X})$ and $\{B_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{L}(\mathcal{X})$ is a net of operators such that $\|B_\lambda\|_{\mathrm{e}} \to 0$ and $\{B_\lambda^{**}\}$ converges to $B_0^{**} \neq 0$ in the weak operator topology on $\mathcal{L}(\mathcal{X}^{**})$. Then there exist a positive integer n and subsets $\{A_1,\dots,A_n\}$ of $\mathcal{A}$ and $\{B_{\lambda_1},\dots,B_{\lambda_n}\}$ of the net $\{B_\lambda\}_{\lambda\in\Lambda}$ such that some combination $C_0 = \sum\limits_{i=1}^{n} A_i B_{\lambda_i}$ satisfies $1 \in \sigma_{\mathrm{p}}(C_0)$ and $\|C_0\|_{\mathrm{e}} < 1/2$. Consequently the same conclusions about C_0 that are stated in Theorem 1.2 obtain.*

Proof. According to Lemma 1.3, there exists a net $\{B_\mu^{**}\}_{\mu\in M}$ in $\mathcal{L}(\mathcal{X}^{**})$ such that $B_\mu^{**} \to B_0^{**}$ in the strong operator topology on $\mathcal{L}(\mathcal{X}^{**})$, $\|B_\mu\|_{\mathrm{e}} \to 0$, and each B_μ is a convex combination of elements of $\{B_\lambda\}_{\lambda\in\Lambda}$. It is easy to see that without loss of generality we may suppose that $\|B_0\| = \|B_0^{**}\| = 1$. We write $\mathcal{Y} = \mathcal{X}^*$ (so $\mathcal{Y}^* = \mathcal{X}^{**}$), choose $\rho_0 > 2$ and define the set $\mathcal{C}^* = \{B_\mu^* A^* \in \mathcal{L}(\mathcal{Y}) : A \in \mathcal{A}, B_\mu \in \{B_\mu\}_{\mu\in M}, \|AB_\mu\|_{\mathrm{e}} \leqslant 1/\rho_0\}$. Let $x_0 \in \mathcal{X}$ be such that $\|x_0\| = 2$ and $\|B_0 x_0\| > 3/2$. Set $y_0 = j(x_0)$. If $\tilde{y}$ in $\mathcal{Y}^*$ satisfies $\|\tilde{y} - y_0\| \leqslant 1$, then $\|B_0^{**}\tilde{y}\| > 1/2$. Since $B_\mu^{**} \to B_0^{**}$ in the strong operator topology and $B_\mu = T_\mu + K_\mu$ with $\|T_\mu\| \to 0$ and $\{K_\mu\} \subset \mathcal{K}(\mathcal{X})$, we have $B_\mu^{**} = T_\mu^{**} + K_\mu^{**}$ with $K_\mu^{**} \in \mathcal{K}(\mathcal{X}^{**})$ satisfying $K_\mu^{**} \to B_0^{**}$ in the strong operator topology. Thus $\|K_\mu^{**}\tilde{y} - B_0^{**}\tilde{y}\| \to 0$. The compacity of the K_μ's implies that range $K_\mu^{**} \subset j(\mathcal{X})$ ([6], p. 482), so $K_\mu^{**}\tilde{y} \in j(\mathcal{X})$ for every μ, and hence $B_0^{**}\tilde{y} \in j(\mathcal{X})$. Say $B_0^{**}\tilde{y} = j(\widehat{x})$. Now let $\varepsilon > 0$. Since $\mathcal{A}$ is transitive, there exists $A \in \mathcal{A}$ such that $\|A\widehat{x} - x_0\| = \|A^{**}B_0^{**}\tilde{y} - y_0\| < \varepsilon$. Since $B_\mu^{**} \to B_0^{**}$ in the strong operator topology, we may choose μ sufficiently large that $\|A^{**}B_\mu^{**}\tilde{y} - y_0\| < \varepsilon$ and $\|AB_\mu\|_{\mathrm{e}} \leqslant 1/\rho_0$. Thus $\|A^{**}B_\mu^{**}\|_{\mathrm{e}} \leqslant 1/\rho_0$ and the hypotheses of Theorem 1.2 are satisfied for $\mathcal{C}^* \subset \mathcal{L}(\mathcal{Y})$. Hence there exists a convex combination $C_0 = \sum\limits_{i=1}^{k} \alpha_i A_i B_{\mu_i}$ of elements of $\mathcal{C}$ such that $1 \in \sigma_{\mathrm{p}}(C_0^*)$. By the definition of $\mathcal{C}$, $\|C_0\|_{\mathrm{e}} \leqslant \sum\limits_{i=1}^{k} \alpha_i \|A_i B_{\mu_i}\|_{\mathrm{e}} \leqslant 1/\rho_0 < 1/2$, and since each B_{μ_i} is a convex combination of elements of $\{B_\lambda\}$, the theorem follows as in the proof of Theorem 1.2. (Along the way the complex coefficients have been eliminated taking into account that $\mathcal{A}$ is an algebra.) ∎

When $\mathcal{X}$ is reflexive, Theorem 1.4 becomes the following.

COROLLARY 1.5. *If $\mathcal{X}$ is a reflexive Banach space, $\mathcal{A}$ is a transitive subalgebra of $\mathcal{L}(\mathcal{X})$, and $\{B_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{L}(\mathcal{X})$ is a net of operators such that $\|B_\lambda\|_{\mathrm{e}} \to 0$ and $B_\lambda \to B_0 \neq 0$ in the weak operator topology on $\mathcal{L}(\mathcal{X})$. Then the conclusion in Theorem 1.4 remains valid.*

Our first priority is to show that, indeed, Theorem 1.1 is an easy consequence of Theorem 1.4.

Proof of Theorem 1.1. Consider the constant sequence $\{K_n = K\}$ in $\mathcal{L}(\mathcal{X})$. Obviously $K_n^{**} \equiv K^{**} \neq 0$ and $\|K_n\|_{\mathrm{e}} \equiv 0$. Thus Theorem 1.4 applies and there exists a combination

$$C_0 = \sum_{i=1}^{n} A_i K_i = \tilde{A} K$$

such that $1 \in \sigma_{\mathrm{p}}(C_0) = \sigma_{\mathrm{p}}(\tilde{A}K)$ and $\tilde{A} \in \mathcal{A}$. ∎

The following may be said to be the main new result of [8], which generalized earlier results in [5] and [9]. The proof to follow is due to the present authors.

THEOREM 1.6. ([8]) *Suppose $\mathcal{B}$ is a subalgebra of $\mathcal{L}(\mathcal{X})$ such that $\mathcal{B}^* = \{B^* : B \in \mathcal{B}\}$ is transitive in $\mathcal{L}(\mathcal{X}^*)$. Suppose also that there exists a net $\{B_\lambda^*\}$ in $\mathcal{B}^*$ that converges to a nonzero B_0 in the weak operator topology on $\mathcal{L}(\mathcal{X}^*)$ and satisfies $\|B_\lambda\|_{\mathrm{e}} \to 0$. Then $\mathcal{B}^*$ is dense in $\mathcal{L}(\mathcal{X}^*)$ in the weak operator topology (equivalently, the strong operator topology).*

Proof. Without loss of generality we may suppose that $\mathcal{B}$ is closed in the norm topology and that $\|B_0\| = 1$. Choose $\rho_0 > 2$ and $y_0 \in \mathcal{X}^*$ such that $\|y_0\| = 2$ and $\|B_0 y_0\| > 3/2$. Then, for each $y \in \mathcal{X}^*$ satisfying $\|y - y_0\| \leqslant 1$, $B_0 y \neq 0$, so fix such a y. Define $\mathcal{C} = \mathcal{B}$, let $\varepsilon > 0$, and choose $B_y \in \mathcal{B}$ such that $\|B_y^*(B_0 y) - y_0\| < \varepsilon$. By Lemma 1.3 we may suppose that the net $\{B_\lambda^*\}$ converges to B_0 in the strong operator topology on $\mathcal{L}(\mathcal{X}^*)$, and thus for λ sufficiently large, $\|B_y^* B_\lambda^* y - y_0\| < \varepsilon$ and $\|B_\lambda B_y\|_{\mathrm{e}} < 1/\rho_0$. Thus, the hypotheses of Theorem 1.2 are satisfied, and by that theorem there exists an operator $C_0 \in \mathcal{B}$ such that $1 \in \sigma_{\mathrm{p}}(C_0^*)$ and the root space corresponding to the eigenvalue 1 of C_0^* is finite dimensional. In this situation, a standard argument (which can be found in the proof of [10], Theorem 6) shows that $\mathcal{B}^*$ contains a nonzero idempotent whose range is the above-mentioned root space for C_0^*, and thus is dense in $\mathcal{L}(\mathcal{X}^*)$ in the strong operator topology. ∎

COROLLARY 1.7. ([8]) *Suppose that $\mathcal{A}$ is a proper subalgebra of $\mathcal{L}(\mathcal{X})$ that is closed in the weak operator topology* (*equivalently, the strong operator topology*). *Then there exist nonzero $z_0 \in \mathcal{X}^{**}$ and $y_0 \in \mathcal{X}^*$ such that*

$$|\langle z_0, A^* y_0 \rangle| \leqslant \|A\|_{\mathrm{e}}, \quad A \in \mathcal{A}.$$

Proof. Consider first the possibility that for every $y \in \mathcal{X}^* \setminus \{0\}$, the set

$$\{A^* y : A \in \mathcal{A}, \|A\|_{\mathrm{e}} \leqslant 1\}$$

is strongly (equivalently, weakly) dense in $\mathcal{X}^*$. Then, of course, $\mathcal{A}^*$ is transitive, and it is easy to see that the hypotheses of Theorem 1.2 are satisfied with $\mathcal{C} = \mathcal{A}$ and $\rho_0 = 3$. Thus, by that theorem, there is an operator $C_0 \in \mathcal{A}$ such that $1 \in \sigma_{\mathrm{p}}(C_0^*)$ and $\|C_0\|_{\mathrm{e}} \leqslant 1/3$. Then, just as in the proof of Theorem 1.6, one shows that there exists a nonzero idempotent of finite rank in the (weak operator topology) closure of the transitive algebra $\mathcal{A}^*$, so $\mathcal{A}^*$ is dense in $\mathcal{L}(\mathcal{X}^*)$ in the strong operator topology. But this forces $\mathcal{A}$ to be dense in $\mathcal{L}(\mathcal{X})$ in the strong operator topology by the Hahn-Banach theorem, which yields a contradiction. Thus, there must exist $y_0 \neq 0$ in $\mathcal{X}^*$ such that the norm-closure of the absolutely convex set

$$\mathcal{C}_{y_0} = \{A^* y_0 : A \in \mathcal{A},\ \|A\|_{\mathrm{e}} \leqslant 1\}$$

is not all of $\mathcal{X}^*$. Thus, by [2], Lemma 16.15, there exists $z_0 \neq 0$ in $\mathcal{X}^{**}$ such that $|\langle z_0, A^* y_0\rangle| \leqslant 1$ for all $A \in \mathcal{A}$ such that $\|A\|_{\mathrm{e}} \leqslant 1$. This shows that $|\langle z_0, A^* y_0\rangle| \leqslant \|A\|_{\mathrm{e}}$ whenever $\|A\|_{\mathrm{e}} \neq 0$. On the other hand, if $A \in \mathcal{A} \cap \mathcal{K}(\mathcal{X})$, then

$$|\langle z_0, (1/r)A^* y_0\rangle| \leqslant 1$$

for all $r \in (0,1)$, which proves the corollary. ∎

The following corollary of either Theorem 1.4 or 1.6 seems to be misstated in [8].

COROLLARY 1.8. ([8]) *Suppose T is a nonscalar operator in $\mathcal{L}(\mathcal{X})$ and there exists a net $\{A_\lambda\}_{\lambda\in\Lambda}$ of operators commuting with T such that $\|A_\lambda\|_{\mathrm{e}} \to 0$ and the net $\{A_\lambda^*\}$ converges to a nonzero operator in the weak operator topology. Then the algebra $(\{T\}')^* = \{X^* \in \mathcal{L}(\mathcal{X}^*) : TX = XT\}$ has a nontrivial invariant subspace. Moreover, if T is a weakly compact operator* (*which is automatic if $\mathcal{X}$ is reflexive*), *then T^* has a nontrivial hyperinvariant subspace.*

The following new result is an improvement of the preceding corollary.

THEOREM 1.9. *If the statement of Corollary* 1.8 *is changed by replacing the hypothesis that $TA_\lambda = A_\lambda T$, $\lambda \in \Lambda$, by the weaker hypothesis that there exists a complex number μ_0 such that $TA_\lambda = \mu_0 A_\lambda T$, $\lambda \in \Lambda$, then the conclusion remains valid.*

Proof. Define the subspace $\mathcal{M} \subset \mathcal{L}(\mathcal{X})$ by $\mathcal{M} = \{A \in \mathcal{L}(\mathcal{X}) : TA = \mu_0 AT\}$, and let $B_0 \neq 0$ be the limit in the weak operator topology of the net $\{A_\lambda^*\}$. By Lemma 1.3, there exists a net $\{B_\nu^*\} \subset \mathcal{M}^*$ which converges to B_0 in the strong operator topology and satisfies $\|B_\nu\|_{\mathrm{e}} \to 0$. Obviously $B_0 T^* = \mu_0 T^* B_0$, and if $\mu_0 = 0$ or $T^* B_0 = 0$, then the kernel of T^* is the desired nontrivial invariant subspace for $(\{T\}')^*$. Thus we may suppose that $B_0T^* \neq 0$ and $\mu_0 \neq 0$. Without loss of generality we may and do suppose that $\|B_0T^*\| = 1$. Define the convex set $\mathcal{C} = \{TA : A \in \mathcal{M}\}$, choose $y_0 \in \mathcal{X}^*$ such that $\|y_0\| = 2$ and $\|B_0T^*y_0\| > 3/2$, and let $\rho_0 > 2$. Fix a vector y in $\mathcal{X}^*$ such that $\|y - y_0\| \leqslant 1$, and note that $B_0T^*y \neq 0$. Suppose next that $(\{T\}')^*$ is transitive, and let ε be an arbitrary positive number. Then there exists an operator T_1 commuting with T such that $\|T_1^*B_0T^*y - y_0\| < \varepsilon$, and since the net $\{B_\nu^*\}$ converges to B_0 in the strong operator topology, we may choose ν_0 sufficiently large that $\|T_1^*B_{\nu_0}^*T^*y - y_0\| < \varepsilon$ and $\|TB_{\nu_0}T_1\|_{\mathrm{e}} < 1/\rho_0$. Since $TB_{\nu_0}T_1 \in \mathcal{C}$, we may apply Theorem 1.2 to $(\mathcal{C}, y_0, \rho_0)$ to obtain an operator A_0 in $\mathcal{M}$ such that $1 \in \sigma_{\mathrm{p}}(TA_0)$ and $\|TA_0\|_{\mathrm{e}} \leqslant 1/\rho_0$. Thus $A_0T = (1/\mu_0)TA_0$ has $(1/\mu_0)$ in its spectrum and since $\sigma(TA_0) \cup \{0\} = \sigma(A_0T) \cup \{0\}$, by iterating we see that μ_0^n and $(1/\mu_0)^n$ belong to $\sigma(TA_0)$ for all positive integers n. Obviously this is possible only if $|\mu_0| = 1$. Furthermore, one knows from Fredholm theory that the numbers $\{\mu_0^n\}$ cannot all be distinct, since the essential spectral radius of TA_0 is less than $1/2$. Thus there exists a positive integer n_0 such that $\mu_0^{n_0} = 1$. But then the commutant of T^{n_0} contains $\mathcal{M}$ and thus contains the net $\{B_\nu\}$. Hence by Corollary 1.8, the algebra $(\{T^{n_0}\}')^*$ has a nontrivial invariant subspace, which contradicts the assumption that $(\{T\}')^*$ is transitive. Thus the proof is complete. ∎

Our last result is a modest generalization of a theorem of Scott Brown ([3]).

Theorem 1.10. *Let $T \in \mathcal{L}(\mathcal{X})$, and for some μ, $\theta \in \mathbb{C}$ with $|\mu| \neq 1$, suppose there exists a net $\{A_\lambda\}_{\lambda\in\Lambda}$ of operators in $\mathcal{L}(\mathcal{X})$ such that $\|A_\lambda\|_{\mathrm{e}} \to 0$ and $\{A_\lambda^*\}$ converges to a nonzero operator in the weak operator topology. If there exists a nonzero operator $S \in \mathcal{L}(\mathcal{X})$ such that*

$$ST = \mu TS \quad \text{and} \quad SA_\lambda = \theta A_\lambda S, \quad \lambda \in \Lambda,$$

then T^* has a nontrivial invariant subspace.

Proof. If $\theta = 0$, then $B_0S^* = 0$, and since $B_0 \neq 0$, the closure of the range of S^* is a proper nonzero subspace $\mathcal{R} \subset \mathcal{X}^*$. Since $T^*(S^*\mathcal{X}^*) = \mu S^*T^*\mathcal{X}^* \subset \mathcal{R}$, $\mathcal{R}$ is the desired invariant subspace for T^*. Thus we may suppose that $\theta \neq 0$, and we fix N large enough so that $|\mu^N\theta| < 1$ if $|\mu| < 1$ and $1/|\mu^N\theta| < 1$ if $|\mu| > 1$. Set $\mathcal{M} = \{A \in \mathcal{L}(\mathcal{X}) : SA = \theta AS\}$. By Lemma 1.3, there exists a net $\{B_\nu^*\} \subset \mathcal{M}^*$ which converges to $B_0 \neq 0$ in the strong operator topology and satisfies $\|B_\nu\|_{\mathrm{e}} \to 0$. Obviously, $B_0S^* = \theta S^*B_0$. If $T^{*N}B_0 = 0$, then the nontrivial kernel of T^* is a nontrivial subspace for T^*. Thus we may suppose that $T^{*N}B_0 \neq 0$, and assume, without loss of generality, that $\|T^{*N}B_0\| = 1$. Choose $y_0 \in \mathcal{X}^*$ such that $\|y_0\| = 2$ and $\|T^{*N}B_0y_0\| > 3/2$, and let $\rho_0 > 2$. Let $\mathcal{C}$ be the convex set generated by

$\{Ap(T)T^N : p$ is a polynomial, $A \in \mathcal{M}\}$. Fix a vector $y \in \mathcal{X}^*$ such that $\|y_0 - y\| \leqslant 1$ and note that $T^{*N} B_0 y \neq 0$. Suppose next that T^* has no nontrivial subspaces, i.e., the algebra $\{p(T^*) : p$ is a polynomial$\}$ is transitive. Then there exists an operator $q(T^*)$ for some polynomial q such that $\|q(T^*){T^*}^N B_0 y - y_0\| < \varepsilon$. Since the net $\{B_\nu^*\}$ converges to B_0 in the strong operator topology, we may choose ν_0 sufficiently large that $\|q(T^*)T^{*N} B_{\nu_0}^* y - y_0\| < \varepsilon$ and $\|q(T^*){T^*}^N B_\nu^*\|_{\mathrm{e}} < 1/\rho_0$. Since $B_{\nu_0} q(T)T^N \in \mathcal{C}$, we may apply Theorem 1.2 to $(\mathcal{C}, y_0, \rho_0)$ to obtain an operator $\widehat{T} = (A_1 p_1(T) + A_2 p_2(T) + \cdots + A_n p_n(T))T^N \in \mathcal{C}$, with the A_i in $\mathcal{M}$, such that $1 \in \sigma_{\mathrm{p}}(\widehat{T})$, $1 \in \sigma_{\mathrm{p}}(\widehat{T}^*)$ and $\|\widehat{T}\|_{\mathrm{e}} \leqslant 1/\rho_0$. Fix $0 \neq x \in \mathcal{X}$ and $0 \neq x^* \in \mathcal{X}^*$ such that $\widehat{T}x = x$ and $\widehat{T}^* x^* = x^*$. Here we have two cases to consider. For every $m \in \mathbb{N}$,

$$S^m x = S^m \widehat{T} x = (\mu^N \theta)^m [(A_1 p_1(\mu^m T) + A_2 p_2(\mu^m T) + \cdots + A_n p_n(\mu^m T))] T^N S^m x.$$

If $|\mu| < 1$, then the sequence $\{\|[A_1 p_1(\mu^m T) + A_2 p_2(\mu^m T) + \cdots + A_n p_n(\mu^m T)]T^N\|\}$ is uniformly bounded in m, and as $m \to \infty$, $\{(\mu^N \theta)^m\}$ tends to zero. Thus for some $m_0 \in \mathbb{N}$, we must have $S^{m_0} x = 0$. Therefore S has a nontrivial kernel, which is invariant for T, which contradicts the assumption that T^* has no nontrivial invariant subspace. On the other hand, for $m \in \mathbb{N}$,

$$\begin{aligned} S^{*m} x^* &= S^{*m} \widehat{T}^* x^* \\ &= (1/\mu^N \theta)^m T^{*N} [(p_1((1/\mu)^m T^*) A_1^* + \cdots + p_n((1/\mu)^m T^*) A_n^*)] S^{*m} x^*. \end{aligned}$$

With an argument similar to that above, one shows that if $|\mu| > 1$, then S^* has a nontrivial kernel which is invariant for T^*, so the proof is complete. ∎

REMARK 1.11. One problem with this entire class of results, of course, is that twenty-five years after [7] appeared, concrete applications are still hard to come by.

The second and the third authors are supported by grants from the National Science Foundation.

REFERENCES

1. S. AXLER, N. JEWELL, A. SHIELDS, The essential norm of an operator and its adjoint, *Trans. Amer. Math. Soc.* **261**(1980), 159–167.
2. A. BROWN, C. PEARCY, *Introduction to Operator Theory I: Elements of Functional Analysis*, Springer-Verlag, 1977.
3. S. BROWN, Connections between an operator and a compact operator that yield hyperinvariant subspaces, *J. Operator Theory* **1**(1979), 117–121.

4. S. BROWN, Lomonosov's theorem and essentially normal operators, *New Zealand J. Math.* **23**(1994), 11–18.
5. D. DECKARD, R. DOUGLAS, C. PEARCY, On invariant subspaces of quasitriangular operators, *Amer. J. Math.* **90**(1969), 637–647.
6. N. DUNFORD, J. SCHWARTZ, *Linear Operators I: General Theory*, Interscience, 1958.
7. V. LOMONOSOV, On invariant subspaces of families of operators commuting with a completely continuous operator, [Russian], *Funktsional Anal. i Prilozhen* **7**(1973), 55–56.
8. V. LOMONOSOV, An extension of Burnside's theorem to infinite dimensional spaces, *Israel J. Math.* **75**(1991), 329-339.
9. C. PEARCY, N. SALINAS, An invariant subspace theorem, *Michigan Math. J.* **20** (1973), 21–31.
10. C. PEARCY, A. SHIELDS, A survey of the Lomonosov technique in the theory of invariant subspaces, *Amer. Math. Soc. Math. Surveys* **13**(1974), 219–230.
11. H. RADJAVI, P. ROSENTHAL, *Invariant Subspaces*, Springer-Verlag, 1973.

BERNARD CHEVREAU
Université de Bordeaux I
U.F.R. de Mathématiques
33405 Talence
FRANCE

E-mail: chevreau@math.u-bordeaux.fr

WING SUET LI
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332
U.S.A.

E-mail: li@math.gatech.edu

CARL PEARCY
Department of Mathematics
Texas A & M University
College Station, TX 77843
U.S.A.

E-mail: pearcy@math.tamu.edu

Received June 17, 1997.