J. OPERATOR THEORY
41(1999), 3–22

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COMPLETELY MULTI-POSITIVE LINEAR MAPS AND REPRESENTATIONS ON HILBERT C^*-MODULES

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Communicated by William B. Arveson

ABSTRACT. We introduce the notion of (completely) multi-positive linear maps between C^*-algebras, and show that a completely multi-positive linear map induces a representation of a C^*-algebra on Hilbert C^*-modules. This generalizes the Stinespring's representation and the representations constructed by Paschke and Kaplan as well as the GNS representation. We also construct the covariant representations on Hilbert C^*-modules for covariant completely positive linear maps. Using representations of C^*-algebras on Hilbert C^*-modules associated with completely multi-positive linear maps we establish another approach about representations associated with completely bounded linear maps.

KEYWORDS: *Completely multi-positive maps, covariant multi-positive maps, covariant representations, Hilbert C^*-module representations, injective C^*-algebras.*

MSC (2000): 46L05, 46L40.

0. INTRODUCTION

For each positive linear functional on a C^*-algebra A, we associate the cyclic representation on a Hilbert space by the Gelfand-Naimark-Segal construction. This fundamental theorem has been generalized by Stinespring ([17]) (respectively, Paschke ([9])) for a completely positive linear map from A into $\mathcal{B}(\mathcal{H})$ (respectively, another C^*-algebra B) to get a representation of A on another Hilbert space $\mathcal{K}$ (respectively, a Hilbert B-module). On the other hand, Kaplan ([5]) introduced the notion of an n-positive linear functional of A, an $n \times n$ matrix of linear functionals which induce a positive linear map from $M_n(A)$ into $M_n(\mathbb{C})$, and got a representation of A on a Hilbert space associated with an n-positive linear functional. The representations on Hilbert spaces are naturally generalized to the representations on Hilbert C^*-modules. But the bounded operators on Hilbert C^*-module do not always have adjoints and closed submodules of a Hilbert C^*-module need not be complemented. The main purpose of this paper is to combine the above two constructions to get a representation of A on a Hilbert C^*-module for completely multi-positive linear maps from A into another C^*-algebra. Using this, we will obtain a representation on a Hilbert C^*-module associated with completely bounded linear maps.

Paulsen ([11]) gave the covariant version of Stinespring's Theorem to show that each of three (rigidly, strongly, weakly) equivalence classes forms a group. Kaplan ([6]) extended to bounded operators on Hilbert C^*-modules to characterize the existence of completely positive liftings for extensions of the algebra of compact operators by certain reduced discrete group C^*-algebras.

An $n \times n$ matrix $[\phi_{ij}]_{i,j=1}^n$ of linear maps from a C^*-algebra A into a C^*-algebra B is called *multi-positive* if $[\phi_{ij}(a_{ij})]$ is positive in $M_n(B)$ whenever $[a_{ij}]$ is a positive element of $M_n(A)$. The map $[\phi_{ij}]_{i,j=1}^n$ is said to be *completely multi-positive* if $[\phi_{ij}] \otimes I_k : M_n(A) \otimes M_k \to M_n(B) \otimes M_k$ is positive for each positive integer k, where $I_k : M_k \to M_k$ denotes the identity map.

In Section 1, we show that the cone $\mathcal{P}_\infty^n[A, B]$ of completely multi-positive linear maps $[\phi_{ij}]_{i,j=1}^n$ from A into B is isomorphic to the cone $\mathcal{P}_\infty[M_n(A), B]$ of usual completely positive linear maps from $M_n(A)$ into B, and the cone $\mathcal{P}_\infty[A, M_n(B)]$ of usual completely positive linear maps from A into $M_n(B)$. We will construct in Section 2 the representation of A on a Hilbert B-module for a completely multi-positive linear map from A into B.

In Section 3, we consider the covariant version to construct a covariant representation on Hilbert C^*-module for a covariant completely multi-positive linear map. Using this, we show that a covariant completely multi-positive linear map from A into B extends to a completely multi-positive linear map from the crossed product $A \times_\alpha G$ into B, which generalizes the result in [6].

1. COMPLETELY MULTI-POSITIVE LINEAR MAPS

For a C^*-algebra A, we denote by $M_n(A)$ the C^*-algebra of all $n \times n$ matrices over A. The C^*-algebra $M_n(\mathbb{C})$ will be denoted by M_n.

DEFINITION 1.1. Let $[\phi_{ij}]_{i,j=1}^n$ be an $n \times n$ matrix of linear maps from a C^*-algebra A into a C^*-algebra B. Then $[\phi_{ij}]_{i,j=1}^n$ may be considered as a linear map from $M_n(A)$ into $M_n(B)$ by

$$[\phi_{ij}] : [a_{ij}] \mapsto [\phi_{ij}(a_{ij})]_{i,j=1}^n, \quad [a_{ij}] \in M_n(A). \tag{1.1}$$

We say that $[\phi_{ij}]_{i,j=1}^n$ is a *multi-positive* (respectively, *k-multi-positive* or *completely multi-positive*) *linear map from A into B* if the linear map $[\phi_{ij}]$ in (1.1) is positive (respectively, k-positive or completely positive). We denote by $\mathcal{P}_k^n[A, B]$ (respectively, $\mathcal{P}_\infty^n[A, B]$) the cone of all k-multi-positive (respectively, completely multi-positive) linear maps. If $n = 1$, then $\mathcal{P}_k^1[A, B]$ and $\mathcal{P}_\infty^1[A, B]$ coincide with $\mathcal{P}_k[A, B]$ and $\mathcal{P}_\infty[A, B]$, respectively, as was introduced in [2]. The following proposition is known.

PROPOSITION 1.2. *Let $[\phi_{ij}]_{i,j=1}^n$ be a multi-positive linear map from a unital C^*-algebra A into a C^*-algebra B. Then we have:*

(i) $\phi_{ij}(a^*) = \phi_{ji}(a)^*$ *for each* $a \in A$ *and* $i, j = 1, \ldots, n$;

(ii) $[\phi_{ij}(a_i^* a^* a a_j)]_{i,j=1}^n \leqslant \|a\|^2 [\phi_{ij}(a_i^* a_j)]_{i,j=1}^n$ *for each* $a_1, \ldots, a_n$, $a \in A$.

Let $\mathcal{B}(A, B)$ denote the space of all bounded linear maps from A into B. We define the linear map $T : M_n(\mathcal{B}(A, B)) \to \mathcal{B}(M_n(A), B)$ by

$$T([\phi_{ij}])([a_{ij}]) = \sum_{i,j=1}^n \phi_{ij}(a_{ij}) \tag{1.2}$$

for $[\phi_{ij}] \in M_n(\mathcal{B}(A, B))$ and $[a_{ij}] \in M_n(A)$.

THEOREM 1.3. *Let A and B be C^*-algebras. Then the linear map T given by* (1.2) *satisfies the following:*

(i) *T is an isomorphism from $M_n(\mathcal{B}(A, B))$ onto $\mathcal{B}(M_n(A), B)$;*

(ii) *T maps $\mathcal{P}_k^n[A, B]$ into $\mathcal{P}_k[M_n(A), B]$, and T^{-1} maps $\mathcal{P}_{kn}[M_n(A), B]$ into $\mathcal{P}_k^n[A, B]$ for each $k = 1, 2, \ldots$;*

(iii) *T is an isomorphism from $\mathcal{P}_\infty^n[A, B]$ onto $\mathcal{P}_\infty[M_n(A), B]$.*

Proof. It is clear that the linear map T is one-to-one. Let $\{E_{ij} \mid i, j = 1, \ldots, n\}$ be the standard matrix units in M_n. Then $a \otimes E_{ij}$ is the $n \times n$ matrix in $M_n(A)$ with a at the (i, j) component and zeros elsewhere. For $\Phi \in \mathcal{B}(M_n(A), B)$, define the linear maps $\phi_{ij} : A \to B$ by $\phi_{ij}(a) = \Phi(a \otimes E_{ij})$ for $a \in A$ and $1 \leqslant i, j \leqslant n$. Then we have

$$T([\phi_{ij}]_{i,j=1}^n)([a_{ij}]_{i,j=1}^n) = \sum_{i,j=1}^n \phi_{ij}(a_{ij}) = \Phi([a_{ij}]_{i,j=1}^n),$$

and so the linear map T is onto.

Let $[\phi_{ij}]_{i,j=1}^n$ be a k-multi-positive linear map from A into B. For a while, we will use the notation ϕ instead of the linear map $[\phi_{ij}]$ from $M_n(A)$ into $M_n(B)$ as was given by (1.1) in order to avoid the confusion. We define the linear map $\Gamma : M_n(B) \to B$ by

$$\Gamma([b_{ij}]) = \sum_{i,j=1}^n b_{ij}, \quad [b_{ij}]_{i,j=1}^n \in M_n(B).$$

Then we have $T([\phi_{ij}]_{i,j=1}^n) = \Gamma \circ \phi$. Since $\phi \in \mathcal{P}_k[M_n(A), M_n(B)]$ and $\Gamma \in \mathcal{P}_\infty[M_n(B), B]$, we see that $T([\phi_{ij}])$ is a k-positive linear map of $M_n(A)$ into B.

In order to show that $T^{-1}(\mathcal{P}_{kn}[M_n(A), B]) \subseteq \mathcal{P}_k^n[A, B]$ for each positive integer k, let $[\phi_{ij}]_{i,j=1}^n = T^{-1}(\Phi)$ for any $\Phi \in \mathcal{B}(M_n(A), B)$. First, suppose that Φ is an n-positive linear map of $M_n(A)$ into B. Define the linear map $\varphi_n : M_n(A) \to M_n(M_n(A))$ by

$$\varphi_n([a_{ij}]) = [a_{ij} \otimes E_{ij}]_{i,j=1}^n, \qquad [a_{ij}]_{i,j=1}^n \in M_n(A).$$

Then φ_n is completely positive, and we have

$$\begin{aligned} ((\Phi \otimes I_n) \circ \varphi_n)([a_{ij}]_{i,j=1}^n) &= (\Phi \otimes I_n)([a_{ij} \otimes E_{ij}]_{i,j=1}^n) \\ &= [\Phi(a_{ij} \otimes E_{ij})]_{i,j=1}^n = \phi([a_{ij}]), \end{aligned}$$

for each $[a_{ij}]_{i,j=1}^n \in M_n(A)$. Since $\Phi \otimes I_n$ and φ_n are positive linear, $[\phi_{ij}]_{i,j=1}^n = (\Phi \otimes I_n) \circ \varphi_n$ is a multi-positive linear map from A into B. From the relation

$$[\phi_{ij}]_{i,j=1}^n \otimes I_k = ((\Phi \otimes I_n) \circ \varphi_n) \otimes I_k = (\Phi \otimes I_{nk}) \circ (\varphi_n \otimes I_k),$$

we see that if $\Phi \in \mathcal{P}_{kn}[M_n(A), B]$ then $[\phi_{ij}] \in \mathcal{P}_k^n[A, B]$.

It only remains to establish the property (iii). Let the map $[\phi_{ij}]_{i,j=1}^n$ be a completely multi-positive linear map from A into B. Then $\phi \in \mathcal{P}_\infty[M_n(A), M_n(B)]$. Since $T([\phi_{ij}]) = \Gamma \circ \phi$ and Γ is completely positive, the linear map $T([\phi_{ij}])$ is completely positive. To show that $T(\mathcal{P}_\infty^n[A, B]) = \mathcal{P}_\infty[M_n(A), B]$, assume that $\Phi \in \mathcal{P}_\infty[M_n(A), B]$. Since $[\phi_{ij}]_{i,j=1}^n = (\Phi \otimes I_n) \circ \varphi_n$ and the linear map φ_n is completely positive, the linear map $[\phi_{ij}]$ is completely multi-positive, which completes the proof. ∎

Define the linear map $S : M_n(\mathcal{B}(A,B)) \to \mathcal{B}(A, M_n(B))$ by

$$S([\psi_{ij}]_{i,j=1}^n)(a) = [\psi_{ij}(a)]_{i,j=1}^n \tag{1.3}$$

for each $[\psi_{ij}] \in M_n(\mathcal{B}(A,B))$ and $a \in A$.

Theorem 1.4. *Let A and B be C^*-algebras. Then the linear map S given by* (1.3) *satisfies the following:*

(i) *S is an isomorphism from $M_n(\mathcal{B}(A,B))$ onto $\mathcal{B}(A, M_n(B))$;*

(ii) *S maps $\mathcal{P}_k^n[A,B]$ into $\mathcal{P}_k[A, M_n(B)]$, and S^{-1} maps $\mathcal{P}_{kn}[A, M_n(B)]$ into $\mathcal{P}_k^n[A,B]$ for each $k = 1, 2, \ldots$;*

(iii) *S is an isomorphism from $\mathcal{P}_\infty^n[A,B]$ onto $\mathcal{P}_\infty[A, M_n(B)]$.*

Proof. Clearly, S is one-to-one. Let $\Psi \in \mathcal{B}(A, M_n(B))$. We denote by $\psi_{ij}(a)$ the (i,j) component of $\Psi(a) \in M_n(B)$ for each $a \in A$ and $i, j = 1, \ldots, n$. Then $[\psi_{ij}]_{i,j=1}^n \in M_n(\mathcal{B}(A,B))$ and

$$S([\psi_{ij}])(a) = [\psi_{ij}(a)]_{i,j=1}^n = \Psi(a), \quad a \in A.$$

Therefore, it follows that S is onto.

Let $[\psi_{ij}]_{i,j=1}^n \in \mathcal{P}_k^n[A,B]$ for each $k = 1, 2, \ldots$. Define the linear map $\Theta : A \to M_n(A)$ by

$$\Theta(a) = \sum_{i,j=1}^n a \otimes E_{ij}, \quad a \in A.$$

Then $S([\psi_{ij}]) = \psi \circ \Theta$, where ψ denotes the linear map from $M_n(A)$ into $M_n(B)$ as was given by (1.1). Since $\Theta \in \mathcal{P}_\infty[A, M_n(A)]$ and $\psi \in \mathcal{P}_k[M_n(A), M_n(B)]$, we see that $S([\psi_{ij}])$ is a k-positive linear map of A into $M_n(B)$.

We shall show that $S^{-1}(\mathcal{P}_{kn}[A, M_n(B)]) \subseteq \mathcal{P}_k^n[A,B]$ for each positive integer k. First, assume that $\Psi \in \mathcal{P}_n[A, M_n(B)]$. Let $[\psi_{ij}]_{i,j=1}^n = S^{-1}(\Psi)$. We define the linear map $\tau_n : M_n(M_n(B)) \to M_n(B)$ by

$$\tau_n\Big(\sum_{i,j=1}^n X_{ij} \otimes E_{ij}\Big) = \sum_{i,j=1}^n x_{ij} \otimes E_{ij}, \quad X_{ij} \in M_n(B),$$

where x_{ij} is the (i,j) component of X_{ij}. Then τ_n is completely positive. For each $[a_{ij}]_{i,j=1}^n \in M_n(A)$, we have

$$\begin{aligned}(\tau_n \circ (\Psi \otimes I_n))([a_{ij}]_{i,j=1}^n) &= \tau_n\Big(\sum_{i,j=1}^n [\psi_{kl}(a_{ij})]_{k,l=1}^n \otimes E_{ij}]\Big)\\ &= \sum_{i,j=1}^n \psi_{ij}(a_{ij}) \otimes E_{ij} = \psi([a_{ij}]).\end{aligned}$$

Thus, it follows that $\psi = \tau_n \circ (\Psi \otimes I_n)$. Since $\Psi \otimes I_n$ and τ_n are positive linear, the linear map $[\psi_{ij}]_{i,j=1}^n$ is a multi-positive linear map from A into B. By the equality

$$[\psi_{ij}]_{i,j=1}^n \otimes I_k = (\tau_n \circ (\Psi \otimes I_n)) \otimes I_k = (\tau_n \otimes I_k) \circ (\Psi \otimes I_{nk}),$$

we obtain that $[\psi_{ij}] \in \mathcal{P}_k^n[A, B]$ whenever $\Psi \in \mathcal{P}_{kn}[A, M_n(B)]$.

Now, it only remains to establish the property (iii). Let $[\psi_{ij}]_{i,j=1}^n$ be a completely multi-positive linear map from A into B. Then $\psi \in \mathcal{P}_\infty[M_n(A), M_n(B)]$. Since $S([\psi_{ij}]_{i,j=1}^n) = \psi \circ \Theta$ and Θ is completely positive, we have $S([\psi_{ij}]) \in \mathcal{P}_\infty[A, M_n(B)]$. If $\Psi \in \mathcal{P}_\infty[A, M_n(B)]$, then we get $[\psi_{ij}] \in \mathcal{P}_\infty^n[A, B]$ since $\psi = \tau_n \circ (\Psi \otimes I_n)$ and τ_n is completely positive. This completes the proof. ∎

COROLLARY 1.5. *The map $V : \mathcal{B}(M_n(A), B) \to \mathcal{B}(A, M_n(B))$ given by $V = S \circ T^{-1}$ is an isomorphism preserving the complete positivity.*

2. REPRESENTATIONS ON HILBERT C^*-MODULES

In this chapter we modify Paschke's and Kaplan's methods ([9] and [5]) to construct a representation of A on a Hilbert B-module associated with completely multi-positive linear map $[\phi_{ij}]_{i,j=1}^n$, from a C^*-algebra A into a C^*-algebra B. We first recall the definition of Hilbert C^*-modules.

Let B be a C^*-algebra with the norm $\|\cdot\|$. A complex vector space X is called a *pre-Hilbert B-module* if X is a right B-module equipped with a B-valued mapping $\langle \cdot, \cdot \rangle : X \times X \to B$ which is linear in the second variable with the properties:

(i) $\langle x, y\rangle = \langle y, x\rangle^*$,
(ii) $\langle x, y \cdot b\rangle = \langle x, y\rangle b$,
(iii) $\langle x, x\rangle \geqslant 0$,
(iv) $\langle x, x\rangle = 0 \Leftrightarrow x = 0$.

The mapping $\langle \cdot, \cdot \rangle$ is called a *B-valued inner product* on X. If, in addition, X is complete with respect to the norm $\|x\|_X = \|\langle x, x\rangle\|^{1/2}$, then X is called a *Hilbert B-module.* Note that the properties (i), (ii), and (iii) of X imply ([4], Lemma 1.1.2) the Cauchy-Schwarz inequality

$$\|\langle x, y\rangle\|^2 \leqslant \|\langle x, x\rangle\| \cdot \|\langle y, y\rangle\|, \quad x, y \in X. \tag{2.1}$$

Throughout this section, B and X denote a C^*-algebra and a Hilbert B-module, respectively, unless specified otherwise.

We denote by $X' = \mathrm{Hom}_B(X, B)$ the set of all bounded B-module maps of X into B. Then X' becomes a right B-module with the operations

$$(f+g)(x) = f(x) + g(x), \quad (\lambda \cdot f)(x) = \overline{\lambda} f(x), \quad (f \cdot b)(x) = b^* f(x)$$

for $x \in X$, $b \in B$, and $\lambda \in \mathbb{C}$. If we endow X' with the norm $\|f\|_{X'}$ of f as a bounded linear map from X into B, then X' becomes a Banach B-module. Note that each $x \in X$ gives rise to the map $x' \in X'$ defined by $x'(y) = \langle x, y\rangle$ for $y \in X$. Since the map $\phi : X \to X'$ given by $\phi(x) = x'$ is an isometric B-module map, we can regard X as a submodule of X' by identifying it with $\phi(X)$. We call X *self-dual* if $X = X'$, that is, every bounded B-module map $f : X \to B$ is of the form $\langle x_f, \cdot\rangle$ for some element $x_f \in X$.

Let X and Y be Hilbert B-modules. We denote by $\mathcal{B}_B(X, Y)$ the space of all bounded B-linear operators of X into Y. We write $\mathcal{B}_B(X)$ for $\mathcal{B}_B(X, X)$. With the operator norm, $\mathcal{B}_B(X)$ is a Banach algebra. We denote by $\mathcal{L}_B(X, Y)$ the set of all B-module maps $T : X \to Y$ for which there is an operator $T^* : Y \to X$, called the *adjoint* of T, such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle, \quad x \in X,\ y \in Y.$$

By the Banach-Steinhaus Theorem, $T \in \mathcal{L}_B(X, Y)$ is bounded. We write $\mathcal{L}_B(X)$ for $\mathcal{L}_B(X, X)$, which becomes a C^*-algebra with the operator norm ([4], Lemma 1.1.7). By a *representation* of a C^*-algebra A on a Hilbert B-module X, we mean a $*$-homomorphism $\pi : A \to \mathcal{L}_B(X)$.

Note that a C^*-algebra B is a Hilbert B-module with the B-valued inner product $\langle\cdot,\cdot\rangle_B$ given by $\langle a, b\rangle_B = a^*b$. Any complex Hilbert space is the Hilbert $\mathbb{C}$-module with the inner product $\langle\cdot,\cdot\rangle$ which is linear in the second variable and conjugate linear in the first variable.

Let A and B be C^*-algebras, and let X be a Hilbert B-module. Given a $*$-homomorphism $\pi : A \to \mathcal{L}_B(X)$ and elements $x_1, \dots, x_n \in X$, we define the linear maps $\phi_{ij} : A \to B$ by $\phi_{ij}(a) = \langle x_i, \pi(a)x_j\rangle$ for every $a \in A$ and $i, j = 1, 2, \dots, n$. Then the linear map $[\phi_{ij}]_{i,j=1}^n$ is completely multi-positive. For $x, y \in X$, we define the linear map $\omega_{x,y} : \mathcal{B}_B(X) \to B$ by

$$\omega_{x,y}(T) = \langle x, Ty\rangle, \quad T \in \mathcal{B}_B(X). \tag{2.2}$$

Theorem 2.1. *Let $[\phi_{ij}]_{i,j=1}^n$ be a completely multi-positive linear map from a unital C^*-algebra A into a unital C^*-algebra B. Then there exist a Hilbert B-module X, a representation π of A on X, and vectors $\mathbf{x}_1, \dots, \mathbf{x}_n \in X$ with the properties:*

(i) $\phi_{ij} = \omega_{\mathbf{x}_i,\mathbf{x}_j} \circ \pi$ *for each* $i, j = 1, \dots, n$,

(ii) $\{\pi(a)(\mathbf{x}_i \cdot b) \mid a \in A,\, b \in B,\, 1 \leqslant i \leqslant n\}$ *spans a dense subspace of* X.

Proof. Let $A \otimes B$ be the algebraic tensor product of A and B, and $(A\otimes B)^n$ the direct sum of n copies of $A \otimes B$. For $\mathbf{a} = (a_1, \dots, a_n) \in A^n$ and $\mathbf{b} = (b_1, \dots, b_n) \in B^n$, we write $\mathbf{a} \otimes \mathbf{b} = (a_1 \otimes b_1, \dots, a_n \otimes b_n) \in (A \otimes B)^n$ by the abuse of notations. Then we see that every element of $(A \otimes B)^n$ is the finite sum of $\mathbf{a} \otimes \mathbf{b}$ with $\mathbf{a} \in A^n$ and $\mathbf{b} \in B^n$.

Now, we define the map $\langle \cdot, \cdot \rangle : (A \otimes B)^n \times (A \otimes B)^n \to B$ by

$$\begin{aligned} \Big\langle \sum_{s=1}^{k} \mathbf{a}_s \otimes \mathbf{b}_s, \sum_{t=1}^{\ell} \mathbf{c}_t \otimes \mathbf{d}_t \Big\rangle &= \sum_{s=1}^{k} \sum_{t=1}^{\ell} \mathbf{b}_s^*([\phi_{ij}](\mathbf{a}_s^* \mathbf{c}_t))\mathbf{d}_t \\ &= \sum_{s=1}^{k} \sum_{t=1}^{\ell} \sum_{i,j=1}^{n} b_{i,s}^* \phi_{ij}(a_{i,s}^* c_{j,t}) d_{j,t}, \end{aligned} \tag{2.3}$$

for $\mathbf{a}_s = (a_{1,s}, \dots, a_{n,s})$, $\mathbf{c}_t = (c_{1,t}, \dots, c_{n,t}) \in A^n$, and $\mathbf{b}_s = (b_{1,s}, \dots, b_{n,s})^{\mathrm{T}}$, $\mathbf{d}_t = (d_{1,t}, \dots, d_{n,t})^{\mathrm{T}} \in B^n$, where T denotes the transpose.

It is clear that $\langle \cdot, \cdot \rangle$ is well-defined and is linear in the second variable and conjugate-linear in the first variable. For any $\mathbf{x} = \sum_{s=1}^{k} \mathbf{a}_s \otimes \mathbf{b}_s \in (A \otimes B)^n$, we have

$$\langle \mathbf{x}, \mathbf{x} \rangle = \sum_{s,t=1}^{k} \mathbf{b}_s^*([\phi_{ij}](\mathbf{a}_s^* \mathbf{a}_t))\mathbf{b}_t = \sum_{s,t=1}^{k} \sum_{i,j=1}^{n} b_{i,s}^* \phi_{ij}(a_{i,s}^* a_{j,t}) b_{j,t} \geqslant 0,$$

since $[\phi_{ij}]$ is completely multi-positive. We will show that $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle^*$ for all $\mathbf{x}, \mathbf{y} \in (A \otimes B)^n$. For any $\mathbf{x} = \sum_{s=1}^{k} \mathbf{a}_s \otimes \mathbf{b}_s$ and $\mathbf{y} = \sum_{t=1}^{\ell} \mathbf{c}_t \otimes \mathbf{d}_t$ in $(A \otimes B)^n$, we have

$$\begin{aligned} \langle \mathbf{x}, \mathbf{y} \rangle &= \sum_{s=1}^{k} \sum_{t=1}^{\ell} \sum_{i,j=1}^{n} b_{i,s}^* \phi_{ij}(a_{i,s}^* c_{j,t}) d_{j,t} \\ &= \Big(\sum_{t=1}^{\ell} \sum_{s=1}^{k} \sum_{i,j=1}^{n} d_{j,t}^* \phi_{ji}(c_{j,t}^* a_{i,s}) b_{i,s} \Big)^* = \langle \mathbf{y}, \mathbf{x} \rangle^*, \end{aligned}$$

where the second equality follows from Proposition 1.2 (i). If we define

$$(\mathbf{a} \otimes \mathbf{b}) \cdot b = \mathbf{a} \otimes \mathbf{b} \cdot b = (a_1 \otimes b_1 b, \dots, a_n \otimes b_n b) \tag{2.4}$$

for each $b \in B$, then $(A \otimes B)^n$ becomes a right B-module. It is also easily checked that $\langle \mathbf{x}, \mathbf{y} \cdot b \rangle = \langle \mathbf{x}, \mathbf{y} \rangle b$ for each $\mathbf{x}, \mathbf{y} \in (A \otimes B)^n$ and $b \in B$.

Let N be the set of all $\mathbf{x} \in (A \otimes B)^n$ with $\langle \mathbf{x}, \mathbf{x} \rangle = 0$. By the Cauchy-Schwarz inequality (2.1), we see that the set

$$\begin{aligned} N &= \{\mathbf{x} \in (A \otimes B)^n \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0\} \\ &= \{\mathbf{x} \in (A \otimes B)^n \mid \langle \mathbf{x}, \mathbf{y} \rangle = 0 \ \text{ for all } \mathbf{y} \in (A \otimes B)^n\} \end{aligned}$$

is a linear subspace of $(A \otimes B)^n$. From $\langle \mathbf{x}, \mathbf{y} \cdot b \rangle = \langle \mathbf{x}, \mathbf{y} \rangle b$ we have the relation $N \cdot B \subseteq N$, and so N is a B-submodule of $(A \otimes B)^n$. By the Cauchy-Schwarz inequality again, we see that the B-valued map $\langle \cdot, \cdot \rangle$ given in (2.3) induces the B-valued inner product on the quotient B-module $X_0 = (A \otimes B)^n/N$, and the completion X of X_0 becomes a Hilbert B-module.

For each $a \in A$, we define the linear operator $\pi(a)$ acting on a pre-Hilbert B-module X_0 by

$$\begin{aligned} \pi(a)\Big(\sum_{s=1}^{k} \mathbf{a}_s \otimes \mathbf{b}_s + N\Big) &= \sum_{s=1}^{k} a \cdot \mathbf{a}_s \otimes \mathbf{b}_s + N \\ &= \sum_{s=1}^{k} (aa_{1,s} \otimes b_{1,s}, \ldots, aa_{n,s} \otimes b_{n,s}) + N \end{aligned} \tag{2.5}$$

for each $\sum_{s=1}^{k} \mathbf{a}_s \otimes \mathbf{b}_s \in (A \otimes B)^n$. It follows immediately from the definition that $\pi(a)$ is a B-module map. Let $\mathbf{x} = \mathbf{a} \otimes \mathbf{b} + N \in X_0$ with $\mathbf{a} \otimes \mathbf{b} = (a_1 \otimes b_1, \ldots, a_n \otimes b_n) \in (A \otimes B)^n$. Then we have

$$\begin{aligned} \langle \pi(a)\mathbf{x}, \pi(a)\mathbf{x} \rangle = \langle a \cdot \mathbf{a} \otimes \mathbf{b} + N, a \cdot \mathbf{a} \otimes \mathbf{b} + N \rangle &= \sum_{i,j=1}^{n} b_i^* \phi_{ij}(a_i^* a^* a a_j) b_j \\ \leqslant \|a\|^2 \sum_{i,j=1}^{n} b_i^* \phi_{ij}(a_i^* a_j) b_j &= \|a\|^2 \langle \mathbf{x}, \mathbf{x} \rangle, \end{aligned}$$

by Proposition 1.2 (ii). Therefore, $\pi(a)$ extends to a bounded B-module map from X to X. We proceed to show that $\pi(a) \in \mathcal{L}_B(X)$.

Take $\mathbf{x} = \mathbf{a} \otimes \mathbf{b} + N$ and $\mathbf{y} = \mathbf{c} \otimes \mathbf{d} + N$ in X_0 with $\mathbf{a} \otimes \mathbf{b} = (a_1 \otimes b_1, \ldots, a_n \otimes b_n)$, $\mathbf{c} \otimes \mathbf{d} = (c_1 \otimes d_1, \ldots, c_n \otimes d_n)$. Then we have

$$\langle \pi(a)\mathbf{x}, \mathbf{y} \rangle = \sum_{i,j=1}^{n} b_i^* \phi_{ij}(a_i^*(a^* c_j)) d_j = \langle \mathbf{a} \otimes \mathbf{b} + N, \pi(a^*)(\mathbf{c} \otimes \mathbf{d} + N) \rangle = \langle \mathbf{x}, \pi(a^*)\mathbf{y} \rangle.$$

Thus, we see that $\pi(a) \in \mathcal{L}_B(X_0)$ for each $a \in A$. It is straightforward to check that $a \mapsto \pi(a)$ is a representation of A on the Hilbert B-module X.

For each $i = 1, 2, \ldots, n$, let $\mathbf{x}_i$ be the vector $\mathbf{e}_i + N \in X$ where $\mathbf{e}_i$ is the element of $(A \otimes B)^n$ whose i-th component is $1_A \otimes 1_B$ and all the other components are 0. Then we have

$$\phi_{ij}(a) = \langle \mathbf{e}_i + N, a \cdot \mathbf{e}_j + N \rangle = \langle \mathbf{e}_i + N, \pi(a)(\mathbf{e}_j + N) \rangle = \langle \mathbf{x}_i, \pi(a)\mathbf{x}_j \rangle$$

for each $a \in A$ and $i, j = 1, 2, \ldots, n$. Therefore, we have $\phi_{ij} = \omega_{\mathbf{x}_i, \mathbf{x}_j} \circ \pi$. It remains to show (ii). For $a \in A$ and $b \in B$, we have

$$\pi(a)(\mathbf{x}_i \cdot b) = \pi(a)((\mathbf{e}_i + N) \cdot b) = (0, \ldots, a \otimes b, \ldots, 0) + N,$$

where $a \otimes b$ is placed in the i-th position. So, the linear span of the set $\{\pi(a)(\mathbf{x}_i \cdot b) \mid a \in A,\ b \in B,\ 1 \leqslant i \leqslant n\}$ is precisely X_0, and this completes the proof. ∎

Our construction may be modified for non unital case as was done by Lin ([8]) for $n = 1$. First, we consider the case when B is not unital. Let X be a Hilbert B-module. The algebraic tensor product $X \otimes B^{**}$ becomes a right B^{**}-module if we set $(x \otimes b) \cdot b' = x \otimes bb'$ for each $x \in X$ and $b, b' \in B^{**}$. We define the conjugate-linear map $\langle \cdot , \cdot \rangle : X \otimes B^{**} \times X \otimes B^{**} \to B^{**}$ by

$$\langle x \otimes b, y \otimes d \rangle = b^* \langle x, y \rangle d, \quad x, y \in X,\ b, d \in B^{**}. \tag{2.6}$$

Then $X \otimes B^{**}/L$ becomes a pre-Hilbert B^{**}-module containing X as a B-submodule, where $L = \{z \in X \otimes B^{**} \mid \langle z, z \rangle = 0\}$. Therefore, we may assume that B is unital.

Suppose that A is not unital. Let $\{u_\lambda\}$ be an approximate identity for A. Let $\mathbf{x}_{i,\lambda}$ be the vector $\mathbf{e}_{i,\lambda} + N$ for each $i = 1, \ldots, n$ where $\mathbf{e}_{i,\lambda}$ is the element of $(A \otimes B)^n$ whose i-th component is $u_\lambda \otimes 1_B$ and all the other components are 0. Since

$$\langle \mathbf{x}_{i,\lambda}, \mathbf{x} \rangle = \sum_{j=1}^{n} \phi_{ij}(u_\lambda a_j) b_j \to \sum_{j=1}^{n} \phi_{ij}(a_j) b_j$$

in norm for each $\mathbf{x} = (a_1 \otimes b_1, \ldots, a_n \otimes b_n) + N \in X$, there are $\mathbf{x}_i \in X'$ such that

$$\langle \mathbf{x}_i, \mathbf{x} \rangle = \lim \langle \mathbf{x}_{i,\lambda}, \mathbf{x} \rangle \quad \text{for all } \mathbf{x} \in X.$$

Therefore, we may take $\mathbf{x}_1, \ldots, \mathbf{x}_n \in X'$.

For a completely multi-positive linear map $[\phi_{ij}]_{i,j=1}^n$ from a C^*-algebra A into a C^*-algebra B, we say that the representation (π, X) constructed in Theorem 2.1 is the *representation associated with* $[\phi_{ij}]$. We call a subset Z of a Hilbert B-module X a *generating set* for X if the closed B-submodule of X generated by Z is the whole of X.

An element u of $\mathcal{L}_B(X,Y)$ is called to be *unitary* if $u^*u = 1_X$, and $uu^* = 1_Y$ where 1_X and 1_Y denotes the identity maps. Let π_1 and π_2 be representations of a C^*-algebra A on Hilbert B-modules X_1 and X_2, respectively. We say that π_1 and π_2 are *unitarily equivalent* if there is an isometric B-module map u of X_1 onto X_2 such that $u\pi_1(a)u^* = \pi_2(a)$ for each a in A. If u is a B-module map from X_1 to X_2, it is well known that u is isometric and surjective if and only if u is a unitary element of $\mathcal{L}_B(X_1, X_2)$.

Theorem 2.2. *Let $[\phi_{ij}]_{i,j=1}^n$ be a completely multi-positive linear map from a unital C^*-algebra A into a unital C^*-algebra B. Then the representation (π, X) associated with $[\phi_{ij}]$ is unique up to an unitary equivalence.*

Proof. Let (π, X) and (π', Y) be the representations associated with $[\phi_{ij}]_{i,j=1}^n$, which have the generating sets $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ and $\{\mathbf{y}_1, \ldots, \mathbf{y}_n\}$, respectively. For each $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$, we have

$$\begin{aligned}\Big\| \sum_{i=1}^n \pi(a_i)(\mathbf{x}_i \cdot b_i)\Big\|_X^2 &= \Big\| \sum_{i,j=1}^n b_i^* \langle \pi(a_i)\mathbf{x}_i, \pi(a_j)\mathbf{x}_j\rangle b_j\Big\| = \Big\| \sum_{i,j=1}^n b_i^* \phi_{ij}(a_i^* a_j) b_j\Big\| \\ &= \Big\| \sum_{i,j=1}^n b_i^* \langle \pi'(a_i)\mathbf{y}_i, \pi'(a_j)\mathbf{y}_j\rangle b_j\Big\| = \Big\| \sum_{i=1}^n \pi'(a_i)(\mathbf{y}_i \cdot b_i)\Big\|_Y^2,\end{aligned}$$

and so the linear map

$$\sum_{i=1}^n \pi(a_i)(\mathbf{x}_i \cdot b_i) \mapsto \sum_{i=1}^n \pi'(a_i)(\mathbf{y}_i \cdot b_i) \tag{2.7}$$

extends to an isometry from X onto Y, which will be denoted by u. It is clear that $u : X \to Y$ is a B-module map, and so u is a unitary of X onto Y.

From the relation

$$\omega_{\mathbf{x}_i,\mathbf{x}_j} \circ \pi = \phi_{ij} = \omega_{\mathbf{y}_i,\mathbf{y}_j} \circ \pi', \quad i,j = 1, \ldots, n,$$

we get $u^*\Big(\sum_{i=1}^n \pi'(a_i)(\mathbf{y}_i \cdot b_i)\Big) = \sum_{i=1}^n \pi(a_i)(\mathbf{x}_i \cdot b_i)$. For each $a, a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$, we have

$$u\pi(a)u^*\Big(\sum_{i=1}^n \pi'(a_i)(\mathbf{y}_i \cdot b_i)\Big) = u\Big(\sum_{i=1}^n \pi(aa_i)(\mathbf{x}_i \cdot b_i)\Big) = \pi'(a)\Big(\sum_{i=1}^n \pi'(a_i)(\mathbf{y}_i \cdot b_i)\Big).$$

Since $u\pi(a)u^*$ and $\pi'(a)$ are bounded and $\{\pi'(a)(\mathbf{y}_i \cdot b) \mid a \in A,\, b \in B,\, 1 \leqslant i \leqslant n\}$ spans a dense subspace of Y, we have $\pi'(a) = u\pi(a)u^*$ for all $a \in A$. Therefore, u sets up the unitary equivalence of π and π'. ∎

The linear map $\phi : A \to B$ is called *completely bounded* if the map $\phi \otimes I_n : A \otimes M_n \to B \otimes M_n$ is bounded for all positive integer n, and set $\|\phi\|_{\mathrm{cb}} = \sup_n \|\phi \otimes I_n\|$. We define a linear map $\phi^* : A \to B$ by $\phi^*(a) = \phi(a^*)^*$ for each $a \in A$. We call a C^*-algebra A *injective* if it has Arveson's extension property. It is known that a C^*-algebra A is injective if and only if there exists a completely positive projection of the algebra $\mathcal{B}(\mathcal{H})$ of operators on a Hilbert space $\mathcal{H}$ onto A. We see that if A is injective, then $A \otimes M_n$ is injective for $n = 1, 2, \dots$.

We say that a closed submodule Y of a Hilbert C^*-module X is *complemented* if $X = Y \oplus Y^\perp$, where $Y^\perp = \{x \in X \mid \langle y, x\rangle = 0, \text{ for all } y \in Y\}$. The following proposition is similar to [13], Theorem 7.3, whose proof depends on the fact that any closed subspace of a Hilbert space is complemented. Note that a closed submodule of a Hilbert C^*-module need not be complemented, and that a closed submodule of a Hilbert C^*-module constructed as above is, in general, not complemented. To get a representation on a Hilbert C^*-module associated with completely bounded maps, we will consider completely multi-positive linear maps.

PROPOSITION 2.3. *Let A and B be C^*-algebras with B injective. If $\phi : A \to B$ is a completely bounded linear map, then there exist a Hilbert B-module X, a representation π of A on X and vectors $\mathbf{x}_1, \mathbf{x}_2 \in X$ with the properties;*

(i) *$\phi(a) = \langle \mathbf{x}_1, \pi(a)\mathbf{x}_2\rangle$ for each $a \in A$;*

(ii) *the set $\{\pi(a)(\mathbf{x}_i \cdot b) \mid a \in A,\, b \in B,\, i = 1, 2\}$ spans a dense subspace of X.*

Proof. By [12], Theorem 7.3, there exist completely positive linear maps $\varphi_i : A \to B$, with $\|\varphi_i\|_{\mathrm{cb}} = \|\phi\|_{\mathrm{cb}}$, $i = 1, 2$, such that the map $\Phi : M_2(A) \to M_2(B)$ given by

$$\Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \varphi_1(a) & \phi(b) \\ \phi^*(c) & \varphi_2(d) \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A)$$

is completely positive. We can consider Φ as a linear map from the C^*-algebra $M_2(A)$ into the C^*-algebra $M_2(B)$, that is,

$$\Phi = \begin{pmatrix} \varphi_1 & \phi \\ \phi^* & \varphi_2 \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \varphi_1(a) & \phi(b) \\ \phi^*(c) & \varphi_2(d) \end{pmatrix} = \begin{pmatrix} \phi_{11}(a) & \phi_{12}(b) \\ \phi_{21}(c) & \phi_{22}(d) \end{pmatrix}.$$

Putting $\Phi = [\phi_{ij}]_{i,j=1}^2$, we see that $[\phi_{ij}]_{i,j=1}^2$ is completely multi-positive. By Theorem 2.1, there exist a Hilbert B-module X, a representation π of A on X, and vectors $\mathbf{x}_1, \mathbf{x}_2 \in X$ such that $\phi_{ij} = \omega_{\mathbf{x}_i, \mathbf{x}_j} \circ \pi$ for $i, j = 1, 2$ and the linear span of $\{\pi(a)(\mathbf{x}_i \cdot b) \mid a \in A,\, b \in B,\, i = 1, 2\}$ is dense in X. Hence we have

$$\phi(a) = \phi_{12}(a) = \omega_{\mathbf{x}_1, \mathbf{x}_2} \circ \pi(a) = \langle \mathbf{x}_1, \pi(a)\mathbf{x}_2\rangle$$

for each $a \in A$, which completes the proof. ∎

Kaplan ([5]) constructed representations of n-positive linear functionals on unital C^*-algebras and Suen ([18]) gave representations of $n \times n$ matrices of linear maps by using the methods of Paulsen ([13], Theorem 7.4). But we can give representations of completely multi-positive maps of C^*-algebras into $\mathcal{B}(\mathcal{H})$ from Theorem 2.1 extended the method of Stinespring ([17]).

Corollary 2.4. *Let A be a C^*-algebra and let $[\phi_{ij}]_{i,j=1}^n$ be a completely multi-positive linear map from A into $\mathcal{B}(\mathcal{H})$. Then there exist a representation π of A on a Hilbert space $\mathcal{K}$ and bounded operators $V_i : \mathcal{H} \to \mathcal{K}$, $i = 1, \ldots, n$ such that $\phi_{ij}(a) = V_i^* \pi(a) V_j$ for each $a \in A$ and $i, j = 1, \ldots, n$.*

In Proposition 2.3, we established the representation for completely bounded maps from C^*-algebras into injective C^*-algebras using the representation on Hilbert C^*-modules for completely multi-positive maps. The following proposition gives a commutant representation on a Hilbert C^*-module for completely bounded maps.

Proposition 2.5. *Let A and B be C^*-algebras with B injective. If $\phi : A \to B$ is a completely bounded linear map, then there exist a Hilbert B-module X, a representation π of A on X, a vector $x \in X$, and a unique operator $T \in \mathcal{B}_B(X, X') \cap \pi(A)'$ with the properties:*

(i) *$\phi(a) = \langle x, T\pi(a)x \rangle$ for each $a \in A$;*

(ii) *the set $\{\pi(a)(x \cdot b) \mid a \in A,\ b \in B\}$ spans a dense subspace of X.*

Proof. By [13], Theorem 7.5, there exists a completely positive map $\psi : A \to B$ with $\|\psi\| = \psi(1) = \|\phi\|_{\mathrm{cb}}$ such that $\psi \pm \mathrm{Re}(\phi)$ and $\psi \pm \mathrm{Im}\,(\phi)$ are completely positive. Since $\psi : A \to B$ is completely positive, by [8], Theorem 2.1, there exist a Hilbert B-module X, a representation π of A on X, and a vector $x \in X$ such that for each $a \in A$, $\psi(a) = \langle x, \pi(a)x \rangle$ and the span of $\{\pi(a)(x \cdot b) \mid a \in A,\ b \in B\}$ is dense in X. Note that $1/2(\psi + \mathrm{Re}\,(\phi))$ and $1/2(\psi + \mathrm{Im}\,(\phi))$ are completely positive. Since the following maps

$$\psi - \frac{1}{2}(\psi + \mathrm{Re}\,(\phi)) = \frac{1}{2}(\psi - \mathrm{Re}\,(\phi)),$$
$$\psi - \frac{1}{2}(\psi + \mathrm{Im}\,(\phi)) = \frac{1}{2}(\psi - \mathrm{Im}\,(\phi))$$

are completely positive, we have $\psi \geqslant 1/2(\psi + \mathrm{Re}\,(\phi))$ and $\psi \geqslant 1/2(\psi + \mathrm{Im}\,(\phi))$. By [8], Theorem 2.2, there exist positive operators $P, Q \in \mathcal{B}_B(X, X')$ which commute with every element in $\pi(A)$ such that for each $a \in A$

$$\frac{1}{2}(\psi + \mathrm{Re}\,(\phi))(a) = \langle x, P\pi(a)x \rangle,$$
$$\frac{1}{2}(\psi + \mathrm{Im}\,(\phi))(a) = \langle x, Q\pi(a)x \rangle.$$

Hence we have that for each $a \in A$,

$$\mathrm{Re}\,(\phi(a)) = 2\langle x, P\pi(a)x\rangle - \psi(a) = \langle x, (2P-1)\pi(a)x\rangle,$$
$$\mathrm{Im}\,(\phi(a)) = 2\langle x, Q\pi(a)x\rangle - \psi(a) = \langle x, (2Q-1)\pi(a)x\rangle.$$

Putting $T_1 = 2P-1$, $T_2 = 2Q-1$ and $T = T_1 + \mathrm{i}T_2$, we have that $T \in \pi(A)'$ and for each $a \in A$

$$\phi(a) = \mathrm{Re}\,(\phi(a)) + \mathrm{i}\,\mathrm{Im}\,(\phi(a)) = \langle x, T_1\pi(a)x\rangle + \langle x, \mathrm{i}\,T_2\pi(a)x\rangle = \langle x, T\pi(a)x\rangle.$$

If T' is another operator which commutes with every elements in $\pi(A)$ such that $\phi(a) = \langle x, T'\pi(a)x\rangle$ for each $a \in A$, letting $T' = T_1' + \mathrm{i}T_2'$ be its cartesian decomposition, we have $(\psi + \mathrm{Re}\,(\phi))(a) = \langle x, (1+T_1')\pi(a)x\rangle$ and $(\psi + \mathrm{Re}\,(\phi))(a) = \langle x, (1+T_2')\pi(a)x\rangle$ which imply that $1 + T_1' = 2P$ and $1 + T_2' = 2Q$. Therefore, we get $T = T'$. ∎

Theorem 7.4 in [13] implies the Stinespring's theorem for completely bounded maps and Suen ([18]) gave the representation for completely multi-bounded maps which is similar to the Stinespring representation for completely bounded maps. Using [1], Theorem 1.4.2, we give another characterization of the representation for completely multi-bounded maps.

COROLLARY 2.6. *Let A be a C^*-algebra and let $[\phi_{ij}]_{i,j=1}^n$ be a completely multi-bounded linear map from A into $\mathcal{B}(\mathcal{H})$. Then there exist a representation π of A on a Hilbert space $\mathcal{K}$, a bounded operator $V : \mathcal{H} \to \mathcal{K}$ and $T_{ij} \in \pi(A)'$, for each $a \in A$.*

Proof. Since each ϕ_{ij} is completely bounded, ϕ_{ij} is expressed by a linear combination of four completely positive linear maps, that is,

$$\phi_{ij} = \sigma_{ij}^+ - \sigma_{ij}^- + \mathrm{i}(\tau_{ij}^+ - \tau_{ij}^-)$$

where $\sigma_{ij}^\pm$ and $\tau_{ij}^\pm$ are completely positive for $i, j = 1, \ldots, n$. Set

$$\varphi = \sum_{i,j=1}^n (\sigma_{ij}^+ + \sigma_{ij}^- + \tau_{ij}^+ + \tau_{ij}^-), \quad i, j = 1, \ldots, n.$$

Clearly, φ is completely positive. Let $(\pi, V, \mathcal{K})$ be the Stinespring representation associated with φ. Since $\varphi \geqslant \sigma_{ij}^\pm, \tau_{ij}^\pm$ for each $i, j = 1, \ldots, n$, by [1], Theorem 1.4.2, there are unique positive operators $P_{ij}^\pm, Q_{ij}^\pm \in \mathcal{B}(\mathcal{K}) \cap \pi(A)'$ such that $P_{ij}^\pm, Q_{ij}^\pm \leqslant 1$ and

$$\sigma_{ij}^\pm(a) = V^*P_{ij}^\pm\pi(a)V, \quad \tau_{ij}^\pm(a) = V^*Q_{ij}^\pm\pi(a)V, \quad i, j = 1, \ldots, n$$

for each $a \in A$. Putting $T_{ij} = P_{ij}^+ - P_{ij}^- + \mathrm{i}(Q_{ij}^+ - Q_{ij}^-)$, we have $\phi_{ij}(a) = V^*T_{ij}\pi(a)V$, $(i, j = 1, \ldots, n)$ for each $a \in A$. ∎

3. COVARIANT REPRESENTATIONS ON HILBERT C^*-MODULES

Let (A, G, α) be a C^*-dynamical system with a locally compact group G. Given a unitary representation $u : G \to \mathcal{U}(B)$ of G into the unitary group of a unital C^*-algebra B, a linear map $\phi : A \to B$ is called *u-covariant* if $\phi(\alpha_g(a)) = u_g \phi(a) u_g^*$ for each $a \in A$, $g \in G$. Let X be a Hilbert B-module. A *covariant representation* of a C^*-dynamical system (A, G, α) is a triple (π, σ, X), where (π, X) is a representation of A on a Hilbert B-module X and (σ, X) is a unitary representation of G into $\mathcal{L}_B(X)$ such that

$$\pi(\alpha_g(a)) = \sigma_g \pi(a) \sigma_g^*, \quad a \in A,\, g \in G. \tag{3.1}$$

The action $\alpha : G \to \mathrm{Aut}\,(A)$ induces the action $\widetilde{\alpha} : G \to \mathrm{Aut}\,(M_n(A))$ by

$$\widetilde{\alpha}_g([a_{ij}]) = [\alpha_g(a_{ij})], \quad [a_{ij}] \in M_n(A). \tag{3.2}$$

Let $[\phi_{ij}]_{i,j=1}^n$ be a multi-positive linear map from A into B. The map $[\phi_{ij}]$ may be considered as a map from $M_n(A)$ into $M_n(B)$ as in (1.1). Let $\widetilde{u}_g \in \mathcal{U}(M_n(B))$ be a diagonal matrix with all the diagonal entries u_g. If the map $[\phi_{ij}] : M_n(A) \to M_n(B)$ is $\widetilde{u}$-covariant with respect to the dynamical system $(M_n(A), G, \widetilde{\alpha})$, we say that $[\phi_{ij}]$ is a *u-covariant multi-positive linear map from A into B*. Note that a multi-positive linear map $[\phi_{ij}]_{i,j=1}^n$ is u-covariant if and only if

$$\phi_{ij}(\alpha_g(a_{ij})) = u_g \phi_{ij}(a_{ij}) u_g^*, \quad i, j = 1, \ldots, n, \tag{3.3}$$

for each $[a_{ij}] \in M_n(A)$ and $g \in G$.

Theorem 3.1. *Let (A, G, α) be a unital C^*-dynamical system and $u : G \to \mathcal{U}(B)$ a unitary representation of G into a unital C^*-algebra B. If $[\phi_{ij}]_{i,j=1}^n$ is a u-covariant completely multi-positive linear map from A into B, then there exist:*

(i) *a Hilbert B-module X;*
(ii) *a covariant representation (π, σ, X) of (A, G, α) into $\mathcal{L}_B(X)$;*
(iii) *n vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in X$;*
(iv) *n elements $v_1, \ldots, v_n \in \mathcal{L}_B(B, X)$; such that:*
  (1) *$\phi_{ij} = \omega_{\mathbf{x}_i, \mathbf{x}_j} \circ \pi$ for $i, j = 1, 2, \ldots, n$;*
  (2) *$\{\pi(a)(\mathbf{x}_i \cdot b) \mid a \in A,\, b \in B,\, 1 \leqslant i \leqslant n\}$ spans a dense subspace of X;*
  (3) *$v_i^* \pi(a) v_j = m_{\phi_{ij}(a)}$, $i, j = 1, \ldots, n$ for each $a \in A$;*
  (4) *$\sigma_g v_i = v_i m_{u_g}$, $i = 1, \ldots, n$ for each $g \in G$;*

where m is a left multiplication operator on B.

Proof. We follow the notation in the proof of Theorem 2.1. By Theorem 2.1, there exist a Hilbert B-module X, a representation $\pi : A \to \mathcal{L}_B(X)$ and n vectors

$\mathbf{x}_1, \ldots, \mathbf{x}_n \in X$ satisfying the properties (1) and (2). Hence it suffices to construct a covariant representation (π, σ, X) of (A, G, α) into $\mathcal{L}_B(X)$ and elements $v_1, \ldots, v_n \in \mathcal{L}_B(B, X)$ satisfying the properties (3) and (4).

For each $g \in G$ and $\mathbf{x} = \sum_{s=1}^{k} \mathbf{a}_s \otimes \mathbf{b}_s + N \in X$, we define the linear map $\sigma : G \to \mathcal{L}_B(X)$ by

$$\sigma_g(\mathbf{x}) = \sum_{s=1}^{k} (\alpha_g(a_{1,s}) \otimes u_g b_{1,s}, \ldots, \alpha_g(a_{n,s}) \otimes u_g b_{n,s}) + N. \tag{3.4}$$

Clearly, σ_g is bounded and linear. Take any $\mathbf{x} = \mathbf{a} \otimes \mathbf{b} + N$, $\mathbf{y} = \mathbf{c} \otimes \mathbf{d} + N \in X$. Then we obtain that for each $g \in G$

$$\begin{aligned}\langle \sigma_g(\mathbf{x}), \mathbf{y} \rangle &= \langle (\alpha_g(a_1) \otimes u_g b_1, \ldots, \alpha_g(a_n) \otimes u_g b_n) + N, (c_1 \otimes d_1, \ldots, c_n \otimes d_n) + N \rangle \\ &= \sum_{i,j=1}^{n} b_i^* \phi_{ij}(a_i^* \alpha_{g^{-1}}(c_j)) u_g^* d_j \\ &= \langle (a_1 \otimes b_1, \ldots, a_n \otimes b_n) + N, (\alpha_{g^{-1}}(c_1) \otimes u_g^* d_1, \ldots, \alpha_{g^{-1}}(c_n) \otimes u_g^* d_n) + N \rangle \\ &= \langle \mathbf{x}, \sigma_{g^{-1}}(\mathbf{y}) \rangle.\end{aligned}$$

Therefore σ_g has the adjoint $\sigma_{g^{-1}}$, and it is straightforward that $g \mapsto \sigma_g$ is a unitary representation on X. For each $\mathbf{x} = \sum_{s=1}^{k} \mathbf{a}_s \otimes \mathbf{b}_s + N \in X$ and $a \in A$, we have

$$\begin{aligned}\sigma_g \pi(a) \sigma_g^*(\mathbf{x}) &= \sigma_g \Big(\sum_{s=1}^{k} (a\alpha_g^*(a_{1,s}) \otimes u_g^* b_{1,s}, \ldots, a\alpha_g^*(a_{n,s}) \otimes u_g^* b_{n,s}) + N \Big) \\ &= \sum_{s=1}^{k} (\alpha_g(a) a_{1,s} \otimes b_{1,s}, \ldots, \alpha_g(a) a_{n,s} \otimes b_{n,s}) + N = \pi(\alpha_g(a))\mathbf{x},\end{aligned}$$

which implies that (π, σ, X) is a covariant representation of (A, G, α) on X.

For each $i = 1, 2, \ldots, n$, we define the B-module map $v_i : B \to X$ by

$$v_i(b) = (0, \ldots, 1_A \otimes b, \ldots, 0) + N, \tag{3.5}$$

that is, the element whose i-th component is $1_A \otimes b$ and all the other components are 0. For each $\mathbf{a} \otimes \mathbf{b} + N \in X$, $b \in B$ and $i = 1, \ldots, n$ we have

$$\begin{aligned}\langle v_i(b), \mathbf{a} \otimes \mathbf{b} + N \rangle &= \langle (0, \ldots, 1_A \otimes b, \ldots, 0) + N, (a_1 \otimes b_1, \ldots, a_n \otimes b_n) + N \rangle \\ &= b^* \sum_{j=1}^{n} \phi_{ij}(a_j) b_j = \Big\langle b, \sum_{j=1}^{n} \phi_{ij}(a_j) b_j \Big\rangle,\end{aligned}$$

which implies that $v_i^*(\mathbf{a}\otimes\mathbf{b}+N)=\sum\limits_{j=1}^{n}\phi_{ij}(a_j)b_j$. For each $i,j=1,2,\dots,n$, we have

$$v_i^*\pi(a)v_j=m_{\phi_{ij}(a)},\quad a\in A.$$

It is straightforward to check that $\sigma_g v_i(b)=v_i(u_g b)=v_i m_{u_g}(b)$ for each $b\in B$ and $i=1,\dots,n$, which completes the proof. ∎

In [6], Kaplan showed that given a discrete unital C^*-dynamical system (A,G,α), a covariant completely positive linear map $\varphi:A\to B$ extends to a completely positive linear map on the crossed product $A\times_\alpha G$. Using Theorem 3.1, we generalize Kaplan's argument in the following proposition where we follow the notations in [12], Section 7.6.

Proposition 3.2. *Let (A,G,α) be a unital C^*-dynamical system, and $u:G\to\mathcal{U}(B)$ a unitary representation of G into a unital C^*-algebra B. If $[\phi_{ij}]_{i,j=1}^n$ is a u-covariant completely multi-positive linear map from A into B, then there exists a completely multi-positive linear map $[\psi_{ij}]_{i,j=1}^n$ from $A\times_\alpha G$ into B uniquely given by*

$$\psi_{ij}(f)=\int\phi_{ij}(f(g))u_g\,\mathrm{d}\mu,\quad f\in K(G,A)\tag{3.6}$$

where $K(G,A)$ is a set of continuous functions from G to A with compact supports.

Proof. By Theorem 3.1, there exist a covariant representation (π,σ,X) of (A,G,α) into $\mathcal{L}_B(X)$ and $v_i\in\mathcal{L}_B(B,X)$ such that $v_i^*\pi(a)v_j=m_{\phi_{ij(a)}}$ and $\sigma_g v_i=v_i m_{u_g}$, $1\leqslant i,j\leqslant n$ for all $a\in A$, $g\in G$. We define $\pi\times\sigma$ by

$$(\pi\times\sigma)(f)=\int\limits_G\pi(f(g))\sigma_g\,\mathrm{d}\mu,\quad f\in K(G,A).\tag{3.7}$$

From [16], Proposition 7.6.4, we see that

$$\begin{aligned}&(\pi\times\sigma)(f^*)=((\pi\times\sigma)(f))^*,\\&\|(\pi\times\sigma)(f)\|\leqslant\|f\|_1,\\&(\pi\times\sigma)(f_1*f_2)=(\pi\times\sigma)(f_1)(\pi\times\sigma)(f_2),\end{aligned}$$

for each $f,f_1,f_2\in K(G,A)$. Hence $\pi\times\sigma$ extends to a representation, again denoted by $\pi\times\sigma$ from $L^1(G,A)$ to $\mathcal{L}_B(X)$. By the universal property of the crossed product $A\times_\alpha G$, the representation $\pi\times\sigma$ extends to a representation of

$A \times_\alpha G$ into $\mathcal{L}_B(X)$, still denoted by $\pi \times \sigma$. For each $i, j = 1, 2, \ldots, n$ we define linear map $\varphi_{ij} : A \times_\alpha G \to \mathcal{L}_B(B)$ by

$$\varphi_{ij}(x) = v_i^*(\pi \times \sigma)(x)v_j, \quad x \in A \times_\alpha G. \tag{3.8}$$

For each $b \in B$ and $f \in \mathrm{K}(G, A)$, we have

$$\begin{aligned}\varphi_{ij}(f)(b) &= v_i^*(\pi \times \sigma)(f)v_j(b) = \int_G v_i^*\pi(f(g))\sigma_g v_j(b)\,\mathrm{d}\mu \\ &= \int_G v_i^*\pi(f(g))v_j(u_g b)\,\mathrm{d}\mu = \int_G \phi_{ij}(f(g))u_g b\,\mathrm{d}\mu.\end{aligned}$$

Since the map $\pi \times \sigma$ is a representation from $A \times_\alpha G$ to $\mathcal{L}_B(X)$, $[\varphi_{ij}]_{i,j=1}^n$ is a completely multi-positive linear map from $A \times_\alpha G$ into $\mathcal{L}_B(B)$. Let $\tau : \mathcal{L}_B(B) \to B$ be a natural isomorphism. Putting $\psi_{ij} = \tau \circ \varphi_{ij}$ $(i, j = 1, \ldots, n)$, the linear map $[\psi_{ij}]$ is completely multi-positive. This completes the proof. ∎

Let (A, G, α) be a unital C^*-dynamical system, and u a unitary representation of G into an injective von Neumann algebra B. Let S be an operator system in the C^*-algebra A which is invariant under the action α of G and $\phi : S \to B$ a u-covariant completely positive map. If G is amenable, then there exists a u-covariant completely positive map $\widetilde{\phi} : A \to B$ extending ϕ. To get a new covariant completely positive map, we use the invariant mean to average. But this averaging usually lies in the weak operator closure of the range. We are not sure about the C^*-case. Under the stronger assumption that G is compact, this would work because the averages are in the norm closure ([15]). Using this and Paulsen's off-diagonalization trick, we can see that the covariant version of Theorem 7.3 in [13] holds for covariant completely bounded maps. The following corollary is a covariant version of Proposition 2.3.

COROLLARY 3.3. *Let (A, G, α) be a unital C^*-dynamical system with G amenable and $u : G \to \mathcal{U}(B)$ a unitary representation of G into a unital injective von Neumann algebra B. If ϕ is a u-covariant completely bounded linear map from A into B, then there exist:*

(i) *a Hilbert B-module X;*
(ii) *a covariant representation (π, σ, X) of (A, G, α) into $\mathcal{L}_B(X)$;*
(iii) *vectors $\mathbf{x}_1, \mathbf{x}_2 \in X$;*
(iv) *elements $v_1, v_2 \in \mathcal{L}_B(B, X)$; such that:*
  (1) *$\phi(a) = \langle \mathbf{x}_1, \pi(a)\mathbf{x}_2 \rangle$ for each $a \in A$;*
  (2) *$\{\pi(a)(\mathbf{x}_i \cdot b) \mid a \in A,\ b \in B,\ i = 1, 2\}$ spans a dense subspace of X;*

(3) $v_1^*\pi(a)v_2 = m_{\phi(a)}$ *for each* $a \in A$;

(4) $\sigma_g v_i = v_i m_{u_g}$, $i = 1, 2$ *for each* $g \in G$;

where m *is a left multiplication operator on* B.

Proof. By the above remark, there exist u-covariant completely positive linear maps φ_1 and φ_2 from A into B such that the map $\Phi : M_2(A) \to M_2(B)$ given by

$$\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \varphi_1(a) & \phi(b) \\ \phi^*(c) & \varphi_2(d) \end{pmatrix}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(A) \tag{3.9}$$

is completely positive. When Φ is considered as a linear map from a C^*-algebra $M_2(A)$ into a C^*-algebra $M_2(B)$, that is,

$$\Phi = \begin{pmatrix} \varphi_1 & \phi \\ \phi^* & \varphi_2 \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \varphi_1(a) & \phi(b) \\ \phi^*(c) & \varphi_2(d) \end{pmatrix} = \begin{pmatrix} \phi_{11}(a) & \phi_{12}(b) \\ \phi_{21}(c) & \phi_{22}(d) \end{pmatrix},$$

it follows from definition that $\Phi = [\phi_{ij}]_{i,j=1}^2$ is u-covariant completely multi-positive. By Theorem 3.1, there exist a Hilbert B-module X, a covariant representation (π, σ, X) of A on X, and vectors $\mathbf{x}_1, \mathbf{x}_2 \in X$ and elements $v_1, v_2 \in \mathcal{L}_B(B, X)$ satisfying properties (1), (2), (3) and (4), which completes the proof. ∎

Acknowledgements. The author is indebted to professor Vern Paulsen for his helpful discussion. He is grateful to the referee for his suggestion. He would like to express his gratitude to professors Sa Ge Lee and Seung-Hyeok Kye for many kind advices.

This work was partially supported by GARC-KOSEF.

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Received July 29, 1996; revised June 24, 1998.