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MODEL THEORY FOR ρ-CONTRACTIONS, $\rho \leqslant 2$

MICHAEL A. DRITSCHEL, SCOTT MCCULLOUGH and HUGO J. WOERDEMAN

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Abstract. Agler's abstract model theory is applied to $\mathcal{C}_\rho$, the family of operators with unitary ρ-dilations, where ρ is a fixed number in $(0, 2]$. The extremals, which are the collection of operators in $\mathcal{C}_\rho$ with the property that the only extensions of them which remain in the family are direct sums, are characterized in a variety of manners. They form a part of any model, and in particular, of the boundary, which is defined as the smallest model for the family. Any model for a family is required to be closed under direct sums, restrictions to reducing subspaces, and unital $*$-representations. In the case of the family $\mathcal{C}_\rho$ with $\rho \in (0, 1) \cup (1, 2]$, this closure is shown to be all of $\mathcal{C}_\rho$.

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0. INTRODUCTION

Throughout this paper $\mathcal{C}_\rho$, ρ a fixed positive constant, stands for the class of Hilbert space operators with unitary ρ-dilations; that is, a bounded linear operator A acting on a Hilbert space $\mathcal{H}$ (notation: $A \in \mathcal{L}(\mathcal{H})$) belongs to $\mathcal{C}_\rho$ if and only if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ isometrically and a unitary operator $U \in \mathcal{L}(\mathcal{K})$ such that

$$A^n = \rho P_{\mathcal{H}} U^n | \mathcal{H}, \quad n = 1, 2, \ldots, \tag{0.1}$$

where $P_{\mathcal{H}}$ is the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$. The elements of $\mathcal{C}_\rho$ are referred to as ρ-contractions. The ρ-contractions were introduced by Holbrook ([12]) and Sz.-Nagy and Foiaş ([15]) as a natural generalization of the usual contractions (which correspond to the case where $\rho = 1$). In this paper, we consider model theory for the class $\mathcal{C}_\rho$ when $0 < \rho \leqslant 2$, extending the work of the first and third authors in [9] on the class $\mathcal{C}_2$, the operators with numerical radius less than or equal to one.

In general, the point of model theory is to study some class of operators through a smaller, better understood collection of operators which have the property that the elements of the original class are the restriction to invariant subspaces of elements of the smaller collection. This is what we mean by "modeling". It is a concept which has been widely applied in operator theory. In order to better understand what is going on in a host of seemingly unrelated examples, Jim Agler codified in [2] an abstract model theory. According to Agler, model theory is applied to a "family" of operators (formal definitions are given below). There are potentially many different models for a family of operators, including the family itself. The idea then is to try to find a best one, normally in the sense of being smallest. This best model is called the "boundary" of the family. All models have certain common properties which Agler discerned through his consideration of diverse examples. For instance, a model should be closed with respect to arbitrary direct sums, restrictions to reducing subspaces, and unital $*$-representations. Agler also discovered that at the heart of every model there lurks a special collection of operators called the "extremals". We say that an operator A is an extremal if whenever we consider an extension

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

of A, the only way this extension can be a member of the family is if $B = 0$; that is, the only extensions of an extremal which remain in the family have the form of a direct sum. One strategy in determining the boundary of a family (and it is the one we shall use in this paper) is to first find the extremals, and then take the closure of the extremals with respect to the various properties which are required of a model. In all familiar cases the extremals and the boundary coincide, though *a priori* there is no reason to believe that taking this closure will not add new elements. Indeed, one of the more striking features of the families we consider in this paper is that the extremals and the boundary are not the same, with a notable exception corresponding to $\rho = 1$ (the ordinary contractions).

The main results of the paper are Theorem 2.3, which gives a number of ways of characterizing the extremals of the family $\mathcal{C}_\rho$ when $\rho \in (0, 2]$, and Theorem 3.1, which shows that for these families, the boundary is the whole family (except when $\rho = 1$). It is in fact the condition that a model needs to be closed with respect to the properties mentioned above which in this case causes the relatively thin set of extremals to expand to the whole family. The ρ-contractions are the first known examples where this happens.

This raises several questions, which may be considered in future work:

(1) Can we conclude that those results for the family $\mathcal{C}_1$ of contractions which have proofs relying on the model being the class of coisometries, cannot be easily generalized to the classes $\mathcal{C}_\rho$, $\rho \in (0,1) \cup (1,2]$? In particular, does this help to explain the relative limitedness of results on $\mathcal{C}_2$, the class of operators with numerical radius less than or equal to one, compared with the abundance of results on $\mathcal{C}_1$?

(2) Should we reexamine the abstract model theory? Because of the way it is defined, it is not difficult to see that a ρ-contraction can be used to define a Hilbert module over the disk algebra. An analogy with the situation for contractions and coisometries would suggest that the extremals should correspond to injective modules. So perhaps it would be fruitful to pursue a theory for "pre-models"; that is models that are not necessarily closed with respect to arbitrary direct sums, restrictions to reducing subspaces, and unital $*$-representations.

The paper is divided into three sections. The first lays out the mathematical tools we will need to prove the main results. This begins with a brief overview of Agler's abstract model theory, and includes some new methods for determining when certain elements of a family belong to a model which may be of independent interest. We then list some results from the theory of operator-valued functions used in the sequel, followed by a cursory description of the Schur complement and some of its relevant properties.

Section 2 concentrates on describing the extremals of a family $\mathcal{C}_\rho$ (Theorem 2.3). Along the way we generalize a description of the family $\mathcal{C}_2$ due to Andô ([3]) to $\mathcal{C}_\rho$ when $\rho \in (0,2]$. As a concrete example, we also take up the case of unitary operators, and show that if U is unitary, then for $\rho \in [1,2]$, U is extremal, and for $\rho \in (0,1]$, $\rho/(2-\rho)U$ is extremal. Several quite different proofs are offered.

The final section is devoted to proving that for $\rho \in (0,2]$ and $\rho \neq 1$, the boundary of the family $\mathcal{C}_\rho$ is all of $\mathcal{C}_\rho$. This highlights the exceptional role played by the ordinary contractions among the ρ-contractions, since for the contractions the extremals and boundary coincide with the coisometries.

1. PRELIMINARIES

1.1. ABSTRACT MODEL THEORY. We briefly outline here Agler's abstract approach to model theory. For a fuller exposition see [2].

A *family* is a collection $\mathcal{F} = \{A \mid A \in \mathcal{L}(\mathcal{H}_A)\}$ of Hilbert space operators which is

(i) bounded (that is, there exists a c such that $A \in \mathcal{F}$ implies $\|A\| \leqslant c$);

(ii) closed with respect to taking arbitrary direct sums;

(iii) closed with respect to taking restrictions to invariant subspaces (in other words, $\mathcal{F}$ is *hereditary*);

(iv) closed with respect to unital $*$-representations (meaning that if π is a unital $*$-representation and $A \in \mathcal{F}$ then $\pi(A) \in \mathcal{F}$).

Familiar examples include contractions, subnormal contractions, and isometries.

For fixed ρ, the class $\mathcal{C}_\rho$ is also a family. The fact that it is bounded, closed with respect to direct sums, and hereditary follows from the Definition 0.1. To see that $\mathcal{C}_\rho$ is also closed with respect to unital $*$-representations, it is useful to consider a characterization due to Sz.-Nagy and Foiaş ([15]):

$$(1.1.1) \qquad A \in \mathcal{C}_\rho \iff 1 - aAA^*|z|^2 + bAz + bA^*\overline{z} \geqslant 0 \quad \forall\, z \in \mathbb{D},$$

where $\mathbb{D} = \{z \in \mathbb{C} \,\big|\, |z| \leqslant 1\}$, $a = (2-\rho)/\rho$ and $b = (\rho-1)/\rho$. Then since unital $*$-representations preserve positivity, closure with respect to such representations is immediate.

By a *model* for a family $\mathcal{F}$, we mean a subset $\mathcal{B}$ of $\mathcal{F}$ which is closed with respect to direct sums, restrictions to reducing subspaces, and unital $*$-representations, and having the property that every element of $\mathcal{F}$ has an extension in $\mathcal{B}$ (where an *extension* of an operator A is an operator $\widetilde{A}$ with an invariant subspace $\mathcal{N}$ such that $\widetilde{A}|\mathcal{N} = A$). In this case, we also say that A *lifts* to $\widetilde{A}$.

In general a family has many models, including the whole family itself. However, models tend to be most useful when their elements have a simpler structure, which will then hopefully shed some light on the more complicated members of the family. Hence we are interested in "small" models. Agler has shown (see, for example, [2]) that every family has a smallest model in the sense of inclusion, termed the *boundary* of $\mathcal{F}$. One way of obtaining the boundary of a family is to take the intersection of all of its models, though in general this is not very practical. An alternative approach is to first determine the *extremals* of the family, which are defined as those operators A in $\mathcal{F}$ with the property that the only extensions of A in $\mathcal{F}$ are direct sums. By Proposition 5.9 of [2] the extremals belong to every model, and by Proposition 5.10 of [2], every element of the family lifts to an extremal. Hence one may take the extremals and close them up with respect to direct sums, restrictions to reducing subspaces, and unital $*$-representation to obtain the boundary. This is exactly what we will do in this paper for the family $\mathcal{C}_\rho$. In the more familiar examples mentioned above, the extremals and the boundary coincide.

Models can be shown to have a number of useful properties in addition to the defining ones. Our first theorem indicates several such which we shall need in our study of the model theory of $\mathcal{C}_\rho$.

THEOREM 1.1. *Let $\mathcal{F}$ be a family with model $\mathcal{B}$.*

(i) *Suppose Λ is a directed set, $\{A_j\}_{j\in\Lambda}$ a net in $\mathcal{B}\cap\mathcal{L}(\mathcal{H})$ and $A\in\mathcal{F}\cap\mathcal{L}(\mathcal{H})$ such that A_j converges strictly to A. Then $A\in\mathcal{B}$.*

(ii) *Let S be a bounded operator, and suppose that S has a unitary representation τ (that is, the C^*-algebra generated by S has a representation τ such that $\tau(S)$ is unitary) with $1\in\sigma(\tau(S))$. If $A\in\mathcal{F}$ and $A\otimes S\in\mathcal{B}$, then $A\in\mathcal{B}$.*

(iii) *Suppose $A\in\mathcal{F}\cap\mathcal{L}(\mathcal{H})$ and that for any orthogonal projection P onto a finite dimensional Hilbert space $\mathcal{G}$, $PAP\in\mathcal{B}\cap\mathcal{L}(\mathcal{G})$. Then $A\in\mathcal{B}$.*

Recall that a sequence of operators $\{A_j\}$ converges strictly to A if both $\{A_j\}$ converges to A and $\{A_j^*\}$ converges to A^* in the strong operator topology.

Proof of Theorem 1.1. We begin with (i). First observe that by being members of a family, the A_j's form a uniformly bounded sequence, and so for any $k\in\mathbb{N}$, A_j^k converges strongly to A^k and A_j^{*k} converges strongly to A^{*k}. Let $\widetilde{A}=\bigoplus_j A_j$. Since $\mathcal{B}$ is closed with respect to direct sums, $\widetilde{A}\in\mathcal{B}\cap\mathcal{L}(\widetilde{\mathcal{H}})$, where $\widetilde{\mathcal{H}}=\bigoplus_j\mathcal{H}$. By an *hereditary polynomial*, we mean a polynomial in two non-commuting variables x and y with complex coefficients of the form

$$p(x,y)=\sum_{s,t}p_{s,t}x^t y^s. \tag{1.1.2}$$

For a bounded operator T on a Hilbert space $\mathcal{H}$, and an hereditary polynomial p as above, set

$$p(T^*,T)=\sum_{s,t}p_{s,t}T^{*s}T^t.$$

Define

$$\mathcal{P}=\{p(\widetilde{A}^*,\widetilde{A})+q(\widetilde{A},\widetilde{A}^*)\mid p,q \text{ hereditary polynomials}\}.$$

Then $\mathcal{P} \subset \mathcal{L}(\widetilde{\mathcal{H}})$ is an operator system (that is, it is self-adjoint and contains the identity). Let M_n denote the algebra of all complex $n \times n$ matrices. Since $\widetilde{A}$ is the direct sum of the A_j's, it is apparent that for any $n \in \mathbb{N}$ and $P, Q \in M_n$,

$$P \otimes p(\widetilde{A}^*, \widetilde{A}) + Q \otimes q(\widetilde{A}, \widetilde{A}^*) \cong \bigoplus_j (P \otimes p(A_j^*, A_j) + Q \otimes q(A_j, A_j^*)).$$

Fix $n \in \mathbb{N}$. Suppose that for each $k = 1, \dots, m$, p_k and q_k are hereditary polynomials, and $P_k, Q_k \in M_n$. Then

$$\sum_k (P_k \otimes p_k(\widetilde{A}^*, \widetilde{A}) + Q_k \otimes q_k(\widetilde{A}, \widetilde{A}^*)) \geqslant 0$$

is equivalent to

$$\sum_k (P_k \otimes p_k(A_j^*, A_j) + Q_k \otimes q_k(A_j, A_j^*)) \geqslant 0 \quad \text{for all } j.$$

Now, if p is an hereditary polynomial of the form (1.1.2) and $f, g \in \mathcal{H}$, then

$$\langle p(A_j^*, A_j)\, f, g\rangle = \sum p_{s,t} \langle A_j^t f, A_j^s g\rangle,$$

which converges to

$$\langle p(A^*, A) f, g\rangle,$$

since A_j^ℓ and $A_j^{*\ell}$ are strongly convergent to A^ℓ and $A^{*\ell}$ for each $\ell \in \mathbb{N}$. The corresponding statement also holds for $q(A_j, A_j^*)$ and $q(A, A^*)$. Thus, if $P = (P_{ab})$, $Q = (Q_{ab})$ are $n \times n$ matrices, p, q are hereditary polynomials, and $h = \oplus h_a$ with $h_1, \dots, h_n \in \mathcal{H}$, then

$$\langle (P \otimes p(A_j^*, A_j) + Q \otimes q(A_j, A_j^*))h, h\rangle = \sum_{a,b} \langle (P_{a,b}\, p(A_j^*, A_j) + Q_{a,b}\, q(A_j, A_j^*))h_a, h_b\rangle$$

converges to

$$\sum_{a,b} \langle (P_{a,b}\, p(A^*, A) + Q_{a,b}\, q(A, A^*))\, h_a, h_b\rangle = \langle (P \otimes p(A^*, A) + Q \otimes q(A, A^*))\, h, h\rangle.$$

It follows that the map $\gamma : \mathcal{P} \to \mathcal{L}(\mathcal{H})$ defined by

$$\gamma(p(\widetilde{A}^*, \widetilde{A}) + q(\widetilde{A}, \widetilde{A}^*)) = p(A, A^*) + q(A^*, A)$$

is completely positive. (Note that it is well-defined since, as naïvely defined, it is positive.) By the Arveson Extension Theorem ([7], [13]), γ extends to a completely positive map from $\mathcal{L}(\widetilde{\mathcal{H}})$ to $\mathcal{L}(\mathcal{H})$, which we also call γ. By the Stinespring

Representation Theorem (see, for example, [13]), there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a unital $*$-homomorphism $\pi : \mathcal{L}(\widetilde{\mathcal{H}}) \to \mathcal{L}(\mathcal{K})$ such that

$$\gamma(T) = P_{\mathcal{H}}\pi(T)|\mathcal{H} \quad \text{for all } T \in \mathcal{L}(\widetilde{\mathcal{H}}).$$

Now

$$P_{\mathcal{H}}\pi(\widetilde{A})^* P_{\mathcal{H}}\pi(\widetilde{A})|\mathcal{H} = \gamma(\widetilde{A}^*)\gamma(\widetilde{A}) = A^*A = \gamma(\widetilde{A}^*\widetilde{A}) = P_{\mathcal{H}}\pi(\widetilde{A})^*\pi(\widetilde{A})|\mathcal{H},$$

which implies that $\pi(\widetilde{A})\mathcal{H} \subseteq \mathcal{H}$. By considering AA^* instead, we have $\pi(\widetilde{A})^*\mathcal{H} \subseteq \mathcal{H}$. In other words, $\mathcal{H}$ reduces $\pi(\widetilde{A})$. Since $\mathcal{B}$ is closed with respect to unital $*$-representations and restriction to reducing subspaces, $\pi(\widetilde{A})$, and consequently $A = P_{\mathcal{H}}\pi(\widetilde{A})|\mathcal{H}$, is in $\mathcal{B}$. This proves (i).

To prove (ii), let $A \in \mathcal{F} \cap \mathcal{L}(\mathcal{H})$, $S \in \mathcal{L}(\mathcal{K})$, and suppose $A \otimes S \in \mathcal{B}$. Let $\mathcal{A}$ and $\mathcal{S}$ denote the C^*-algebras generated by A and S, respectively. By assumption, there is a representation τ of $\mathcal{S}$ such that $\tau(S)$ is unitary and $1 \in \sigma(\tau(S))$.

Now $\mathcal{A} \otimes \mathcal{S} \subset \mathcal{L}(\mathcal{H}\widehat{\otimes}\mathcal{K})$, where $\mathcal{H}\widehat{\otimes}\mathcal{K}$ is the Hilbert space completion of $\mathcal{H} \otimes \mathcal{K}$. Notice that since $\mathcal{H} \otimes \mathcal{K}$ is dense in $\mathcal{H}\widehat{\otimes}\mathcal{K}$, $A \otimes S$ extends continuously to an operator in $\mathcal{L}(\mathcal{H}\widehat{\otimes}\mathcal{K})$, which we also denote by $A \otimes S$. Complete $\mathcal{A} \otimes \mathcal{S}$ to a C^*-algebra $\mathcal{A} \otimes_* \mathcal{S} \subset \mathcal{L}(\mathcal{H}\widehat{\otimes}\mathcal{K})$, and consider the C^*-subalgebra generated by $A \otimes S$. Let $\psi = \varphi \otimes \varphi'$, where φ is the identity representation on $\mathcal{A}$ and φ' is the one-dimensional representation on $\mathcal{S}$ with $\varphi'(S) = 1$ (note that φ' exists since the algebra generated by $\tau(S)$ is commutative and by assumption, $1 \in \sigma(\tau(S))$). Then ψ extends continuously to a representation of $\mathcal{A} \otimes_* \mathcal{S}$ and $\psi(\mathcal{A} \otimes \mathcal{S}) = \mathcal{A} \otimes \mathbb{C} \cong \mathcal{A}$. The map ψ is completely positive, so by the Arveson Extension Theorem, ψ extends to a completely positive map (which we also call ψ) of $\mathcal{L}(\mathcal{H}\widehat{\otimes}\mathcal{K})$ into $\mathcal{L}(\mathcal{H})$. From the Stinespring Representation Theorem, there exists a Hilbert space $\mathcal{L}(\mathcal{M})$, a representation $\pi : \mathcal{L}(\mathcal{H}\widehat{\otimes}\mathcal{K}) \to \mathcal{L}(\mathcal{M})$ such that

$$\psi(T) = P_{\mathcal{H}}\pi(T)|\mathcal{H} \quad \text{for all } \ T \in \mathcal{L}(\mathcal{H}\widehat{\otimes}\mathcal{K}).$$

Observing that $\psi(A \otimes S) = A$, we argue as before that A is the restriction of $\pi(A \otimes S)$ to a reducing subspace, and so since $A \otimes S$ is assumed to be in $\mathcal{B}$, A is also in $\mathcal{B}$.

Finally, consider (iii). Let $A \in \mathcal{F} \cap \mathcal{L}(\mathcal{H})$ such that for any orthogonal projection P onto a finite dimensional subspace $\mathcal{G}$, $PAP \in \mathcal{B} \cap \mathcal{L}(\mathcal{G})$. We consider complex polynomials p in two noncommuting variables and define $p(A, A^*)$ in the usual way.

For $\mathcal{G}$ a subspace of $\mathcal{H}$ of dimension $k < \infty$, set $\mathcal{H}_{n,\mathcal{G}} = \bigvee p(A, A^*)\mathcal{G}$, where the span is over all polynomials p of degree less than or equal to n. Note that

$\dim \mathcal{H}_{n,\mathcal{G}} \leqslant k(2^{n+1}-1) < \infty$, and that $\mathcal{G} \subseteq \mathcal{H}_{n,\mathcal{G}}$. Denote by $P_{n,\mathcal{G}}$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_{n,\mathcal{G}}$, and set $A_{n,\mathcal{G}} = P_{n,\mathcal{G}} A|\mathcal{H}_{n,\mathcal{G}}$. By construction $p(A,A^*)f = p(A_{n,\mathcal{G}}, A^*_{n,\mathcal{G}})f$ whenever p is a polynomial of degree less than or equal to n and $f \in \mathcal{G}$.

Next set $\widetilde{A} = \bigoplus_{n,\mathcal{G}} A_{n,\mathcal{G}}$ on $\widetilde{\mathcal{H}} = \bigoplus_{n,\mathcal{G}} \mathcal{H}_{n,\mathcal{G}}$, where we are summing over nonnegative integers n and finite dimensional subspaces $\mathcal{G}$ of $\mathcal{H}$. By assumption, $A_{n,\mathcal{G}} \in \mathcal{B}$, and so $\widetilde{A} \in \mathcal{B}$. Define $\mathcal{R}_n = \bigvee_{\deg p \leqslant n} p(\widetilde{A}, \widetilde{A}^*)$, and $\mathcal{R} = \bigcup_n \mathcal{R}_n$. Then $\mathcal{R}$ is an operator system in $\mathcal{L}(\widetilde{\mathcal{H}})$.

Define $\gamma : \mathcal{R} \to \mathcal{L}(\mathcal{H})$ by $\gamma(p(\widetilde{A}, \widetilde{A}^*)) = p(A, A^*)$ for polynomials p. We show that γ is completely positive, from which it will follow that it is well-defined. Let $M = (p_{ij})$ be a $k \times k$ matrix of polynomials, all of degree less than or equal to n, and suppose that $M(\widetilde{A}, \widetilde{A}^*) = (p(\widetilde{A}, \widetilde{A}^*)) \geqslant 0$. Then for all nonnegative integers n and finite dimensional subspaces $\mathcal{G}$, $M(A_{n,\mathcal{G}}, A^*_{n,\mathcal{G}}) \geqslant 0$. In particular, if $f_1, \dots, f_k \in \mathcal{H}$ and $\mathcal{G} = \bigvee f_k$, then

$$0 \leqslant \sum_{i,j} \langle p_{ij}(A_{n,\mathcal{G}}, A^*_{n,\mathcal{G}}) f_i, f_j \rangle = \sum_{i,j} \langle p_{ij}(A, A^*) f_i, f_j \rangle,$$

and so $M(A, A^*) \geqslant 0$. Hence γ is completely positive. The proof now proceeds using the Arveson Extension Theorem and the Stinespring Representation Theorem as in the previous two cases. ∎

1.2. Factorization of operator-valued functions. Given a Hilbert space $\mathcal{H}$ let $H^2_{\mathcal{H}}(\mathbb{D})$ denote the Hardy space of $\mathcal{H}$-valued functions which are analytic in the unit disk with square integrable boundary values. These functions will be identified with their boundary values whenever convenient. Given a pair of Hilbert spaces $\mathcal{H}, \mathcal{K}$, let $H^\infty_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathbb{D})$ stand for the set of all bounded analytic $\mathcal{L}(\mathcal{H},\mathcal{K})$-valued functions on $\mathbb{D}$. For $F \in H^\infty_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathbb{D})$, we associate the operator $M_F : H^2_{\mathcal{H}}(\mathbb{D}) \to H^2_{\mathcal{K}}(\mathbb{D})$ of multiplication by F; that is, $M_F g(z) = F(z)g(z)$. We say that F is *outer* if the corresponding multiplication operator M_F has dense range in $H^2_{\mathcal{M}}(\mathbb{D})$ for some subspace $\mathcal{M}$ of $\mathcal{K}$. In this case, if we write $F(z) = \sum F_k z^k$, $F_k \in \mathcal{L}(\mathcal{H},\mathcal{K})$, then $\mathcal{M} = \overline{\operatorname{ran}}\, F_0$ (this is a special case of Theorem B, p. 98 of [14]). Furthermore, $\operatorname{ran} F_k \subseteq \mathcal{M}$ for all k. We will be exclusively interested in the situation where F is a polynomial. Under these circumstances, we say that F is $*$-*outer* if $z^n F(1/z)$ is outer, where n is the degree of F.

The following two results summarize well known facts about outer factorizations of operator-valued functions.

Operator Fejér-Riesz Theorem 1.2. *Let* $Q(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{-n}^{n} Q_k \, \mathrm{e}^{\mathrm{i}k\theta}$ *with coefficients in* $\mathcal{L}(\mathcal{H})$ *such that* $Q(\mathrm{e}^{\mathrm{i}\theta}) \geqslant 0$ *for* $\theta \in [0, 2\pi)$*. Then* $Q(\mathrm{e}^{\mathrm{i}\theta}) = F(\mathrm{e}^{\mathrm{i}\theta})^* F(\mathrm{e}^{\mathrm{i}\theta})$ *for all* θ*, where* $F(z) = \sum_{0}^{n} F_k z^k$ *is an operator-valued outer function on the unit disk with coefficients in* $\mathcal{L}(\mathcal{H})$*.*

The polynomial in Theorem 1.2 is uniquely determined if we require $F_0 \geqslant 0$.

Theorem 1.3. *Suppose* $\mathcal{H}$ *and* $\mathcal{K}$ *are Hilbert spaces,* $F_k \in \mathcal{L}(\mathcal{H}, \mathcal{K})$*,* $k = 0, 1, \ldots, n$*. Then* $F(z) = \sum F_k z^k$ *is outer if and only if* $F_0^* F_0 \geqslant G_0^* G_0$ *for all* $G \in H^\infty_{\mathcal{L}(\mathcal{H},\mathcal{K})}(\mathbb{D})$ *such that* $F(\mathrm{e}^{\mathrm{i}\theta})^* F(\mathrm{e}^{\mathrm{i}\theta}) = G(\mathrm{e}^{\mathrm{i}\theta})^* G(\mathrm{e}^{\mathrm{i}\theta})$ *for all* $\theta \in [0, 2\pi)$*. If* F *is outer and* $Q(\mathrm{e}^{\mathrm{i}\theta}) = F(\mathrm{e}^{\mathrm{i}\theta})^* F(\mathrm{e}^{\mathrm{i}\theta})$*, and if* $\mathcal{G}$ *is a Hilbert space and there exists operators* $G_j : \mathcal{H} \to \mathcal{G}$ *such that* $G(\mathrm{e}^{\mathrm{i}\theta}) = \sum_j G_j \mathrm{e}^{\mathrm{i}j\theta}$ *satisfies* $Q(\mathrm{e}^{\mathrm{i}\theta}) = G(\mathrm{e}^{\mathrm{i}\theta})^* G(\mathrm{e}^{\mathrm{i}\theta})$*, then*

$$G(z)^* G(z) \leqslant F(z)^* F(z) \quad z \in \mathbb{D},$$

and for each $n = 0, 1, 2, \ldots$*,*

$$\sum_0^n F_k^* F_k \geqslant \sum_0^n G_k^* G_k.$$

If $G_0^* G_0 = F_0^* F_0$*, then* G *is outer and there is a partial isometry* $R \in \mathcal{L}(\mathcal{K}, \mathcal{G})$ *mapping* $\overline{\operatorname{ran}}\, F_0$ *onto* $\overline{\operatorname{ran}}\, G_0$ *such that* $G = RF$*.*

Proofs of Theorems 1.2 *and* 1.3. A discussion of factorization of operator-valued functions may be found in [14]. In particular, Theorem 1.2 may be found on p. 118 in [14] and Theorem 1.3 is a combination of Theorem A, Section 5.9 (p. 102 in [14]), and Leech's theorem, which is given on p. 107 in [14]. For the last part of Theorem 1.3, if $F_0^* F_0 = G_0^* G_0$, then by the first part, G is also outer. The last statement is then Theorem B(ii), Section 5.8 (p. 101) of [14]. ∎

Observe that by the last two theorems, if $Q(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{-n}^{n} Q_k \mathrm{e}^{\mathrm{i}k\theta}$ is nonnegative for all θ, then for any outer factorization $Q(\mathrm{e}^{\mathrm{i}\theta}) = F(\mathrm{e}^{\mathrm{i}\theta})^* F(\mathrm{e}^{\mathrm{i}\theta})$, F will be a polynomial of degree n. If on the other hand, F is $*$-outer, it is also a polynomial of degree n. To see this, first note that Q has a $*$-outer factorization of degree n, which can be constructed as follows. Let

$$\widehat{Q}(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{-n}^{n} Q_k^* \mathrm{e}^{\mathrm{i}k\theta} = \widehat{F}(\mathrm{e}^{\mathrm{i}\theta})^* \widehat{F}(\mathrm{e}^{\mathrm{i}\theta}) \geqslant 0,$$

where $\widehat{F}(z) = z^n F(1/z)$ and F any polynomial of degree n satisfying $Q(\mathrm{e}^{\mathrm{i}\theta}) = F(\mathrm{e}^{\mathrm{i}\theta})^* F(\mathrm{e}^{\mathrm{i}\theta})$ (for example, F could come from an outer factorization of Q). It is not difficult to verify that $\widehat{Q}$ is independent of the choice of F. Now suppose $\widehat{Q}(\mathrm{e}^{\mathrm{i}\theta}) = \widehat{F}(\mathrm{e}^{\mathrm{i}\theta})^* \widehat{F}(\mathrm{e}^{\mathrm{i}\theta})$, where $\widehat{F}(z)$ is outer, and set $G(z) = z^n \widehat{F}(1/z)$. Then $Q(\mathrm{e}^{\mathrm{i}\theta}) = G(\mathrm{e}^{\mathrm{i}\theta})^* G(\mathrm{e}^{\mathrm{i}\theta})$, and by definition, G is $*$-outer and clearly has degree n. Now suppose $\widetilde{G}$ is a $*$-outer polynomial of degree m such that $Q(\mathrm{e}^{\mathrm{i}\theta}) = \widetilde{G}(\mathrm{e}^{\mathrm{i}\theta})^* \widetilde{G}(\mathrm{e}^{\mathrm{i}\theta})$. Then $\widehat{Q}(\mathrm{e}^{\mathrm{i}\theta}) = \widehat{G}(\mathrm{e}^{\mathrm{i}\theta})^* \widehat{G}(\mathrm{e}^{\mathrm{i}\theta})$, where $\widehat{G} = z^m \widetilde{G}(1/z)$. But by Theorem 1.3, all outer factorizations of $\widehat{Q}$ are unitarily equivalent, and so $m = n$.

THEOREM 1.4. *Let $Q(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{-n}^{n} Q_k \mathrm{e}^{\mathrm{i}k\theta}$ be a trigonometric polynomial with coefficients in $\mathcal{L}(\mathcal{H})$ such that $Q(\mathrm{e}^{\mathrm{i}\theta}) \geqslant 0$ for all $\theta \in [0, 2\pi)$. Then for $0 \leqslant j, k \leqslant n$, the set*

$$\mathcal{F}_{j,k} = \left\{ F_j^* F_k \;\middle|\; F(z) = \sum_{k=0}^{n} F_k z^k \ \text{and} \ F(\mathrm{e}^{\mathrm{i}\theta})^* F(\mathrm{e}^{\mathrm{i}\theta}) = Q(\mathrm{e}^{\mathrm{i}\theta}) \right\}$$

is convex, where for each F, the coefficients F_k are in $\mathcal{L}(\mathcal{H}, \mathcal{K}_F)$, the Hilbert space $\mathcal{K}_F$ depending on F.

Proof. Let $F(z) = \sum_{k=0}^{n} F_k z^k$, $F_k \in \mathcal{L}(\mathcal{H}, \mathcal{K}_F)$, where $F(\mathrm{e}^{\mathrm{i}\theta})^* F(\mathrm{e}^{\mathrm{i}\theta}) = Q(\mathrm{e}^{\mathrm{i}\theta})$. Set

$$\widetilde{F} = \begin{pmatrix} F_0^* \\ \vdots \\ F_n^* \end{pmatrix} \begin{pmatrix} F_0 & \cdots & F_n \end{pmatrix}.$$

Then the coefficient of z^k in $Q(z)$ is $\sum_{j,k+j \geqslant 0} F_j^* F_{k+j}$; that is, the sum of the elements on the k^{th} diagonal of $\widetilde{F}$, counted from bottom left to upper right, the main diagonal being in the zeroth position. Since this sum is independent of the factorization, if $Q(\mathrm{e}^{\mathrm{i}\theta}) = G(\mathrm{e}^{\mathrm{i}\theta})^* G(\mathrm{e}^{\mathrm{i}\theta})$ is another such factorization and $\widetilde{G}$ is formed as $\widetilde{F}$, then for fixed $0 \leqslant s \leqslant 1$,

$$\widetilde{H} = s\widetilde{F} + (1-s)\widetilde{G}$$

also has the property that the coefficient of z^k in $Q(z)$ is $\sum_{j,k+j \geqslant 0} \widetilde{H}_{j,k+j}$. The operators $\widetilde{F}$ and $\widetilde{G}$ are nonnegative, so $\widetilde{H}$ is as well, and consequently it has a square root $\widetilde{H}^{1/2}$ in $\mathcal{L}\Big(\bigoplus_0^n \mathcal{H}\Big)$. Let H_k be the restriction of $\widetilde{H}^{1/2}$ to the k^{th} copy of $\mathcal{H}$ in $\bigoplus_0^n \mathcal{H}$, and set $H(z) = \sum_0^n H_k z^k$. Then it is easy to see that $Q(\mathrm{e}^{\mathrm{i}\theta}) = H(\mathrm{e}^{\mathrm{i}\theta})^* H(\mathrm{e}^{\mathrm{i}\theta})$. Furthermore, for all $0 \leqslant j, k \leqslant n$,

$$H_j^* H_k = \widetilde{H}_{jk} = s\widetilde{F}_{j,k} + (1-s)\widetilde{G}_{j,k} = sF_j^* F_k + (1-s) G_j^* G_k,$$

showing that $\mathcal{F}$ is convex. ∎

It is apparent that in place of $\mathcal{F}_{j,k}$, one could consider sets of affine combinations of the $F_j^* F_k$'s, and in this manner also obtain convex sets. A special case of this is the next result, which follows in a straightforward manner from the last theorem and Theorem 1.3.

Corollary 1.5. *Let $Q(\mathrm{e}^{\mathrm{i}\theta}) = \sum\limits_{-n}^{n} Q_k \mathrm{e}^{\mathrm{i}k\theta}$ be a trigonometric polynomial with coefficients in $\mathcal{L}(\mathcal{H})$ such that $Q(\mathrm{e}^{\mathrm{i}\theta}) \geqslant 0$ for all $\theta \in [0, 2\pi)$. Then for $0 \leqslant m \leqslant n$, the set*

$$\mathcal{F}_m = \Big\{ \sum_{k=0}^{m} F_k^* F_k \,\Big|\, F(z) = \sum_{k=0}^{n} F_k z^k \text{ and } F(\mathrm{e}^{\mathrm{i}\theta})^* F(\mathrm{e}^{\mathrm{i}\theta}) = Q(\mathrm{e}^{\mathrm{i}\theta}) \Big\}$$

is convex, where for each F, the coefficients F_k are in $\mathcal{L}(\mathcal{H}, \mathcal{K}_F)$, the Hilbert space $\mathcal{K}_F$ depending on F. In addition, $\mathcal{F}_m$ has a maximal and minimal element obtained from outer and $*$-outer factorizations of $Q(\mathrm{e}^{\mathrm{i}\theta})$, respectively.

The only part of the statement of the above corollary which is perhaps not obvious is that the minimal element should correspond to the $*$-outer factorization of $Q(\mathrm{e}^{\mathrm{i}\theta})$. By Theorem 1.3 and the definition of $*$-outer functions, if F is a $*$-outer polynomial of degree n, then $F_n^* F_n \geqslant G_n^* G_n$ for all polynomials G such that $Q(\mathrm{e}^{\mathrm{i}\theta}) = G(\mathrm{e}^{\mathrm{i}\theta})^* G(\mathrm{e}^{\mathrm{i}\theta})$. Indeed, in this case, $\sum\limits_{j}^{n} F_k^* F_k \geqslant \sum\limits_{j}^{n} G_k^* G_k$ for $0 \leqslant j \leqslant n$. Since $\sum\limits_{0}^{n} F_k^* F_k = \sum\limits_{0}^{n} G_k^* G_k$, it follows that $\sum\limits_{0}^{j} F_k^* F_k \leqslant \sum\limits_{0}^{j} G_k^* G_k$ for $0 \leqslant j \leqslant n$.

The next proposition describes when outer polynomials are also $*$-outer.

Proposition 1.6. *Suppose $F(z) = \sum\limits_{k=0}^{n} F_k z^k$ with coefficients in $\mathcal{L}(\mathcal{H}, \mathcal{K})$ is outer, and $Q(\mathrm{e}^{\mathrm{i}\theta}) = F(\mathrm{e}^{\mathrm{i}\theta})^* F(\mathrm{e}^{\mathrm{i}\theta})$. Then the following are equivalent:*

(i) *F is $*$-outer;*

(ii) *for every $G(z) = \sum\limits_{k=0}^{n} G_k z^k$ with coefficients in $\mathcal{L}(\mathcal{H}, \mathcal{G})$ such that $G(\mathrm{e}^{\mathrm{i}\theta})^* G(\mathrm{e}^{\mathrm{i}\theta}) = Q(\mathrm{e}^{\mathrm{i}\theta})$, there exists a partial isometry R mapping $\overline{\operatorname{ran}}\, F(0)$ onto $\overline{\operatorname{ran}}\, G(0)$ such that $G(z) = RF(z)$;*

(iii) *for every $G(z) = \sum\limits_{k=0}^{n} G_k z^k$ with coefficients in $\mathcal{L}(\mathcal{H}, \mathcal{G})$ such that $G(\mathrm{e}^{\mathrm{i}\theta})^* G(\mathrm{e}^{\mathrm{i}\theta}) = Q(\mathrm{e}^{\mathrm{i}\theta})$, $G_0^* G_0 = F_0^* F_0$;*

(iv) *for every $G(z) = \sum\limits_{k=0}^{n} G_k z^k$ with coefficients in $\mathcal{L}(\mathcal{H}, \mathcal{G})$ such that $G(\mathrm{e}^{\mathrm{i}\theta})^* G(\mathrm{e}^{\mathrm{i}\theta}) = Q(\mathrm{e}^{\mathrm{i}\theta})$, G is both outer and $*$-outer.*

Proof. The equivalence of (i) and (iv) follows directly from the last corollary. That (iv) implies (iii) and (iii) implies (ii) is a simple consequence of Theorem 1.3.

To prove (ii) implies (i), let $G_*(\mathrm{e}^{\mathrm{i}\theta})^* G_*(\mathrm{e}^{\mathrm{i}\theta}) = \widehat{Q}(\mathrm{e}^{\mathrm{i}\theta})$ be an outer factorization of $\widehat{Q}(\mathrm{e}^{\mathrm{i}\theta}) = \widehat{F}(\mathrm{e}^{\mathrm{i}\theta})^* \widehat{F}(\mathrm{e}^{\mathrm{i}\theta})$, where $\widehat{F}(z) = z^n F(1/z)$. By the operator Fejér-Riesz theorem, G_* is a polynomial of the form $G_*(z) = \sum\limits_{k=0}^{n} G_{*k} z^k$. Since (ii) is assumed to hold, for $\widehat{G}_* = z^n G_*(1/z)$, there is a partial isometry R such that $\widehat{G}_* = RF$. Equivalently, $G_* = R\widehat{F}$. Since G_* is outer, it follows from Theorem 1.3 that $\widehat{F}$ is outer. ∎

1.3. Schur complements. Recall that if $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and

$$M = \begin{pmatrix} P & Q \\ Q^* & R \end{pmatrix} : \mathcal{H} \oplus \mathcal{K} \to \mathcal{H} \oplus \mathcal{K} \tag{1.3.1}$$

is a nonnegative operator, then there exists a unique contraction $G : \overline{\operatorname{ran}}\,(R) \to \overline{\operatorname{ran}}\,(P)$ such that $Q = P^{1/2} G R^{1/2}$. The *Schur complement S of P in M* is defined to be the operator $R^{1/2}(1 - G^*G)R^{1/2}$, and the *Schur complement of R in M* is defined to be the operator $P^{1/2}(1 - GG^*)P^{1/2}$. The identities in these expressions are the identities on $\overline{\operatorname{ran}}\,(R)$ and $\overline{\operatorname{ran}}\,(P)$, respectively. An alternative way to define the Schur complement of P in M is via

$$\langle Sg, g\rangle = \inf \left\{ \left\langle \begin{pmatrix} P & Q \\ Q^* & R \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle, \begin{pmatrix} f \\ g \end{pmatrix} \,\middle|\, f \in \mathcal{H} \right\};$$

that is, it is the largest positive operator which may be subtracted from R in (1.3.1) such that the resulting operator matrix is positive.

2. The extremals of the family $\mathcal{C}_\rho$

In studying the families $\mathcal{C}_\rho$, one discovers that there is a major difference between the case when $\rho \leqslant 2$ and $\rho > 2$. Namely, as is indicated by the following proposition, it is only necessary to check the validity of

$$A \in \mathcal{C}_\rho \iff 1 - aAA^*|z|^2 + bAz + bA^*\overline{z} \geqslant 0 \quad \forall\, z \in \mathbb{D} \tag{2.1}$$

for $|z| = 1$ to be able to conclude membership on $\mathcal{C}_\rho$ when $\rho \leqslant 2$.

Lemma 2.1. *Let $\rho \in (0, 2]$. Then $A \in \mathcal{C}_\rho$ if and only if*

$$1 - aAA^* + bA\mathrm{e}^{\mathrm{i}\theta} + bA^*\mathrm{e}^{-\mathrm{i}\theta} \geqslant 0 \quad \forall\, \theta \in [0, 2\pi), \tag{2.2}$$

where $a = (2 - \rho)/\rho$ and $b = (\rho - 1)/\rho$.

Proof. Note that when $0 < r \leqslant 1$ we have that

$$\Big(\frac{1}{r} - aAA^*r\Big) - (1 - aAA^*) = \Big(\frac{1}{r} - 1\Big) + a(1-r)AA^* \geqslant 0,$$

since $a \geqslant 0$. But then (2.2) implies that

$$\frac{1}{r} - aAA^*r + bA\mathrm{e}^{\mathrm{i}\theta} + bA^*\mathrm{e}^{-\mathrm{i}\theta} \geqslant 0 \quad \forall\, r \in (0,1],\ \forall\, \theta \in (0, 2\pi].$$

Multiplying by r and taking $z = r\mathrm{e}^{\mathrm{i}\theta}$ yields the right side of (2.1) (observe that when $r = 0$, (2.1) is vacuous). But then $A \in \mathcal{C}_\rho$ follows. ∎

Another characterization for $\mathcal{C}_\rho$, $\rho \in (0,2]$, is the following generalization of a result in [3] for operators with numerical radius less than or equal to one (corresponding to the family $\mathcal{C}_2$).

Proposition 2.2. *Let $\rho \in (0,2]$ and*

$$\mathcal{Z}_\rho(A) = \left\{ Z \;\middle|\; \begin{pmatrix} 1 - aAA^* + Z & 2bA \\ 2bA^* & 1 - aAA^* - Z \end{pmatrix} \geqslant 0 \right\}$$
$$= \left\{ Z \;\middle|\; \begin{pmatrix} 1 + Z & A \\ A^* & 1 - Z \end{pmatrix} - a \begin{pmatrix} AA^* & A \\ A^* & AA^* \end{pmatrix} \geqslant 0 \right\},$$

where $a = (2-\rho)/\rho$ and $b = (\rho-1)/\rho$. An operator A is in $\mathcal{C}_\rho$ if and only if $\mathcal{Z}_\rho(A) \neq \emptyset$. Moreover, in this case there exist a largest and a smallest Z (with respect to the Loewner ordering) in $\mathcal{Z}_\rho(A)$, denoted by $Z_+^\rho(A)$ and $Z_-^\rho(A)$, respectively. In addition, every element $Z \in \mathcal{Z}_\rho(A)$ is obtained from a factorization

$$1 - aAA^* + bA\mathrm{e}^{\mathrm{i}\theta} + bA^*\mathrm{e}^{-\mathrm{i}\theta} = F(\mathrm{e}^{\mathrm{i}\theta})^* F(\mathrm{e}^{\mathrm{i}\theta}), \tag{2.3}$$

where F is of the form $F(z) = V + Wz$, via $Z = 2V^*V - 1 + aAA^*$. The extremes $Z_+^\rho(A)$ and $Z_-^\rho(A)$ correspond to taking an outer and $*$-outer factorization in (2.3)*, respectively. In particular, $\mathcal{Z}_\rho(A)$ is a singleton if and only if every outer factorization in* (2.3) *is $*$-outer.*

In the case that $\rho = 1$ we get that $A \in \mathcal{C}_1$ if and only if $1 - AA^* \geqslant 0$. When $\rho = 2$, we have that $A \in \mathcal{C}_2$ if and only if there exist a Z such that

$$\begin{pmatrix} 1 + Z & A \\ A^* & 1 - Z \end{pmatrix} \geqslant 0,$$

which is Andô's criterion for A to be an operator with numerical radius less than or equal to one ([3]). The more general cases are implicit in factorization results for operators in $\mathcal{C}_\rho$ due to Durszt ([10]) and Andô and Okuba ([6]).

Proof of Proposition 2.2. Suppose that $\rho \in (0,2]$ and that $A \in \mathcal{C}_\rho$. By Lemma 2.1, this is equivalent to

$$1 - aAA^* + bA\mathrm{e}^{\mathrm{i}\theta} + bA^*\mathrm{e}^{-\mathrm{i}\theta} \geqslant 0 \quad \forall\, \theta \in (0, 2\pi].$$

Now by Theorem 1.2, there is a factorization as in (2.3) where F has the required form. Putting $Z = 2V^*V - 1 + aAA^*$, we get

$$0 \leqslant 2\begin{pmatrix} V^* \\ W^* \end{pmatrix}\begin{pmatrix} V & W \end{pmatrix} = \begin{pmatrix} 1 - aAA^* + Z & 2bA \\ 2bA^* & 1 - aAA^* - Z \end{pmatrix}. \tag{2.4}$$

Hence $Z \in \mathcal{Z}_\rho(A)$.

On the other hand, if $Z \in \mathcal{Z}_\rho(A)$, then since the matrix in (2.4) is nonnegative, it has a factorization of the form

$$2\begin{pmatrix} V^* \\ W^* \end{pmatrix}\begin{pmatrix} V & W \end{pmatrix}$$

(for example, take $\begin{pmatrix} V & W \end{pmatrix}$ to be the square root of the matrix in (2.4), with V and W restrictions of the resulting operator to suitable subspaces). But then (2.3) is seen to hold for $F(z) = V + Wz$, and so by Lemma 2.1, $A \in \mathcal{C}_\rho$.

The remainder follows from Corollary 1.5. ∎

Our first major theorem gives various characterizations of the collection of extremals $\partial^{\mathrm{e}}_\rho$ for the family of ρ-contractions, $\mathcal{C}_\rho$. In the special case of $\rho = 2$ the equivalence (i) $\Leftrightarrow$ (ii) was proved in the finite dimensional case in [9], and subsequently in the infinite dimensional case in [11]. For the finite dimensional case additional characterizations of extremal elements in $\mathcal{C}_2$ may be found in [9].

THEOREM 2.3. *Fix $\rho \in (0,1) \cup (1,2]$ and let $A \in \mathcal{C}_\rho$. Set $a = (2-\rho)/\rho$ and $b = (\rho - 1)/\rho$. Then the following are equivalent:*

(i) $A \in \partial^{\mathrm{e}}_\rho$;

(ii) $Z^\rho_-(A) = Z^\rho_+(A)$;

(iii) *some (and hence any) outer factorization of*

$$1 - aAA^* + bA\mathrm{e}^{\mathrm{i}\theta} + bA^*\mathrm{e}^{-\mathrm{i}\theta}$$

is $*$-outer.

(iv) *for any factorization*

$$1 - aAA^* + bA\mathrm{e}^{\mathrm{i}\theta} + bA^*\mathrm{e}^{-\mathrm{i}\theta} = F(\mathrm{e}^{\mathrm{i}\theta})^*F(\mathrm{e}^{\mathrm{i}\theta}),$$

where $F(z) = V + Wz$,

(a) *it obtains that* $\overline{\operatorname{ran}}\, V \supseteq \overline{\operatorname{ran}}\, W$*;*
(b) *it obtains that* $\overline{\operatorname{ran}}\, V \subseteq \overline{\operatorname{ran}}\, W$*;*
(c) *it obtains that* $\overline{\operatorname{ran}}\, V = \overline{\operatorname{ran}}\, W$*;*

(v) *for all* $Z \in \mathcal{Z}_\rho(A)$*, in the matrix*

$$\begin{pmatrix} 1 - aAA^* + Z & 2bA \\ 2bA^* & 1 - aAA^* - Z \end{pmatrix}, \tag{2.5}$$

(a) *the Schur complement of* $1 - aAA^* - Z$ *is zero;*
(b) *the Schur complement of* $1 - aAA^* + Z$ *is zero;*
(c) *the Schur complements of both* $1 - aAA^* - Z$ *and* $1 - aAA^* + Z$ *are zero;*

(vi) *for all* $Z \in \mathcal{Z}_\rho(A)$ *it holds that*

$$A = \frac{1}{2b}[1 - aAA^* + Z]^{\frac{1}{2}} G [1 - aAA^* - Z]^{\frac{1}{2}},$$

with $G : \overline{\operatorname{ran}}\,(1 - aAA^* - Z) \to \overline{\operatorname{ran}}\,(1 - aAA^* + Z)$
(a) *coisometric;*
(b) *isometric;*
(c) *unitary.*

Order of the proof:

$$\begin{tikzcd}
 & & \text{(vi)(c)} \arrow[d, <->] & \\
 & \text{(vi)(a)} \arrow[d, <->] & \text{(v)(c)} \arrow[dl] \arrow[dr] & \text{(vi)(b)} \arrow[d, <->] \\
 & \text{(v)(a)} \arrow[dr] & & \text{(v)(b)} \arrow[dl] \\
\text{(i)} \arrow[ur] & \text{(iv)(b)} \arrow[l] & \text{(ii)} \arrow[uu] \arrow[d, <->] & \\
\text{(iv)(a)} \arrow[u] & \text{(iv)(c)} \arrow[l] \arrow[u] & \text{(iii)} \arrow[l] &
\end{tikzcd}$$

Proof of Theorem 2.3. (iv)(c) $\Rightarrow$ (iv)(a), (iv)(c) $\Rightarrow$ (iv)(b), (v)(c) $\Rightarrow$ (v)(a), and (v)(c) $\Rightarrow$ (v)(b) are trivial. (v)(a) $\Leftrightarrow$ (vi)(a), (v)(b) $\Leftrightarrow$ (vi)(b), and (v)(c) $\Leftrightarrow$ (vi)(c) follow directly from the definition of the Schur complement. (ii) $\Leftrightarrow$ (iii) follows directly from Proposition 2.2. (iii) $\Rightarrow$ (iv)(c) is a consequence of Proposition 2.2 and the observation that for $F = V + Wz$ outer $\overline{\operatorname{ran}}\, V \supseteq \overline{\operatorname{ran}}\, W$, and when F is $*$-outer, $\overline{\operatorname{ran}}\, W \supseteq \overline{\operatorname{ran}}\, V$. This leaves the following parts of the above diagram to prove:

$$
\begin{array}{ccccccc}
 & & & & (v)(c) & & \\
 & & (v)(a) & & \uparrow & & (v)(b) \\
 & \nearrow & & \searrow & & \swarrow & \\
(i) & \leftarrow & (iv)(b) & & (ii) & & \\
\uparrow & & & & & & \\
(iv)(a). & & & & & &
\end{array}
$$

(ii) $\Rightarrow$ (v)(c). Let $S \geqslant 0$ be the Schur complement of $1 - aAA^* - Z$ in (2.5). Then

$$
\begin{aligned}
&\begin{pmatrix} 1 - aAA^* + Z - S & 2bA \\ 2bA^* & 1 - aAA^* - Z + S \end{pmatrix} \\
&\qquad \geqslant \begin{pmatrix} 1 - aAA^* + Z - S & 2bA \\ 2bA^* & 1 - aAA^* - Z \end{pmatrix} \geqslant 0,
\end{aligned}
$$

and so $Z - S$ is in $\mathcal{Z}_\rho(A)$. Since by assumption $Z \in \mathcal{Z}_\rho(A)$ is unique, $S = 0$. An identical argument shows that the Schur complement of $1 - aAA^* + Z$ in (2.5) is also zero.

(v)(a) $\Rightarrow$ (ii). Write Z_+ and Z_- for the largest and smallest elements of $\mathcal{Z}_\rho(A)$, and set $Z = (1/2)(Z_+ + Z_-)$. Then

$$
\begin{aligned}
\begin{pmatrix} 1-aAA^*+Z & 2bA \\ 2bA^* & 1 - aAA^*-Z \end{pmatrix} = \frac{1}{2}\Bigg[&\begin{pmatrix} 1-aAA^*+Z_+ & 2bA \\ 2bA^* & 1-aAA^*-Z_+ \end{pmatrix} \\
&+ \begin{pmatrix} 1-aAA^*+Z_- & 2bA \\ 2bA^* & 1-aAA^*-Z_- \end{pmatrix}\Bigg].
\end{aligned} \tag{2.6}
$$

Denote the Schur complement of the (2,2)-entry in the first matrix in (2.6) by R. By assumption, $R = 0$. We also have that

$$
\begin{aligned}
2\langle Rx, x\rangle = \inf_y \Bigg\{ &\Bigg\langle \begin{pmatrix} 1 - aAA^* + Z_+ & 2bA \\ 2bA^* & 1 - aAA^* - Z_+ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \Bigg\rangle \\
&+ \Bigg\langle \begin{pmatrix} 1 - aAA^* + Z_- & 2bA \\ 2bA^* & 1 - aAA^* - Z_- \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \Bigg\rangle \Bigg\}.
\end{aligned} \tag{2.7}
$$

Fix x. Since $\langle Rx, x\rangle = 0$, the positivity of the two matrices in (2.7) implies that there is a sequence $\{y_n\}$ such that

$$
\Bigg\langle \begin{pmatrix} 1 - aAA^* + Z_\pm & 2bA \\ 2bA^* & 1 - aAA^* - Z_\pm . \end{pmatrix} \begin{pmatrix} x \\ y_n \end{pmatrix}, \begin{pmatrix} x \\ y_n \end{pmatrix} \Bigg\rangle \to 0. \tag{2.8}
$$

By the positivity of the two matrices in (2.7) we also know that there are contractions $G_\pm : \overline{\operatorname{ran}}\,(1 - aAA^* - Z_\pm) \to \overline{\operatorname{ran}}\,(1 - aAA^* + Z_\pm)$ such that

$$A = \frac{1}{2b}[1 - aAA^* + Z_\pm]^{\frac12} G_\pm [1 - aAA^* - Z_\pm]^{\frac12}.$$

So,

$$\begin{aligned}|\langle Ay_n, x\rangle| &= \frac{1}{2|b|}\big|\big\langle [1 - aAA^* + Z_\pm]^{\frac12} G_\pm [1 - aAA^* - Z_\pm]^{\frac12} y_n x\big\rangle\big| \\ &\leqslant \frac{1}{2|b|}\big\|[1 - aAA^* + Z_\pm]^{\frac12} x\big\|\,\big\|[1 - aAA^* - Z_\pm]^{\frac12} y_n\big\|.\end{aligned}$$

Thus, since $\langle 2bAy_n, x\rangle \geqslant -2|b||\langle Ay_n, x\rangle|$, we get that

$$\begin{aligned}&\langle [1 - aAA^* + Z_\pm]x, x\rangle + 2\mathrm{Re}\,\langle 2bAy_n, x\rangle + \langle [1 - aAA^* - Z_\pm]y_n, y_n\rangle \\ &\quad \geqslant \big[\big\|[1 - aAA^* + Z_\pm]^{\frac12} x\big\| - \big\|[1 - aAA^* - Z_\pm]^{\frac12} y_n\big\|\big]^2 \geqslant 0.\end{aligned}$$

Since the left side tends to 0 (by (2.7) and the fact that $R = 0$), we find that

$$\big\|[1 - aAA^* - Z_\pm]^{\frac12} y_n\big\| \to \big\|[1 - aAA^* + Z_\pm]^{\frac12} x\big\|.$$

Now (2.8) translates into

$$\mathrm{Re}\,\langle 2bAy_n, x\rangle \to -\langle [1 - aAA^* + Z_\pm]x, x\rangle,$$

yielding

$$\langle [1 - aAA^* + Z_+]x, x\rangle = \langle [1 - aAA^* + Z_-]x, x\rangle.$$

Thus $\langle (Z_+ - Z_-)x, x\rangle = 0$, and since $Z_+ - Z_- \geqslant 0$, we obtain $Z_+x = Z_-x$. Since x was arbitrary, it follows that $Z_+ = Z_-$.

(v)(b) $\Rightarrow$ (ii) follows from an identical argument.

(i) $\Rightarrow$ (v)(a). Let $Z \in \mathcal{Z}_\rho(A)$, and let YY^* be the Schur complement of $1 - aAA^* - Z$ in (2.5). Set

$$\varepsilon = \frac{1}{\sqrt{4b^2 + 2a}}.$$

Since $4b^2 + 2a = 4b^2 - 4b + 2 > 0$ for all $\rho > 0$, it follows that $\varepsilon > 0$ for all $\rho > 0$. Using the fact that

$$1 - 2a\varepsilon^2 = \left[\frac{1}{\varepsilon^2} - 2a\right]\varepsilon^2 = [4b^2 + 2a - 2a]\varepsilon^2 = (2b\varepsilon)^2,$$

we have

$$\begin{pmatrix} (1 - 2a\varepsilon^2)YY^* & 2b\varepsilon Y \\ 2b\varepsilon Y^* & 1 \end{pmatrix} = \begin{pmatrix} 2b\varepsilon Y \\ 1 \end{pmatrix} \begin{pmatrix} 2b\varepsilon Y^* & 1 \end{pmatrix} \geqslant 0.$$

By definition of the Schur complement,

$$\begin{pmatrix} 1 - aAA^* + Z - YY^* & 2bA \\ 2bA^* & 1 - aAA^* - Z \end{pmatrix} \geqslant 0,$$

so for

$$Q_1 = 1 - aAA^* + [Z - a\varepsilon^2 YY^*] - a\varepsilon^2 YY^*$$
$$Q_2 = 1 - aAA^* - [Z - a\varepsilon^2 YY^*] - a\varepsilon^2 YY^*,$$

it follows that

$$\begin{pmatrix} Q_1 & 0 & 2bA & 2b\varepsilon Y \\ 0 & 1 & 0 & 0 \\ 2bA^* & 0 & Q_2 & 0 \\ 2b\varepsilon Y^* & 0 & 0 & 1 \end{pmatrix} \geqslant \begin{pmatrix} (1 - 2a\varepsilon^2)YY^* & 0 & 0 & 2b\varepsilon Y \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2b\varepsilon Y^* & 0 & 0 & 1 \end{pmatrix} \geqslant 0.$$

Therefore, by Proposition 2.2, the operator

$$\begin{pmatrix} A & \varepsilon Y \\ 0 & 0 \end{pmatrix}$$

is in $\mathcal{C}_\rho$ (the element corresponding to Z in that proposition being

$$\begin{pmatrix} Z - a\varepsilon^2 YY^* & 0 \\ 0 & 0 \end{pmatrix}$$

in this case). But since $A \in \partial^{\mathrm{e}}_\rho$, Y must be zero, and so the Schur complement is zero.

(iv)(b) $\Rightarrow$ (i). We first consider the case when $\rho < 2$. Suppose

$$1 - aAA^* + bA\mathrm{e}^{\mathrm{i}\theta} + bA^*\mathrm{e}^{-\mathrm{i}\theta} = F(\mathrm{e}^{\mathrm{i}\theta})^* F(\mathrm{e}^{\mathrm{i}\theta}),$$

where $F(z) = V + Wz$. Then

$$V^*W = bA$$
$$V^*V + W^*W = 1 - aAA^*.$$

Let

$$\widetilde{A} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathcal{C}_\rho,$$

and $\widetilde{F} = \widetilde{V} + \widetilde{W}z$, where

$$\widetilde{V} = \begin{pmatrix} V_1 & V_2 \end{pmatrix} \quad \text{and} \quad \widetilde{W} = \begin{pmatrix} W_1 & W_2 \end{pmatrix},$$

and

$$\widetilde{F}(e^{i\theta})^* \widetilde{F}(e^{i\theta}) = 1 - a\widetilde{A}\widetilde{A}^* + b\widetilde{A}e^{i\theta} + b\widetilde{A}^* e^{-i\theta}.$$

Then $V_1^* W_1 = bA$ and $V_1^* V_1 + W_1^* W_1 = 1 - a(AA^* + BB^*)$. Since $0 < \rho < 2$, we have $aBB^* \geqslant 0$. Now set

$$\widehat{V} = \begin{pmatrix} V_1 \\ \sqrt{a}B^* \end{pmatrix} \quad \text{and} \quad \widehat{W} = \begin{pmatrix} W_1 \\ 0 \end{pmatrix}. \tag{2.9}$$

It follows that

$$\begin{aligned} &\widehat{V}^* \widehat{W} = bA \\ &\widehat{V}^* \widehat{V} + \widehat{W}^* \widehat{W} = 1 - aAA^*. \end{aligned}$$

By our assumptions, $\overline{\mathrm{ran}}\, \widehat{V} \subseteq \overline{\mathrm{ran}}\, \widehat{W}$, and since $a \neq 0$, it follows that $B = 0$, and so $A \in \partial_\rho^{\mathrm{e}}$.

Now suppose $\rho = 2$, again with F as in the previous case. Then

$$V^* W = A \quad \text{and} \quad V^* V + W^* W = 1.$$

Let $\widetilde{A}$, $\widetilde{F}$, $\widetilde{V}$, and $\widetilde{W}$ be as above. Then

$$\begin{aligned} &\widetilde{V}^* \widetilde{W} = \frac{1}{2} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \\ &\widetilde{V}^* \widetilde{V} + \widetilde{W}^* \widetilde{W} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{2.10}$$

Let

$$\widetilde{V} = \begin{pmatrix} V_1 & V_2 \end{pmatrix} \quad \text{and} \quad \widetilde{W} = \begin{pmatrix} W_1 & W_2 \end{pmatrix},$$

and set $\widehat{F} = V_1 + W_1 z$. Then $V_1^* W_1 = (1/2)A$ and $V_1^* V_1 + W_1^* W_1 = 1$. By the first equation in (2.10), $V_2^* W_1 = 0$, so $\overline{\mathrm{ran}}\, V_2 \perp \overline{\mathrm{ran}}\, W_1$, and by assumption $\overline{\mathrm{ran}}\, V_1 \subseteq \overline{\mathrm{ran}}\, W_1$. Thus $V_2^* V_1 = 0$, and so by the second equation in (2.10), $W_2^* W_1 = 0$. It follows that $\overline{\mathrm{ran}}\, W_2 \perp \overline{\mathrm{ran}}\, W_1 \supseteq \overline{\mathrm{ran}}\, V_1$, and so $B = (1/2) V_1^* W_2 = 0$. That is, $A \in \partial_\rho^{\mathrm{e}}$.

(iv)(a) $\Rightarrow$ (i). As before, we first look at the case $\rho < 2$. The argument proceeds in an identical fashion, except that at (2.9), we use

$$\widehat{V} = \begin{pmatrix} V_1 \\ 0 \end{pmatrix} \quad \text{and} \quad \widehat{W} = \begin{pmatrix} W_1 \\ \sqrt{a}B^* \end{pmatrix}.$$

Then as before

$$\begin{aligned} &\widehat{V}^* \widehat{W} = bA \\ &\widehat{V}^* \widehat{V} + \widehat{W}^* \widehat{W} = 1 - aAA^*. \end{aligned}$$

Now the assumption $\overline{\operatorname{ran}}\,\widehat{V} \supseteq \overline{\operatorname{ran}}\,\widehat{W}$ and the fact that $a \neq 0$ implies $B = 0$, and so $A \in \partial_\rho^{\mathrm{e}}$.

Finally, consider the case $\rho = 2$. As before, we assume

$$1 + \frac{1}{2}(A\mathrm{e}^{\mathrm{i}\theta} + A^*\mathrm{e}^{-\mathrm{i}\theta}) = F(\mathrm{e}^{\mathrm{i}\theta})^* F(\mathrm{e}^{\mathrm{i}\theta}),$$

where $F(z) = V + Wz$; and so

$$V^*W = \frac{1}{2}A \quad \text{and} \quad V^*V + W^*W = 1.$$

Let

$$\widetilde{A} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathcal{C}_2,$$

where without loss of generality, we may assume that A, B, and C all act on the same space. Then by the remark following the proof of Proposition 2.1 of [9], $\widetilde{A} \in \mathcal{C}_2$ if and only if

$$\begin{pmatrix} A^* & B \\ 0 & C^* \end{pmatrix} \in \mathcal{C}_2.$$

Since $\mathcal{C}_2$ is closed under unitary similarity and taking adjoints, we have $\widetilde{A} \in \mathcal{C}_2$ if and only if

$$\widehat{A} = \begin{pmatrix} C & B^* \\ 0 & A \end{pmatrix} \in \mathcal{C}_2.$$

Let $\widetilde{F} = \widetilde{V} + \widetilde{W}z$, where

$$\widetilde{V} = \begin{pmatrix} V_2 & V_1 \end{pmatrix} \quad \text{and} \quad \widetilde{W} = \begin{pmatrix} W_2 & W_1 \end{pmatrix},$$

and

$$\widetilde{F}(\mathrm{e}^{\mathrm{i}\theta})^* \widetilde{F}(\mathrm{e}^{\mathrm{i}\theta}) = 1 + \frac{1}{2}(\widehat{A}\mathrm{e}^{\mathrm{i}\theta} + \widehat{A}^*\mathrm{e}^{-\mathrm{i}\theta}).$$

Then (2.10) becomes

$$\begin{aligned} \widetilde{V}^*\widetilde{W} &= \frac{1}{2}\widehat{A} \\ \widetilde{V}^*\widetilde{V} + \widetilde{W}^*\widetilde{W} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

As before $V_1^*W_1 = (1/2)A$ and $V_1^*V_1 + W_1^*W_1 = 1$, but now $V_1^*W_2 = 0$. Consequently $\overline{\operatorname{ran}}\,W_2 \perp \overline{\operatorname{ran}}\,V_1 \supseteq \overline{\operatorname{ran}}\,W_1$, and hence $W_2^*W_1 = 0$, which implies $V_2^*V_1 = 0$. It follows that $\overline{\operatorname{ran}}\,V_2 \perp \overline{\operatorname{ran}}\,V_1 \supseteq \overline{\operatorname{ran}}\,W_1$, and so $B^* = (1/2)V_2^*W_1 = 0$. Once again, $A \in \partial_\rho^{\mathrm{e}}$. ∎

With regards to statement (vi)(c) in the last theorem, it is not true that if $A = (1/2b)(1 - aAA^* + Z)^{1/2}U(1 - aAA^* - Z)^{1/2}$ for some $Z \in \mathcal{Z}_\rho(A)$ with $U : \overline{\mathrm{ran}}\,(1 - aAA^* - Z) \to \overline{\mathrm{ran}}\,(1 - aAA^* + Z)$ unitary, we have $A \in \partial_\rho^{\mathrm{e}}$, as might initially be hoped. This is a consequence of the next lemma, contractions being in $\mathcal{C}_\rho$ for $\rho > 1$, and the fact that contractions have nontrivial coisometric extensions. (When $\rho < 1$, one must instead consider the operator ball of radius $1/a$.)

Lemma 2.4. *If $A \in \mathcal{C}_\rho \cap \mathcal{L}(\mathcal{H})$, $\rho \neq 1$, is right invertible (left invertible), then there is a polynomial $F(z) = V + Wz$ with*

$$F(\mathrm{e}^{\mathrm{i}\theta})^* F(\mathrm{e}^{\mathrm{i}\theta}) = 1 - aAA^* + bA\mathrm{e}^{\mathrm{i}\theta} + bA^*\mathrm{e}^{-\mathrm{i}\theta}$$

such that both $V, W \in \mathcal{L}(\mathcal{H})$ with $\mathrm{ran}\,V = \mathrm{ran}\,W = \mathcal{H}$, V invertible (left invertible), W right invertible (invertible) and $A = (1/2b)(1-aAA^*+Z)^{1/2}U(1-aAA^*-Z)^{1/2}$, where $U : \overline{\mathrm{ran}}\,(1 - aAA^* - Z) \to \overline{\mathrm{ran}}\,(1 - aAA^* + Z)$ is unitary and $Z = 2V^*V - 1 + aAA^*$.

Proof. Assume A is right invertible. Take F coming from an outer factorization of $1 - aAA^* + bA\mathrm{e}^{\mathrm{i}\theta} + bA^*\mathrm{e}^{-\mathrm{i}\theta}$. Then $A = (1/b)V^*W$, $V^*V = (1/2)(1 - aAA^* + Z)$, and $W^*W = (1/2)(1 - aAA^* - Z)$. Using polar decompositions we have

$$\begin{aligned} A &= \frac{1}{b}V^*W = \frac{1}{b}(V^*V)^{\frac{1}{2}}U_V U_W^*(W^*W)^{\frac{1}{2}} \\ &= \frac{1}{2b}(1 - aAA^* + Z)^{\frac{1}{2}}U(1 - aAA^* - Z)^{\frac{1}{2}}, \end{aligned}$$

where $U = U_V U_W^*$ is an isometry since $\overline{\mathrm{ran}}\,V \supseteq \overline{\mathrm{ran}}\,W$, which stems from taking F to be outer.

In fact, we see that the assumption that A is right invertible implies that $\mathrm{ran}\,(V^*V)^{1/2} = \mathrm{ran}\,V^* = \mathcal{H}$, so $(V^*V)^{1/2}$ is invertible. But then $\mathrm{ran}\,U = \mathcal{H}$, and so U is a unitary operator from $\overline{\mathrm{ran}}\,(1 - aAA^* - Z)$ to $\overline{\mathrm{ran}}\,(1 - aAA^* + Z)$. This implies that $\mathrm{ran}\,V = \mathrm{ran}\,W$ is closed and isomorphic as a Hilbert space to $\mathcal{H}$. Hence, without loss of generality, we may take V to be invertible and W right invertible.

The case where A is left invertible follows from an identical argument, though with F $*$-outer rather than outer, so that $\overline{\mathrm{ran}}\,W \supseteq \overline{\mathrm{ran}}\,V$. ∎

The left and right invertible operators considered in the last lemma are ubiquitous in any $\mathcal{C}_\rho$, as the next result shows.

LEMMA 2.5. *For fixed* $0 < \rho \leqslant 2$, $\mathcal{C}_\rho \cap \mathcal{L}(\mathcal{H})$ *is the closure of its interior in the operator norm topology. Consequently, the set of operators in* $\mathcal{C}_\rho \cap \mathcal{L}(\mathcal{H})$ *which are either left or right invertible are norm dense in* $\mathcal{C}_\rho \cap \mathcal{L}(\mathcal{H})$.

Proof. The statement of the lemma is true for ordinary contractions, and since the ρ-contractions for $0 < \rho \leqslant 2$ correspond to the unit ball in a norm which is equivalent to the operator norm [12], the result is true in this case as well. ∎

The proof of the following corollary to Theorem 2.3 shows that 0 *can* be the unique Z for an element of $\partial_\rho^{\mathrm{e}}$; namely if that element is unitary. We give several different proofs that unitary operators are extremal based on the various characterizations of $\partial_\rho^{\mathrm{e}}$ found in the statement of that theorem. For the case when $\rho = 2$ this result was proved in [8] by different methods. The second proof we will give is similar to Hara's proof, which was also in the case where $\rho = 2$ ([11]). The first proof features techniques used in the proof of our main result (Theorem 3.1), the ideas being more transparent here due to the simpler circumstances.

COROLLARY 2.6. *Let* $U \in \mathcal{L}(\mathcal{H})$ *be unitary. If* $\rho \geqslant 1$, *then* $U \in \partial_\rho^{\mathrm{e}}$. *If* $\rho < 1$, *then* $(1/a)U \in \partial_\rho^{\mathrm{e}}$.

Observe that when $\rho < 1$ and U is unitary, $U \notin \mathcal{C}_\rho$, since $\|U\| = 1 \nleqslant \rho$.

First proof of Corollary 2.6. First take $\rho \geqslant 1$. Consider the factorization

$$\begin{aligned} Q(\mathrm{e}^{\mathrm{i}\theta}) &\equiv 1 - aUU^* + bU\mathrm{e}^{\mathrm{i}\theta} + bU^*\mathrm{e}^{-\mathrm{i}\theta} \\ &= 2b\left[1 + \left(\frac{1}{2}\mathrm{e}^{\mathrm{i}\theta}U + \frac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}U^*\right)\right] = F^*(\mathrm{e}^{\mathrm{i}\theta})F(\mathrm{e}^{\mathrm{i}\theta}), \end{aligned}$$

where $F(z) = \sqrt{b}(1 + Uz)$. We show that this is both outer and $*$-outer.

We first prove that $\operatorname{ran} M_F$, multiplication by F, is dense in $H^2_{\mathcal{H}}(\mathbb{D})$, and so F is outer. So suppose $g = \sum_{k=0}^{\infty} g_k z^k \in H^2_{\mathcal{H}}(\mathbb{D})$ is orthogonal to $\operatorname{ran} M_F$. Equivalently, assume that for all $h \in \mathcal{H}$ and $k = 0, 1, 2, \ldots$,

$$0 = \langle g, F(z)hz^k \rangle = \sqrt{b}[\langle g_k, h\rangle + \langle g_{k+1}, Uh\rangle].$$

Then for all k, $U^* g_{k+1} = -g_k$, and so

$$g_k = (-1)^k U^k g_0 \quad k = 1, 2, \ldots,$$

which means that $\|g_k\| = \|g_0\|$ for all k. Since $\sum \|g_k\|^2 < \infty$, g must be zero.

An identical argument shows that F is $*$-outer.

Now suppose $\rho < 1$. By a result of Andô and Nishio ([5]), an operator A is in $\mathcal{C}_\rho$, $0 < \rho < 1$, if and only if $((2 - \rho)/\rho)A$ is in $\mathcal{C}_{2-\rho}$; and so this case could be obtained from the one we just proved. However, since it is only a few lines, we offer a direct proof. To this end, let $A = (1/a)U$. Then

$$1 - aAA^* + bAe^{i\theta} + bA^*e^{-i\theta} = \frac{-2b}{a}\Big[1 - \Big(\frac{1}{2}e^{i\theta}U + \frac{1}{2}e^{-i\theta}U^*\Big)\Big] \geqslant 0,$$

so A is in $\mathcal{C}_\rho$. Setting $F(z) = \sqrt{-b/a}(1 - Uz)$, we have

$$F^*(e^{i\theta})F(e^{i\theta}) = 1 - aAA^* + bAe^{i\theta} + bA^*e^{-i\theta}.$$

An identical argument to the one used for the case where $\rho \geqslant 1$ shows that F is both outer and $*$-outer. ∎

It follows from Proposition 2.2 that we may choose

$$Z = \begin{cases} 2b - 1 + aUU^* = 0, & \text{for } \rho \geqslant 1, \\ -2\frac{b}{a} - 1 + \frac{1}{a}UU^* = 0, & \text{for } \rho < 1; \end{cases}$$

since $V^2 = b$ in the first case, and $-b/a$ in the second.

Second proof of Corollary 2.6. Write $Z_\pm$ for $Z_\pm^\rho(U)$. First consider the case where $\rho \geqslant 1$. Now

$$\begin{pmatrix} 1 - aUU^* + Z_- & 2bU \\ 2bU^* & 1 - aUU^* + Z_- \end{pmatrix} = 2b\begin{pmatrix} 1 + Z_- & U \\ U^* & 1 + \frac{1}{2b}Z_- \end{pmatrix} \geqslant 0.$$

So for real λ we have

$$\begin{aligned} 0 &\leqslant \Big\langle \begin{pmatrix} 1-aUU^*+Z_- & 2bU \\ 2bU^* & 1-aUU^*-Z_- \end{pmatrix}\begin{pmatrix} x \\ -\lambda U^*x \end{pmatrix}, \begin{pmatrix} x \\ -\lambda U^*x \end{pmatrix}\Big\rangle \\ &= \|x\|^2(1-a) + \langle Z_-x, x\rangle - 4b\lambda\|x\|^2 + \lambda^2(1-a)\big(\|x\|^2 - \langle UZ_-U^*x, x\rangle\big). \end{aligned} \tag{2.12}$$

When $\lambda = 1$ (use $b = (1/2)(1-a)$) we obtain from (2.12) that $Z_- \geqslant UZ_-U^*$. Further,

$$\begin{aligned} 0 &\leqslant \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}\begin{pmatrix} 1 - aUU^* + Z_- & 2bU \\ 2bU^* & 1 - aUU^* - Z_- \end{pmatrix}\begin{pmatrix} U^* & 0 \\ 0 & U^* \end{pmatrix} \\ &= \begin{pmatrix} 1 - aUU^* + UZ_-U^* & 2bU \\ 2bU^* & 1 - aUU^* - UZ_-U^* \end{pmatrix}. \end{aligned}$$

So $UZ_-U^* \in \mathcal{Z}_\rho(U)$. But then since Z_- is the smallest element in $\mathcal{Z}_\rho(U)$ we must have $Z_- = UZ_-U^*$.

Since inequality (2.12) holds for all real λ, we obtain using elementary calculus that

$$16b^2\|x\|^4 - 4\big[\|x\|^2(1-a)+\langle Z_-x,x\rangle\big]\big[\|x\|^2(1-a)-\langle UZ_-U^*x,x\rangle\big] = 4(\langle Z_-x,x\rangle)^2 \leqslant 0,$$

where we used that $Z_- = UZ_-U^*$. Thus $\langle Z_-x,x\rangle = 0$. Since x is arbitrary this implies that $Z_- = 0$. Similarly one shows that $Z_+ = 0$. But then by Theorem 2.3, $U \in \partial_\rho^{\mathrm{e}}$.

The case $\rho < 1$ follows by identical arguments or from [5]. ∎

Third proof of Corollary 2.6. It is also possible to prove this result using limiting schemes, since it can be shown that

$$X_1 = 1 - aAA^*, \quad X_{i+1} = 1 - aAA^* - b^2A^*X_i^{-1}A$$

converges to $(1/2)(1 - aAA^* + Z_+(A))$, and

$$Y_1 = 1 - aAA^*, \quad Y_{i+1} = 1 - aAA^* - b^2AY_i^{-1}A^*$$

converges to $(1/2)(1 - aAA^* - Z_-(A))$ (see [3] for the case $\rho = 2$). ∎

3. THE BOUNDARY OF THE FAMILY $\mathcal{C}_\rho$

Recall that the boundary of $\mathcal{C}_\rho$, denoted ∂_ρ, is the smallest model for this family. It is also the smallest subcollection of $\mathcal{C}_\rho$ containing $\partial_\rho^{\mathrm{e}}$ which is closed under unital $*$-representations, direct sums, and restrictions to reducing subspaces. We use this in the following theorem to show that $\partial_\rho = \mathcal{C}_\rho$ when $\rho \neq 1$. As was noted earlier, the case when $\rho = 1$ (that is, the contractions in the operator norm) is exceptional, with ∂_1 being the coisometries by the Sz.-Nagy dilation theorem.

THEOREM 3.1. *Let $\rho \in (0,1) \cup (1,2]$. Then $\mathcal{C}_\rho = \partial_\rho$.*

Proof. We first consider the case where $A \in \mathcal{C}_\rho \cap \mathcal{L}(\mathcal{H})$ is right invertible. The general case will then be shown to follow from Lemma 2.5 and Theorem 1.1.

So assume A is right invertible and consider the positive operator function

$$Q(\mathrm{e}^{\mathrm{i}\theta}) = 1 - aAA^* + bA\mathrm{e}^{\mathrm{i}\theta} + bA^*\mathrm{e}^{-\mathrm{i}\theta},$$

where as usual, $a = 2/\rho - 1$ and $b = 1 - 1/\rho$. Let $Q(\mathrm{e}^{\mathrm{i}\theta}) = F(\mathrm{e}^{\mathrm{i}\theta})^*F(\mathrm{e}^{\mathrm{i}\theta})$ be a factorization where F has the form $F(z) = V + Wz$ (such factorizations exist by Theorem 1.2). By Lemma 2.4, we may assume $V, W \in \mathcal{L}(\mathcal{H})$ with V invertible and W right invertible. We may also assume $V \geqslant 0$.

Let $\widetilde{\mathcal{H}} = \bigoplus\limits_{-\infty}^{\infty} \mathcal{H}$, and define for $M \in \mathbb{N}$, $\mathbf{V}_M, \mathbf{W}_M$, and $\mathbf{A}_M$ in $\mathcal{L}(\widetilde{\mathcal{H}})$ by the following (here the boxed entries are in the $(-M,-M)$, (0,0), and (M,M) positions and all unspecified entries are zero):

$$\mathbf{V}_M = \begin{pmatrix} \ddots &&&&&&&&&&& \\ & c &&&&&&&&&& \\ && c &&&&&&&&& \\ &&& \boxed{\tilde{c}X} &&&&&&&& \\ &&&& V &&&&&&& \\ &&&&& \ddots &&&&&& \\ &&&&&& \boxed{V} &&&&& \\ &&&&&&& \ddots &&&& \\ &&&&&&&& \boxed{V} &&& \\ &&&&&&&&& V && \\ &&&&&&&&&& \tilde{c}\widetilde{Y} & \\ &&&&&&&&&&& c \\ &&&&&&&&&&&& c \\ &&&&&&&&&&&&& \ddots \end{pmatrix},$$

$$\mathbf{W}_M = \begin{pmatrix} \ddots &&&&&&&&&&& \\ & c & \ddots &&&&&&&&& \\ && c & \boxed{0} &&&&&&&& \\ &&& W & \ddots &&&&&&& \\ &&&& \ddots & \boxed{0} &&&&&& \\ &&&&& W & \ddots &&&&& \\ &&&&&& \ddots & \boxed{0} &&&& \\ &&&&&&& W & \ddots &&& \\ &&&&&&&& W &&& \\ &&&&&&&&& c && \\ &&&&&&&&&& c & \\ &&&&&&&&&&& \ddots \end{pmatrix},$$

$$\mathbf{A}_M = \frac{1}{b}\mathbf{V}_M\mathbf{W}_M = \begin{pmatrix} \ddots &&&&&&&&&&&& \\ & k &&&&&&&&&&& \\ && k & \ddots &&&&&&&&& \\ &&& X & \boxed{0} &&&&&&&& \\ &&&& A & \ddots &&&&&&& \\ &&&&& \ddots & \boxed{0} &&&&&& \\ &&&&&& A & \ddots &&&&& \\ &&&&&&& \ddots & \boxed{0} &&&& \\ &&&&&&&& A & \ddots &&& \\ &&&&&&&&& Y &&& \\ &&&&&&&&&& k && \\ &&&&&&&&&&& k & \\ &&&&&&&&&&&& \ddots \end{pmatrix}.$$

In the above matrices,

$$X = \frac{\rho}{\widetilde{k}}(1 - W^*W)^{\frac{1}{2}}, \quad \widetilde{Y} = \frac{1}{\widetilde{k}}(\widetilde{c}^{\,2} + aWW^*)^{-\frac{1}{2}}, \quad \text{and} \quad Y = \frac{1}{c}\widetilde{Y}W,$$

where if $0 < \rho < 1$, $\widetilde{k} = \sqrt{2-\rho}$, $c = \sqrt{-b/a} = \sqrt{(1-\rho)/(2-\rho)}$, $k = -1/a = \rho/(\rho-2)$, and $\widetilde{c} = b/c = \sqrt{[(1-\rho)(2-\rho)]/\rho^2}$, while if $1 < \rho \leqslant 2$, $\widetilde{k} = \sqrt{\rho}$, $c = \widetilde{c} = \sqrt{b} = \sqrt{(\rho-1)/\rho}$, and $k = 1$. Note that $1 - W^*W = V^2 + aAA^*$, which is positive and invertible, since $a \geqslant 0$ for $\rho \leqslant 2$ and V is invertible, and for $\rho \neq 1$, $b \neq 0$, and so $b^2 + aWW^* > 0$. Hence X, $\widetilde{Y}$, and Y are well-defined and bounded operators with bounded inverses.

It is clear that $\mathbf{V}_M \geqslant 0$ and invertible, and that $\mathbf{W}_M$ is right invertible. A straightforward though tedious calculation shows that

$$\mathbf{V}_M^2 + \mathbf{W}_M^*\mathbf{W}_M = 1_{\widetilde{\mathcal{H}}} - a\mathbf{A}_M\mathbf{A}_M^*,$$

so if we set $\mathbf{F}_M(z) = \mathbf{V}_M + \mathbf{W}_M z$, then

$$\mathbf{F}_M(\mathrm{e}^{\mathrm{i}\theta})^*\mathbf{F}_M(\mathrm{e}^{\mathrm{i}\theta}) = 1 - a\mathbf{A}_M\mathbf{A}_M^* + b\mathbf{A}_M\mathrm{e}^{\mathrm{i}\theta} + b\mathbf{A}_M^*\mathrm{e}^{-\mathrm{i}\theta} \geqslant 0,$$

and consequently, $\mathbf{A}_M \in \mathcal{C}_\rho$.

We wish to show that for each M, $\mathbf{A}_M$ is extremal. By Theorem 2.3, it suffices to show that $\mathbf{F}_M$ is both outer and $*$-outer. Since $\mathbf{V}_M$ is invertible and $\mathbf{W}_M$ is right invertible, this amounts to verifying that if $g \in H^2_{\widetilde{\mathcal{H}}}(\mathbb{D})$ is orthogonal to the range of the multiplication operator corresponding to either $\mathbf{V}_M + \mathbf{W}_M z$ or $\mathbf{W}_M + \mathbf{V}_M z$, then it must be zero.

Let us consider the first case. Write $g = \sum_0^\infty g_k z^k$, where $g_k \in \widetilde{\mathcal{H}}$ for all k. Each g_k has the form

$$g_k = (\cdots \quad g_{k,-2} \quad g_{k,-1} \quad g_{k,0} \quad g_{k,1} \quad g_{k,2} \quad \cdots)^{\mathrm{t}},$$

where "t" stands for transpose, and $g_{k,0}$ is in the zeroth position. For g to be orthogonal to the range of the multiplication operator corresponding to $\mathbf{V}_M + \mathbf{W}_M z$, it is necessary and sufficient that for all $h \in \widetilde{\mathcal{H}}$ and all $k \in \mathbb{N} \cup \{0\}$,

$$0 = \langle g, (\mathbf{V}_M + \mathbf{W}_M z) h z^k \rangle = \langle \mathbf{V}_M^* g_k, h \rangle + \langle \mathbf{W}_M^* g_{k+1}, h \rangle;$$

that is,

$$\mathbf{V}_M^* g_k = -\mathbf{W}_M^* g_{k+1}, \quad k = 0, 1, 2, \ldots.$$

Explicitly,

$$\begin{pmatrix} \vdots \\ c g_{k,-M-1} \\ \boxed{\widetilde{c} X g_{k,-M}} \\ V g_{k,-M+1} \\ \vdots \\ \boxed{V g_{k,0}} \\ \vdots \\ \boxed{V g_{k,M}} \\ V g_{k,M+1} \\ \widetilde{c} \widetilde{Y} g_{k,M+2} \\ c g_{k,M+3} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ -c g_{k+1,-M} \\ \boxed{-W^* g_{k+1,-M+1}} \\ -W^* g_{k+1,-M+2} \\ \vdots \\ \boxed{-W^* g_{k+1,1}} \\ \vdots \\ \boxed{-W^* g_{k+1,M+1}} \\ -W^* g_{k+1,M+2} \\ -c g_{k+1,M+3} \\ -c g_{k+1,M+4} \\ \vdots \end{pmatrix}, \quad k = 0, 1, 2, \ldots. \tag{3.1}$$

So for $k = 0, 1, 2, \ldots$, $\ell = 3, 4, \ldots$, and $j = 1, 2, \ldots$, we have $g_{k+j,M+j+\ell} = (-1)^j g_{k,M+\ell}$, and since $\sum_j \|g_{k+j,j+M+\ell}\|^2$ is finite for every k, it is required that $g_{k,M+\ell} = 0$ for all $k \geqslant 0$ and $\ell \geqslant 3$. Since all of the operators appearing on the left in (3.1) are invertible, we find first that $g_{k,M+2} = 0$ for all k, which in turn gives $g_{k,M+1} = 0$ for all k, and so on. The final result is that $g_{k,j} = 0$ for all k, j. Thus $g_k = 0$ for all k, proving that $\mathbf{V}_M + \mathbf{W}_M z$ is outer.

To show that it is $*$-outer, we need to demonstrate that if $g \in H^2_{\widetilde{\mathcal{H}}}(\mathbb{D})$ is orthogonal to the range of $\mathbf{W}_M + \mathbf{V}_M z$, then $g = 0$. We write $g = \sum_k g_k$ and express the g_k's as before. Then g orthogonal to the range of $\mathbf{W}_M + \mathbf{V}_M z$ is equivalent to

$$\begin{pmatrix} \vdots \\ -cg_{k,-M} \\ \boxed{-W^* g_{k,-M+1}} \\ -W^* g_{k,-M+2} \\ \vdots \\ \boxed{-W^* g_{k,1}} \\ \vdots \\ \boxed{-W^* g_{k,M+1}} \\ -W^* g_{k,M+2} \\ -cg_{k,M+3} \\ -cg_{k,M+4} \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ cg_{k+1,-M-1} \\ \boxed{\widetilde{c}X g_{k+1,-M}} \\ V g_{k+1,-M+1} \\ \vdots \\ \boxed{V g_{k+1,0}} \\ \vdots \\ \boxed{V g_{k+1,M}} \\ V g_{k+1,M+1} \\ \widetilde{c}\widetilde{Y} g_{k+1,M+2} \\ cg_{k+1,M+3} \\ \vdots \end{pmatrix}, \quad k = 0, 1, 2, \dots.$$

Arguing as before, but now starting at the top, $g_{k+j,-M-j-\ell} = (-1)^j g_{k,-M-\ell}$ for $k, \ell = 0, 1, 2, \dots$ and $j = 1, 2, \dots$, and so $g_{k,-M-\ell} = 0$ for all k and for $\ell = 0, 1, 2, \dots$. Now using the fact that all of the operators on the left are left invertible, we have first that $g_{k,-M+1} = -cW^{*-1} X g_{k+1,-M} = 0$ for all k, which gives $g_{k,-M+2} = -W^{*-1} V g_{k+1,-M+1} = 0$ for all k, and so on. Consequently $g_{k,j} = 0$ for all k, j, or equivalently, g is zero.

By Theorem 2.3, $\mathbf{A}_M$ is extremal for each M. Let $\mathbf{A}$ denote the operator on $\widetilde{\mathcal{H}}$ which has A in the $(j+1, j)$ entries and 0 elsewhere, relative to the decomposition $\bigoplus_{-\infty}^{\infty} \mathcal{H}$; that is, the tensor product of A with the bilateral shift. By Theorem 1.1, our proof will be complete once we show $\mathbf{A}_M$ and $\mathbf{A}_M^*$ converge to $\mathbf{A}$ and $\mathbf{A}^*$, respectively, in the strong operator topology.

For $m < M - 1$, and a vector $h = \bigoplus_{-m}^{m} h_j$, we have

$$\mathbf{A}_M h = \bigoplus_{-m+1}^{m+1} A h_{j-1} = \mathbf{A} h$$

and

$$\mathbf{A}_M^* h = \bigoplus_{-m-1}^{m-1} A^* h_{j+1} = \mathbf{A}^* h.$$

Since vectors of the form h are dense in $\widetilde{\mathcal{H}}$, given $f \in \widetilde{\mathcal{H}}$ there exists h as above with $\|f - h\|$ small. Choosing M larger than $m + 1$ obtains

$$\|(\mathbf{A}_M - \mathbf{A}) f\| = \|(\mathbf{A}_M - \mathbf{A})(f - h)\|.$$

Now, the quantity on the right side is at most $2\rho\|f-h\|$, since $\|\mathbf{A}_M\|$ and $\|\mathbf{A}\|$ have norm at most ρ. Hence $\mathbf{A}_M f$ converges to $\mathbf{A}f$. The same sort of argument shows that $\mathbf{A}_M^* f$ converges to $\mathbf{A}^* f$.

We now prove the general case. Suppose $A \in \mathcal{C}_\rho \cap \mathcal{L}(\mathcal{H})$ and $\mathcal{G} \subseteq \mathcal{H}$ is a closed subspace with $P_\mathcal{G}$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{G}$. Then the compression of A to $\mathcal{G}$, $A_\mathcal{G} = P_\mathcal{G} A|\mathcal{G}$, is easily seen to be in $\mathcal{C}_\rho \cap \mathcal{L}(\mathcal{G})$ by using the formula in Lemma 2.1. Lemma 2.5 and what has been shown so far imply that if $\mathcal{G}$ is finite dimensional, then the set of invertible elements of $\mathcal{C}_\rho \cap \mathcal{L}(\mathcal{G})$ is norm dense in $\partial_\rho \cap \mathcal{L}(\mathcal{G})$. Consequently, by Theorem 1.1, any element of $\mathcal{C}_\rho \cap \mathcal{L}(\mathcal{G})$ is in the boundary. In particular then, all compressions of A to finite dimensional subspaces are in the boundary, and so by another application of Theorem 1.1, A is in the boundary. This concludes the proof. ∎

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MICHAEL A. DRITSCHEL
Department of Mathematics
Purdue University
West Lafayette
Indiana 47907–1395
U.S.A.

SCOTT MCCULLOUGH
Department of Mathematics
The University of Florida
Gainesville
Florida 32611–2082
U.S.A.

HUGO J. WOERDEMAN
Department of Mathematics
The College of William & Mary
Williamsburg
Virginia 23185–8795
U.S.A.

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