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EMBEDDING $*$-ALGEBRAS INTO C^*-ALGEBRAS AND C^*-IDEALS GENERATED BY WORDS

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Abstract. A definition is given for C^*-ideals related to embedding $*$-algebras in C^*-algebras. The shrinking algorithm on words is defined and related to the question of which words algebraically generate C^*-ideals. It is proved that all words in the free unital $*$-algebra with one generator generate C^*-ideals. Finally, the C^*-ideals generated by some specific words are described.

Keywords: *C^*-algebra, relations, $*$-algebra, embed, word.*

MSC (2000): 46L.

0. INTRODUCTION

There has been some consideration of C^*-algebras given by a set of generators with relations (see [1], [2], [3], [4], [5], [6], [7], [8] and [9]). Given a $*$-algebra, with generators and relations, it may not be possible to embed it into a C^*-algebra. This paper considers which $*$-algebras may be embedded into C^*-algebras, and specifically, what relations, involving words being equated to zero, generate $*$-algebras which may be embedded in C^*-algebras.

1. C^*-IDEALS

In Section 1 we shall define C^*-ideals of $*$-algebras, and give some simple related results.

NOTATION 1.1. Let A_1 be the free unital $*$-algebra (algebra with an involution $*$) over $\mathbb{C}$ generated by the element x. Let W_1 be the set of finite words which may be formed from the elements x and x^*, (the element 1 is also considered to be a word in W_1). The elements of A_1 are finite linear combinations of words from W_1. So, if $y \in A_1$ then $y = \sum\limits_{w \in W_1} y_w \cdot w$ where each $y_w \in \mathbb{C}$ and only finitely many are non-zero. An ideal of A_1 is said to be a $*$-ideal of A_1 if it is closed under the involution operation. Given a subset S of A_1 we shall use $\langle S \rangle$ to denote the two-sided ideal of A_1 generated by S. We shall use $\langle S \rangle_*$ to denote the two-sided $*$-ideal of A_1 generated by S. Note that

$$\langle S \rangle_* = \langle S \rangle + \langle S^* \rangle = \langle S \cup S^* \rangle.$$

Given a $*$-algebra A, we say that a functional $\nu : A \to \mathbb{R}^+$ is a C^*-*seminorm* on A if ν is a sub-multiplicative seminorm which fulfills the C^*-condition $\nu(y^*y) = \nu(y)^2$ for all y in A. Let N_A be the set of all C^*-seminorms on A, and let N_A be partially ordered by saying that $\nu \leqslant \nu'$ if $\nu(y) \leqslant \nu'(y)$ for all y in A. We shall use "$*$-polynomial" to mean a finite linear combination of words in non-commuting elements and their adjoints. We will allow C^*-seminorms to send the identity to zero. Note that N_A must be non-empty (as it contains the zero functional).

LEMMA 1.2. *If A is a $*$-algebra and I is a $*$-ideal of A then the following are equivalent:*

(i) *the quotient $*$-algebra A/I may be embedded in a C^*-algebra;*

(ii) *there exists a C^*-norm on A/I;*

(iii) *there exist some C^*-seminorm ν on A with $I = \ker(\nu)$;*

(iv) *there exist some $*$-representation $\varphi : A \to B(H)$ (H a Hilbert space) with $I = \ker(\varphi)$;*

(v) *there exist some C^*-seminorm ν on A with I closed in A with respect to ν.*

Proof. Given (ii), then we may embed A/I into the completion of A/I with respect to its C^*-norm which will be a C^*-algebra, which gives (i). Given (iii), ν induces a C^*-norm on A/I, which gives (ii). Given (iv), the C^*-norm on $B(H)$ induces, via φ, a C^*-seminorm on A, which gives (iii). Given (i), and using the Gelfand-Naimark Theorem, the embedding lifts to a $*$-representation on A as required for (iv). So (i) to (iv) are equivalent. Clearly (iii) inplies (v). Given (v), let ν' be the quotient semi-norm ν/I on A/I. Now, $\ker(\nu) \subseteq I$, as, if there exists y not in I with $\nu(y) = 0$, then I cannot be closed with respect to ν. So, ν' is a C^*-norm on A/I and we have (ii). ∎

DEFINITION 1.3. Given a $*$-algebra A and a $*$-ideal I of A, we say that I is a *C^*-ideal* of A if any of the equivalent conditions in Lemma 1.2 hold.

EXAMPLES 1.4. For some examples we shall consider some ideals of A_1. The $*$-ideal $\langle x^*x\rangle$ is not a C^*-ideal, as if it were then, by condition (iii) of Lemma 1.2, there would exist some C^*-seminorm ν on A_1 with $\ker(\nu) = \langle x^*x\rangle$. But, in this case $0 = \nu(x^*x) = \nu(x)^2$ and so $x \in \ker(\nu)$ but $x \notin \langle x^*x\rangle$. However $\langle x^*x - 1\rangle$ is a C^*-ideal of A_1 and this follows from a result of Coburn ([2]). Also, it is a result of Goodearl and Menal ([3]) that the $*$-algebra A_1 may be embedded into a C^*-algebra, so $\langle 0\rangle$ is a C^*-ideal of A_1.

DEFINITION 1.5. If A is a $*$-algebra with the property that for all y in A the set $\{\nu(y) \mid \nu \in N_A\}$ is bounded then say that A is *compact*. In this case we may define $\mu : A \to \mathbb{R}^+$ by $\mu(y) = \sup\{\nu(y) \mid \nu \in N_A\}$. It is easy to see that μ will be the maximal C^*-seminorm on A, that is, the maximal element of N_A.

PROPOSITION 1.6. *If A is a compact $*$-algebra and S is a subset of A then there exists a minimum C^*-ideal in A containing S. This C^*-ideal is the closure of $\langle S\rangle_*$ in A with respect to the maximal C^*-seminorm on A.*

Proof. This is standard. ∎

DEFINITION 1.7. Given a subset S of a $*$-algebra A, if the set of C^*-ideals of A containing S has a minimum element then call it *the C^*-ideal generated by* S and write it $\langle S\rangle_{C^*}$. Proposition 1.6 shows that when A is a compact $*$-algebra any subset of A will have this property. In a general $*$-algebra this property may not hold, and so we may not always be able to define the C^*-ideal generated by a particular subset of a particular $*$-algebra.

2. THE SHRINKING ALGORITHM ON WORDS

The rest of this paper will consider which words in W_1 generate C^*-ideals in A_1, and related results. So, from now on, objects should be taken to be in A_1 and, if we call an element a word, we shall mean that it is an element of W_1.

NOTATION 2.1. Given words v and w, say that $v \geqslant w$ if v is in the $*$-ideal generated by w, that is, if $v \in \langle w \rangle_*$. So then $v \geqslant w$ if and only if $v = u_1 \cdot w \cdot u_2$ or $v = u_1 \cdot w^* \cdot u_2$ for some words u_1 and u_2. When we use an ordering on W_1 without comment, we shall mean this ordering. Let $|w|$ denote the length of word w. For example $|x^2x^{*3}x| = 6$ and $|1| = 0$.

Now, if I is a C^*-ideal of A_1 with $u^*u \cdot v$ in I, where u and v are words, then $(uv)^* \cdot uv \in I$ and so $uv \in I$ (see Examples 1.4). To express this implication we will write $u^*uv \rightsquigarrow uv$ to mean that any C^*-ideal containing u^*uv must also contain uv, and we shall say that u^*uv shrinks to uv. This leads us to the following definition.

DEFINITION 2.2. For a word w we define the *shrinking algorithm* on w as follows:

(0) Let $w' = w$.

(1) If $w' = u^*u \cdot v$ for any words u and v, with $u \neq 1$, then let $w' = uv$, and go back to (1).

(2) If $w' = u \cdot vv^*$ for any words u and v, with $v \neq 1$, then let $w' = uv$, and go back to (2).

(3) Terminate the algorithm.

Let $\widetilde{w}$ denote the final result of applying the shrinking algorithm to the word w. We say that w is *unshrinkable* if $w = \widetilde{w}$. Let $\widetilde{W_1}$ be the set of unshrinkable words in W_1. The next result shows that these are good definitions.

THEOREM 2.3. *If w is a word then:*

(i) *If we apply the shrinking algorithm to w we shall always get the same result at termination no matter what choices we make in the process, that is $\widetilde{.} : W \to W$ is a well-defined function;*

(ii) *The set $\{\widetilde{w}, \widetilde{w}^*\}$ is exactly the set of maximal unshrinkable words less than or equal to w;*

(iii) *If I is a C^*-ideal of A_1 then $\widetilde{w}$ is in I if and only if w is in I.*

Proof. (i) The reason there is something to prove is that given some word w' occurring during the algorithm, it may be that $w' = u_1^*u_1 \cdot v_1 = u_2^*u_2 \cdot v_2$ for some words u_1, v_1, u_2 and v_2, with $u_1 \neq u_2$. In this case we could choose to proceed in the algorithm in two (or more) possible ways, and this could effect the final outcome. This is effectively the only place it could go wrong so we need only consider this case.

We may assume without loss of generality that $v_1 = sv_2$ for some word s. So $u_2^* \geqslant u_1^*$. We shall show that, if we choose to remove u_1^* rather than all of u_2^*, then the algorithm will eventually remove the rest of u_2^* as well. As $v_1 = sv_2$, we have $u_1^* u_1 s = u_2^* u_2$. As $u_2^* u_2$ is hermitian we must have $u_1^* u_1 s = s^* u_1^* u_1$. By comparing the right-hand ends of these two expressions, as $|u_1 s| \geqslant |u_1|$, we must have $u_1 s = t \cdot u_1$ for some word t. Thus $u_2^* u_2 = u_1^* t u_1$. As this is hermitian it follows that $u_1^* t u_1 = u_1^* t^* u_1$, and thus $t = t^*$. Write $t = r^* r$ for some word r. So $u_2^* u_2 = u_1^* r^* \cdot r u_1$ and thus $u_2 = r \cdot u_1$. Thus, if we choose to shrink our word by removing u_1^* we will be left with $u_1 v_1 = u_1 s \cdot v_2 = t u_1 \cdot v_2 = r^* r \cdot u_1 v_2$.

Thus, in the next iteration of the shrinking algorithm we still have the option to get to $r u_1 \cdot v_2 = u_2 v_2$. Again, we may have more than one option, but applying the same argument repeatedly we may see that either the algorithm will reduce to something less than $u_2 v_2$, or we will eventually be left with $u_2 v_2$ as the algorithm only terminates when all the options are gone. So, if there is ever more than one word we could remove in the shrinking algorithm, the algorithm will eventually remove the largest of them no matter which one we choose at this point. From this we may deduce that the shrinking algorithm will always terminate with the same answer for a given starting word.

(ii) Clearly $\widetilde{w}$ is an unshrinkable word less than or equal to w. Write w as $w = w_1 w_2 \cdots w_l$ where each $w_j = x$ or x^*. Clearly, $\widetilde{w} = w_{s_1} \cdots w_{s_2}$ for some s_1 and s_2 with $1 \leqslant s_1 \leqslant s_2 \leqslant l$. If v is an unshrinkable word less than or equal to w then $v = w_{t_1} \cdots w_{t_2}$, or $v^* = w_{t_1} \cdots w_{t_2}$, for some t_1 and t_2 with $1 \leqslant t_1 \leqslant t_2 \leqslant l$. If $t_1 < s_1$ then there must be some point in the shrinking algorithm where $w' = w_j \cdots w_l$, with $j \leqslant t_1$ and the next step in the algorithm will shrink past t_1. That is $w' = c^* c \cdot v'$ with $vu = d \cdot cv'$ for some word u and some word $d \neq 1$. So then $c^* = ed$ for some word e and $vu = dd^* \cdot e^* v'$. As v is unshrinkable, this means that $d = 1$ which is a contradiction. So we must have $t_1 \geqslant s_1$. Similarly, we may deduce that $t_2 \leqslant s_2$, so $v \leqslant \widetilde{w}$. Thus $\{\widetilde{w}, \widetilde{w}^*\}$ is exactly the set of maximal unshrinkable words less than or equal to w.

(iii) This is clear from the comments before Definition 2.2, along with the observation that $\widetilde{w} \leqslant w$. ∎

Not all unshrinkable words are of the form x^n or x^{*n} for some n, although these certainly are unshrinkable. For example, other unshrinkable words are $x^2 x^* x^2$, $x^4 x^{*2} x^3$ and $x^6 x^{*2} x x^* x^3$. Shrinkable words can look quite similar to unshrinkable words, as an example consider $x^6 x^{*2} x x^* x^2 \rightsquigarrow x^6 x^{*2} x \rightsquigarrow x^6 x^{*2} \rightsquigarrow x^6$.

COMMENT 2.4. If $I = \langle w\rangle_*$ for some shrinkable word w then I cannot be a C^*-ideal as, if it were, we would have $\widetilde{w} \in I$ (by Theorem 2.3 (iii)) but $\widetilde{w} \notin I$ as w is shrinkable. This observation leads us to ask if $\langle w\rangle_*$ is a C^*-ideal when w is an unshrinkable word. At this point we might conjecture that $\langle w\rangle_{C^*} = \langle \widetilde{w}\rangle_*$ for all words w. The last two sections are concerned with trying to prove this conjecture.

3. THE EXISTENCE OF C^*-IDEALS GENERATED BY WORDS

In this section we shall prove that we may define the C^*-ideals generated by words; first we shall need a few results to set up some machinery. Note that, if $y \in A_1$ and B is any $*$-algebra, then y may be considered to act as an operator on B. If $b \in B$ then we obtain $y(b)$ by substituting b for x in the expression $y = \sum y_w \cdot w$. So for example, if $y = x^2 + 2x^*x$ then $y(b) = b^2 + 2b^*b$.

DEFINITION 3.1. Let $\varphi : A_1 \to B(H)$ and $\psi : A_1 \to B(G)$ be representation of A_1 over Hilbert spaces H and G. Define $\varphi \odot \psi : A_1 \to B(H \otimes G)$ to be the $*$-representation given by $(\varphi \odot \psi)x = \varphi(x) \otimes \psi(x)$.

Note that this is not the same as $\varphi \otimes \psi$ as this acts on the $*$-algebra $A_1 \otimes A_1$. Also note that if $y \in A_1$ then it may not be true that $(\varphi \odot \psi)y = \varphi(y) \otimes \psi(y)$. For example consider $(\varphi \odot \psi)(2x) = 2\varphi(x) \otimes \psi(x)$ but $\varphi(2x) \otimes \psi(2x) = 4\varphi(x) \otimes \psi(x)$.

THEOREM 3.2 *Take φ and ψ to be $*$-representations of A_1 as above. Let w be a word. Then $(\varphi \odot \psi)(w) = \varphi(w) \otimes \psi(w)$.*

Proof. If $(a \otimes b)$ and $(c \otimes d)$ are in $B(H \otimes G)$ then $(a \otimes b)^* = a^* \otimes b^*$ and $(a \otimes b) \cdot (c \otimes d) = (ac \otimes bd)$. So then $w(a \otimes b) = w(a) \otimes w(b)$. So, $(\varphi \odot \psi)(w) = w((\varphi \odot \psi)x) = w(\varphi(x)) \otimes w(\psi(x)) = \varphi(w) \otimes \psi(w)$. ∎

CORROLARY 3.3. *Take φ and ψ to be $*$-representations of A_1 as above. If $y = \sum y_w \cdot w$ in A_1 then $(\varphi \odot \psi)(y) = \sum y_m \cdot (\varphi(w) \otimes \psi(w))$.*

We shall use $\odot$ later to combine $*$-representations in order to prove results about ideals generated by words. First we must define a rather complicated looking $*$-representation with some useful properties.

DEFINITION 3.4. Let w be a word and let $m = |w|$. We may write $w = \cdots x^{r_3} x^{*r_2} x^{r_1}$. Let $H = l^2(m+1)$ with basis $\{\varepsilon_0, \ldots, \varepsilon_m\}$. Also, for all j, let $n_j = \sum_{k=1}^{j} r_k$. Define the representation $\mathfrak{S}_w : A_1 \to B(H)$ to be the unital $*$-homomorphism defined as follows:

$$\mathfrak{S}_w(x)\varepsilon_j = \begin{cases} \frac{\varepsilon_{j+1}}{2} & \text{for } 0 \leqslant j < n_1, n_2 < j < n_3, \ldots, \\ 0 & \text{for } j = n_1, n_3, \ldots, \\ \frac{\varepsilon_{j-1}}{2} & \text{for } n_1 < j < n_2, n_3 < j < n_4, \ldots, \\ \frac{\varepsilon_{j-1}+\varepsilon_{j+1}}{2} & \text{for } j = n_2, n_4, \ldots; \end{cases}$$

so,

$$\mathfrak{S}_w(x^*)\varepsilon_j = \begin{cases} \frac{\varepsilon_{j-1}}{2} & \text{for } 0 \leqslant j < n_1, n_2 < j < n_3, \ldots, \\ \frac{\varepsilon_{j-1}+\varepsilon_{j+1}}{2} & \text{for } j = n_1, n_3, \ldots, \\ \frac{\varepsilon_{j+1}}{2} & \text{for } n_1 < j < n_2, n_3 < j < n_4, \ldots, \\ 0 & \text{for } j = n_2, n_4, \ldots. \end{cases}$$

Informally, the operator $\mathfrak{S}_w(x)$ acts like a weighted shift on H but shifts different parts of it in different directions. Note that $\|\mathfrak{S}_w(x)\| < 1$.

LEMMA 3.5. *If $y = \sum y_v \cdot v \neq 0$ is in A_1 then there exists a word v with non-zero coefficient in y and points $\varepsilon, \varepsilon' \in l^2(|v|+1)$ such that $\langle \mathfrak{S}_v(v)\varepsilon, \varepsilon'\rangle \neq 0$ but $\langle y_u \mathfrak{S}_v(u)\varepsilon, \varepsilon'\rangle = 0$ for all other words u.*

Proof. Let $m = \max\{|v| \,|\, y_v \neq 0\}$ and let v be a word of length m with $y_v \neq 0$. Let $\varepsilon = \varepsilon_0$ and $\varepsilon' = \varepsilon_m$. Given α in H, let $d(\alpha) = \max\{j \mid \langle \varepsilon_j, \alpha\rangle \neq 0\}$ Informally, this represents the distance along the basis for which α contains information. Write $\mathfrak{S}_v(x) = t$. Considering the action of t on α in H we see that both t and t^* can only move information along to the right by at most one basis vector or, more formally, $d(t\alpha) \leqslant d(\alpha) + 1$ and $d(t^*\alpha) \leqslant d(\alpha) + 1$. If u is a word then $d(u(t)\varepsilon_0) \leqslant |u|$ with equality only being attained if each letter of the word u increases d. The operator $\mathfrak{S}_v$ is defined in such a way that $d(\mathfrak{S}_v(v)\varepsilon_0) = m$. Let u be a word other than v. If $y_u = 0$ then clearly $\langle y_u \mathfrak{S}(u)\varepsilon, \varepsilon'\rangle = 0$. If $y_u \neq 0$ then either $|u| < m$, in which case $d(\mathfrak{S}_v(u)\varepsilon_0) < m$, or $|u| = m$. It is not hard to see that if $|u| = m$ and $u \neq v$ then we again have $d(\mathfrak{S}_v(u)\varepsilon_0) < m$ (informally, in this case the operator turns back, or goes to zero, at some point along the basis) and we have finished. ∎

DEFINITION 3.6. If $w_1, \ldots, w_r$ are words then let $\mu_{w_1,\ldots,w_r}$ be the maximal C^*-seminorm on A_1 with $\mu_{w_1,\ldots,w_r}(x) \leqslant 1$ and $\mu_{w_1,\ldots,w_r}(w_j) = 0$ for all $1 \leqslant j \leqslant r$. So $\mu_{w_1,\ldots,w_r} = \sup\{\nu \mid \nu$ is a C^*-seminorm on A_1 with $\nu(x) \leqslant 1$ and $\nu(w_j) = 0$ for all $1 \leqslant j \leqslant r\}$ which is a C^*-seminorm. Also, given $\lambda \in \mathbb{R}$ let ψ_λ be the $*$-representation of A_1 given by $\psi_\lambda : A_1 \to \mathbb{C}$ $(x \mapsto \lambda)$. If φ is a $*$-representation of A_1 and w is a word then $(\psi_\lambda \odot \varphi)(w) = 1 \otimes \lambda^{|w|}\varphi(w)$.

LEMMA 3.7. *Let $w_1, \ldots, w_r$ and v be words. If there exists a C^*-seminorm ν on A_1 with $\nu(w_j) = 0$ for all $1 \leqslant j \leqslant r$ and $\nu(v) > 0$ then $v \notin \ker(\mu_{w_1,\ldots,w_r})$.*

Proof. If $v = 1$ then the result holds as the C^*-seminorm given by ψ_0 must be less than or equal to $\mu_{w_1,\ldots,w_r}$ (as $w_j \neq 1$ for all j) and $\psi_0(1) = 1$. So take v to be not equal to 1. Let ν be a C^*-seminorm with the required properties. As $v \neq 1$ and $\nu(v) > 0$ we must have $\nu(x) \neq 0$. Let $\lambda = 1/\nu(x)$. Let φ be a $*$-representation with $\|\varphi(y)\| = \nu(y)$ for all $y \in A_1$. Let $\varphi' = \varphi \odot \psi_\lambda$ and let $\nu'(y) = \|\varphi'(y)\|$ for all $y \in A_1$. Then ν' is a C^*-seminorm with $\nu'(x) = \nu(x)/\nu(x) = 1$ and $\nu'(w_j) = \nu(w_j)/\nu(x)^{|w_j|} = 0$. So $\nu' \leqslant \mu_{w_1,\ldots,w_r}$ and $\nu'(v) = \nu(v)/\nu(x)^{|v|} > 0$, therefore $\mu_{w_1,\ldots,w_r}(v) > 0$. ∎

THEOREM 3.8. *Let $w_1, \ldots, w_r$ be words. If $y \in \ker(\mu_{w_1,\ldots,w_r})$ then y is a finite linear combination of words in $\ker(\mu_{w_1,\ldots,w_r})$.*

Proof. Let $y \in \ker(\mu_{w_1,\ldots,w_r})$. In order to get a contradiction assume y has a non-zero coefficient for a word not in $\ker(\mu_{w_1,\ldots,w_r})$. Let $y' = \sum\{y_v \cdot v \mid v \in \ker(\mu_{w_1,\ldots,w_r})\}$. As $\ker(\mu_{w_1,\ldots,w_r})$ is an ideal, $y' \in \ker(\mu_{w_1,\ldots,w_r})$ so $y - y' \in \ker(\mu_{w_1,\ldots,w_r})$. Let $z = y - y'$. Because of our assumption about y the element z must be non-zero and is a linear combination of words not in $\ker(\mu_{w_1,\ldots,w_r})$. By Lemma 3.5 there exists a word v and vectors ε and ε' in $l^2(|v|+1)$ such that $z_v \neq 0$ (so $v \notin \ker(\mu_{w_1,\ldots,w_r})$) and $\langle \mathfrak{S}_v(v)\varepsilon, \varepsilon'\rangle \neq 0$ but $\langle z_u \mathfrak{S}_v(u)\varepsilon, \varepsilon'\rangle = 0$ for all words $u \neq v$. Let $\varphi : A_1 \to B(G)$ be a $*$-representation of A_1 onto the Hilbert space G with $\|\varphi(\cdot)\| = \mu_{w_1,\ldots,w_r}(\cdot)$. We have $\varphi(v) \neq 0$, therefore there exist δ and δ' in G such that $\langle \varphi(v)\delta, \delta'\rangle \neq 0$. Now let $\pi = \varphi \odot \mathfrak{S}_v$ so $\pi : A_1 \to B(G \otimes l^2(|v|+1))$. By Corollary 3.3, we have $\pi(z) = \sum z_u \varphi(u) \otimes \mathfrak{S}_v(u)$ but $\langle z_u \mathfrak{S}_v(u)\varepsilon, \varepsilon'\rangle = 0$ for all $u \neq v$. Therefore

$$\langle \pi(z)(\delta \otimes \varepsilon), \delta' \otimes \varepsilon'\rangle = \langle z_v \pi(v)(\delta \otimes \varepsilon), \delta' \otimes \varepsilon'\rangle = z_v \langle \varphi(v)\delta, \delta'\rangle \langle \mathfrak{S}_v(v)\varepsilon, \varepsilon'\rangle \neq 0.$$

So $\pi(z) \neq 0$. But, if we let ν be the C^*-seminorm on A_1 given by π then $\nu(x) \leqslant 1$. Also, $\nu(w_j) = \|\varphi(w_j) \otimes \mathfrak{S}_v(w_j)\| = 0$, as $\varphi(w_j) = 0$, for all j. Therefore $\nu \leqslant \mu_{w_1,\ldots,w_r}$ but $\mu_{w_1,\ldots,w_r}(z) = 0$, which contradicts $\pi(z) \neq 0$, and the result is proved. ∎

COROLLARY 3.9. *If w is a word then $\langle w\rangle_{C^*}$ exists (see Definition 1.7) and is equal to the kernel of μ_w.*

Proof. Clearly $\ker(\mu_w)$ is a C^*-ideal. Let I be a C^*-ideal of A_1 containing w and let $y \in \ker(\mu_w)$. Let ν be a C^*-seminorm of A_1 with $\ker(\nu) = I$. By Theorem 3.8, the element y is a finite linear combination of words in $\ker(\mu_w)$. If v is a word in $\ker(\mu_w)$ then v must be in $\ker(\nu)$ by Lemma 3.7. Therefore y is a finite linear combination of words in $\ker(\nu)$ which is an ideal, so $y \in \ker(\nu)$. That is, $y \in I$. So $\ker(\mu_w)$ is contained in any C^*-ideal containing w and is $\langle w \rangle_{C^*}$. ∎

So, for any word in A_1, we may define the C^*-ideal generated by it. Proposition 1.6 stated that any subset of a compact $*$-algebra generates a C^*-ideal of that $*$-algebra. This last result shows that it is not always necessary for a $*$-algebra to be compact to be able to define the C^*-ideal generated by certain elements (or subsets) of that $*$-algebra. In fact the C^*-seminorms on A_1 which are zero for the word w are unbounded whenever $x \notin \langle w \rangle_{C^*}$. To see this, let φ be any $*$-representation of A_1 with $\varphi(x) \neq 0$ and $\varphi(w) = 0$ (which must exists as $x \notin \langle w \rangle_{C^*}$). For any $\lambda \in \mathbb{R}$ we have $(\varphi \odot \lambda)w = 0$ and $\|(\varphi \odot \lambda)x\| = \|\varphi(x)\|\,|\lambda|$ which we may take to be as large as we like. Before the final section we shall give some simple corollaries from Theorem 3.8 and Corollary 3.9.

Corollary 3.10. *Let w be a word. Then $\langle w \rangle_* = \langle w \rangle_{C^*}$ if and only if $W_1 \cap \langle w \rangle_* = W_1 \cap \langle w \rangle_{C^*}$.*

Proof. The "only if" part is clear. Assume $W_1 \cap \langle w \rangle_* = W_1 \cap \langle w \rangle_{C^*}$. Clearly $\langle w \rangle_* \subseteq \langle w \rangle_{C^*}$. Let y be in $\langle w \rangle_{C^*}$. By Corollary 3.9, $y \in \ker(\mu_w)$ and, by Theorem 3.8, y is a finite linear combination of words in $\ker(\mu_w)$ which by the assumption are in $\langle w \rangle_*$, and the result is proved. ∎

So, if we have a C^*-ideal generated by a word, and we know which words it contains, then we know everything it contains. This greatly simplifies problems about the C^*-ideals generated by words.

Corollary 3.11. *Let w be a word. Then $\langle w \rangle_* = \langle w \rangle_{C^*}$ if and only if $\widetilde{W_1} \cap \langle w \rangle_* = \widetilde{W_1} \cap \langle w \rangle_{C^*}$ (see Definition 2.2).*

Proof. Again the "only if" part is clear. Assume $\widetilde{W_1} \cap \langle w \rangle_* = \widetilde{W_1} \cap \langle w \rangle_{C^*}$. By Theorem 2.3 (iii), if v is a word in $\langle w \rangle_{C^*}$ then $\widetilde{v} \in \langle w \rangle_{C^*}$. So, by our assumption $\widetilde{v} \in \langle w \rangle_*$ and as $\widetilde{v} \leqslant v$ we have $v \in \langle w \rangle_*$. This gives us $W_1 \cap \langle w \rangle_* = W_1 \cap \langle w \rangle_{C^*}$ and by Corollary 3.10 we have finished. ∎

4. THE C^*-IDEALS OF SOME SPECIFIC WORDS

In this final section we shall prove the conjecture that $\langle w\rangle_{C^*} = \langle\widetilde{w}\rangle_*$ (see Comment 2.4), for some particular words w. Finally we shall consider C^*-ideals generated by more than one word.

NOTATION 4.1. We shall call the elements of the set $\{x^r, x^{*r} \mid r \in \mathbb{N}\}$ *syllables*. Any word w may be written as a product of syllables. We shall say the number of syllables of w is the least number of syllables it takes to make up w. For example, the number of syllables of $w = x^3x^{*2}x^4$ is three and they are x^3, x^{*2} and x^4. Also, let $A^*\{x : R_1, \ldots, R_n\}$ denote the universal unital $*$-algebra generated by one element, x, subject to the relation $R_1, \ldots, R_n$.

THEOREM 4.2. *If n is in $\mathbb{N}$ then $\langle x^n\rangle_{C^*} = \langle x^n\rangle_*$. So $A^*\{x : x^n = 0\}$ may be embedded into a C^*-algebra.*

Proof. Take n in $\mathbb{N}$. By Corollary 3.10 it is sufficient to prove that any word in $\langle x^n\rangle_{C^*}$ is also in $\langle x^n\rangle_*$. Let w be a word which is not in $\langle x^n\rangle_*$. Define a representation φ of A_1 by

$$\varphi(x) = \begin{pmatrix} 0 & 0 & 0 & \cdot & \cdot & 0 \\ 1 & 0 & 0 & \cdot & \cdot & 0 \\ 1 & 1 & 0 & 0 & \cdot & 0 \\ \cdot & 1 & 1 & 0 & & 0 \\ 1 & \cdot & & & & \cdot \\ 1 & 1 & \cdot & 1 & 1 & 0 \end{pmatrix}$$

with this being an n by n square matrix. If X is a matrix we shall use $X[j,k]$ to denote the entry of X at position $[j,k]$. It is not hard to see that φ sends x^n to zero and that $\varphi(x^r)[n,1] \geqslant 1$ for all $r \in \mathbb{N}$ with $r < n$. Hence, $\varphi(x^{*r})[1,n] \geqslant 1$ also. We shall show by induction on the number of syllables of w, that if v is a word which has non-zero image under φ and $w = v \cdot x^r$ with $r < n$ then $\varphi(w)[1,1] > 0$ or $\varphi(w)[n,1] > 0$, and that if $w = v \cdot x^{*r}$ then $\varphi(w)[1,n] > 0$ or $\varphi(w)[n,n] > 0$.

First, if $w = x^r$ then we may see that $\varphi(w)[n,1] \geqslant 1 > 0$ as $r < n$. Similarly, if $w = x^{*r}$ then $\varphi(w)[1,n] \geqslant 1 > 0$. Now, if $w = v' \cdot x^{*s}x^r$ then $s < n$ (as $\varphi(v'x^{*s}) \neq 0$) and so we may apply the induction hypothesis to get $\varphi(v'x^{*s})[1,n] > 0$ or $\varphi(v'x^{*s})[n,n] > 0$. As all the entries of $\varphi(x)$ are non-negative, and $\varphi(x^r)[n,1] \geqslant 1$, we have $\varphi(w)[1,1] > 0$ or $\varphi(w)[n,1] > 0$. A similar argument works for $w = v \cdot x^s x^{*r}$. So then, $\varphi(w) \neq 0$ and thus w is not in $\langle x^n\rangle_{C^*}$ as this must be contained in the kernel of φ and we have finished. ∎

Any syllable x^n is an unshrinkable word and we have just proved that the $*$-ideal it generates is a C^*-ideal. However not all unshrinkable words are syllables. The shortest unshrinkable word with more than one syllable is $x^2x^*x^2$ and so we go on to prove the following theorem.

THEOREM 4.3. *The $*$-ideal $\langle x^2x^*x^2\rangle_*$ is a C^*-ideal. So $A^*\{x : x^2x^*x^2 = 0\}$ may be embedded into a C^*-algebra.*

Proof. By Corollary 3.10, we need only show that $x^2x^*x^2$ generates algebraically any of the words in its C^*-ideal. Define φ to be the representation of A given by $\varphi : x \mapsto X$ where

$$X = \begin{pmatrix} 0 & -3 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that $X^2X^*X^2 = 0$. Note that for all $n > 1$,

$$X^n = X^2 = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

So, if w is a word then $\varphi(w) = \varphi(w')$ where w' is a word with syllables only out of $\{1, x, x^*, x^2, x^{*2}\}$. We are seeking to show that $w \notin \langle x^2x^*x^2\rangle_*$ implies that $\varphi(w) \neq 0$. By Corollary 3.11 we may take w to be unshrinkable. As φ applied to any syllables is non-zero we need only consider words consisting of more than one syllable. Any such word whose first or last syllable is x or x^* will be shrinkable. Also, any word ending in x^2x^{*2} or $x^{*2}x^2$ is shrinkable and any word ending in $x^2x^*x^2$ or $x^{*2}xx^{*2}$ is in $\langle x^2x^*x^2\rangle_*$. So, we need only to consider words in one of following four forms:

(i) $w = v \cdot x^{*2} \cdot (xx^*)^r \cdot x^2$,
(ii) $w = v \cdot x^2 \cdot x^*(xx^*)^r \cdot x^2$,
(iii) $w = v \cdot x^2 \cdot (x^*x)^r \cdot x^{*2}$, or
(iv) $w = v \cdot x^{*2} \cdot x(x^*x)^r \cdot x^{*2}$,

where $v \in W$ and $r \in \{1, 2, 3, \ldots\}$.

We shall prove that any such word w has $\varphi(w) \neq 0$ by induction on the number of syllables of w. Take w to be in one of the four forms listed above and assume, for cases (i) and (iv) that $\varphi(v \cdot x^{*2}) \neq 0$, and for cases (ii) and (iii) that $\varphi(v \cdot x^2) \neq 0$.

By using induction, (and noting that $(XX^*)^r XX^* = XX^*(XX^*)^r$), one may check that for any $r \in \{1, 2, \ldots\}$ we have

$$(XX^*)^r = \begin{pmatrix} d & e & e \\ e & f & f \\ e & f & f \end{pmatrix}$$

for some $d > 2e, 2f$ and $e, f > 0$. Similarly, one may see that

$$(X^*X)^r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & d' & -e' \\ 0 & -e' & f' \end{pmatrix}$$

for some d', e', f' with $d', e', f' > 0$. It is also easy to see that for any word v, we will have

$$\varphi(v) \cdot X^2 = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & c \end{pmatrix}$$

for some $a, b, c \in \mathbb{R}$, and

$$\varphi(v) \cdot X^{*2} = \begin{pmatrix} -2a' & a' & a' \\ -2b' & b' & b' \\ -2c' & c' & c' \end{pmatrix}$$

for some $a', b', c' \in \mathbb{R}$.

We may now consider the four cases mentioned earlier:

(i) We take $w = v \cdot x^{*2} \cdot (xx^*)^r \cdot x^2$ and assume that $\varphi(v \cdot x^{*2}) \neq 0$. So,

$$\varphi(w) = \begin{pmatrix} 0 & 0 & 4a'(d + f - 2e) \\ 0 & 0 & 4b'(d + f - 2e) \\ 0 & 0 & 4c'(d + f - 2e) \end{pmatrix}$$

with $\{a', b', c'\} \neq \{0\}$ and $d + f - 2e > d - 2e > 0$ thus $\varphi(w) \neq 0$.

Similarly, in case (ii) we have

$$\varphi(w) = \begin{pmatrix} 0 & 0 & 2a(2f - d - e) \\ 0 & 0 & 2b(2f - d - e) \\ 0 & 0 & 2c(2f - d - e) \end{pmatrix}$$

with $\{a, b, c\} \neq \{0\}$ and $2f - d - e < 2f - d < 0$.

In case (iii) we have

$$\varphi(w) = \begin{pmatrix} -2af' & af' & af' \\ -2bf' & bf' & bf' \\ -2cf' & cf' & cf' \end{pmatrix}$$

with $\{a, b, c\} \neq \{0\}$ and $f' > 0$.

Finally, in case (iv) we have

$$\varphi(w) = \begin{pmatrix} 12a'e' & -6a'e' & -6a'e' \\ 12b'e' & -6b'e' & -6b'e' \\ 12c'e' & -6c'e' & -6c'e' \end{pmatrix}$$

with $\{a', b', c'\} \neq \{0\}$ and $e' > 0$. So, in each case we have $\varphi(w) \neq 0$ and have finished our proof. ∎

Theorem 4.4. *Let $w_1, \ldots, w_r$ be words. Then $\langle w_1, \ldots, w_r \rangle_{C^*} = \langle w_1 \rangle_{C^*} + \cdots + \langle w_r \rangle_{C^*}$.*

Proof. The ideal $\langle w_1, \ldots, w_r \rangle_{C^*}$ exists by an easy extension of the proof of Corollary 3.9. Let $I = \langle w_1, \ldots, w_r \rangle_{C^*}$ and $J = \langle w_1 \rangle_{C^*} + \cdots + \langle w_r \rangle_{C^*}$. As $\langle w_j \rangle_{C^*}$ is the smallest C^*-ideal containing w_j, and I is a C^*-ideal containing w_j, we have $\langle w_j \rangle_{C^*} \subseteq I$. Therefore $J \subseteq I$ and we need to prove the reverse inclusion. For all w_j, let φ_{w_j} be a $*$-representation with kernel $\langle w_j \rangle_{C^*}$. Let $\psi = \varphi_{w_1} \odot \cdots \odot \varphi_{w_r}$. Clearly $\psi(w_j) = 0$ for all $j \in \{1, \ldots, r\}$ As $\ker(\psi)$ is a C^*-ideal we have $I \subseteq \ker(\psi)$. For any word v, its image $\psi(v)$ is equal to $\varphi_{w_1}(v) \otimes \cdots \otimes \varphi_{w_r}(v)$, which is zero only if $\varphi_{w_j}(v) = 0$ for some j. Therefore $W_1 \cap I \subseteq W_1 \cap \ker(\psi) \subseteq W_1 \cap J$. Now let y be in I with $y = \sum y_u \cdot u$. We are seeking to show that y is in J. Similarly to the proof of Theorem 3.8, as $W_1 \cap I \subseteq W_1 \cap J$, we need only to consider the possibility that y is a non-zero linear combination of words which are not in J. So $u \notin J$ for all $y_u \neq 0$. Let $\pi = \psi \odot \mathfrak{S}_v$ where v is a word with $y_v \neq 0$ and $\langle \mathfrak{S}_v(v)\varepsilon, \varepsilon' \rangle \neq 0$ but $\langle y_u \mathfrak{S}_v(v)\varepsilon, \varepsilon' \rangle = 0$ for all words $u \neq v$. We may assume that ψ maps A_1 into $B(G)$ where G is a Hilbert space. As $\psi(v) \neq 0$ we may let δ, δ' be in G with $\langle \psi(v)\delta, \delta' \rangle = 0$. Similarly to the proof of Theorem 3.8 this gives us $\langle \pi(y)(\delta \otimes \varepsilon), \delta' \otimes \varepsilon \rangle \neq 0$. But $I \subseteq \ker(\pi)$ (as $\pi(w_j) = 0$ for all j) so $\pi(y) = 0$ which gives a contradiction. So y cannot be a non-zero linear combination of words not in J, and the result is proved. ∎

This last result shows that we may answer any question about the C^*-ideals generated by more than one word by knowing about the C^*-ideals generated by single words. This leaves us still wanting to prove the conjecture that $\langle w \rangle_{C^*} = \langle \widetilde{w} \rangle_*$, for all words w, or, equivalently, the $*$-ideal generated by a word is a C^*-ideal if and only if the word is unshrinkable. The "only if" part is proved and we have seen that the conjecture holds for any syllable and for $x^2 x^* x^2$ but it is open for longer unshrinkable words although we have proved that they do generate C^*-ideals.

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