STANDARD MODELS UNDER
POLYNOMIAL POSITIVITY CONDITIONS

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Abstract. We develop standard models for commuting tuples of bounded linear operators on a Hilbert space under certain polynomial positivity conditions, generalizing the work of V. Müller and F.-H. Vasilescu in [6], [14]. As a consequence of the model, we prove a von Neumann-type inequality for such tuples. Up to similarity, we obtain the existence of in a certain sense “unitary” dilations.

Keywords: Multivariable spectral theory, weighted multishifts, standard models, dilations, functional calculus.


1. INTRODUCTION

Let \( \mathcal{H} \) be a separable Hilbert space and \( T = (T_1, \ldots, T_n) \) a commuting tuple of bounded linear operators on \( \mathcal{H} \). \( T \) is called a spherical contraction, if \( \sum_{i=1}^{n} T_i^* T_i \leq 1_\mathcal{H} \), and a spherical unitary, if \( \sum_{i=1}^{n} T_i^* T_i = 1_\mathcal{H} \) and in addition, all components of \( T \) are normal. We say that \( T \) has a spherical dilation if there is a spherical unitary \( U \) which dilates \( T \), i.e. \( T^\alpha = P_\mathcal{H} U^\alpha | \mathcal{H} \) for all \( \alpha \in \mathbb{N}_0^n \). There is no easy generalization of the famous Dilation Theorem for contractions of Sz.-Nagy (see [12]) to spherical contractions: in general, spherical contractions have no spherical dilations, and there is not even a von Neumann-type inequality over the unit ball in \( \mathbb{C}^n \) for spherical contractions ([3]). Athavale has shown in [1] that under certain additional positivity conditions a spherical contraction \( T \) has a spherical dilation, and Müller and Vasilescu have developed a model for \( T \) under these conditions.
which reproduces this result ([6], [14]). This model consists of a spherical unitary part and a weighted backward multishift part which for suitable order coincides with the adjoint of the tuple of multiplication operators with the coordinates on a Hardy space over the unit ball in $\mathbb{C}^n$. For $n = 1$, this is just the well-known coisometric extension for contractions.

In the current paper, we will develop a model for a commuting tuple $T$ under certain polynomial positivity conditions. We call $T$ a $P$-contraction, where $P = \sum_{\gamma \in \mathbb{N}^n_0} a_{\gamma} x^\gamma$ is a polynomial with non-negative coefficients of a certain type, if $\sum_{\gamma \in \mathbb{N}^n_0} a_{\gamma} T^*\gamma T^\gamma \leq 1_H$, and a $P$-unitary if $\sum_{\gamma \in \mathbb{N}^n_0} a_{\gamma} T^*\gamma T^\gamma = 1_H$, $T_1, \ldots, T_n$ normal. We will show that $P$-contractions satisfying additional positivity conditions of suitable order have a model consisting of a $P$-unitary part and a weighted backward multishift part, which may be identified topologically with the adjoint of the multiplication tuple on a Bergman space. In particular, up to topological equivalence, $T$ has a $P$-unitary dilation and therefore a rich functional calculus.

The crucial tools in identifying the weighted backward multishift with the adjoint Bergman space multiplication tuple are a theorem of A. Cumenge from complex analysis which allows to extend Bergman space functions on a complex submanifold $\mathcal{M}$ to Hardy space functions on a strictly pseudoconvex set containing $\mathcal{M}$ and the simple idea of regarding a $P$-contraction as a spherical contraction in a higher dimension.

2. PRELIMINARIES AND NOTATION

A commuting tuple $T = (T_1, \ldots, T_n)$ of bounded linear operators on the separable Hilbert space $\mathcal{H}$ will be called a commuting multiplier or just a multiplier. For $A \in \mathcal{L}(\mathcal{H})$, let $C_A$ be the bounded linear map

$$\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}), \quad X \mapsto A^*XA,$$

and for a commuting tuple $T = (T_1, \ldots, T_n) \in \mathcal{L}(\mathcal{H})^n$ let $C_T = (C_{T_1}, \ldots, C_{T_n})$. If $P = \sum_{\gamma \in \mathbb{N}^n_0} a_{\gamma} x^\gamma \in \mathcal{C}[X_1, \ldots, X_n]$ is a polynomial, then $P(C_T)$ is the bounded linear map $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}), \quad X \mapsto \sum_{\gamma \in \mathbb{N}^n_0} a_{\gamma} T^*\gamma XT^\gamma$. This map is well-defined, since $T_1, \ldots, T_n$ commute.

If $T = (T_1, \ldots, T_n)$ is a commuting multiplier on $\mathcal{H}$, $S = (S_1, \ldots, S_n)$ a commuting multiplier on some Hilbert space $\mathcal{H}'$ and $A : \mathcal{H} \to \mathcal{H}'$ is a linear map, then we will write $AT = SA$ for the identity $AT_i = S_iA$, $i = 1, \ldots, n$. In this situation, we call $T$ and $S$ topologically equivalent or similar if $A$ is a
We will call a commuting multioperator normal in case all components are normal.

For \( z = (z_1, \ldots, z_n) \), \( w = (w_1, \ldots, w_n) \in \mathbb{C}^n \), we will denote the tuple \((\bar{z}_1w_1, \ldots, \bar{z}_nw_n)\) by \( \bar{z}w \) and the tuple \((|z_1|^2, \ldots, |z_n|^2)\) by \( |z|^2 \).

Let us introduce the class of polynomials from which our positivity conditions are obtained. A polynomial \( P \in \mathbb{C}[X_1, \ldots, X_n] \) is said to be positive regular, if

(i) the constant term is 0;

(ii) \( P \) has non-negative coefficients;

(iii) the coefficients of the linear terms \( X_1, \ldots, X_n \) are all different from 0.

There is a complete Reinhardt domain in \( \mathbb{C}^n \) associated to each positive regular polynomial \( P \), namely

\[
\mathcal{P} = \{ z \in \mathbb{C}^n \mid P(|z|^2) < 1 \}
\]

which we call the \( P \)-ball. For \( P = \sum_{i=1}^{n} x_i \), the \( P \)-ball is just the unit ball \( \mathbb{B}^n \) in \( \mathbb{C}^n \).

For a positive regular polynomial \( P, X \in \mathcal{L}(\mathcal{H}) \) positive and \( m \in \mathbb{N} \), we will call a commuting multioperator \( T \) \((P, m)\)-positive for \( X \), if

\[
\Delta^{(1)}_{(P, m)}(X) := (1 - P)(C_T)(X) \geq 0
\]

and

\[
\Delta^{(m)}_{(P, m)}(X) := (1 - P)^m(C_T)(X) \geq 0.
\]

In this case,

\[
\Delta^{(k)}_{P}(X) := (1 - P)^k(C_T)(X) \geq 0 \quad \text{for } 1 \leq k \leq m,
\]

as one obtains completely analogously to Lemma 2 in [6]. The tuple \( T \) is said to be \((P, m)\)-positive, if it is \((P, m)\)-positive for \( 1_{\mathcal{H}} \). Furthermore, we call \( T \) a \( P \)-isometry, if \( \Delta^{(1)}_{P}(1_{\mathcal{H}}) = 0 \), and a \( P \)-unitary, if in addition \( T \) is normal.

For \( P = \sum_{i=1}^{n} x_i \), the \((P, 1)\)-positive operators are just the spherical contractions.
3. STANDARD MODELS

We will now develop in analogy to [6] a standard model for \((P, m)\)-positive commuting tuples, consisting of a part which is the adjoint of a multiplication tuple — or, equivalently, a weighted backward multishift — and a \(P\)-unitary part.

For \(|P(x)| < 1\), we have

\[
\frac{1}{(1 - P(x))^m} = \left( \sum_{j=0}^{\infty} P^j(x) \right)^m.
\]

Therefore the function \(x \mapsto 1/(1 - P(x))^m\) has a power series representation which converges compactly on \(\{x \mid |P(x)| < 1\}\) and coincides with the Taylor series expansion at 0. For positive regular \(P\), all Taylor coefficients are positive.

**Definition 3.1.** Let \(P\) be a positive regular polynomial in \(n\) variables and let \(m \in \mathbb{N}\). For each \(\alpha \in \mathbb{N}_n^0\), let \(\rho_m^P(\alpha)\) be the Taylor coefficient at index \(\alpha\) of the function \(x \mapsto 1/(1 - P(x))^m\) at 0.

We will denote the coefficients \(\rho_m^P(\alpha)\), \(\alpha \in \mathbb{N}_n^0\), as \((P, m)\)-weights.

Now let \(H^2(\rho_m^P)\) be the linear space of all formal power series \(\sum_{\alpha \in \mathbb{N}_n^0} c_\alpha z^\alpha\) such that \(\sum_{\alpha \in \mathbb{N}_n^0} |c_\alpha|^2/\rho_m^P(\alpha) < \infty\). The space \(H^2(\rho_m^P)\) is obviously a Hilbert space with the inner product

\[
\langle \sum_{\alpha \in \mathbb{N}_n^0} c_\alpha z^\alpha, \sum_{\alpha' \in \mathbb{N}_n^0} b_{\alpha'} z^{\alpha'} \rangle = \sum_{\alpha \in \mathbb{N}_n^0} c_\alpha \overline{b}_{\alpha} \frac{1}{\rho_m^P(\alpha)}.
\]

It can be regarded as a space of holomorphic functions on the \(P\)-ball \(P\), and there is an obvious reproducing kernel:

**Lemma 3.2.** The elements of \(H^2(\rho_m^P)\) define holomorphic functions on the \(P\)-ball \(P\). Furthermore, let

\[
\forall : P \times P \to \mathbb{C}, \quad \forall(z, w) = \frac{1}{(1 - P(\overline{z}w))^m}.
\]

For each \(z \in P\), the function \(\forall_z = \forall(z, \cdot)\) is a holomorphic function on \(P\) and by identification with its Taylor series expansion at 0 an element of \(H^2(\rho_m^P)\) such that

\[
(f, \forall_z) = f(z), \quad f \in H^2(\rho_m^P).
\]

We have \(\|\forall_z\| = (1/(1 - P(|z|^2))^m)^{1/2}\) for \(z \in P\).
Proof. For \( f = \sum_{\alpha \in \mathbb{N}_0^n} c_{\alpha} w^\alpha \in H^2(\rho^m_P) \) and \( z \in \mathcal{P} \), we have

\[
\sum_{\alpha \in \mathbb{N}_0^n} |c_{\alpha} z^\alpha| \leq \left( \sum_{\alpha \in \mathbb{N}_0^n} |c_{\alpha}|^2 \frac{1}{\rho^m_P(\alpha)} \right)^{1/2} \left( \sum_{\alpha \in \mathbb{N}_0^n} \rho^m_P(\alpha)|z^\alpha|^2 \right)^{1/2} = \frac{1}{(1 - P(|z|^2))^{m/2}} \|f\|.
\]

Thus \( f \) converges uniformly on compact subsets of \( \mathcal{P} \) and defines a holomorphic function on \( \mathcal{P} \) (see [9], Corollaries 1.16 and 1.17), which we again call \( f \). Furthermore, one obtains for \( z \in \mathcal{P} \)

\[
\|e_z\|^2 = \left\| \sum_{\alpha \in \mathbb{N}_0^n} \rho^m_P(\alpha) z^\alpha w^\alpha \right\|^2 = \sum_{\alpha \in \mathbb{N}_0^n} |z^\alpha|^2 \rho^m_P(\alpha) = \frac{1}{(1 - P(|z|^2))^{m/2}} < \infty
\]

and \( \langle f, e_z \rangle = \sum_{\alpha \in \mathbb{N}_0^n} c_{\alpha} z^\alpha = f(z) \).

We define multiplication operators \( M_{z_i} \), \( i = 1, \ldots, n \), on \( H^2(\rho^m_P) \) by

\[
M_{z_i} \sum_{\alpha \in \mathbb{N}_0^n} c_{\alpha} z^\alpha = \sum_{\alpha \in \mathbb{N}_0^n} c_{\alpha} z^\alpha + e_i.
\]

For the study of the multiplication operators and the construction of the model, we need more information about the \((P, m)\)-weights \((\rho^m_P(\alpha))\). Thus we give a more explicit form and a recursion formula for the weights.

Let us first introduce some notation. For a given positive regular polynomial \( P \), let \( I_P = \{ \gamma \in \mathbb{N}_0^n \mid a_\gamma > 0 \} \) and \( \text{mult}(P) = |I_P| \) be the number of nontrivial coefficients in \( P \). We form the vector of the coefficients of \( P \), \( A = (a_\gamma)_{\gamma \in I_P} \in \mathbb{C}^{I_P} \). Furthermore, let for \( K = (k_\gamma)_{\gamma \in I_P}, L = (l_\gamma)_{\gamma \in I_P} \in \mathbb{C}^{I_P} \)

\[
A^K := \prod_{\gamma \in I_P} a_\gamma^{k_\gamma}, \quad |K| := \sum_{\gamma \in I_P} k_\gamma,
\]

\[
\binom{|K|}{K} := \frac{|K|!}{\prod_{\gamma \in I_P} k_\gamma!}, \quad (L \choose K) := \prod_{\gamma \in I_P} \binom{l_\gamma}{k_\gamma}
\]

and

\[
[K] := ([K], \ldots, [K]_n), \quad \text{where } [K]_i := \sum_{\gamma \in I_P} \gamma_i k_\gamma \text{ for } i \in \{1, \ldots, n\}.
\]

Write \( K \triangleq L \) if \( k_\gamma \leq l_\gamma \) for all \( \gamma \in I_P \). We need some combinatorial results:
Lemma 3.3. For $L \in N_0^{|P|}$ and $m \in N$,

$$(3.9) \quad \binom{|L|}{L} \binom{|L| + m}{m} = \sum_{\substack{K \in N_0^{|P|} \atop K \leq L}} \binom{|L| - K}{L - K} \binom{|K|}{K} \binom{|K| + m - 1}{m - 1}.$$ 

Proof. We obtain the identity

$$(3.10) \quad \sum_{\substack{K \leq L \atop |K| = r}} \binom{L}{K} = \binom{|L|}{r} \quad \text{for } r = 0, \ldots, |L|$$

by induction over the number of nontrivial coefficients $|I_P|$ of $P$ and the well-known fact

$$(3.11) \quad \sum_{q=0}^{r} \binom{|L| - l}{q} \binom{l}{r - q} = \binom{|L|}{r} \quad \text{for } 0 \leq l \leq |L|.$$ 

Now, we have

$$\sum_{\substack{K \leq L \atop |K| = r}} \binom{|L| - K}{L - K} \binom{|K|}{K} \binom{|K| + m - 1}{m - 1}$$

$$= \sum_{r=0}^{L} \left[ \sum_{\substack{K \leq L \atop |K| = r}} \binom{|L| - r}{L - K} \binom{r}{K} \binom{r + m - 1}{m - 1} \right]$$

$$= \binom{|L|}{L} \sum_{r=0}^{L} \frac{(|L| - r)!}{|L|!} \frac{r + m - 1}{r} \frac{m - 1}{m - 1} \sum_{\substack{K \leq L \atop |K| = r}} \binom{L}{K}$$

$$= \binom{|L|}{L} \sum_{r=0}^{L} \binom{r + m - 1}{m - 1}.$$ 

It remains to show that $\sum_{r=0}^{L} \binom{r + m - 1}{m - 1} = \binom{|L| + m}{m}$ for $m \in N$, which is an easy induction. 

Furthermore, Equation (3.10) yields the identity

$$(3.12) \quad \sum_{\substack{K \leq L \atop |K| = r}} \binom{r}{K} \binom{|L| - K}{L - K} = \frac{r!(|L| - r)!}{|L|!} \binom{|L|}{L} \sum_{\substack{K \leq L \atop |K| = r}} \binom{L}{K} = \binom{|L|}{L}$$

for $0 \leq r \leq |L|$. Now we can characterize the $(P, m)$-weights more explicitly.
Lemma 3.4. Let $P$ be a positive regular polynomial and $m \in \mathbb{N}$. Then

$$\rho_P^m(\alpha) = \sum_{K \in \mathbb{N}^n_0} A^K \left( \frac{|K| + m - 1}{|K|} \right) \binom{|K|}{K} \text{ for } \alpha \in \mathbb{N}^n_0.$$  

Proof. For $m = 1$ and $|P(x)| < 1$, we have

$$\frac{1}{1 - P(x)} = \sum_{j=0}^\infty P(x)^j = \sum_{j=0}^\infty \left[ \sum_{|K|=j} \binom{|K|}{K} \prod_{\gamma \in I_P} a_{\gamma_j}^j (x^\gamma)^{k_\gamma} \right]$$

(3.15) $= \sum_{j=0}^\infty \left[ \sum_{|K|=j} A^K \left( \frac{|K|}{K} \right) x^{|K|} \right] = \sum_{\alpha \in \mathbb{N}^n_0} x^\alpha \sum_{|K|=\alpha} A^K \left( \frac{|K|}{K} \right).$

So, by uniqueness of the coefficients, (3.14) holds for $m = 1$. Now let (3.14) be valid for an arbitrary $m \in \mathbb{N}$. Then we obtain again by uniqueness and by Lemma 3.3 the identity for $m + 1$:

$$\frac{1}{(1 - P(x))^{m+1}} = \left( \sum_{\alpha \in \mathbb{N}^n_0} \rho_P^m(\alpha)x^\alpha \right) \left( \sum_{\alpha \in \mathbb{N}^n_0} \rho_P^1(\alpha)x^\alpha \right)$$

$$= \left( \sum_{K \in \mathbb{N}^n_0} \binom{|K|}{K} \left( \frac{|K| + m - 1}{m - 1} \right) A^K x^{|K|} \right) \left( \sum_{J \in \mathbb{N}^n_0} \binom{|J|}{J} A^J x^{|J|} \right)$$

(3.16) $= \sum_{L \in \mathbb{N}^n_0} \left[ A^L x^{|L|} \sum_{K \in \mathbb{N}^n_0} \binom{|L - K|}{L - K} \left( \frac{|K| + m - 1}{m - 1} \right) \binom{|K|}{K} \right]$

$$= \sum_{\alpha \in \mathbb{N}^n_0} x^\alpha \left[ \sum_{L \in \mathbb{N}^n_0} A^L \left( \frac{|L|}{L} \right) \left( \frac{|L| + m}{m} \right) \right].$

Let from now on $\rho_P^m(\alpha) = 0$ for $\alpha \in \mathbb{Z}^n \setminus \mathbb{N}^n_0$. Then we obtain the following recursion formulae for the $(P, m)$-weights:

Remark 3.5. Let $P = \sum_{\gamma \in \mathbb{N}^n_0} a_\gamma x^\gamma$ be a positive regular polynomial and let $Q = 1 - (1 - P)^m = \sum_{\gamma \in \mathbb{N}^n_0} b_\gamma x^\gamma$. Then

$$\rho_P^m(\alpha) = \sum_{\gamma \in \mathbb{N}^n_0} b_\gamma \rho_P^m(\alpha - \gamma), \text{ } \alpha \in \mathbb{N}^n_0.$$
and for $m > 1$,

$$\rho^m_P(\alpha) = \rho^{m-1}_P(\alpha) + \sum_{\gamma \in \mathbb{N}_0^n} a_\gamma \rho^m_P(\alpha - \gamma). \tag{3.18}$$

**Proof.** For $\alpha \in \mathbb{N}_0^n$, $\sum_{\gamma \in \mathbb{N}_0^n} b_\gamma \rho^m_P(\alpha - \gamma)$ is the coefficient at index $\alpha$ of the product power series $\left( \sum_{\alpha \in \mathbb{N}_0^n} \rho^m_P(\alpha) x^\alpha \right) \left( \sum_{\gamma \in \mathbb{N}_0^n} b_\gamma x^\gamma \right)$. We obtain Equation (3.17) by comparison of coefficients, since for $|P(x)| < 1$ we have

$$\sum_{\alpha \in \mathbb{N}_0^n} x^\alpha \sum_{\gamma \in \mathbb{N}_0^n} b_\gamma \rho^m_P(\alpha - \gamma) = (1 - P(x))^{-m} (1 - (1 - P(x))^m)$$

$$= \sum_{\alpha \in \mathbb{N}_0^n} \rho^m_P(\alpha) x^\alpha - 1. \tag{3.19}$$

Similarly, $\sum_{\gamma \in \mathbb{N}_0^n} a_\gamma \rho^m_P(\alpha - \gamma)$ is the $\alpha$-coefficient of the product power series

$$\left( \sum_{\alpha \in \mathbb{N}_0^n} \rho^m_P(\alpha) x^\alpha \right) \left( \sum_{\gamma \in \mathbb{N}_0^n} a_\gamma x^\gamma \right),$$

and we obtain for $|P(x)| < 1$, $m > 1$

$$\sum_{\alpha \in \mathbb{N}_0^n} x^\alpha \left( \rho^m_P(\alpha) - \sum_{\gamma \in \mathbb{N}_0^n} a_\gamma \rho^m_P(\alpha - \gamma) \right) = 1$$

$$= (1 - P(x))^{-m} - (1 - P(x))^{-m} P(x) - 1$$

$$= (1 - P(x))^{-m+1} - 1 = \sum_{\alpha \in \mathbb{N}_0^n} \rho^{m-1}_P(\alpha) x^\alpha - 1 \tag{3.20}$$

implying (3.18).

Now we can prove that the multiplication operators are well-defined bounded operators on $H^2(\rho^m_P)$.

**Lemma 3.6.** $M_{z_1}, \ldots, M_{z_n} \in \mathcal{L}(H^2(\rho^m_P))$.

**Proof.** Let $e_i$ be the $i$th unit vector in $\mathbb{C}^n$, $i = 1, \ldots, n$. It is sufficient to show that for some constant $c > 0$, $\rho^m_P(\alpha + e_i) \geq c \rho^m_P(\alpha)$ for all $\alpha \in \mathbb{N}_0^n$. But by Remark 3.5,

$$\rho^m_P(\alpha + e_i) \geq \sum_{\gamma \in \mathbb{N}_0^n} a_\gamma \rho^m_P(\alpha + e_i - \gamma) \geq a_n \rho^m_P(\alpha) \tag{3.21}$$

for $\alpha \in \mathbb{N}_0^n$, which proves the lemma.
The multiplication operators are obviously commuting.

For the separable Hilbert space $\mathcal{H}$, we can consider the Hilbert space tensor product $\mathcal{H} \otimes H^2(\rho^m_P) =: H^2_H(\rho^m_P)$. This space can obviously be identified with the space of formal power series with coefficients in $\mathcal{H}$, \[ \sum_{\alpha \in \mathbb{N}^n_0} h_\alpha z^\alpha \] with $h_\alpha \in \mathcal{H}$ for $\alpha \in \mathbb{N}^n_0$, such that \[ \sum_{\alpha \in \mathbb{N}^n_0} \|h_\alpha\|^2 (1/\rho^m_P(\alpha)) < \infty \]. The inner product on $H^2_H(\rho^m_P)$ is then given by

\[ \langle \sum_{\alpha \in \mathbb{N}^n_0} h_\alpha z^\alpha, \sum_{\alpha' \in \mathbb{N}^n_0} h'_\alpha z^{\alpha'} \rangle = \sum_{\alpha \in \mathbb{N}^n_0} \langle h_\alpha, h'_\alpha \rangle \frac{1}{\rho^m_P(\alpha)}. \]

We can view $H^2_H(\rho^m_P)$ as a space of $\mathcal{H}$-valued holomorphic functions on $\mathcal{P}$. From now on, we will denote the multiplication operators with the coordinates on $H^2_H(\rho^m_P)$ as well as the ones on $H^2(\rho^m_P)$ by $M_z^1, \ldots, M_z^n$. By Lemma 3.6, these operators are also well-defined and bounded on $H^2_H(\rho^m_P)$.

As in the case of spherical contractions, the spectrum of a $(P,1)$-positive multioperator is contained in the closure of the $P$-ball:

**Lemma 3.7.** Let $P$ be a positive regular polynomial and $T$ a $(P,1)$-positive commuting multioperator. Then the Taylor spectrum $\sigma(T)$ of $T$ is contained in the closure $\overline{P}$ of the $P$-ball.

**Proof.** This lemma is a special case of a more general result ([11], Theorem 1.12). We give a more elementary proof for our situation.

Let $\lambda \in \mathbb{C}^n \setminus \overline{P}$. We will show that $\lambda$ is not contained in the joint spectrum of $T$ relative to the closed commutative subalgebra $\mathcal{A}$ of $\mathcal{L}(\mathcal{H})$ generated by $T_1, \ldots, T_n$, i.e. we will show that the ideal $I$ generated by $\lambda_1 \mathbf{1}_H - T_1, \ldots, \lambda_n \mathbf{1}_H - T_n$ in $\mathcal{A}$ is equal to $\mathcal{A}$. Since the Taylor spectrum $\sigma(T)$ of $T$ is contained in the joint spectrum of $T$ relative to any closed commutative subalgebra of $\mathcal{L}(\mathcal{H})$ containing $T$, this means that $\lambda$ is not in $\sigma(T)$.

Let $Q_\lambda(z) = (1/P(|\lambda|^2)) P(\lambda z)$. Then $Q_\lambda(\lambda) = 1$, and for $h \in \mathcal{H}$, $\|h\| \leq 1$,

\[
\|Q_\lambda(T)h\| = \frac{1}{P(|\lambda|^2)} \|P(\lambda T)h\| 
\leq \frac{1}{P(|\lambda|^2)} \left( \sum_{\gamma \in I_P} a_{\gamma} |\lambda^\gamma|^2 \right)^{1/2} \left( \sum_{\gamma \in I_P} a_{\gamma} \|T^\gamma h\|^2 \right)^{1/2} 
= \frac{1}{P(|\lambda|^2)^{1/2}} \langle P(CT)(\mathbf{1}_P)h, h \rangle^{1/2} \leq \frac{1}{P(|\lambda|^2)^{1/2}} < 1
\]
by definition of \( P \). Thus \( \|Q_\lambda(T)\| < 1 \), and \( 1_H - Q_\lambda(T) \) is invertible in \( A \). On the other hand, one easily verifies that

\[
(3.23) \quad 1_H - Q_\lambda(T) = Q_\lambda(\lambda) - Q_\lambda(T) = \frac{1}{P(|\lambda|^{2})} \sum_{\gamma \in I_P} a_{\gamma}X(\lambda^m 1_H - T^m) \in I,
\]

which finishes the proof. □

We are now in the situation to state our model theorem:

**Theorem 3.8.** Let \( P \) be a positive regular polynomial in \( n \) variables, \( T = (T_1, \ldots, T_n) \) a commuting multioperator on the separable Hilbert space \( H \) and \( m \in \mathbb{N} \). Then the following are equivalent:

(i) \( T \) is \((P, m)\)-positive;

(ii) there exist a Hilbert space \( N \), a \( P \)-unitary operator \( N = (N_1, \ldots, N_n) \in \mathcal{L}(N)^n \) and an isometry \( V = V_1 \oplus V_2 : H \to H^2_\rho(\rho_P^m) \oplus N \) such that \( VT = (M_\alpha^2 \oplus N)V \).

**Proof.** First we prove (i) \( \Rightarrow \) (ii).

**Claim 1.** Let \( T \) be \((P, 1)\)-positive for the positive operator \( X \in \mathcal{L}(H) \). Then the sequence \((P(C_T)^k(X))_{k \in \mathbb{N}}\) converges to some positive operator \( \tilde{P}_X \) in the strong operator topology (SOT) on \( \mathcal{L}(H) \).

**Proof.** Since \( P \) is positive regular, \((P(C_T)^k(X))_{k \in \mathbb{N}}\) is a sequence of positive operators and thus bounded below by 0. Moreover, the sequence is decreasing because of

\[
P(C_T)^k(X) - P(C_T)^{k+1}(X) = P(C_T)^k(1 - P(C_T))(X) \geq 0
\]

and consequently converging to some positive operator \( \tilde{P}_X \) in the SOT-topology. □

Now define for \( X \in \mathcal{L}(H), \ X \geq 0 \), and \( T \) \((P, m)\)-positive for \( X \) the map

\[
V_{1}^{X} : H \to H^2_\rho(\rho_P^m), \ h \mapsto \sum_{\alpha \in \mathcal{N}_m} \rho_P^m(\alpha) ((1 - P(C_T))^m(X))^{1/2} T^\alpha h \zeta^\alpha.
\]

As one proves by induction completely analogously to [6], Lemmas 4 and 5 (see also [11], 2.1 and 2.8), we have

\[
\sum_{j=0}^{k} \binom{j + m - 1}{m - 1} P(C_T)^j(1 - P(C_T))^m
\]

\[
= 1 - \sum_{j=0}^{m-1} \binom{k + j}{j} P(C_T)^{k+1}(1 - P(C_T))^j, \quad k \in \mathbb{N}
\]
and

\begin{equation}
\lim_{k \to \infty} \left( \begin{array}{c} k + j \\ j \end{array} \right) (P(C_T)^{k+1} (1 - P(C_T))^j (X) h, h) = 0, \quad h \in \mathcal{H},
\end{equation}

for \( j = 1, \ldots, m - 1 \). We obtain

\begin{equation}
\|V_1^X h\|^2 = \|h\|^2 - \lim_{k \to \infty} \langle P(C_T)^k (X) h, h \rangle = \|h\|^2 - \langle \tilde{P}_X h, h \rangle, \quad h \in \mathcal{H}
\end{equation}

by

\begin{align}
\|V_1^X h\|^2 &= \sum_{\alpha \in \mathbb{N}_0^m} \rho_{m}^\alpha (\alpha) \langle (1 - P(C_T))^m(X) T^\alpha h, T^\alpha h \rangle \\
&= \sum_{\alpha \in \mathbb{N}_0^m} \left[ \sum_{K \in \mathbb{N}_0^{|K| - 1}} \left( \begin{array}{c} |K| + m - 1 \\ m - 1 \end{array} \right) \right] A^K \langle C_T^m (1 - P(C_T))^m(X) h, h \rangle \\
&= \sum_{j=0}^\infty \sum_{K \in \mathbb{N}_0^{|K| = j}} \left( \begin{array}{c} j + m - 1 \\ m - 1 \end{array} \right) A^K \langle C_T^j (1 - P(C_T))^m(X) h, h \rangle \\
&= \sum_{j=0}^\infty (P(C_T)^j (1 - P(C_T))^m(X) h, h)
\end{align}

according to (3.24) and (3.25), with the limits existing because of Claim 1. For \( T \) \((P, m)\)-positive and \( V_1 = V_1^{1\gamma} \), one gets

\begin{equation}
V_1 T h = \sum_{\alpha \in \mathbb{N}_0^m} \rho_{m}^\alpha (\alpha) ((1 - P(C_T))^m (1_{\mathcal{H}}))^{1/2} T^{\alpha+\varepsilon} h z^\alpha
\end{equation}

\begin{align}
&= \sum_{\alpha \in \mathbb{N}_0^m} \frac{\rho_{m}^\alpha (\alpha)}{\rho_{m}^\alpha (\alpha + \varepsilon)} \rho_{m}^\alpha (\alpha + \varepsilon) ((1 - P(C_T))^m (1_{\mathcal{H}}))^{1/2} T^{\alpha+\varepsilon} h z^{\alpha+\varepsilon} \\
&= M_\varepsilon^* \left( \sum_{\alpha \in \mathbb{N}_0^m} \rho_{m}^\alpha (\alpha + \varepsilon) ((1 - P(C_T))^m (1_{\mathcal{H}}))^{1/2} T^{\alpha+\varepsilon} h z^{\alpha+\varepsilon} \right) \\
&= M_\varepsilon^* V_1 T h.
\end{align}

So we have constructed the first part of our model. In a second step we construct the \( P \)-unitary part, using the fact that \( \tilde{P} = \tilde{P}_{1\gamma} \) is invariant under \( P(C_T) \). In the following, we write \( s \)-lim for the limits in the strong operator topology on \( \mathcal{L}(\mathcal{H}) \).
Lemma 3.9. Let $T$ be a $(P,1)$-positive commuting multioperator on $\mathcal{H}$ and
\[ \tilde{P} = \lim_{k \to \infty} P(C_T)^k(1_{\mathcal{H}}). \]
Then there exist a Hilbert space $\mathcal{N}$, a $P$-unitary multioperator $N \in \mathcal{L}(\mathcal{N})^n$ and a contractive linear mapping $V_2 : \mathcal{H} \to \mathcal{N}$ such that $\|V_2 h\|^2 = \langle \tilde{P} h, h \rangle$ for $h \in \mathcal{H}$ and $V_2 T = NV_2$.

Proof. Let $\mathcal{K} = \overline{P^{1/2}\mathcal{H}}$ and $V_2 : \mathcal{H} \to \mathcal{K}$, $h \mapsto \tilde{P}^{1/2} h$. For $i = 1, \ldots, n$, the linear map
\[ W_i : \tilde{P}^{1/2} \mathcal{H} \to \mathcal{K}, \quad h \mapsto V_2 T_i h \]
is well-defined and bounded, since
\[ \|W_i V_2 h\|^2 = \langle T_i^* \tilde{P} T_i h, h \rangle \leq a_{e_i}^{-1} (P(C_T)(\tilde{P})h, h) = a_{e_i}^{-1} \|V_2 h\|^2, \quad h \in \mathcal{H}. \]

So we can extend $W_i$ to a bounded linear map $\mathcal{K} \to \mathcal{K}$, which we also call $W_i$. By (3.29) and continuity, we have $W V_2 = V_2 T$ for $W = (W_1, \ldots, W_n)$ and consequently
\[ V_2^* (P(C_W)(1_{\mathcal{K}})) V_2 = P(C_T)(V_2^* V_2) = P(C_T)(\tilde{P}) = V_2^* V_2 \]
because of the SOT-continuity of $P(C_T)$.

Now $P(C_W)(1_{\mathcal{K}}) = 1_{\mathcal{K}}$, since $V_2 \mathcal{H}$ is dense in $\mathcal{K}$. Thus $W$ is a $P$-isometry. To replace $W$ by a $P$-unitary tuple, we need the following lemma:

Lemma 3.10. Every $P$-isometry is subnormal, and its minimal normal extension is a $P$-unitary.

Proof. Let $W \in \mathcal{L}(\mathcal{W})^n$ be a $P$-isometry. Then the tuple $(a_{\gamma}^{1/2} W_{\gamma})_{\gamma \in I_P}$ is a spherical isometry and consequently by [1], Proposition 2, a subnormal tuple. Since $a_{e_1}, \ldots, a_{e_n}$ are all not 0, in particular the tuple $W = (W_1, \ldots, W_n)$ is subnormal. Let $N = (N_1, \ldots, N_n)$ be its minimal normal extension on the Hilbert space $\mathcal{N} \supseteq \mathcal{K}$. Then $(a_{\gamma}^{1/2} N_{\gamma})_{\gamma \in I_P}$ is the minimal normal extension of the tuple $(a_{\gamma}^{1/2} W_{\gamma})_{\gamma \in I_P}$ and by [1] also a spherical isometry, which implies that $N$ is a $P$-unitary.

Now let for a $(P, m)$-positive multioperator $T$ on $\mathcal{H}$
\[ V = V_1 \oplus V_2 : \mathcal{H} \to H^n_{\mathcal{H}}(\rho_P^m) \oplus \mathcal{N}. \]
The mapping $V$ is an isometry, and $VT = (M_z^* \oplus N)V$. Note that only the first part of the model depends on $m$.

For the proof of the reverse direction, we have only to show that $M_z^* \in \mathcal{L}(H^2(\rho_p^m))^n$ is $(P, m)$-positive for arbitrary $m$. Then the $(P, m)$-positivity of $M_z^*$ on $H^2_N(\rho_p^m)$ follows, and we obtain the $(P, m)$-positivity of $T$ by the fact that any $P$-unitary is $(P, m)$-positive for every $m$ and that $(P, m)$-positivity is preserved under the direct sum $M_z^* \oplus N$, the restriction to the invariant subspace $VH$ and the unitary transformation $H \to VH$.

**Lemma 3.11.** For every $m \in \mathbb{N}$, the commuting multioperator $M_z^* \in \mathcal{L}(H^2(\rho_p^m))^n$ is $(P, m)$-positive. Moreover, $(1 - P(C_{M_z}))(1)$ is the orthogonal projection onto the subspace of constants in $H^2(\rho_p^m)$.

**Proof.** For $\alpha, \beta \in \mathbb{N}_0^n$, we have

$$M_z^\beta M_z^\alpha z^\alpha = \begin{cases} \rho_p^m(\alpha - \beta) z^\alpha & \text{if } \beta \leq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

So obviously $(1 - P(C_{M_z}))(1)z^\alpha = z^\alpha$ for $\alpha = 0$. Let as before $\rho_p^m(\alpha) = 0$ for $\alpha \in \mathbb{Z}^n \setminus \mathbb{N}_0^n$. Since the spaces $\mathbb{C} \cdot z^\alpha$ are invariant under $M_z^\beta M_z^\alpha$, thus also invariant under $(1 - P(C_{M_z}))(1)$ and $(1 - P(C_{M_z}))^m(1)$, it remains to show that

$$\langle (1 - P(C_{M_z}))(1)z^\alpha, z^\alpha \rangle \geq 0, \quad \alpha \geq 0$$

and

$$\langle (1 - P(C_{M_z}))^m(1)z^\alpha, z^\alpha \rangle = 0, \quad \alpha \geq 0, \alpha \neq 0.$$

By Equation (3.33), we have

$$\langle (1 - P(C_{M_z}))(1)z^\alpha, z^\alpha \rangle = \frac{1}{\rho_p^m(\alpha)^2} \left( \rho_p^m(\alpha) - \sum_{\gamma \in I_P} a_\gamma \rho_p^m(\alpha - \gamma) \right)$$

and

$$\langle (1 - P(C_{M_z}))^m(1)z^\alpha, z^\alpha \rangle = \frac{1}{\rho_p^m(\alpha)^2} \left( \rho_p^m(\alpha) - \sum_{\gamma \in \mathbb{N}_0^n} b_\gamma \rho_p^m(\alpha - \gamma) \right),$$

where $\sum_{\gamma \in \mathbb{N}_0^n} b_\gamma x^\gamma$ is the polynomial $1 - (1 - P)^m$. The rest of the proof now results from Remark 3.5.

This finishes the proof of Theorem 3.8.
Via the isometric isomorphism

\[(3.38) \quad H^2_H(\rho_m^P) \rightarrow l^2(N_0^\infty, H), \quad \sum_{\alpha \in N_0^\infty} h_\alpha \bar{z}^\alpha \mapsto \left( \frac{1}{\rho_m^P(\alpha)^{1/2}} h_\alpha \right)_{\alpha \in N_0^\infty}, \]

the multioperator \( M^* \) may be looked upon as a weighted multi-backward shift. So \( V_1 H \subseteq H^2_H(\rho_m^P) \) may be regarded as the shift part of our model, and \( V_2 H \subseteq N \) is the \( P \)-unitary part.

In case \( m = n = 1 \) and \( P = x \), the \((P, m)\)-positive operators are just the contractions, and our model is the well-known coisometric extension for contractions.

If \( P \) is the polynomial \( n \sum_{i=1}^n x_i \), the \( P \)-ball \( P = \{ z \in \mathbb{C}^n \mid P(|z|^2) < 1 \} \) is just the unit ball \( B_n \) of \( \mathbb{C}^n \), and the \( P \)-unitaries are just the spherical unitaries. For this case, Theorem 3.8 was proved by V. Müller and F.-H. Vasilescu in [6]. The positivity conditions \( \Delta^m_P \geq 0, 1 \leq m \leq n \), were examined earlier by A. Athavale, who showed in [1], Remark 1 to Proposition 4, that the tuple \( T \) then has a spherical dilation.

The standard model of Müller and Vasilescu reproduces this result: as one easily verifies, for the above \( P \) the space \( H^2(\rho_m^P) \) is just the Hardy space \( H^2(\mathbb{B}) = \left\{ f : \mathbb{B}^n \rightarrow \mathbb{C} \text{ holomorphic} \mid \|f\|_2 := \sup_{0 < r < 1} \int_{\partial \mathbb{B}^n} |f(rz)|^2 d\sigma < \infty \right\} \),

where \( \sigma \) is the normalized surface measure on \( \partial \mathbb{B}^n \), since

\[ \int_{\partial \mathbb{B}^n} |z|^\alpha d\sigma = (n-1)!\alpha!/(n-1-|\alpha|)! \]

for \( \alpha \in N_0^n \) (see e.g. [10], Proposition 1.4.9). The adjoint of the multiplication tuple here of course has a spherical dilation, for example the multioperator \( M^*_z \) \( \in \mathcal{L}(L^2(\partial \mathbb{B}^n, \sigma))^n \) via the isometric inclusion \( H^2(\mathbb{B}) \hookrightarrow L^2(\partial \mathbb{B}^n, \sigma) \). Thus \( M^*_z \oplus N \), where \( N \) is a spherical unitary, has a spherical dilation, and \( T \), being unitarily equivalent to the restriction of \( M^*_z \oplus N \) to an invariant subspace, has a spherical dilation, too.

The existence of a spherical dilation implies a von Neumann-type inequality over \( \mathbb{B}^n \) and consequently the existence of a contractive \( \mathcal{A}(\mathbb{B}^n) \)-functional calculus for \( T \), where \( \mathcal{A}(\mathbb{B}^n) = \{ f : \mathbb{B}^n \rightarrow \mathbb{C} \text{ continuous} \mid f \mid \mathbb{B}^n \text{ holomorphic} \} \).

But since the multioperator \( M_z^* \in \mathcal{L}(H^2_H(\rho_m^P))^n = \mathcal{L}(H^2(\mathbb{B}))^n \) has an obvious \( H^\infty(\mathbb{B}^n) \)-functional calculus defined by

\[(3.39) \quad f(M_z^*) = (M_j)^*, \quad f \in H^\infty(\mathbb{B}^n)\]
with \( \bar{f}(z) = \overline{f(\overline{z})} \), every \((P, n)\)-positive operator for which the model given by Theorem 3.8 consists only of the first part has even an \( H^\infty(\mathbb{B}^n) \)-functional calculus. So, according to Lemma 3.9 in the proof of Theorem 3.8, every \((P, n)\)-positive multioperator \( T \) with \( \lim_{k \to \infty} P(C_T)^k I_{1T} = 0 \) has a \( H^\infty(B_n) \)-functional calculus. This result is contained in [6] and may also be obtained by means of an operator-valued Poisson integral formula ([14]).

So for general positive regular polynomials \( P \), a natural question to ask is whether \( H^2(\rho^m P) \) may be identified for suitable \( m \) with a well-known Hilbert space of holomorphic functions on the \( P \)-ball \( P \) and thus one can obtain a rich functional calculus for \( M_{z}^* \in L(H^2(\rho^m P)) \) (and consequently for \((P, m)\)-positive \( T \)) by this identification.

In the next section, we will show that such an identification is possible by passing to an equivalent norm.

4. THE FUNCTIONAL MODEL

**Theorem 4.1.** Let \( P \) be a positive regular polynomial and \( m = \text{mult}(P) > n \). Furthermore, let \( \mu \) be the normalization of the positive measure \( (1 - P(|z|^2))^{m-n-1} \, d\lambda \) on \( P \), where \( d\lambda \) denotes Lebesgue measure. Then the space \( H^2(\rho^m P) \) and the Bergman space \( B^2(P, \mu) = \{ f : P \to \mathbb{C} \text{ holomorphic} \mid \int_{P} |f(z)|^2 \, d\mu < \infty \} \) coincide as sets of functions on \( P \), and the identifying map \( \text{id} : B^2(P, \mu) \to H^2(\rho^m P) \) is a topological isomorphism.

**Proof.** Let us first introduce some notations. With \( P = \sum_{\gamma \in \mathbb{N}_n^0} a_\gamma x^\gamma \), \( I_P = \{ \gamma \in \mathbb{N}_n^m \mid a_\gamma > 0 \} \) and \( |I_P| = \text{mult}(P) = m \), identify \( \mathbb{C}^m \) with \( \mathbb{C}^{|I_P|} \) and denote the elements of \( \mathbb{C}^m \) by \( w = (w_\gamma)_{\gamma \in I_P} \). Let \( \tau : \mathbb{C}^m \to \mathbb{C}^n \), \( w = (w_\gamma)_{\gamma \in I_P} \mapsto (w_{e_1}, \ldots, w_{e_n}) \), and \( \kappa : \mathbb{C}^m \to \mathbb{C}^n \), \( w = (w_\gamma)_{\gamma \in I_P} \mapsto (a_1^{-1/2}w_{e_1}, \ldots, a_n^{-1/2}w_{e_n}) \). Now define the holomorphic map

\[
\varphi : \mathbb{C}^m \to \mathbb{C}^m, \quad \varphi(w)_\gamma = \begin{cases} a_\gamma^{1/2}w_\gamma & \text{if } \gamma \in e_1, \ldots, e_n, \\ w_\gamma + a_\gamma^{1/2}\tau(w)_\gamma & \text{otherwise}. \end{cases}
\]

The map \( \varphi \) is biholomorphic, since

\[
\varphi^{-1} : \mathbb{C}^m \to \mathbb{C}^m, \quad \varphi^{-1}(w)_\gamma = \begin{cases} a_\gamma^{-1/2}w_\gamma & \text{if } \gamma \in e_1, \ldots, e_n, \\ w_\gamma - a_\gamma^{-1/2}\kappa(w)_\gamma & \text{otherwise}; \end{cases}
\]
Sandra Pott is obviously a holomorphic inverse map. Let $D = \varphi^{-1}(\mathbb{B}^m)$. Then $D$ is strictly pseudoconvex, since $\mathbb{B}^m$ is strictly pseudoconvex (see e.g. [9], II.2.7), and we have

$$D \cap (\mathbb{C}^n \times \{0\} \times \cdots \times \{0\}) = \left\{ w \in \mathbb{C}^m \mid w_\gamma = 0 \text{ for } \gamma \notin \{e_1, \ldots, e_n\}, \sum_{\gamma \in I_P} a_\gamma |r(w)_\gamma|^2 < 1 \right\} = \mathcal{P} \times \{0\} \times \cdots \times \{0\}.$$  

Moreover, $M = \varphi(\mathcal{P})$ is a complex submanifold of $\mathbb{B}^m$ such that $M = \left\{ w \in \mathbb{B}^m \mid \sum_{\gamma \in I_P} \gamma \cdot P_a |r(w)_\gamma|^2 < 1 \right\}$.

Let $Q$ be the polynomial in $m$ variables that corresponds to the unit ball, $Q \in \mathbb{C}[(X_\gamma)_{\gamma \in I_P}]$, $Q = \sum_{\gamma \in I_P} x_\gamma$.

We will now construct the identifying map $B^2(\mathcal{P}, \mu) \rightarrow H^2(\rho^m_P)$ in several steps.

**Step 1. The restriction.** As in (3.8), let $[·] : \mathbb{N}_0^m = \mathbb{N}_0^I \rightarrow \mathbb{N}_0^m, [\beta], = \sum_{\gamma \in I_P} \gamma_i \beta_i$.

**Lemma 4.2.** With $A = (a_\gamma)_{\gamma \in I_P}$ and the notation in (3.6), the map

$$\pi : H^2(\mathbb{B}^m) \rightarrow H^2(\rho^m_P), \quad \sum_{\beta \in \mathbb{N}_0^m} c_\beta w^\beta \mapsto \sum_{\beta \in \mathbb{N}_0^m} c_\beta A^{\beta/2} z^{[\beta]}$$

is well-defined, surjective, linear and has norm 1.

**Proof.** First notice that the $(P, m)$-weights may be expressed in terms of $(Q, m)$-weights: For $\alpha \in \mathbb{N}_0^m$, we have

$$\rho^m_P(\alpha) = \sum_{\beta \in \mathbb{N}_0^m_{[\beta]=\alpha}} A^\beta \binom{|\beta| + m - 1}{m - 1} \binom{|\beta|}{\beta} = \sum_{\beta \in \mathbb{N}_0^m_{[\beta]=\alpha}} A^\beta \rho^m_Q(\beta).$$

As one shows easily by induction over $r$, for any $a_1, \ldots, a_r, a_1, \ldots, b_r \in \mathbb{R}$ with $a_1, \ldots, a_r \geq 0$ and $b_1, \ldots, b_r > 0$ one has

$$\frac{\left( \sum_{i=1}^r a_i \right)^2}{\sum_{i=1}^r b_i} \leq \sum_{i=1}^r \frac{a_i^2}{b_i}.$$
Consequently we obtain for arbitrary $f = \sum_{\beta \in \mathbb{N}_0^m} c_\beta w^\beta \in H^2(\mathbb{B}^m)$, $\alpha \in \mathbb{N}_0^n$

$$\left| \frac{\sum_{\beta \in \mathbb{N}_0^m} A^{3/2}c_\beta}{\rho_P^2(\alpha)} \right|^2 \leq \left( \sum_{\beta \in \mathbb{N}_0^m \mid |\beta| = \alpha} A^{3/2}|c_\beta| \right)^2 \leq \sum_{\beta \in \mathbb{N}_0^m \mid |\beta| = \alpha} |c_\beta|^2 \rho_Q^2(\beta)$$

(4.7)

and

$$\|\pi(f)\|^2 = \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\rho_P^2(\alpha)} \left| \sum_{\beta \in \mathbb{N}_0^m \mid |\beta| = \alpha} A^{3/2}c_\beta \right|^2 \leq \|f\|^2.$$  

(4.8)

To show the surjectivity of $\pi$, consider the map $\iota : H^2(\rho_P^m) \rightarrow H^2(\mathbb{B}^m)$, $g = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha \mapsto \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha \sum_{\beta \in \mathbb{N}_0^m \mid |\beta| = \alpha} A^{3/2}(\rho_Q^m(\beta)/\rho_P^m(\alpha))w^\beta$. Then $\iota$ is well-defined and isometric, since $\iota(g) \in H^2(\mathbb{B}^m)$ with $||\iota(g)||^2 = \sum_{\alpha \in \mathbb{N}_0^n} |c_\alpha|^2 \sum_{\beta \in \mathbb{N}_0^m \mid |\beta| = \alpha} A^2(\rho_Q^m(\beta)/\rho_P^m(\alpha))^2 = \|g\|^2$ by Equation (4.5), and $\pi \circ \iota = 1$.  

Thus the map $\pi$ can be regarded as the orthogonal projection from $H^2(\mathbb{B}^m)$ onto the closed subspace $H^2(\rho_P^m)$. This close relationship between $H^2(\rho_P^m)$ and $H^2(\mathbb{B}^m)$ and the definitions of $\varphi$ and $\pi$ become clearer by considering the following idea:

Let $T = (T_1, \ldots, T_n)$ be a $(P, m)$-positive multioperator on $H$ and let $V_1 : H \rightarrow H^2_T(\rho_P^m)$ be the map constructed in Theorem 3.8. Let $W$ be the commuting $m$-tuple $(W_\gamma)_{\gamma \in I^m}$, $W_\gamma = \alpha_\gamma^{1/2}T^\gamma$. Then

$$\left(1-P\right)(C_T) = \left(1-Q\right)(C_W)$$

(4.9)

and thus $W$ is $(Q, m)$-positive. Again by Theorem 3.8, now applied to the $m$-tuple $W$, we obtain the map $\tilde{V}_1 : H \rightarrow H^2(\mathbb{B}^m)$ as first part of the model for the tuple $W$. Therefore

$$(1_H \otimes \pi) \circ \tilde{V}_1(h) = (1_H \otimes \pi) \left( \sum_{\alpha \in \mathbb{N}_0^n} \rho_Q^m(\beta)((1-Q)^m(C_W)(1_H))^{1/2}T^\beta h w^\beta \right)$$

$$= \sum_{\alpha \in \mathbb{N}_0^n} \sum_{\beta \in \mathbb{N}_0^m \mid |\beta| = \alpha} \rho_Q^m(\beta)A^\beta((1-P)^m(C_T)(1_H))^{1/2}T^\beta h z^\alpha$$

(4.10)

$$= \sum_{\alpha \in \mathbb{N}_0^n} \rho_P^m(\alpha)((1-P)^m(C_T)(1_H))^{1/2}T^\alpha h z^\alpha = V_1(h)$$
for \( h \in \mathcal{H} \), and we have

\[
(1_{\mathcal{H}} \otimes \pi) \circ \tilde{V}_1 = V_1.
\]

In particular, the map \( 1_{\mathcal{H}} \circ \pi \) is isometric on \( \tilde{V}_1 \mathcal{H} \), since

\[
\|V_1 h\|^2 = \lim_{k \to \infty} \langle P(C_T)^k(1_{\mathcal{H}}) h, h \rangle = \lim_{k \to \infty} \langle Q(C_W)^k(1_{\mathcal{H}}) h, h \rangle = \|\tilde{V}_1 h\|^2.
\]

The submanifold \( \mathcal{M} = \{ w \in \mathbb{B}^m \mid w_\gamma = a^{1/2}_\gamma \kappa(w) \} \) corresponds to the identities \( W_\gamma = a^{1/2}_\gamma T_\gamma \). The map \( \pi \) may be regarded as the restriction of functions in \( H^2(\mathbb{B}^m) \) to the submanifold \( \mathcal{M} \), up to the biholomorphic map \( \varphi \). For \( z \in \mathcal{P} \) and \( f = \sum_{\beta \in \mathbb{N}_m^0} c_\beta w^\beta \in H^2(\mathbb{B}^m) \), we have

\[
\begin{align*}
f \circ \varphi(z) &= \sum_{\beta \in \mathbb{N}_m^0} c_\beta (\varphi(z))^\beta = \sum_{\beta \in \mathbb{N}_m^0} c_\beta \prod_{\gamma \in I_P} a^{\beta/2}_\gamma(z)^\gamma \\
&= \sum_{\beta \in \mathbb{N}_m^0} c_\beta A^{\beta/2} z^\beta = \pi(f)(z).
\end{align*}
\]

Altogether, we have the following commutative diagram.

\[
\begin{array}{ccc}
\tilde{V}_1 \mathcal{H} & \hookrightarrow & H^2(\mathbb{B}^m) \\
\downarrow & & \downarrow 1 \otimes \iota & 1_{\mathcal{H}} \otimes \pi = \circ \varphi |_\mathcal{P} \\
\mathcal{H} & \xrightarrow{V_1} & V_1 \mathcal{H} & \hookrightarrow & H^2_0(\rho^m)
\end{array}
\]

**Step 2. The transformation.** Recall that the Hardy space \( H^p(\Omega) \), \( 1 < p < \infty \), over a bounded strictly pseudoconvex set \( \Omega \subseteq \mathbb{C}^n \) with \( C^2 \)-boundary can be obtained in the following way (see e.g. [5], Section 8.3):

Let \( \varphi : U \to \mathbb{R} \) be a strictly plurisubharmonic defining \( C^2 \)-function for \( \Omega \), defined on some region \( U \supset \overline{\Omega} \). That means,

\[
\Omega = \{ z \in U \mid \varphi(z) < 0 \}.
\]

Now for \( \varepsilon > 0 \) let \( \Omega_\varepsilon = \{ z \in U \mid \varphi(z) < \varepsilon \} \). For sufficiently small \( \varepsilon_0 \), \( \partial \Omega_\varepsilon \) is a real \( C^2 \)-manifold for each \( \varepsilon \) with \( 0 < \varepsilon < \varepsilon_0 \). Let \( \sigma_\varepsilon \) be the surface measure on \( \partial \Omega_\varepsilon \) and define

\[
H^p(\Omega) = \left\{ f : \Omega \to \mathbb{C} \text{ holomorphic} \mid \|f\|_p = \left( \sup_{\varepsilon_0 > \varepsilon > 0} \int_{\partial \Omega_\varepsilon} |f(z)|^p \, d\sigma_\varepsilon \right)^{1/p} < \infty \right\}.
\]
Then $H^p(\Omega, \| \cdot \|_p)$ is a Banach space. The space $H^p(\Omega)$ is independent of the choice of the defining function $\varrho$ in the sense that any two plurisubharmonic defining $C^2$-functions for $\Omega$ induce equivalent norms on $H^p(\Omega)$. Furthermore, by passing to nontangential boundary values $H^p(\Omega)$ may be embedded topologically into $L^p(\partial \Omega, \sigma)$, where $\sigma$ is the surface measure on $\partial \Omega$.

Our aim is to show that the biholomorphic map $\varphi : D \to \mathbb{B}^m$ induces a topological isomorphism

$$U_\varphi : H^2(\mathbb{B}^m) \to H^2(D), \quad f \mapsto f \circ \varphi. \quad (4.17)$$

This can be done by using the transformation formula and looking at the Jacobi-matrix for $\varphi$ on $\partial D$, but an alternative characterization of $H^p(\Omega)$ and an equivalent norm to $\| \cdot \|_p$ give a much shorter and less technical proof. We have

$$H^p(\Omega) = \{ f : \Omega \to \mathbb{C} \text{ holomorphic} \mid |f|^p \text{ has a harmonic majorant on } \Omega \}, \quad (4.18)$$

and if $\Omega$ is connected, for any $z \in \Omega$

$$\| f \|_{p,z} = \left( \inf \{ g(z) \mid g : \Omega \to \mathbb{R} \text{ harmonic}, g \geq |f|^p \} \right)^{1/p} \quad (4.19)$$

defines an equivalent norm to $\| \cdot \|_p$ on $H^p(\Omega)$ (see e.g. [15], Section 2.2).

Since composition with the biholomorphic map $\varphi$ maps the class of real-valued harmonic functions on $\mathbb{B}^m$ bijectively onto the class of real-valued harmonic functions on $D$, for any fixed $z_0 \in D$ and any $f \in H^2(\mathbb{B}^m)$ we have

$$\| f \circ \varphi \|_{p,z_0}^2 = \inf \{ g(z_0) \mid g : D \to \mathbb{R} \text{ harmonic}, g \geq |f \circ \varphi|^2 \}$$

$$= \inf \{ g(\varphi(z_0)) \mid g : \mathbb{B}^m \to \mathbb{R} \text{ harmonic}, g \geq |f|^2 \}$$

$$= \| f \|_{2,\varphi(z_0)}^2, \quad (4.20)$$

and $U_\varphi$ in (4.17) is thus a topological isomorphism with inverse $U_{\varphi^{-1}}$.

**Step 3. The Extension.** Now we come to the main step of our construction of the identification $B^2(\mathcal{P}, \mu) \to H^2(\mu_{\mathcal{P}})$, using a theorem of A. Cumenge.

We will show that for a measure $\tilde{\mu}$ equivalent to $\mu$, there is a bounded linear extension operator $E : B^2(\mathcal{P}, \tilde{\mu}) \to H^2(D)$ and that the restriction $R : H^2(D) \to B^2(\mathcal{P}, \tilde{\mu})$ is well-defined, bounded and surjective. To apply the theorem of Cumenge, we first have to show that $\mathcal{P}$ may be extended to a complex manifold transverse to $\partial D$, i.e. that there is a complex submanifold $\tilde{\mathcal{P}}$ of $\mathbb{C}^m$ intersecting $\partial D$ transversally such that $\mathcal{P} = D \cap \tilde{\mathcal{P}}$. 
Let $\tilde{\mathcal{P}} = \mathbb{C}^n \times \{0\} \times \cdots \times \{0\}$. Then $\mathcal{P} = D \cap \tilde{\mathcal{P}}$ by (4.3). The function $r : \mathbb{C}^m \to \mathbb{R}$, $r(z) = \sum_{\gamma \in I_F} |z_\gamma|^2 - 1$, is a strictly plurisubharmonic defining $C^\infty$-function for $\mathbb{B}^m$. Thus $g = \varphi \circ r$ is a strictly plurisubharmonic defining $C^\infty$-function for $D$.

To prove that $\tilde{\mathcal{P}}$ intersects $\partial D$ transversally, we have to show that

$$\tag{4.21} d\rho(z) \wedge \left( \bigwedge_{\gamma \in I_F \setminus \{e_1, \ldots, e_n\}} d\tau_{\gamma} \right) \neq 0 \quad \text{for all } z \in \tilde{\mathcal{P}} \cap \partial D$$

(see e.g. [9], p. 118). So it suffices to prove that for every $z \in \tilde{\mathcal{P}} \cap \partial D$, there is an $i \in \{1, \ldots, n\}$ such that $\partial \rho/\partial z_i(z) \neq 0$. On $\tilde{\mathcal{P}}$, identify $z$ with $\tilde{z} = \tau(z) \in \mathbb{C}^n$ to obtain $\rho(z) = \sum_{\gamma \in I_F} a_{\gamma_i} |z_\gamma|^2$. Now let $z \in \tilde{\mathcal{P}} \cap \partial D$. Since $0 \notin \partial D$, there is an $i$ with $\tau(z)_i \neq 0$, and we obtain

$$\frac{\partial \rho}{\partial z_i}(z) = \frac{\partial \rho}{\partial z_i}(\tilde{z}) = a_{e_i} \tau(z)_i + \sum_{\gamma \in I_F \setminus \{e_1, \ldots, e_n\}, \gamma_i \neq 0} \gamma_i (a_{\gamma_i} \tau(z)_i)^{\tau(z)_i} \gamma_i^{\tau(z)_i - e_i}$$

$$= \tau(z)_i \left( a_{e_i} + \sum_{\gamma \in I_F \setminus \{e_1, \ldots, e_n\}, \gamma_i \neq 0} \gamma_i (a_{\gamma_i} |\tau(z)_{\gamma_i}|)^2 \right) \neq 0,$$

since the second factor is strictly positive.

Now $\mathcal{P}$ is a complex submanifold of codimension $m - n$ of the smoothly bounded strictly pseudoconvex set $D$. Thus we are in the situation of Theorem 0.1 in [2]: let $\tilde{\mu}$ be the measure $\text{dist}(z, \partial D) d\lambda$ on $\mathcal{P}$. Then $f|\mathcal{P} \in B^2(\mathcal{P}, \tilde{\mu})$ for every $f \in H^2(\partial D)$, and there exists a bounded linear extension operator $E : B^2(\mathcal{P}, \tilde{\mu}) \to H^2(D)$, $E|\mathcal{P} = g$ for $g \in B^2(\mathcal{P}, \tilde{\mu})$.

Moreover, the restriction operator $R : H^2(D) \to B^2(\mathcal{P}, \tilde{\mu})$ is bounded since $\tilde{\mu}$ is a Carleson measure on $D$ by Hörmander’s formulation of Carleson’s Theorem and by Lemme II.1.1 in [2] (see [2], Section II.1, and [4], Theorem 4.3). It is surjective since $R \circ E = 1_B^2(\mathcal{P}, \tilde{\mu})$. The map $\pi \circ U_{\varphi^{-1}} \circ E : B^2(\mathcal{P}, \tilde{\mu}) \to H^2(\varphi^m_D)$ now maps each function $g \in B^2(\mathcal{P}, \tilde{\mu})$ onto itself. It is bounded by construction and has the bounded inverse $R \circ U_{\varphi \circ \iota}$. Altogether, we have the following commutative diagram:

$$\begin{array}{ccc}
H^2(D) & \xrightarrow{U_{\varphi^{-1}}} & H^2(\mathbb{B})^m \\
E \uparrow & & \downarrow \\
B^2(\mathcal{P}, \tilde{\mu}) & \xrightarrow{\sim} & H^2(\varphi^m_D)
\end{array}$$
It remains to compare $\mu$ and $\tilde{\mu}$.

**Step 4. The equivalence of the measures.** It suffices to show that there are constants $c_1, c_2 > 0$ such that

$$c_1 \text{dist}(z, \partial D) \leq 1 - P(|z_1|^2, \ldots, |z_n|^2) \leq c_2 \text{dist}(z, \partial D), \quad z \in \partial P.$$  

Then $B^2(\mathcal{P}, \mu)$ and $B^2(\mathcal{P}, \tilde{\mu})$ coincide as sets and carry equivalent norms.

The second inequality just follows by the Lipschitz continuity of the map $z \mapsto P(|z_1|^2, \ldots, |z_n|^2)$ on the compact set $\overline{P}$. For the first inequality, choose for $z \in P$ some $w \in \partial P$ such that $z = \lambda w$ for a suitable $\lambda \in [0, 1)$. Then

$$1 - P(|z_1|^2, \ldots, |z_n|^2) = \sum_{\gamma \in \mathcal{I}_P} a_{\gamma}(|w_\gamma|^2 - |z_\gamma|^2)$$

$$\geq (1 - \lambda^2) \sum_{i=1}^{n} a_i |w_i|^2 \geq c(1 - \lambda) \|w\|^2$$

$$\geq c_1 (1 - \lambda) \|w\| = c_1 \|w - z\| \geq c_1 \text{dist}(z, \partial P)$$

for suitable constants $c, c_1 > 0$, since $\partial P$ is bounded away from 0. Thus we obtain (4.23), which finishes the proof of the theorem.

5. **DILATIONS**

The identifying map $B^2(\mathcal{P}, \mu) \to H^2(\rho^m_P)$ obviously intertwines the multiplication operators with the coordinate functions on $B^2(\mathcal{P}, \mu)$ and $H^2(\rho^m_P)$. So its adjoint intertwines the adjoints of the multiplication operators, and we obtain the following easy consequence of Theorem 3.8 and Theorem 4.1. Let as before $P$ be a positive regular polynomial with $m = \text{mult}(P) > n$, $\mu$ the normalization of the measure $(1 - P(|z_1|^2, \ldots, |z_n|^2))^{m-n-1} d\lambda$ on $\mathcal{P}$ and let $M = (M_1, \ldots, M_n)$ be the tuple of multiplication operators with the coordinate functions on $B^2_2(\mathcal{P}, \mu)$.

**Corollary 5.1.** The following are equivalent:

(i) $T$ is topologically equivalent to a $(P, m)$-positive multioperator;

(ii) $T$ is topologically equivalent to the restriction of $M^* \oplus N \in \mathcal{L}(B^2_2(\mathcal{P}, \mu) \oplus N)^n$ to an invariant subspace, where $N$ is a $P$-unitary operator on some separable Hilbert space $N$.

Moreover, the functional model for a $(P, m)$-positive multioperator $T$ implies — up to topological equivalence — the existence of a $P$-unitary dilation for $T$. Unlike the situation of the unit ball, we cannot obtain a $P$-unitary dilation directly. We have to check the complete boundedness of the map $q \mapsto q(T)$ on the algebra of polynomials, equipped with the supremum norm on $\mathcal{P}$.
Theorem 5.2. Let $T$ be a $(P,m)$-positive commuting multioperator. Then $T$ is topologically equivalent to a multioperator $S$ which has a $P$-unitary dilation.

Proof. By Corollary 5.1, $T$ is topologically equivalent to the restriction of $M^* \oplus N$ to an invariant subspace. Thus it is sufficient to show that $M^*$ has a $P$-unitary dilation.

The algebra $\mathbb{C}[X_1,\ldots,X_n]$ carries an operator algebra structure as a subalgebra of the commutative $C^*$-algebra $\mathcal{C}(\partial \mathcal{P})$ of continuous functions on $\partial \mathcal{P}$. We denote this operator algebra by $\text{Pol}(\mathcal{P})$.

Remark 5.3. The algebra homomorphism

\begin{equation}
\Phi : \text{Pol}(\mathcal{P}) \rightarrow \mathcal{L}(B^2_R(\mathcal{P},\mu)), \quad q \mapsto q(M^*)
\end{equation}

is completely contractive.

Proof. Let $M_n(\mathcal{L}(B^2_R(\mathcal{P},\mu)))$ be the $C^*$-algebra of $n \times n$-matrices over $\mathcal{L}(B^2_R(\mathcal{P},\mu))$ and let $M_n(\text{Pol}(\mathcal{P}))$ be the algebra of $n \times n$-matrices over $\text{Pol}(\mathcal{P})$, carrying the norm $\|(q_{i,j})\|_n = \sup\{\|(q_{i,j}(z))\| \in \mathcal{P} \}$, where $\|(q_{i,j}(z))\|$ denotes the usual operator norm of the complex $n \times n$-matrix $(q_{i,j}(z))$. We have to show that for each $n$, the map

\begin{equation}
\Phi^{(n)} : M_n(\text{Pol}(\mathcal{P})) \rightarrow M_n(\mathcal{L}(B^2_R(\mathcal{P},\mu))), \quad (q_{i,j}) \mapsto (q_{i,j}(M^*)�
\end{equation}

is a contraction.

For $q \in \mathbb{C}[X_1,\ldots,X_n]$, let $\check{q}$ be the polynomial obtained by complex conjugation of the coefficients of $q$. Then for $(q_{i,j}) \in M_n(\text{Pol}(\mathcal{P}))$, $\|\Phi^{(n)}((q_{i,j}))\| = \|(q_{i,j}(M^*))\| = \|\check{q}_{j,i}(M)\|$, and for $f = (f_1,\ldots,f_n) \in B^2_R(\mathcal{P},\mu)^n = B^2_R^\mathcal{P}(\mathcal{P},\mu)$ we have

\begin{equation}
\|\check{q}_{j,i}(M)f\|^2 = \int_{\mathcal{P}}\|((\check{q}_{j,i}(M))f)(z)\|^2 \, d\mu = \int_{\mathcal{P}}\|((\check{q}_{j,i}(z))B^2_R(\mathcal{P},\mu))f(z)\|^2 \, d\mu \\
\leq \int_{\mathcal{P}}\|\check{q}_{j,i}(z)f(z)\|^2 \, d\mu \leq \|(q_{j,i})\|^2 \|f\|^2.
\end{equation}

Thus $\Phi^{(n)}$ is a contraction, and the remark is proved. 

To finish the proof of the theorem, note that by a corollary to Arveson’s Extension Theorem (see [7], Corollary 6.7) the map $\Phi$ dilates to a homomorphism $\Psi : \mathcal{C}(\mathcal{P}) \rightarrow \mathcal{L}(\mathcal{K})$ with some Hilbert space $\mathcal{K} \supseteq B^2_R(\mathcal{P},\mu)$. Then the tuple $K = (\Psi(z_1),\ldots,\Psi(z_n))$ is a normal multioperator dilating $M^*$, and the Taylor spectrum
of $K$ is contained in $\partial P$. By the Spectral Theorem for normal multioperators (see [13], Theorem 7.26), we have

\begin{equation}
P(C_K)(1_K) = \int_{\partial P} P(|z|^2) \, dE = 1_K,
\end{equation}

where $E$ is the spectral measure for the tuple $K$ on $K$.

In particular, Theorem 5.2 implies that each $(P,m)$-positive multioperator satisfies a von Neumann-type inequality with respect to the $P$-ball $P$. Let $A(P)$ be the Banach algebra of complex-valued continuous functions on $P$ which are holomorphic on $P$, together with the supremum norm on $P$.

**Corollary 5.4.** Let $T$ be a $(P,m)$-positive multioperator. Then $T$ has a continuous $A(P)$-functional calculus. In particular, there is a constant $c > 0$ such that

\begin{equation}
\|q(T)\| \leq c \sup \{ |q(z)| \mid z \in P \} \text{ for } q \in \mathbb{C}[X_1, \ldots, X_n].
\end{equation}

**Proof.** As one easily sees by the Spectral Theorem for normal multioperators (see [13], Theorem 7.26) and by Lemma 3.7, a $P$-unitary multioperator $U$ satisfies the von Neumann-inequality

\begin{equation}
\|q(U)\| \leq \sup \{ |q(z)| \mid z \in P \} \text{ for } q \in \mathbb{C}[X_1, \ldots, X_n].
\end{equation}

The corollary now follows from Theorem 3.8, since the polynomials are dense in $A(P)$.

In case the model for $T$ provided by Theorem 5.2 consists only of the multiplication operator part, i.e. in case $P(C_T)(1_H)$ converges strongly to 0 for $s \to \infty$, we can strengthen this result. Let $A : H^2_H(\rho^m_P) \to B^2_P(\mathcal{P}, \mu)$ be the isomorphism intertwining $M_z^*$ on $H^2_H(\rho^m_P)$ and $M^*$ on $B^2_P(\mathcal{P}, \mu)$ mentioned in the beginning of this paragraph. Then

\begin{equation}
H^\infty(\mathcal{P}) \to \mathcal{L}(\mathcal{H}), \quad f \mapsto V^* A^{-1} M_f^* AV,
\end{equation}

where $V : \mathcal{H} \to H^2_H(\rho^m_P)$ is the isometry constructed in Theorem 3.8, $\tilde{f}$ is the holomorphic map $z \mapsto \tilde{f}(\overline{z})$ on $\mathcal{P}$ and $M_f$ is the bounded operator of multiplication with $f$ on $B^2_P(\mathcal{P}, \mu)$, defines a continuous algebra homomorphism with norm less or equal to $\|A\| \|A^{-1}\|$, mapping the coordinate functions to the components of $T$. Thus (5.7) gives a continuous $H^\infty(\mathcal{P})$-functional calculus for $T$. 

In a forthcoming paper ([9]), the developed standard model for \((P, m)\)-positive multioperators \(T\) will be applied to give necessary conditions for the existence of non-trivial joint invariant subspaces of \(T\).

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