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PURELY INFINITE SIMPLE TOEPLITZ ALGEBRAS

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ABSTRACT. The Toeplitz C^*-algebras associated to quasi-lattice ordered groups (G, P) studied by Nica in [12] were shown by Laca and Raeburn ([7]) to be crossed products of an abelian C^*-algebra B_P by a semigroup of endomorphisms. Here we define a natural boundary for the semigroup P as a subset of the maximal ideal space (or spectrum) of B_P and prove that the Toeplitz C^*-algebra associated to P is simple exactly when this boundary is all of the spectrum of B_P, in which case the Toeplitz C^*-algebra is actually purely infinite. We also prove that when the boundary is a proper subset of the spectrum, it induces an ideal of the Toeplitz C^*-algebra which is maximal among induced ideals.

KEYWORDS: *Toeplitz algebras, quasi-lattice order, semigroup of isometries, semigroup crossed product.*

MSC (2000): 46L55.

1. INTRODUCTION

Suppose G is a group and $P \subset G$ is a unital subsemigroup with no inverses, that is, $P^2 \subset P$ and $P \cap P^{-1} = \{e\}$. Then P induces a left-invariant partial order on G via $x \leqslant y$ if $x^{-1}y \in P$; according to Nica the pair (G, P) is a *quasi-lattice ordered* group if every finite subset of G having an upper bound in P has a least upper bound in P. The notation $x \vee y$ is used for the least common upper bound of x and y, with the convention $x \vee y = \infty$ when there is no common upper bound in P. The reader is referred to [12] and [7] for the basic properties and examples of quasi-lattice ordered groups.

Nica associated two C^*-algebras to every (G, P): a regular Toeplitz (or Wiener-Hopf) C^*-algebra $\mathcal{T}(G, P)$, generated by the left regular representation of P on $\ell^2(P)$, and a C^*-algebra $C^*(G,P)$ which is universal for isometric representations satisfying the covariance condition

$$i(x)^*i(y) = i(x^{-1}(x \vee y))i(y^{-1}(x \vee y))^*,$$

where $\{i(x) \mid x \in P\}$ is a semigroup of isometries such that for any covariant semigroup of isometries $\{V_x \mid x \in P\}$, the map $i(x) \mapsto V_x$ extends to a homomorphism of $C^*(G,P)$ onto $C^*(\{V_x \mid x \in P\})$. This universal algebra is clearly unique up to isomorphisms and, when the quasi-lattice order is amenable, it is isomorphic to the Toeplitz C^*-algebra, ([12], Section 4 (see also [7], Section 2)).

In [7], $C^*(G,P)$ was shown to be a semigroup crossed product of an abelian C^*-algebra by a semigroup of endomorphisms. Specifically, if 1_x denotes the characteristic function of the set $\{y \in P \mid y \geqslant x\}$, then the family of projections $\{1_x \in \ell^\infty(P) \mid x \in P\}$ is closed under multiplication and thus

$$B_P = \overline{\operatorname{span}}\{1_x \mid x \in P\}$$

is a commutative unital C^*-algebra, on which the semigroup P acts by (left translation) endomorphisms given by $\alpha_x : 1_y \mapsto 1_{xy}$. The semigroup crossed product $B_P \rtimes P$ is naturally isomorphic to $C^*(G,P)$ ([7], Corollary 2.4), and this led in [7] to a characterization of faithful representations and to uniqueness results for Toeplitz algebras which unify the treatment of various C^*-algebras generated by semigroups of isometries. The main result of this note, Theorem 5.4, gives necessary and sufficient conditions for simplicity and pure infiniteness of generalized Toeplitz C^*-algebras in terms of the quasi-lattice order structure.

When applied to specific free product quasi-lattice orders this gives as special cases Cuntz's result that $\mathcal{O}_\infty$ is purely infinite simple as well as a result of Dinh on the spectral C^*-algebras of discrete product systems.

In their study of noncommutative disk algebras, Davidson and Popescu have independently obtained a related result ([3]), see Remark 5.7 below.

2. THE MAXIMAL IDEAL SPACE OF B_P

Following [12] we will say that a subset A of P is *hereditary* if $x \leqslant y \in A$ implies $x \in A$, and that it is *directed* if any pair $x, y \in A$ has a common upper bound in A. The collection of all nonempty hereditary directed subsets of P will be denoted by Ω and endowed with the topology inherited from $\{0,1\}^P$ by identifying subsets of P with their characteristic functions. Thus, a net $\{A_\lambda\}$ converges to A in Ω if and only if $\chi_{A_\lambda}(x)$ converges to $\chi_A(x)$ for every $x \in P$.

There is a homeomorphism between the maximal ideal space of B_P and Ω obtained by sending a multiplicative linear functional ω on B_P to the set $A_\omega = \{x \in P \mid \omega(1_x) = 1\}$ ([12], Proposition 6.2.1). The inverse of this homeomorphism sends a hereditary directed set A into the functional $\widehat{A}$ defined by $\widehat{A}(f) = \lim\limits_{x \in A} f(x)$ for $f \in B_P$; this limit exists for every $f \in B_P$ because B_P is the closed linear span of functions of the form 1_x. We will identify B_P and $C(\Omega)$ and write $f(A)$ instead of $\widehat{A}(f)$.

The semigroup P embeds as a dense subset of Ω by $x \mapsto [e,x] = \{y \in P \mid e \leqslant y \leqslant x\}$, so Ω may be seen as a compactification of P coming from the order structure. The following proposition gives a basis for the topology of Ω consisting of closed and open neighborhoods.

Proposition 2.1. *If H is a finite subset of P and $a \leqslant h$ for all $h \in H$, the set*

$$V(a,H) := \{B \in \Omega \mid a \in B, h \notin B \ (\forall) h \in H\}, \tag{2.1}$$

is closed and open. Furthermore, the collection of all such subsets is a basis for the topology, and Ω is totally disconnected.

Proof. Notice first that

$$V(a,H) = \Big\{B \in \Omega \Big| \prod_{h \in H} (1_a - 1_h)(B) = 1\Big\} = \Big\{B \in \Omega \Big| \prod_{h \in H} (1_a - 1_h)(B) > 0\Big\}.$$

Since the function $\prod\limits_{h \in H} (1_a - 1_h)$ is continuous on Ω, the set $V(a,H)$ is clopen.

A typical basic open neighborhood around $A \in \Omega$ is $N(A;F) = \{B \in \Omega \mid \chi_B(x) = \chi_A(x) \text{ for } x \in F\}$, where F is a finite subset of P. Thus, in order to prove that the sets from (2.1) form a base for the topology, it suffices to show that for any $N(A;F)$ there exist a and H such that $A \in V(a,H) \subset N(A;F)$.

If $F \cap A$ is not empty, let a be its least upper bound, which exists because A is directed; otherwise let $a = e$. Let $H = \{a \vee x \in P \mid x \in F \setminus A\}$, thus $a \leqslant h$

for each $h \in H$. Since A is hereditary $h \notin A$ for every $h \in H$ because if $a \vee x \in A$, then $x \in A$. Thus $A \in V(a, H)$.

Suppose now $B \in V(a, H)$. If $x \in F \cap A$ then $x \leqslant a$. Since B is hereditary and $a \in B$ it follows that $x \in B$. If $x \in F \setminus A$, then $x \notin A$. Hence either a and x have no common upper bound, in which case $x \notin B$, or else $a \vee x = h \in H$ in which case $x \notin B$ as well. This proves that $\chi_A(x) = \chi_B(x)$ for every $x \in F$, hence $B \in N(A; F)$. ∎

Next we define an action of P by left multiplication as partial homeomorphisms of Ω. For $t \in P$ and $A \in \Omega$ we consider the smallest hereditary subset of P containing tA, i.e. the set

$$[e, tA] = \{y \in P \mid y \leqslant ta \text{ for some } a \in A\}.$$

This set is indeed hereditary because if $x \leqslant y \in [e, tA]$ then $x \leqslant ta$ for some $a \in A$, so $x \in [e, tA]$ and it is also directed because if $x \leqslant ta$ and $y \leqslant tb$ for $a, b \in A$, then $x \vee y \leqslant t(a \vee b)$ and $a \vee b \in A$ so $x \vee y \in [e, tA]$. Thus $[e, tA] \in \Omega$.

PROPOSITION 2.2. *For each $t \in P$ the map $\theta_t : \Omega \to \Omega$ defined by $\theta_t(A) = [e, tA]$, is a homeomorphism of Ω onto the clopen subset $\Omega_t = \{B \in \Omega \mid t \in B\}$, with inverse given by $\theta_t^{-1}(B) = t^{-1}(t \vee B) := \{t^{-1}(t \vee b) \mid b \in B\}$. Furthermore, $\theta_s \circ \theta_t = \theta_{st}$.*

Proof. The function $1_t : B \in \Omega \mapsto \chi_B(t)$ is continuous for each $t \in P$, hence the set $\Omega_t = \{B \in \Omega \mid \chi_B(t) = 1\} = \{B \in \Omega \mid \chi_B(t) > 0\}$ is closed and open.

Suppose $B \in \Omega_t$, that is, $t \in B$. The set $B_0 = \{t^{-1}(t \vee b) \mid b \in B\}$ is hereditary because if $x \leqslant y \in B_0$ then $tx \leqslant ty \in B$ so $tx \in B$ and $x = t^{-1}(t \vee tx) \in B_0$; it is also directed because if $x, y \in B_0$ then $t(x \vee y) = tx \vee ty \in B$ so $x \vee y \in B_0$. Therefore B_0 is in Ω and it is easy to verify that $\theta_t(B_0) = B$, hence that $\theta_t : \Omega \to \Omega_t$ is surjective.

The map θ_t is also injective, with inverse given by $\theta_t^{-1}(B) = t^{-1}(t \vee B)$ because $A = t^{-1}(t \vee \theta_t(A))$.

Recall that a net $\{A_\lambda \mid \lambda \in \Lambda\}$ converges to A in Ω if and only if $\chi_{A_\lambda}(x)$ converges to $\chi_A(x)$ in $\mathbb{C}$ for each $x \in P$. To prove that θ_t is continuous, suppose $A_\lambda \to A$ and fix $x \in P$; it is enough to show that $\chi_{[e,tA_\lambda]}(x)$ converges to $\chi_{[e,tA]}(x)$. There are two cases to be considered, depending on whether x is in $[e, tA]$ or not. If $x \in [e, tA]$, there is an $a \in A$ with $x \leqslant ta$, hence $\chi_{A_\lambda}(a) = 1$ for λ in a cofinal subset Λ_1 of Λ. Thus $x \in [e, tA_\lambda]$ for λ in Λ_1, hence $\chi_{[e,tA_\lambda]}(x)$ converges to 1.

If $x \notin [e, tA]$, then $\chi_{A_\lambda}(a) = 0$ for a cofinal subset of Λ. Thus $x \notin [e, tA_\lambda]$ for this cofinal subset of Λ, hence $\chi_{[e,tA_\lambda]}(x)$ converges to 0.

Since θ_t is a continuous bijection from a compact Hausdorff space onto another, its inverse is continuous as well. ∎

Recall from [7], Section 2, that the embeddings i_{B_P} of B_P and i_P of P into the crossed product $B_P \rtimes P$ are covariant for α in the sense that $i_{B_P}(\alpha_x(f)) = i_P(x) i_{B_P}(f) i_P(x)^*$. Since i_{B_P} is injective we will identify B_P with its image $i_{B_P}(B_P)$ and simply use f in place of $i_{B_P}(f)$. The endomorphism α_x has a left inverse α_x^{-1} given by $\alpha_x^{-1}(f) \mapsto i_P(x)^* f i_P(x)$. Next we see that at the level of Ω the maps corresponding to these endomorphisms α_x are given by the partial homeomorphisms θ_x.

Proposition 2.3. *Let θ be as in Proposition* 2.2 *and suppose $f \in C(\Omega)$ and $A \in \Omega$.*

(i) *If $x \in A$ then $\alpha_x(f)(A) = f(\theta_x^{-1}(A))$.*

(ii) *If $x \notin A$ then $\alpha_x(f)(A) = 0$.*

(iii) *If $x \in P$ then $(i(x)^* f i(x))(A) = f(\theta_x(A))$.*

Proof. For fixed A both sides of (i)–(iii) are bounded linear functionals on $f \in C(\Omega) = \overline{\operatorname{span}}\{1_y \mid y \in P\}$, so it suffices to prove the claims for $f = 1_y$ for each $y \in P$.

By definition of α, $(\alpha_x(1_y))(A) = (1_{xy})(A) = \chi_A(xy)$. Since $x \notin A$ implies $xy \notin A$, (ii) follows. If $x \in A$ then $xy \in A$ if and only if $y \in x^{-1}(x \vee A) = \theta_x^{-1}(A)$ so $\chi_A(xy) = \chi_{\theta_x^{-1}(A)}(y)$, which proves (i).

If x and y do not have a common upper bound, then $i(x)^* 1_y i(x) = 0$ so the left hand side in (iii) vanishes. Since $\theta_x(A)$ is directed and contains x, it cannot contain y, thus $\chi_{\theta_x(A)}(y) = 0$, and the right hand side vanishes as well.

If x and y do have a common upper bound, then $i(x)^* 1_y i(x) = 1_{x^{-1}(x\vee y)}$ by covariance, thus

$$\begin{aligned}(i(x)^* 1_y i(x))(A) &= (1_{x^{-1}(x\vee y)})(A) = \chi_A(x^{-1}(x \vee y)) \\ &= \chi_{xA}(x \vee y) = \chi_{[e,xA]}(y) = 1_y([e, xA]),\end{aligned}$$

finishing the proof of (iii). ∎

Remark 2.4. After having identified the maximal ideal space of B_P and Ω, it is easy to compute the partial action from [14], Theorem 6.6 explicitly in terms of partial homeomorphisms:

$$\theta_x = \begin{cases} \theta_{\sigma(x)} \theta_{\tau(x)^{-1}} & \text{if } x \in PP^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

We recall that for $x \in PP^{-1}$, the element $\sigma(x) \in P$ is the least upper bound of x, and $\tau(x) = x^{-1}\sigma(x)$ is the least upper bound of x^{-1}, so that $x = \sigma(x)\tau(x)^{-1}$ is the "most efficient" way to write x as st^{-1} with $s, t \in P$.

REMARK 2.5. At first sight it looks a bit un-natural that the action of $t \in P$ on $A \in \Omega$ does not give simply tA but the hereditary subset generated by tA; also, the result of the action of t^{-1} on $B \in \Omega$ has the somewhat surprising formula of $t^{-1}(t \vee B)$. The reason for this lack of naturality is our insistence on representing the points in the spectrum of B_P as subsets of P, which is the form originally given by Nica ([12]).

The formulas can be made more transparent if one is willing to modify the presentation of the spectrum slightly and consider, for every $A \in \Omega$ its *tail-set*:

$$\mathrm{Tail}(A) := \{x \in G \mid x \text{ has an upper bound in } A\},$$

which is a subset of G from which A can be recovered easily as $A = \mathrm{Tail}(A) \cap \Omega$. Then not only the action of t and t^{-1} (for $t \in P$), but in general the action θ_x for $x \in PP^{-1}$ is described by the simple formula

$$\mathrm{Tail}(\theta_x(A)) = x\,\mathrm{Tail}(A),$$

defined on $\{A \in \Omega \mid x^{-1} \in \mathrm{Tail}(A)\}$ and having image equal to $\{B \in \Omega \mid x \in \mathrm{Tail}(B)\}$.

These natural formulas for the partial homeomorphisms have been used to extend the action $(\theta_x)_{x \in PP^{-1}}$ to the inverse semigroup of transformations of P generated by left translations, cf. [13].

3. INVARIANT SUBSETS AND THE BOUNDARY OF P

Next we review a few basic facts about θ-invariant subsets of Ω and introduce the boundary ∂P of P, which will give rise to the maximal induced ideal.

DEFINITION 3.1. ([12], Section 6) A subset K of Ω is *θ-invariant* or simply *invariant* if $\theta_x(K) \subset K$ and $\theta_x(\Omega \setminus K) \subset \Omega \setminus K$ for every $x \in P$, equivalently, if K contains both $\theta_x(K)$ and $\theta_x^{-1}(K \cap \Omega_x)$ for each $x \in P$.

An ideal $\mathcal{I}$ of B_P is called *invariant* if both $\alpha_x(\mathcal{I})$ and $i(x)^*\mathcal{I}i(x)$ are contained in $\mathcal{I}$ for each $x \in P$.

PROPOSITION 3.2. *A closed subset K of Ω is invariant if and only if the associated ideal $\mathcal{I}_K = \{f \in B_P \mid f(A) = 0\ (\forall) A \in K\}$ is invariant.*

Proof. Direct application of Proposition 2.3. ∎

DEFINITION 3.3. An element A of Ω is *maximal* if $A \subset B \in \Omega$ implies $A = B$. We will denote by Ω_∞ the set of maximal elements of Ω (i.e. the collection of all maximal hereditary directed subsets of P). The closure of Ω_∞ in Ω will be the *boundary* ∂P of P.

We first show that maximal elements always exist.

LEMMA 3.4. *If $A \in \Omega$ there exists $B \in \Omega_\infty$ such that $A \subset B$.*

Proof. The collection $\Omega_A = \{B \in \Omega \mid A \subset B\}$ is partially ordered by inclusion. Since the union of a linearly ordered family of hereditary directed subsets of P is hereditary and directed, Zorn's lemma applies and gives the existence of at least one maximal element in Ω_A, i.e. there exists $B \in \Omega_\infty$ with $A \subset B$. ∎

PROPOSITION 3.5. (i) *If the semigroup P is directed, in the sense that any two elements have a common upper bound, then $\Omega_\infty = \{P\}$ and conversely, if Ω_∞ consists of a single point then P is directed.*

(ii) *If the semigroup P is not directed then for each $x \in P$ there is a continuum of elements $A \in \Omega_\infty$ containing x.*

Proof. The semigroup P is always hereditary, if it is also directed, then clearly $\Omega_\infty = \{P\}$. If there is only one element in Ω_∞, it must be P because of Lemma 3.4. This proves (i).

Suppose P is not directed, i.e. there exist a and b in P such that $a \vee b = \infty$. Let $x \in P$ and consider the product π_s associated to a sequence s of a's and b's of length n. If $\pi_s = \pi_{s'}$ then the first factors, of s and of s', have a common upper bound and hence coincide; by induction, it follows that $s = s'$. Thus different sequences give different products.

The previous lemma, when applied to $A = [e, x\pi_s]$, gives 2^n different points in Ω_∞. They are different because if $s \neq s'$ then $\pi_s \vee \pi_{s'} = \infty$ and hence also $x\pi_s \vee x\pi_{s'} = \infty$, so $x\pi_s$ and $x\pi_{s'}$ cannot both be in the same directed subset of P.

For each infinite sequence λ of a's and b's consider the net $\{x\pi_{\lambda_1\cdots\lambda_n}\}$ determined by the finite initial subsequences. By compactness, this net has an accumulation point, and different sequences λ give different points because of the preceding argument. Although these accumulation points need not be in Ω_∞, Lemma 3.4 gives a subset of Ω_∞ with cardinality c. ∎

LEMMA 3.6. *The set Ω_∞ is θ-invariant.*

Proof. We must show that θ_t leaves Ω_∞ and $\Omega\backslash\Omega_\infty$ invariant for every $t \in P$. Let $A \in \Omega_\infty$; if $\theta_t(A) \subset B \in \Omega$ then $t \in B$ and the set $\theta_t^{-1}(B) = \{x \in P \mid tx \in B\}$ is in Ω and contains A as a subset. By maximality $A = \theta_t^{-1}(B)$. Thus $\theta_t(A) = B$ and $\theta_t(A)$ is maximal, proving that θ_t leaves Ω_∞ invariant.

Assume now $A \notin \Omega_\infty$, then A is properly contained in some $B \in \Omega$. But then $\theta_t(A)$ is properly contained in $\theta_t(B)$, hence $\theta_t(A) \notin \Omega_\infty$. ∎

The following theorem gives a key characteristic property of the boundary. The maximality of the elements of Ω_∞ translates into a minimality property of its closure under the action of P.

THEOREM 3.7. *The boundary ∂P is the smallest nonempty closed invariant subset of Ω and the collection of functions vanishing on ∂P is the largest proper invariant ideal of B_P.*

Proof. Suppose $x \in P$. Since θ_x is continuous and leaves Ω_∞ invariant, it leaves $\partial P := \overline{\Omega}_\infty$ invariant. If $A \notin \overline{\Omega}_\infty$, there is an open set V around A disjoint from Ω_∞. Since θ_x is open and leaves $\Omega \setminus \Omega_\infty$ invariant, the set $\theta_x(V)$ is an open neighborhood of $\theta_x(A)$ which is disjoint from Ω_∞ by the previous lemma. Thus ∂P is a nonempty closed invariant set.

If a closed invariant subset of Ω contains an element B, then it also contains $\lim_{t\in A} \theta_t(B)$ whenever it exists. Thus, to prove that ∂P is contained in every closed invariant subset, it suffices to prove that if $A \in \Omega_\infty$ then the net $\{\theta_t(B)\}_{t\in A}$ converges to A for every $B \in \Omega$.

Let A_1 be a cofinal subset of A and suppose the subnet $\{\theta_x(B) \mid x \in A_1\}$ converges to $C \in \Omega$. If $a \in A$ then $a \in \theta_x(B)$ for every $x \in A_1$ with $x \geqslant a$. Since $1_{\theta_x(B)}(a) \to 1_C(a)$, we conclude that $a \in C$. So $A \subset C$ and by maximality $A = C$. Thus $\theta_x(B)$ converges to A because Ω is compact and A is the only possible limit point for convergent subnets.

By Proposition 2.3 invariant ideals correspond to invariant closed sets so the second claim follows from the first. ∎

REMARK 3.8. It is interesting to compare our ∂P with the collection of *unbounded* elements of Ω. We have seen that ∂P is the singleton $\{P\}$ for every directed semigroup, but for instance for the directed semigroup $\mathbb{N}^2 \subset \mathbb{Z}^2$ the unbounded elements in $\Omega(\mathbb{N}^2)$ contain two copies of $\mathbb{N}$ besides the boundary point $\mathbb{N}^2$ itself. (These copies of $\mathbb{N}$ arise from taking limits along vertical and horizontal lines in $B_{\mathbb{N}^2}$.)

REMARK 3.9. In the case of a free group with finitely many generators, the boundary corresponds to the space of infinite words in the generators, endowed with the product topology. The free group with infinitely many generators is more interesting: in this case the space of infinite words is Ω_∞, but the boundary is all of Ω. To see this simply observe that a typical neighborhood of $[e, e] \in \Omega$ involves only finitely many restrictions, voiding a finite collection of first letters, so it must contain infinite words which start with other letters. This apparently simple fact underlines the advantage of Nica's approach to define the topology on Ω using the partial order structure on P rather than a product space construction. This allows for a unified treatment of the diagonal subalgebras $B_{\mathbb{F}_n^+}$ of $\mathcal{TO}_n$ for n finite or infinite, while infinite product space techniques fail to deal with infinite n.

Remark 3.10. The compactification Ω of P coincides with the unit space of the Wiener-Hopf groupoid of Muhly and Renault ([9], Section 3). Indeed, the Toeplitz C^*-algebras of quasi-lattice ordered groups can be obtained as groupoid C^*-algebras, cf. [13], Section 3. Moreover, the notion of invariance used here corresponds to invariance in the groupoid sense, so our discussion about invariant ideals of the Toeplitz algebras could also be placed in the framework of Renault's theory — see e.g. [15], Propositions 4.5 and 4.6. The context of Muhly and Renault is more general than the present one in the sense that they deal with locally compact semigroups, not just discrete ones. However, we point out that, in general, quasi-lattice ordered semigroups do not satisfy the normality assumption ([9], 3.1 (iv)). In fact, normal semigroups can be intrinsically characterized as the cancellative semigroups P for which $tP = Pt$ for every $t \in P$ ([6], Remark 1.2 (i)), which, in turn, implies that P is directed, so normal quasi-lattice ordered semigroups have trivial boundary by Proposition 3.5.

4. THE LARGEST INDUCED IDEAL

Denote by $\Phi : C^*(G,P) \to C(\Omega)$ the positive conditional expectation extending the map

$$i(x)i(y)^* \mapsto \delta_{x,y} i(x)i(y)^*$$

by linearity and continuity ([12], [7]). This conditional expectation can be used to lift (or induce) invariant ideals from the fixed point algebra to ideals in $C^*(G,P)$ as in [12]. Specifically, if $\mathcal{I}$ is an invariant ideal in B_P, the set

$$J = \{X \in C^*(G,P) \mid \Phi(XX^*) \in \mathcal{I}\}$$

is an ideal in $B_P \rtimes P$, which is said to be *induced* from $\mathcal{I}$. It is also possible to induce from ideals of B_P which are not invariant, but this does not give any new induced ideals, see [12], Equation (22). There is a correspondence between induced ideals of $C^*(G,P)$, invariant ideals of B_P and closed invariant subsets of Ω.

LEMMA 4.1. *If $X \in C^*(G,P)$ and $x, y \in P$ then*

$$\Phi(i(x)i(y)^*Xi(y)i(x)^*) = i(x)i(y)^*\Phi(X)i(y)i(x)^*.$$

Proof. Fix x and y; since both sides are bounded linear maps on X, it suffices to prove it for the elements spanning a dense set. With $X = i(s)i(t)^*$ both sides are zero if $s \neq t$, while if $s = t$ both sides are $i(x)i(y)^*i(s)i(t)^*i(y)i(x)^*$. ∎

PROPOSITION 4.2. *If $\Phi(J) \subset J$ for an ideal J of $C^*(G,P)$, then $\Phi(J)$ is an invariant ideal of $B_P = C(\Omega)$.*

Proof. $\Phi(J)$ is an ideal because $B_P\Phi(J) = \Phi(B_P\Phi(J)) \subset \Phi(B_PJ) \subset \Phi(J)$, and it is invariant because if $X \in J$ then $i(x)i(y)^*Xi(y)i(x)^* \in J$, so by Lemma 4.1 $i(x)i(y)^*\Phi(X)i(y)i(x)^* \in \Phi(J)$. ∎

If (G,P) has the approximation property for positive definite functions ([12], Definition 4.5.2), then $\mathcal{J}$ coincides with the ideal induced from $\Phi(\mathcal{J})$ ([12], Corollary 6.1). But even if this is not the case, there is a largest element in the class of proper ideals closed under the conditional expectation, as shown in the following proposition.

PROPOSITION 4.3. *If K is an invariant closed subset of Ω, then*

$$\mathcal{J}_K = \{X \in C^*(G,P) \mid (\Phi(XX^*))(A) = 0 \ (\forall) A \in K\}$$

is an ideal in $C^*(G,P)$, which is proper if and only if $K \neq \emptyset$.

If J is a proper ideal in $C^*(G,P)$ with $\Phi(J) \subset J$, then $J \subset \mathcal{J}_{\partial P}$.

Proof. It suffices to show that both $Xi(x)i(y)^*$ and $i(x)i(y)^*X$ are in $\mathcal{J}_K$ whenever $X \in \mathcal{J}_K$ because $C^*(G,P) = \overline{\operatorname{span}}\{i(x)i(y)^* \mid x, y \in P\}$. Since ideals are hereditary, the first claim follows from

$$\Phi(Xi(x)i(y)^*i(y)i(x)^*X^*) = \Phi(Xi(x)i(x)^*X^*) \leqslant \Phi(XX^*). \tag{4.1}$$

For the second claim recall that since K is invariant, the associated ideal of $C(\Omega)$,

$$\mathcal{I}_K = \{f \in C(\Omega) \mid f(A) = 0 \ (\forall) A \in K\},$$

is invariant so $i(x)i(y)^*\Phi(XX^*)i(y)i(x)^*$ vanishes on K. Because of Lemma 4.1

$$\Phi(i(x)i(y)^*XX^*i(y)i(x)^*) = i(x)i(y)^*\Phi(XX^*)i(y)i(x)^*, \tag{4.2}$$

thus $i(x)i(y)^*X \in \mathcal{J}_K$ as well, proving that $\mathcal{J}_K$ is a two sided ideal which is closed because Φ and $\widehat{A}$ are continuous.

If $X \in \mathcal{J}_K^+$ then $X^{1/2} \in \mathcal{J}_K$, hence $\Phi(X) = \Phi(X^{1/2}X^{1/2})$ vanishes on K. Thus $\Phi(\Phi(X)\Phi(X)) = \Phi(X)^2$ also vanishes on K proving that $\Phi(X)$ is in $\mathcal{J}_K$. Since every element in $\mathcal{J}_K$ is a linear combination of four positive elements in $\mathcal{J}_K$ and since Φ is linear, we have $\Phi(\mathcal{J}_K) \subset \mathcal{J}_K$.

Suppose now J is an ideal of $C^*(G,P)$ with $\Phi(J) \subset J$. Then $\Phi(J)$ is an invariant ideal in $C(\Omega)$, and there is a corresponding invariant closed set $K \subset \Omega$, such that $\Phi(J) = \{f \in B_P \mid f(K) = (0)\}$. If J is proper, so is $\Phi(J)$ and by Theorem 3.7 for all $X \in J$ the function $\Phi(XX^*)$ vanishes on Ω_∞ which causes $J \subset \mathcal{J}_{\partial P}$. ∎

REMARK 4.4. If ideals are closed under the conditional expectation and the quasi-lattice order has the approximation property of Nica ([12], Section 6), then every ideal of the Toeplitz C^*-algebra is induced. In such case, denoting also by α the action of P by endomorphisms of $C(\partial P)$, it follows from Proposition 4.3 that $C(\partial P)\rtimes_\alpha P$ is simple.

5. SIMPLICITY AND PURE INFINITENESS

When the conditional expectation Φ is faithful as a positive map, i.e. when $\mathcal{J}_\Omega = (0)$, the quasi-lattice ordered group (G,P) is said to be amenable and it follows from [7], Theorem 3.7 that every ideal in $C^*(G,P)$ contains a projection of the form $\prod\limits_{i=1}^{n}(1_a - 1_{x_i})$ for some $a \in P$ and $x_i \geqslant a$. The following lemma specifies conditions under which such a projection is large enough to force the identity into the ideal.

LEMMA 5.1. *Let F be a finite subset of P and assume $\prod\limits_{x\in F}(1_a - 1_x)$ does not vanish at a point A of Ω_∞. Then $\prod\limits_{x\in F}(1_a - 1_x)$ dominates a projection of the form 1_y for some $y \geqslant a$.*

Proof. Suppose for every $x \in a \vee A$ there exists $z_x \in F$ such that $x \vee z_x \in P$. Since F is finite, the net $x \vee z_x$ has a subnet with $z_x = z$ constant. Passing to a further subnet if necessary, we can assume $\lim\limits_{\Lambda}[e, \lambda \vee z_\lambda]$ exists. Since Λ is cofinal in A, the limit contains A, and since A is maximal they must coincide. Thus $x \vee z \in A$ and since A is hereditary, $z \in A$ which constitutes a contradiction. Hence there must exist some $y \in a \vee A$ for which $y \vee z = \infty$ for all $z \in F$, which implies $1_y \leqslant \prod\limits_{F}(1_a - 1_x)$. ∎

LEMMA 5.2. *Suppose (G, P) is a quasi-lattice ordered group. The following are equivalent:*

(i) *The subset of maximal elements Ω_∞ is dense in Ω (i.e. $\partial P = \Omega$).*

(ii) *For every finite subset F of $P \setminus \{e\}$ there exists $z \in P$ such that $z \vee x = \infty$ for all $x \in F$.*

(iii) *For every finite subset F of $P \setminus \{e\}$ and every $a \in P$ the projection $\prod_{x \in F} (1_a - 1_{ax})$ dominates 1_y for some $y \geqslant a$.*

Proof. Assume (ii) holds and let $a \in P$ and F be finite subset $P \setminus \{e\}$. If $y \geqslant az$ then $y \geqslant a$ and $y \not\geqslant ax$ for all $x \in F$, so $1_{az} \leqslant \prod_{x \in F} (1_a - 1_{ax})$, giving (iii). Conversely, if F is a finite subset of P, it suffices to apply (iii) with $a = e$ to obtain (ii).

To prove that (ii) implies (i) it suffices to show that the basic clopen set $V(a, aF)$ intersects Ω_∞. By Proposition 3.5 there exists at least one $A \in \Omega_\infty$ containing $[e, az]$. Since $ax \vee az = \infty$ for all $x \in F$ and A is directed, $ax \notin A$ so A is in $V(a, aF)$.

Conversely, if Ω_∞ is dense, the projection $\prod_{x \in F} (1 - 1_x)$ does not vanish at a boundary point and Lemma 5.1 gives $z \in P$ such that $z \vee x = \infty$ for all $x \in F$. Thus (i) implies (ii). ∎

REMARK 5.3. Condition (ii) above is closely related but slightly stronger than assuming that no finite set contains a lower bound for every element in $P \setminus \{e\}$ or, equivalently, that the compacts are not contained in the Wiener-Hopf algebra ([12], Proposition 6.3).

If there is a finite set of lower bounds then condition (ii) obviously fails to hold, but the absence of such a finite set of lower bounds does not imply (ii), as shown by the example of a totally ordered dense subgroup (Γ, Γ^+) of the reals with the usual order. The existence of a finite set of lower bounds means that the semigroup P is contained in finitely many branches (originating from those lower bounds), while the existence of a set F such that for every $y \in P$ there is $x \in F$ with $x \vee y \in P$ corresponds to a finite collection of branches eventually intersecting any given branch.

THEOREM 5.4. *Suppose (G, P) is a quasi-lattice ordered group. The following are equivalent:*

(i) *(G, P) is amenable and $\partial P = \Omega$.*

(ii) *$C^*(G, P)$ is simple.*

(iii) *For every nonzero $A \in C^*(G, P)$ there exist $B, C \in C^*(G, P)$ such that $BAC = 1$.*

Proof. For the proof of (i) $\Rightarrow$ (iii) we borrow a familiar argument of Cuntz ([2]). Let $A \in C^*(G,P)$, $A \neq 0$. It suffices to produce elements $B, C \in C^*(G,P)$ such that BAC is invertible. There is no loss of generality in assuming $\|A\| = 1$. By amenability $\|\Phi(AA^*)\| = c > 0$, and there is an element of the form $X = \sum_{x,y\in F} \lambda_{x,y} i(x)i(y)^*$, with $\|X\| = 1$ such that $\|A - X\| < c/4$. Thus $\|AA^* - XX^*\| < c/2$ and, since Φ is a contraction, $\|\Phi(XX^*)\| > c/2$. Let $\mu = \|\Phi(XX^*)\|^{-1}$, so that $\mu c/2 < 1/2$.

The projection Q corresponding to XX^* as in [7], Lemma 3.2 is of the form $\prod_{x\in F} (1_a - 1_{ax})$ for a finite subset F of P, and since Ω_∞ is assumed to be dense, part (ii) of Lemma 5.2 gives $z \in P$ such that $i(1_z)Q = i(1_z)$. Since Q is a subprojection of the projection on which $\Phi(XX^*)$ attains its norm, $QXX^*Q = Q\Phi(XX^*)Q = \|\Phi(XX^*)\|Q$, which gives

$$i(z)^* i(1_z) QXX^*Qi(z) = i(z)^* \|\Phi(XX^*)\| i(1_z) i(z) = \|\Phi(XX^*)\| I.$$

Since $\|i(z)^* i(1_z)Q\| = \|Qi(z)\| = 1$ it follows that

$$\|\mu i(z)^* i(1_z) QAA^*Qi(z) - I\| = \mu \|i(z)^* i(1_z) Q(AA^* - XX^*)Qi(z)\| < \frac{\mu c}{2} < \frac{1}{2},$$

which implies that $\mu i(z)^* i(1_z) QAA^*Qi(z)$ is invertible.

(ii) $\Rightarrow$ (i). If $C^*(G,P)$ is simple, then every representation is faithful so (G,P) is amenable. Moreover, the ideal $\mathcal{J}_{\partial P}$ induced from $\partial P := \overline{\Omega}_\infty$ is trivial so every function vanishing on Ω_∞ must vanish everywhere, which is enough to conclude that Ω_∞ is dense.

(iii) $\Rightarrow$ (ii) is trivial. ∎

Remark 5.5. If $C^*(G,P)$ is simple, then it is purely infinite.

Corollary 5.6. *The Toeplitz C^*-algebra of the free product of two or more countable ordered subgroups of $\mathbb{R}$, at least one of which is dense, is purely infinite simple.*

Proof. Such a free product is amenable as a quasi-lattice ordered group by [7], Theorem 4.4 and it satisfies the conditions of Lemma 5.2 by [7], Corollary 5.3, so Theorem 5.4 above gives the result. ∎

Remark 5.7. When all the factors are isomorphic the result follows from [4], Theorem 4.3. When there are finitely many nonisomorphic factors, it has been established independently in current work of Davidson and Popescu ([3]).

Since the situation studied here fits naturally into the framework of partial actions, it seems interesting to explore the extent to which recent results on purely infinite C^*-algebras from group boundary actions ([8]), and from free products of cyclic groups ([16]) also hold for partial actions. In particular the action on ∂P ought to be a boundary action in a sense generalizing that of [8].

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