EXPONENTIAL ORDERING ON BOUNDED SELF-ADJOINT OPERATORS

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Communicated by Şerban Strâtilă

Abstract. Reducibility property is proved for bounded self-adjoint operators satisfying the exponential ordering.

Keywords: Exponential ordering, Löwner-Heinz inequality, operator inequality, perturbation of linear operators.

MSC (2000): 47A, 47B.

1. MAIN RESULTS

Let $A$, $B$ be bounded self-adjoint operators on a Hilbert space $H$. In [2], the notion of exponential ordering is introduced as the one defined by $e^A \leq e^B$. In this article, we deal with an infinitesimal version of it. Consider the condition

$$e^{\kappa A} \leq e^{\kappa B} \quad \text{for some } \kappa > 0,$$

which is equivalent to the following one by Löwner-Heinz' inequality ([3], [4]): there is a positive real $\kappa_0$ such that

$$e^{\kappa A} \leq e^{\kappa B} \quad \text{for all } 0 \leq \kappa \leq \kappa_0.$$

By the last expression, we see that the condition in fact defines an order relation in the set of bounded selfadjoint operators, which is weaker than the exponential ordering in [2] and will be referred to as infinitesimal exponential ordering in what follows.
By power series expansion in the exponential functions, the last condition is further equivalent to
\[ B - A + \frac{\kappa}{2}(B^2 - A^2) + \frac{\kappa^2}{3!}(B^3 - A^3) + \cdots \geq 0 \]
for sufficiently small \( \kappa > 0 \), which particularly implies the operator inequality \( A \leq B \): the infinitesimal exponential ordering is finer than the ordinary ordering.

If \( B - A \) is invertible, the converse implication is apparently true as remarked in [1].

We here deal with the case when the kernel of \( B - A \) is non-trivial and prove the following:

**Theorem.** Let \( A \) and \( B \) be bounded self-adjoint operators on a Hilbert space. Then the operator inequality \( e^{\kappa A} \leq e^{\kappa B} \) for some \( \kappa > 0 \) forces simultaneous decomposability of operators \( A, B \) with respect to the orthogonal projection \( P \) to the kernel of \( B - A \), i.e., \( AP = PA \) and \( PB = BP \).

**Corollary.** If the range of \( B - A \) is closed, then the operator inequality \( e^{\kappa A} \leq e^{\kappa B} \) for some \( \kappa > 0 \) is equivalent to require \( A \leq B, AP = PA \) and \( BP = PB \).

**Proof.** The condition is necessary by the theorem. Conversely, if the range of \( B - A \) is closed and the condition is satisfied, then the reduced operators \( A' = (1 - P)A = A(1 - P) \) and \( B' = (1 - P)B = B(1 - P) \) on the closed subspace \( (1 - P)\mathcal{H} = (\ker(B - A))^\perp \) satisfy the inequality \( A' \leq B' \) and \( B' - A' \) is invertible because \( B' - A' \) is injective with the closed range. Then, as remarked above, we know the inequality \( e^{\kappa A'} \leq e^{\kappa B'} \) for some (small) \( \kappa > 0 \). Now the assertion follows if we notice \( e^{\kappa B} - e^{\kappa A} = (e^{\kappa B'} - e^{\kappa A'}) \oplus 0 \).

When the Hilbert space \( \mathcal{H} \) is finite-dimensional, the closedness of the range is automatic and we obtain the following characterization of the infinitesimal exponential ordering. Let \( A \) and \( B \) be hermitian \( n \times n \) matrices. The condition
\[ e^{\kappa A} \leq e^{\kappa B} \]
for some \( \kappa > 0 \)
is then equivalent to require \( PA = AP, PB = BP \) and \( A \leq B \).

Since a generic operator inequality \( A \leq B \) (under the assumption that \( \ker(B - A) \neq 0 \)) does not satisfy the reducing property \( PA = AP, PB = BP \), we have plenty of examples of operator inequality \( A \leq B \) without satisfying the infinitesimal exponential order relation.
2. PROOF OF THEOREM

We use the notation in the previous section and set \( Q = 1 - P \), the range projection of \( B - A \).

Then, for sufficiently small \( t > 0 \), we have the positivity of the operator 
\[
(2P + tQ)(e^{tB} - e^{tA})(2P + tQ)/t
\]
and by Taylor expansion of exponential function, the operator inequality
\[
0 \leq (2P + tQ)(B - A)(2P + tQ) + \frac{t}{2}(2P + tQ)(B^2 - A^2)(2P + tQ)
\]
\[
+ \frac{t^2}{6}(2P + tQ)(B^3 - A^3)(2P + tQ)
\]
\[
+ \sum_{n \geq 4} \frac{t^{n-1}Q}{n!}(2P + tQ)(B^n - A^n)(2P + tQ).
\]

Since \((B - A)P = 0 = P(B - A)\), we have
\[
(B^2 - A^2)P = (B - A)AP = (B - A)QAP,
\]
\[
P(B^2 - A^2) = PA(B - A) = PAQ(B - A),
\]
\[
P(B^2 - A^2)P = 0,
\]
\[
P(B^3 - A^3)P = PA(B - A)AP = PAQ(B - A)QAP,
\]
which is used in the above inequality to get
\[
0 \leq t^2 \left( Q(B - A)Q + Q(B - A)QAP + PAQ(B - A)Q + \frac{2}{3}PAQ(B - A)QAP \right)
\]
\[
+ \frac{t^3}{2}Q(B^2 - A^2)Q + \frac{t^3}{3}(Q(B^3 - A^3)P + P(B^3 - A^3)Q)
\]
\[
+ \frac{t^4}{6}Q(B^3 - A^3)Q + \sum_{n \geq 4} \frac{t^{n-1}}{n!}(2P + tQ)(B^n - A^n)(2P + tQ).
\]

Dividing by \( t^2 \) and taking the limit \( t \to +0 \), we obtain the inequality
\[
0 \leq Q(B - A)Q + Q(B - A)QAP + PAQ(B - A)Q + \frac{2}{3}PAQ(B - A)QAP
\]
\[
= (Q + PAQ)(B - A)(Q + QAP) - \frac{1}{3}PA(B - A)AP.
\]

Taking into account the identities
\[
(Q + QAP)(P - QAP) = 0, \quad AP(P - QAP) = AP,
\]
it follows by simultaneous right multiplication by $P - QAP$ and left multiplication by $(P - QAP)^*$ that
\[-\frac{1}{3}PA(B - A)AP \geq 0.\]
Since $B - A \geq 0$, the above inequality forces the equality $PA(B - A)AP = 0$ and hence
\[(B - A)AP = (B - A)^{1/2}(B - A)^{1/2}AP = 0.\]
Thus the range of $AP$ is contained in the subspace $P\mathcal{H}$ and we have
\[AP = PAP,\]
which yields the commutativity $AP = PA$. Since $BP = AP$ and $PB = PA$, the reducibility of $B$ also follows.

3. EXAMPLES

For a pair of bounded self-adjoint operators $(A, B)$ satisfying $A \leq B$, we set
\[\kappa(A, B) = \sup\{\kappa \geq 0 : e^{\kappa A} \leq e^{\kappa B}\},\]
which has the following obvious properties:

\[
\begin{cases} 
\kappa(A + c1, B + c1) = \kappa(A, B) & \text{if } c \text{ is a real number;} \\
\kappa(cA, cB) = \frac{1}{c}\kappa(A, B) & \text{if } c \text{ is a positive real;} \\
\kappa(UAU^*, UBU^*) = \kappa(A, B) & \text{if } U \text{ is a unitary operator.}
\end{cases}
\]

When $A$ and $B$ are $2 \times 2$ hermitian matrices, after the composition of these three operations, the pair $(A, B)$ takes the form
\[A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \lambda \left( \begin{array}{cc} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{array} \right) + \mu \left( \begin{array}{cc} \sin^2 \theta & -\cos \theta \sin \theta \\ -\cos \theta \sin \theta & \cos^2 \theta \end{array} \right),\]
with $\lambda, \mu$ reals except for the trivial case that $A$ is a scalar matrix. (Use the angle representation of two projections.)

The condition of majorization $A \leq B$ is then equivalent to
\[0 \leq \lambda \sin^2 \theta + \mu \cos^2 \theta \leq \lambda \mu,\]
which particularly implies $\lambda \geq 0, \mu \geq 0$.

Now the following is easy to check:
Proposition. Assume that \( \cos \theta \sin \theta \neq 0 \). Then, for \( \lambda \geq 0, \mu \geq 0 \), we have

\[
\begin{align*}
\kappa(A, B) = +\infty & \quad \text{if and only if } \lambda \geq 1 \text{ and } \mu \geq 1; \\
0 < \kappa(A, B) < +\infty & \quad \text{if and only if } (\lambda - 1)(\mu - 1) < 0, \lambda \sin^2 \theta + \mu \cos^2 \theta < \lambda \mu; \\
\kappa(A, B) = 0 & \quad \text{if and only if } (\lambda - 1)(\mu - 1) < 0, \lambda \sin^2 \theta + \mu \cos^2 \theta = \lambda \mu.
\end{align*}
\]

For example, choose \( \sin \theta = \cos \theta = 1/\sqrt{2} \) and

\[
\lambda_n^{-1} = \frac{1}{2} - \frac{1}{n}, \quad \mu_n^{-1} = \frac{3}{2} - \frac{1}{n}
\]

for \( n \geq 3 \). Then

\[
B_n = \frac{2n}{(n-2)(3n-2)} \begin{pmatrix} 2n-2 & n \\ n & 2n-2 \end{pmatrix}
\]

majorates \( A \) with the limit

\[
B = \lim_{n \to \infty} B_n = \frac{2}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}
\]

and these satisfy \( \kappa(A, B_n) > 0, \kappa(A, B) = 0. \)
Now we are ready to construct an example of bounded self-adjoint operators $A' \leq B'$ with no infinitesimal exponential order relation and having the trivial kernel for the difference $B' - A'$. Let $A' \leq B'$ be defined on the Hilbert space $\bigoplus_{n \geq 3} \mathbb{C}^2$ by

$$A' = \bigoplus_{n \geq 3} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B' = \bigoplus_{n \geq 3} B_n.$$

Then clearly $\ker(B' - A') = \{0\}$. If $\kappa = \kappa(A', B') = \inf \{\kappa(A, B_n) : n \geq 3\}$ is strictly positive, $e^{\kappa A} \leq e^{\kappa B_n}$ for any $n \geq 3$ and therefore, by taking the limit $n \to \infty$, $e^{\kappa A} \leq e^{\kappa B}$, which is impossible because $\kappa(A, B) = 0$.

Acknowledgements. The authors are grateful to the referee for the significant simplification in the proof of the main theorem as presented in the text. We record here that the original proof was fairly involved and was given by analysing the limit behaviour of analytic perturbations of bounded self-adjoint operators.

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Received August 28, 1997; revised May 30, 1998.