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HYPERCYCLICITY OF THE OPERATOR ALGEBRA FOR A SEPARABLE HILBERT SPACE

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Abstract. If X is a topological vector space and $T : X \to X$ is a continuous linear mapping, then T is said to be *hypercyclic* when there is a vector f in X such that the set $\{T^n f : n \geqslant 0\}$ is dense in X. When X is a separable Fréchet space, Gethner and Shapiro obtained a sufficient condition for the mapping T to be hypercyclic. In the present paper, we obtain an analogous sufficient condition when X is a particular nonmetrizable space, namely the operator algebra for a separable infinite dimensional Hilbert space H, endowed with the strong operator topology. Using our result, we further provide a sufficient condition for a mapping T on H to have a closed infinite dimensional subspace of hypercyclic vectors. This condition was first found by Montes-Rodríguez for a general Banach space, but the approach that we take is entirely different and simpler.

Keywords: *Operator algebras, separable Hilbert spaces, strong operator topology, hypercyclic vectors.*

MSC (2000): Primary 47D25; Secondary 47B48.

1. INTRODUCTION

In 1929 Birkhoff ([1]) proved a remarkable theorem that there exists an entire function $f(z)$ whose successive translates $f(z), f(z+1), f(z+2), \ldots$ are dense in the space of all entire functions, endowed with the topology of uniform convergence on compact sets. Analogous to Birkhoff's theorem, G.R. MacLane ([11]) proved in 1952 that there exists an entire function f whose successive derivatives $f, f', f'', \ldots$ are dense in the space of all entire functions. The results of Birkhoff and MacLane can be rephrased in operator theory terms. To explain that, we let

X be a topological vector space and $T : X \to X$ be a continuous linear mapping. Corresponding to T, the *orbit* of a vector x in X is the set

$$\mathrm{orb}(T, x) = \{x, Tx, T^2x, T^3x, \ldots\}.$$

DEFINITION 1.1. A vector x is said to be a *hypercyclic vector* of T if its orbit $\mathrm{orb}(T, x)$ is dense in X. The mapping T is said to be *hypercyclic* if T has a hypercyclic vector.

When X is the Fréchet space of all entire functions, Birkhoff's Theorem states that the translation operator $T : f(z) \mapsto f(z + 1)$ is hypercyclic, and MacLane's Theorem states that the differentiation operator $T : f \mapsto f'$ is hypercyclic. When X is a separable Fréchet space, Gethner and Shapiro ([5]) showed in 1987 a sufficient condition for a continuous linear mapping $T : X \to X$ to be hypercyclic. Their condition is analogous to the condition obtained in 1982 by Kitai ([7]) for the case when X is a Banach space. To state the result of Gethner and Shapiro, we use d to denote the metric of a Fréchet space X.

THEOREM 1.2. ([5]) *Suppose X is a separable Fréchet space, and $T : X \to X$ is a continuous linear mapping. If there exists a dense subset E of X and a right inverse W of T such that*

$$\lim_{n\to\infty} d(0, T^n f) = 0 \quad \textit{and} \quad \lim_{n\to\infty} d(0, W^n f) = 0$$

for every $f \in E$, then T has a hypercyclic vector.

Applying Theorem 1.2 to the Fréchet space of entire functions, Gethner and Shapiro derived the theorems of Birkhoff and MacLane. Besides this space, some hypercyclicity-like phenomena have been found in other interesting spaces; for example, the Banach space of continuous functions on a compact interval studied by Luh ([10]), and also the Fréchet space of all analytic functions on the open unit disk studied by Seidel and Walsh ([13]). Hypercyclicity can also occur in a separable infinite dimensional complex Hilbert space H, as it is well-known ([6], Problem 168) that if we let $B : H \to H$ be the unilateral backward shift, then the operator $2B$ is hypercyclic. In view of this example, we raise the following question for the operator algebra $B(H)$ which, by definition, consists of all bounded linear operators from H to H.

QUESTION 1.3. Can hypercyclicity occur in the operator algebra $B(H)$ of a separable infinite dimensional complex Hilbert space H?

The answer for the above question is of course negative if we use the operator norm topology of the algebra $B(H)$, because no countable subset of $B(H)$ can be dense in that topology. In order for the above question to make sense, we introduce the following definition.

DEFINITION 1.4. Let $L : B(H) \to B(H)$ be a bounded linear mapping. An operator T in $B(H)$ is said to be a *hypercyclic vector* of L if its orbit $\operatorname{orb}(L, T)$ is dense in $B(H)$ in the strong operator topology. Furthermore, the mapping L is said to be *hypercyclic* when it has a hypercyclic vector.

In Section 2 below, we prove the main theorem of the paper, which is a sufficient condition for the mapping $L : B(H) \to B(H)$ to be hypercyclic, analogous to the condition given by Theorem 1.2. Using the main theorem we give examples of a hypercyclic mapping L. Then we further show that for every hypercyclic mapping L there exists an invariant infinite dimensional linear subspace of $B(H)$ consisting entirely, except for the zero operator, of hypercyclic vectors of L. Furthermore, this subspace is even dense in the strong operator topology of $B(H)$. This result originates from the work of P.S. Bourdon ([2]) who showed this phenomenon for the Hilbert space H. Then, in Section 3, we discuss the relation between the main theorem and Theorem 1.2. In particular, we show that if $T \in B(H)$ satisfies the hypotheses of Theorem 1.2, then the left multiplication linear mapping $L_T : B(H) \to B(H)$ defined by $L_T(V) = TV$ is hypercyclic. This in turn implies that the operator T itself is hypercyclic. Motivated by this result, we further show that H can have a closed infinite dimensional subspace consisting entirely, except for zero, of hypercyclic vectors of T, when T satisfies an extra hypothesis. The existence of such closed subspaces was first shown by A. Montes-Rodríguez ([12]), and our work provides a different approach to this phenomenon.

2. HYPERCYCLICITY

For a separable infinite dimensional complex Hilbert space H, the operator algebra $B(H)$ naturally has many topologies, but in this paper we use only two, namely the operator norm topology and the strong operator topology. To distinguish the two, we use the convention that when a topological term is used for $B(H)$ it always refers to the operator norm topology, otherwise we add the prefix "SOT" in front of the term with reference to the strong operator topology.

With our convention, we remark that the operator algebra $B(H)$ is SOT-separable. There are many ways to see that, but we now offer one that relates to one of our later arguments. To begin, we let $\{e_k : k \geqslant 1\}$ be an orthonormal basis of H, let $P_n : H \to H$ be the orthogonal projection onto $\operatorname{span}\{e_k : 1 \leqslant k \leqslant n\}$, and let $I : H \to H$ be the identity mapping. Then, for any vector f in H and any operator T in $B(H)$,

$$\begin{aligned}\|P_nTP_nf - Tf\| &\leqslant \|P_nT(P_nf - f)\| + \|(P_n - I)(Tf)\|,\\ &\leqslant \|T\|\,\|(P_n - I)f\| + \|(P_n - I)(Tf)\|,\end{aligned}$$

which goes to zero as $n \to \infty$. In other words, $P_nTP_n \to T$ in the strong operator topology. Each operator P_nTP_n can be represented uniquely as an $n \times n$ matrix with scalar entries with respect to the basis $\{e_k\}$, and so P_nTP_n can be approximated arbitrarily close, in the operator norm, by an $n \times n$ matrix with rational entries. Thus the countable set of all finite square matrices with rational entries is SOT-dense in $B(H)$.

The SOT-separability of $B(H)$ makes one wonder whether a countable SOT-dense set of $B(H)$ can be the orbit of an operator in $B(H)$ under a linear mapping $L : B(H) \to B(H)$. For that to happen, we now prove a sufficient condition analogous to Theorem 1.2, based on some techniques used by Kitai [7], Theorem 1.4, in showing the Banach space version of Theorem 1.2.

Theorem 2.1. *A bounded linear mapping $L : B(H) \to B(H)$ is hypercyclic if there exist a linear mapping $A : B(H) \to B(H)$ and a countable* SOT*-dense subset D of $B(H)$ such that LA is the identity mapping on $B(H)$ and for each operator V in D,*

$$\lim_{n\to\infty} \|A^n(V)\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|L^n(V)\| = 0.$$

Proof. Denote the countable SOT-dense subset D by $D = \{T_k : k \geqslant 1\}$. Using the hypothesis on D, we now determine a sequence of integers $\{n_k : k \geqslant 1\}$.

Let n_1 be a positive integer such that $\|A^{n_1}(T_1)\| < 2^{-1}$, and determine a positive integer n_2 such that the following three inequalities hold:

$$\|A^{n_1+n_2}(T_2)\| < 2^{-1}2^{-2}, \quad \|A^{n_2}(T_2)\| < 2^{-2} \quad \text{and} \quad \|L^{n_2}(T_1)\| < 2^{-2}.$$

With n_1 and n_2 determined, we can find a positive integer n_3 such that

$$\|A^{n_1+n_2+n_3}(T_3)\| < 2^{-1}2^{-2}2^{-3}, \quad \|A^{n_2+n_3}(T_3)\| < 2^{-2}2^{-3}, \quad \|A^{n_3}(T_3)\| < 2^{-3},$$

and also

$$\|L^{n_2+n_3}(T_1)\| < 2^{-2}2^{-3} \quad \text{and} \quad \|L^{n_3}(T_2)\| < 2^{-3}.$$

In general, we determine a positive integer n_k such that for all integers $m = 1, 2, \ldots, k$,

$$\|A^{n_m+n_{m+1}+\cdots+n_k}(T_k)\| < 2^{-m}2^{-m-1}\cdots 2^{-k}, \tag{2.1}$$

and for all integers $l = 2, 3, \ldots, k$,

$$\|L^{n_l+n_{l+1}+\cdots+n_k}(T_{l-1})\| < 2^{-l}2^{-l-1}\cdots 2^{-k}. \tag{2.2}$$

In terms of the sequence $\{n_k\}$, we define T by

$$T = \sum_{k=1}^{\infty} A^{n_1+n_2+\cdots+n_k}(T_k),$$

in which every summand satisfies

$$\|A^{n_1+n_2+\cdots+n_k}(T_k)\| < 2^{-1}2^{-2}\cdots 2^{-k}.$$

Thus the series expression of T is absolutely convergent and so it defines an operator T in $B(H)$.

We now show that T is a hypercyclic vector of L, by first observing the following: for any integer $m \geqslant 2$,

$$\begin{aligned} L^{n_1+n_2+\cdots+n_m}(T) &= \sum_{k=1}^{\infty} L^{n_1+n_2+\cdots+n_m} A^{n_1+n_2+\cdots+n_k}(T_k), \text{ because } L \text{ is bounded} \\ &= \sum_{k=1}^{m-1} L^{n_{k+1}+\cdots+n_m}(T_k) + T_m + \sum_{k=m+1}^{\infty} A^{n_{m+1}+\cdots+n_k}(T_k), \end{aligned}$$

from which it follows that

$$\begin{aligned}
&\|L^{n_1+n_2+\cdots+n_m}(T) - T_m\| \\
&\quad\leqslant \sum_{k=1}^{m-1} \|L^{n_{k+1}+\cdots+n_m}(T_k)\| + \sum_{k=m+1}^{\infty} \|A^{n_{m+1}+\cdots+n_k}(T_k)\| \\
&\quad\leqslant \sum_{k=1}^{m-1} 2^{-k-1}\cdots 2^{-m} + \sum_{k=m+1}^{\infty} 2^{-m-1}\cdots 2^{-k}, \qquad \text{by (2.1) and (2.2)} \\
&\quad< 2^{-m}\sum_{k=0}^{\infty} 2^{-k} + 2^{-m-1}\sum_{k=0}^{\infty} 2^{-k} = 2^{-m+1} + 2^{-m},
\end{aligned}$$

and hence

$$\lim_{m\to\infty} \|L^{n_1+n_2+\cdots+n_m}(T) - T_m\| = 0. \tag{2.3}$$

We now use Equation (2.3) to show that every basic SOT-open set U contains an operator of the form $L^{n_1+n_2+\cdots+n_\alpha}(T)$, for some integer α, and hence $\mathrm{orb}(L,T)$ is SOT-dense in $B(H)$. By the definition of the strong operator topology, U can be written as

$$\begin{aligned}
U &= U(V_0, \varepsilon\,;\, f_1, f_2, \ldots, f_N) \\
&= \{V \in B(H) : \|(V - V_0)f_i\| < \varepsilon, \text{ for all } i = 1, \ldots, N\},
\end{aligned}$$

where $\varepsilon > 0$, V_0 is an operator in $B(H)$, and $f_1, f_2, \ldots, f_N$ are N vectors in H. If every f_i is the zero vector then $U = B(H)$, and thus we can assume that some f_i is nonzero in our proof. We denote $C = \max\{\|f_i\| : 1 \leqslant i \leqslant N\}$, for which Equation (2.3) gives an integer M such that for all integers $m \geqslant M$,

$$\|L^{n_1+n_2+\cdots+n_m}(T) - T_m\| < (2C)^{-1}\varepsilon.$$

Hence if $1 \leqslant i \leqslant N$ and $m \geqslant M$, then

$$\|L^{n_1+n_2+\cdots+n_m}T(f_i) - T_m(f_i)\| < (2C)^{-1}\varepsilon\|f_i\| \leqslant \frac{\varepsilon}{2}. \tag{2.4}$$

On the other hand, the set D is SOT-dense in $B(H)$ and so the set $\{T_m : m \geqslant M\}$ is also SOT-dense in $B(H)$. Thus we can choose an integer $\alpha \geqslant M$ such that $T_\alpha \in U(V_0, \varepsilon/2\,;\, f_1, f_2, \ldots, f_N)$; that is, for all $i = 1, 2, \ldots, N$,

$$\|(T_\alpha - V_0)f_i\| < \frac{\varepsilon}{2}. \tag{2.5}$$

Hence, for that particular integer α and for all f_i,

$$\begin{aligned}
&\|L^{n_1+n_2+\cdots+n_\alpha}T(f_i) - V_0(f_i)\| \\
&\qquad\leqslant \|L^{n_1+n_2+\cdots+n_\alpha}T(f_i) - T_\alpha(f_i)\| + \|T_\alpha(f_i) - V_0(f_i)\| \\
&\qquad< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

by Inequalities (2.4) and (2.5). This implies that $L^{n_1+n_2+\cdots+n_\alpha}(T) \in U$ and completes the proof. ∎

We now present two examples showing that the sufficient condition in Theorem 2.1 can be satisfied. These two examples involve the unilateral forward and backward shifts, which are probably the best studied Hilbert space operators in the literature.

EXAMPLE 2.2. Let $\{e_n : n \geqslant 1\}$ be an orthonormal basis of H and let $S \in B(H)$ be the unilateral forward shift, which is defined by $Se_n = e_{n+1}$ for all integers $n \geqslant 1$. The Hilbert space adjoint S^* is the unilateral backward shift, which satisfies $S^*e_1 = 0$ and $S^*e_n = e_{n-1}$ for all integers $n \geqslant 2$. Define a linear mapping $L : B(H) \to B(H)$ by $L(T) = 2S^*T$ for all T in $B(H)$. Since $\|L(T)\| \leqslant 2\|S^*\|\,\|T\| = 2\|S\|\,\|T\| = 2\|T\|$, the linear mapping L is bounded. In addition, we define a linear mapping $A : B(H) \to B(H)$ by $A(T) = 2^{-1}ST$, giving the identity $LA = I$. Furthermore, we let D denote the set of all operators T in $B(H)$ such that T has a positive integer N with the property that $Te_k = 0$ for all integers $k > N$, and also that each of the vectors $Te_1, Te_2, \dots, Te_N$ is a linear combination of $e_1, e_2, \dots, e_N$ with rational coefficients. In other words, D consists of all operators that are represented as finite square matrices with rational entries, with respect to the basis $\{e_n\}$. The set D is a countable SOT-dense subset of $B(H)$, as indicated in the second paragraph of this section. Moreover, if T is an operator in D that can be represented as an $n \times n$ matrix, then one can check that $L^{n+1}(T) = 0$, and that $\|A^k(T)\| \leqslant 2^{-k}\|S\|\,\|T\| = 2^{-k}\|T\|$. Hence the linear mapping L is hypercyclic, by Theorem 2.1.

EXAMPLE 2.3. With S, S^* and D given as in Example 2.2, we define bounded linear mappings $L, A : B(H) \to B(H)$ by $L(T) = 2S^*TS$ and $A(T) = 2^{-1}STS^*$, for all T in $B(H)$. It is easy to check that $LA = I$, and furthermore if T is an operator in D that can be represented as an $n \times n$ matrix with rational entries, then $S^{*(n+1)}T = 0$, and hence $L^{n+1}(T) = 0$. Since $\|A^n(T)\| \leqslant 2^{-n}\|S^*\|\,\|T\|\,\|S\| = 2^{-n}\|T\|$, the linear mapping L is hypercyclic by Theorem 2.1.

Having given two examples based on the sufficient condition for hypercyclicity provided by Theorem 2.1, we now discuss a simple necessary condition. It is clear from the definition that if $L : B(H) \to B(H)$ is hypercyclic, then L must have an SOT-dense range in $B(H)$. Though this necessary condition for hypercyclicity is not at all related to the sufficient condition given by Theorem 2.1, it can be strengthened to the condition given in the following lemma, which in turn yields a result better than Theorem 2.1.

LEMMA 2.4. *If $L : B(H) \to B(H)$ is hypercyclic then for every scalar λ, the mapping $L - \lambda$ has an* SOT-*dense range.*

Proof. Let L^* be the Banach space adjoint of L. We remark that every nonzero SOT-continuous linear functional S on $B(H)$ is necessarily norm-continuous. For that S, we claim that if there exists a scalar λ such that $L^*S = \lambda S$ then L is not hypercyclic. To prove the claim, we simply observe that for every operator T in $B(H)$ and for every positive integer n,

$$S(L^n T) = (L^{*n} S)(T) = \lambda^n S(T).$$

This implies that the orbit $\mathrm{orb}(L, T)$ is not SOT-dense in $B(H)$.

From our claim we deduce that if L is hypercyclic, then for every nonzero SOT-continuous linear functional S on $B(H)$ and for every scalar λ, there exists an operator T in $B(H)$ such that $S((L-\lambda)T) \neq 0$. Thus $(L-\lambda)B(H)$ is SOT-dense in $B(H)$, because $B(H)$ is a locally convex topological vector space in the strong operator topology. ∎

The phenomenon in Lemma 2.4 happens also in a Banach space setting. In fact, Kitai ([7], Theorem 2.3) showed that if $T : X \to X$ is a hypercyclic operator on a Banach space X and if λ is a complex number, then $T - \lambda$ has a dense range. This, in turn, implies that the adjoint operator T^* has no eigenvalue.

We now turn our attention to the density of hypercyclic vectors of a bounded linear mapping $L : B(H) \to B(H)$. We claim that if L has one hypercyclic vector T, then L must have an SOT-dense set of hypercyclic vectors. The reason is that if the set $\{T, L(T), L^2(T), L^3(T), \ldots\}$ is SOT-dense in $B(H)$, then the set $\{L^n(T), L^{n+1}(T), L^{n+2}(T), \ldots\}$ is also SOT-dense and thus every operator $L^n(T)$ is a hypercyclic vector of L. Furthermore, those vectors in the orbit actually span an SOT-dense linear subspace of hypercyclic vectors of L, as indicated in the proof of the next theorem. The existence of a dense linear subspace of hypercyclic vectors in a space was first found by P.S. Bourdon ([2]), who showed that if a bounded linear operator $T : H \to H$ has a hypercyclic vector in H, then H must have a dense linear subspace that is invariant under T and consists entirely, except for zero, of hypercyclic vectors of T. This result is surprising because in general the sum of two hypercyclic vectors may not be a hypercyclic vector. The argument of Bourdon works for our setting and we include it here for the completeness of our discussion.

PROPOSITION 2.5. *If $L : B(H) \to B(H)$ is a hypercyclic bounded linear mapping, then $B(H)$ has an* SOT*-dense linear subspace that is invariant under L and consists entirely, except for the zero operator, of hypercyclic vectors of L.*

Proof. For a hypercyclic vector T of L, the linear subspace $\{p(L)T : p$ is a polynomial$\}$ contains the orbit $\mathrm{orb}(L, T)$ and so it is SOT-dense in $B(H)$. We let $p(L)T$ be a nonzero operator in that linear subspace, and now show that $p(L)T$ is a hypercyclic vector of L.

Since $p(L)$ commutes with L, we see that $\mathrm{orb}(L, p(L)T) = p(L)\mathrm{orb}(L, T)$. Note that $\mathrm{orb}(L, T)$ is SOT-dense in $B(H)$, and also that $p(L)$ has an SOT-dense range, due to Lemma 2.4 and the linear factorization of the polynomial p. Thus $p(L)\mathrm{orb}(L, T)$ is SOT-dense in $B(H)$ and so is $\mathrm{orb}(L, p(L)T)$. This completes our proof. ∎

To conclude this section, we remark that the set of hypercyclic vectors of L may not be dense in $B(H)$ in the operator norm topology. To illustrate this point, we take the linear mapping L in Example 2.2 defined by $L(T) = 2S^*T$, and use G_r to denote the set of all right invertible operators in $B(H)$. We claim that every operator T in G_r is not a hypercyclic vector of L. To show that, let W be an operator in $B(H)$ such that $TW = I$, the identity operator on H. Since $TWe_1 = e_1$, we see that $\|We_1\| > 0$ and so

$$U = \left\{ V \in B(H) : \|(V - I)(We_1)\| < \frac{\|We_1\|}{2} \right\}$$

is a basic SOT-open set containing I. Note that $L^n(T) \notin U$ for every integer $n \geqslant 1$, because $\|(L^n(T) - I)(We_1)\| = \|2^n S^{*n} TWe_1 - We_1\| = \|2^n S^{*n} e_1 - We_1\| = \|We_1\|$. Hence the set $\{L^n(T) : n \geqslant 1\}$ is not SOT-dense in $B(H)$ and so the orbit $\{L^n(T) : n \geqslant 0\}$ of T under L cannot be SOT-dense either. Since G_r is open in the operator norm topology, the hypercyclic vectors of L cannot be dense in $B(H)$ in that topology.

3. LEFT MULTIPLICATION

In this section, we continue to discuss hypercyclic mappings on the algebra $B(H)$ of a separable infinite dimensional complex Hilbert space H, in particular the relation between Theorems 1.2 and 2.1. To begin our discussion, we use $\{e_k : k \geqslant 1\}$ to denote an orthonormal basis of H. In addition, for any subset E of H we use $D(E)$ to denote the set of all operators T in $B(H)$ satisfying the following condition: there exists a positive integer n, depending on T, such that $Te_k = 0$ for all integers $k > n$ and that $Te_k \in E$ for all integers $k \leqslant n$.

LEMMA 3.1. *If E is dense in H, then $D(E)$ is* SOT*-dense in $B(H)$.*

Proof. Let $P_n : H \to H$ be the orthogonal projection onto $\operatorname{span}\{e_k : k \leqslant n\}$, and let Y be an operator in $B(H)$. Then, for any vector f in H, one can use the triangle inequality to deduce that

$$\|P_n Y P_n f - Yf\| \leqslant \|P_n Y\| \, \|P_n f - f\| + \|(P_n - I)(Yf)\|,$$

from which it follows that the SOT-limit of $P_n Y P_n$ is Y as $n \to \infty$. The operator $P_n Y P_n$ takes e_k to the zero vector if $k > n$, and takes e_k to a vector in $\operatorname{span}\{e_j : j \leqslant n\}$ if $k \leqslant n$. Using this property of $P_n Y P_n$ and the density of E in H, we deduce that $P_n Y P_n$ can be approximated arbitrarily close, in the operator norm, by an operator T in $D(E)$. Thus Y is in the SOT-closure of $D(E)$, and so $D(E)$ is SOT-dense in $B(H)$. ∎

Lemma 3.1 provides a way to connect a dense subset of H to an SOT-dense subset of $B(H)$. We now want to connect an operator on H to a bounded linear mapping on $B(H)$ through the following definition.

DEFINITION 3.2. Corresponding to any operator $T \in B(H)$, we define a left multiplication mapping $L_T : B(H) \to B(H)$ by $L_T(V) = TV$, for all V in $B(H)$.

Suppose T and W are two operators in $B(H)$ such that TW is the identity operator on H, then $L_T L_W$ is the identity mapping on $B(H)$. Furthermore, we now show that if T and W satisfy the hypotheses of Theorem 1.2 then L_T and L_W satisfy the hypotheses of Theorem 2.1.

PROPOSITION 3.3. *Suppose $T, W \in B(H)$. If there is a dense subset E of H such that for every $f \in E$,*

$$\lim_{n\to\infty} \|T^n f\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|W^n f\| = 0,$$

then there is a countable SOT*-dense subset D of $B(H)$ such that for every operator V in D,*

$$\lim_{n\to\infty} \|L_T^n(V)\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|L_W^n(V)\| = 0.$$

Proof. Since E contains a countable subset E' that is dense in H, we can let $D = D(E')$ be the countable set of operators that we discuss in Lemma 3.1. By the definition of D, every operator V in D has an integer m such that $Ve_k = 0$ whenever $k > m$. Hence, if f is a vector represented as

$$f = \sum_1^\infty a_k e_k$$

with scalars a_k satisfying $\sum |a_k|^2 < \infty$, then

$$V(f) = \sum_1^m a_k V(e_k).$$

With this equation, we deduce that

$$\begin{aligned}\|L_T^n(V)(f)\|^2 &= \Big\| \sum_{k=1}^m a_k T^n V(e_k) \Big\|^2 \leqslant \Big(\sum_{k=1}^m |a_k| \, \|T^n V(e_k)\| \Big)^2 \\ &\leqslant \Big(\sum_{k=1}^m |a_k|^2 \Big) \Big(\sum_{k=1}^m \|T^n V(e_k)\|^2 \Big), \text{ by the Cauchy-Schwarz Inequality} \\ &\leqslant \|f\|^2 \sum_{k=1}^m \|T^n V(e_k)\|^2.\end{aligned}$$

It follows that

$$\|L_T^n(V)\|^2 \leqslant \sum_{k=1}^m \|T^n V(e_k)\|^2,$$

in which every term $V(e_k)$ is in E' because V is an operator in D. Thus $\|T^n V(e_k)\| \to 0$ as $n \to \infty$, and hence

$$\lim_{n\to\infty} \|L_T^n(V)\| = 0.$$

By repeating the above argument we prove that $\|L_W^n(V)\| \to 0$. ∎

From Proposition 3.3 and Theorem 2.1, we easily deduce the following.

Corollary 3.4. *If $T : H \to H$ is a bounded linear operator satisfying the hypotheses of Theorem* 1.2 *then the corresponding left multiplication mapping $L_T : B(H) \to B(H)$ is hypercyclic.*

To illustrate the phenomenon that occurs in Corollary 3.4, one may take Example 2.2. As an application to the hypercyclicity of the Hilbert space H, we now show that we can use a hypercyclic vector of L_T to construct hypercyclic vectors of T, strengthening the hypercyclicity relation between T and L_T.

Proposition 3.5. *If L_T has a hypercyclic vector V and if f is any nonzero vector in H, then the vector Vf is a hypercyclic vector of T.*

Proof. It suffices to show that if $g \in H$ then there exists a sequence of positive integers $\{n_k\}$ such that $T^{n_k}(Vf) \to g$, as $k \to \infty$. For that vector g, we consider the bounded linear operator $S_g : H \to H$ defined by

$$S_g(h) = \frac{\langle h, f \rangle}{\|f\|^2} g.$$

Since V is a hypercyclic vector of L_T, there exists a sequence of integers $\{n_k\}$ such that $L_T^{n_k}(V) = T^{n_k}(V) \to S_g$ in the strong operator topology. Thus $T^{n_k}(Vf) \to S_g(f) = g$ in the norm topology of H. ∎

Since the zero vector of H cannot be a hypercyclic vector of any operator, Proposition 3.5 implies that every hypercyclic vector V of L_T is a one-to-one operator. Hence VH must be an infinite dimensional vector subspace. It is natural to ask whether VH can be a closed infinite dimensional subspace. When that happens, Proposition 3.5 implies that H has a closed infinite dimensional subspace of hypercyclic vectors of T, except for the zero vector. A result of this kind was first obtained by A. Montes-Rodríguez ([12]), who showed that such subspaces exist if T is an operator on a separable Banach space X satisfying the hypotheses in Theorem 1.2, and if X has a closed infinite dimensional subspace X_0 with the property that $\lim \|T^n f\| = 0$ for every vector f in X_0. Furthermore, F. León-Saavedra and A. Montes-Rodríguez ([8]) showed that the operator T can be taken to be a compact perturbation of an operator with norm no more than 1. The result of A. Montes-Rodríguez in [12] requires the existence of X_0, but this is not superfluous because this existence condition is proved recently by F. León-Saavedra and A. Montes-Rodríguez ([9]) to be essential. We now offer a simple proof for the Hilbert space version of the result of A. Montes-Rodríguez ([12]).

THEOREM 3.6. *Suppose $T : H \to H$ is a bounded linear operator satisfying the hypotheses of Theorem* 1.2. *If H has a closed infinite dimensional subspace K such that for every vector f in K*

$$\lim_{n\to\infty} \|T^n f\| = 0,$$

then H has an infinite dimensional closed subspace consisting entirely, except for zero, of hypercyclic vectors of T.

Proof. To prove the theorem, it suffices to show that the bounded linear mapping $L_T : B(H) \to B(H)$ has a hypercyclic vector that is bounded below, in view of Proposition 3.5 and the discussion immediately after its proof.

Since K is a closed infinite dimensional subspace of H, there is a Hilbert space isomorphism U from H onto K. By our hypothesis on K, we see that if h is a vector in H then $\|T^n U h\| \to 0$ as $n \to \infty$.

On the other hand, L_T has a hypercyclic vector V, by Corollary 3.4. Since any scalar multiple of V is also a hypercyclic vector of L_T, we can assume that V has norm $1/2$. Thus the operator $U + V$ is bounded below by $1/2$. Now one can easily check that the operator $U + V$ is a hypercyclic vector of L_T, because $L_T^n U \to 0$ in the strong operator topology. ∎

To conclude this paper, we remark that our results in this paper, except Theorem 3.6, can readily be generalized to any Banach space X that has a Schauder basis, though these results are stated and proved here for a Hilbert space H. This is because a *Schauder basis* of X is a sequence $\{x_k \in X : k \geqslant 1\}$ with the property that for every vector x in X, there is a unique sequence of scalars $\{\alpha_k : k \geqslant 1\}$ such that

$$\lim_{n\to\infty} \|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n - x\| = 0.$$

This property is shared by an orthonormal basis of the Hilbert space, and is the only property that we use to make all the proofs in the present paper work. To this end, the only problem that we appear to have is the use of the inner product in Proposition 3.5, but we can replace that by a continuous projection onto a one dimensional subspace.

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