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ALGEBRAS OF MULTIPLICATION OPERATORS IN BANACH FUNCTION SPACES

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Abstract. Let E be a Banach function space based on a Maharam measure μ. For each $\varphi \in L^\infty(\mu)$, the linear operator M_φ of multiplication by φ is continuous on E. Let $\mathfrak{U}$ be a subalgebra of $L^\infty(\mu)$. We make a detailed study of the relationship between $\mathfrak{M}_E(\mathfrak{U}) = \{M_\varphi : \varphi \in \mathfrak{U}\}$, the weak operator closed algebra $\overline{\mathfrak{M}_E(\mathfrak{U})^{\mathrm{w}}}$ it generates, the bicommutant algebra $\mathfrak{M}_E(\mathfrak{U})^{\mathrm{cc}}$, and the algebra $\mathfrak{M}_E(\overline{\mathfrak{U}^*})$, where $\overline{\mathfrak{U}^*}$ is the weak-$*$ closure of $\mathfrak{U}$ in $L^\infty(\mu)$. When E is fully symmetric it is shown that

$$\mathfrak{M}_E(\mathfrak{U}) \subseteq \overline{\mathfrak{M}_E(\mathfrak{U})^{\mathrm{w}}} \subseteq \mathfrak{M}_E(\mathfrak{U})^{\mathrm{cc}} \subseteq \mathfrak{M}_E(\overline{\mathfrak{U}^*}) \subseteq \mathfrak{M}_E(L^\infty(\mu)).$$

The inclusion $\mathfrak{M}_E(\mathfrak{U})^{\mathrm{cc}} \subseteq \mathfrak{M}_E(\overline{\mathfrak{U}^*})$ may fail if E is not fully symmetric.

Keywords: *Fully symmetric Banach function space, multiplication operator, bicommutant.*

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1. INTRODUCTION

Let (Ω, Σ, μ) be a Maharam measure space (i.e. μ is localizable and has the finite subset property). Let E denote anyone of the Banach spaces $L^p(\mu), 1 \leqslant p \leqslant \infty$. For a unital, norm-closed subalgebra $\mathfrak{U} \subseteq L^\infty(\mu)$ let $\mathfrak{M}_E(\mathfrak{U})$ denote the algebra of all multiplication operators in E by elements from $\mathfrak{U}$. For the moment $\mathfrak{U}$ is assumed to be conjugate closed. There are four natural commutative subalgebras of the space $\mathcal{L}(E)$ of all bounded operators on E which are of interest. There is $\mathfrak{M}_E(\mathfrak{U})$ itself, the strong (resp. weak) operator closed subalgebra $\overline{\mathfrak{M}_E(\mathfrak{U})^{\mathrm{s}}}$ (resp. $\overline{\mathfrak{M}_E(\mathfrak{U})^{\mathrm{w}}}$) of $\mathcal{L}(E)$ generated by $\mathfrak{M}_E(\mathfrak{U})$, the bicommutant $\mathfrak{M}_E(\mathfrak{U})^{\mathrm{cc}}$ of $\mathfrak{M}_E(\mathfrak{U})$, and the algebra $\mathfrak{M}_E(\overline{\mathfrak{U}^*}) \subseteq \mathcal{L}(E)$ of all multiplication operators by elements from the weak-star closure $\overline{\mathfrak{U}^*} \subseteq L^\infty(\mu)$ of $\mathfrak{U}$. What are the connections between these algebras? For $1 \leqslant p < \infty$ it is shown in [18] that

$$\mathfrak{M}_{L^p(\mu)}(\mathfrak{U}) \subseteq \overline{\mathfrak{M}_{L^p(\mu)}(\mathfrak{U})}^{\mathrm{w}} = \mathfrak{M}_{L^p(\mu)}(\mathfrak{U})^{\mathrm{cc}} = \mathfrak{M}_{L^p(\mu)}(\overline{\mathfrak{U}^*}). \tag{1.1}$$

Of course, the case $p = 2$ is well-known and corresponds to a particular version of von Neumann's bicommutant theorem.

The case $p = \infty$ is distinctly different. It turns out that

$$\mathfrak{M}_{L^\infty(\mu)}(\mathfrak{U})^{\mathrm{cc}} = \mathfrak{M}_{L^\infty(\mu)}(\mathfrak{R}(\mathfrak{U})) \tag{1.2}$$

where $\mathfrak{R}(\mathfrak{U})$ is the Dedekind closure of $\mathfrak{U}$, formed in the Dedekind complete Riesz space $L^\infty(\mu)$; see [18] for the details. For instance, if μ is Lebesgue measure in $\Omega = [0,1]$ and $\mathfrak{U} = C([0,1])$, then $\overline{\mathfrak{U}^*} = L^\infty([0,1])$ whereas $\mathfrak{R}(\mathfrak{U})$ is the space of all (bounded) Riemann integrable functions on $[0,1]$. It is routine to check that the operator norm and strong operator topologies of $\mathcal{L}(L^\infty(\mu))$ coincide on $\mathfrak{M}_{L^\infty(\mu)}(\mathfrak{U})$ and hence, $\mathfrak{M}_{L^\infty(\mu)}(\mathfrak{U}) = \overline{\mathfrak{M}_{L^\infty(\mu)}(\mathfrak{U})}^{\mathrm{s}}$. Furthermore, it is immediate from the definition of Dedekind closure that $\mathfrak{U} \subseteq \mathfrak{R}(\mathfrak{U})$ and it is not difficult to check that $\mathfrak{R}(\mathfrak{U}) \subseteq \overline{\mathfrak{U}^*}$; see Section 2. So, for $E = L^p(\mu), 1 \leqslant p \leqslant \infty$, we see immediately that

$$\mathfrak{M}_E(\mathfrak{U}) \subseteq \overline{\mathfrak{M}_E(\mathfrak{U})}^{\mathrm{w}} \subseteq \mathfrak{M}_E(\mathfrak{U})^{\mathrm{cc}} \subseteq \mathfrak{M}_E(\overline{\mathfrak{U}^*}) \subseteq \mathfrak{M}_E(L^\infty(\mu)). \tag{1.3}$$

Of course, for $1 \leqslant p < \infty$ the second and third containments are actually equalities (cf. (1.1)). For $p = \infty$ the first containment is an equality. However, the example mentioned above shows that the second and third containments may be *strict* in the L^∞-setting.

The aim of this article is to investigate further the possibility of (1.3) being satisfied for a class of Banach spaces E which is larger than just the L^p-spaces, thereby developing further and extending the results of [18]. A natural class of spaces E to consider in this context is the class of Banach function spaces, firstly because algebras of multiplication operators by elements from a unital, norm-closed subalgebra $\mathfrak{U} \subseteq L^\infty(\mu)$, where E is based on (Ω, Σ, μ), are defined in such spaces and secondly (as for L^p-spaces), the algebra $\mathfrak{M}_E(L^\infty(\mu))$ is maximal abelian in $\mathcal{L}(E)$, i.e. $\mathfrak{M}_E(L^\infty(\mu))^{\mathrm{c}} = \mathfrak{M}_E(L^\infty(\mu))$; see Proposition 2.2. It was noted above that always $\mathfrak{U} \subseteq \mathfrak{R}(\mathfrak{U}) \subseteq \overline{\mathfrak{U}^*} \subseteq L^\infty(\mu)$ and so

$$\mathfrak{M}_E(\mathfrak{U}) \subseteq \mathfrak{M}_E(\mathfrak{R}(\mathfrak{U})) \subseteq \mathfrak{M}_E(\overline{\mathfrak{U}^*}) \subseteq \mathfrak{M}_E(L^\infty(\mu)). \tag{1.4 (a)}$$

It is shown in Section 2 that the additional containments

$$\text{(1.4 (b))} \qquad \mathfrak{M}_E(\mathfrak{U}) \subseteq \overline{\mathfrak{M}_E(\mathfrak{U})}^{\mathrm{w}} \subseteq \mathfrak{M}_E(\mathfrak{U})^{\mathrm{cc}} \subseteq \mathfrak{M}_E(L^\infty(\mu))$$

and

$$\text{(1.4 (c))} \qquad \overline{\mathfrak{M}_E(\mathfrak{U})}^{\mathrm{w}} \subseteq \mathfrak{M}_E(\overline{\mathfrak{U}^*})$$

are also satisfied in any Banach function space E. Moreover, if E has order continuous norm, then (1.4 (c)) is an equality.

An examination of (1.4 (a))–(1.4 (c)) shows that they do "not quite" imply (1.3). There is good reason for this since the inclusion $\mathfrak{M}_E(\mathfrak{U})^{\mathrm{cc}} \subseteq \mathfrak{M}_E(\overline{\mathfrak{U}^*})$ *fails in general.* Indeed, in Section 3 we exhibit a Banach function space E (based on a finite measure μ) and a unital, weak-star closed subalgebra $\mathfrak{U} \subseteq L^\infty(\mu)$ such that $\mathfrak{M}_E(\mathfrak{U}) = \mathfrak{M}_E(\overline{\mathfrak{U}^*})$ is a *proper* subalgebra of $\mathfrak{M}_E(\mathfrak{U})^{\mathrm{cc}}$. So, the class of all Banach function spaces is too large to accomodate (1.3). However, it is shown in Section 3 that the subclass of *fully symmetric* Banach function spaces (i.e. the exact (L^1, L^∞)-interpolation spaces) has the property that

$$\text{(1.5)} \qquad \mathfrak{M}_E(\mathfrak{U})^{\mathrm{cc}} \subseteq \mathfrak{M}_E(\overline{\mathfrak{U}^*}).$$

Combining this with (1.4 (b)) shows that (1.3) is indeed satisfied for all fully symmetric Banach function spaces E. We note that all spaces $L^p(\mu), 1 \leqslant p \leqslant \infty$, are fully symmetric Banach function spaces. Moreover, if $1 \leqslant p < \infty$, then $L^p(\mu)$ has order continuous norm and so it follows from (1.3) and the remark immediately after (1.4 (c)) that $\overline{\mathfrak{M}_{L^p(\mu)}(\mathfrak{U})}^{\mathrm{w}} = \mathfrak{M}_{L^p(\mu)}(\mathfrak{U})^{\mathrm{cc}} = \mathfrak{M}_{L^p(\mu)}(\overline{\mathfrak{U}^*})$, i.e. we recover (1.1) as a special case.

The final section deals with the special spaces $(L^1 + L^\infty)(\mu)$ and $(L^1 \cap L^\infty)(\mu)$. The aim is to exhibit conditions on the subalgebra $\mathfrak{U} \subseteq L^\infty(\mu)$ which can be used to determine when the various inclusions in (1.3) are actually equalities (or are strict) and which can also be used to determine explicitly the various subalgebras $\overline{\mathfrak{M}_E(\mathfrak{U})}^{\mathrm{w}}, \mathfrak{M}_E(\mathfrak{U})^{\mathrm{cc}}, \mathfrak{M}_E(\mathfrak{R}(\mathfrak{U}))$ and $\mathfrak{M}_E(\overline{\mathfrak{U}^*})$ for specific measures μ and subalgebras $\mathfrak{U}$.

2. PRELIMINARIES

In this section we establish some terminology and collect some facts which will be used throughout the paper. It is always assumed that (Ω, Σ, μ) is a *Maharam measure space* (in the sense of [7]), i.e., the associated measure algebra is a complete Boolean algebra (so μ is localizable) and for every $A \in \Sigma$ with $\mu(A) > 0$ there exists $B \in \Sigma$ such that $B \subseteq A$ and $0 < \mu(B) < \infty$ (i.e., μ has the finite subset property). By $L^0(\mu)$ we denote the space of all (equivalence classes of) μ-a.e. finitely-valued measurable functions on Ω. If it is necessary to distinguish explicitly between $\mathbb{C}$ or $\mathbb{R}$-valued functions we will denote the corresponding spaces by $L^0_{\mathbb{C}}(\mu)$ and $L^0_{\mathbb{R}}(\mu)$, respectively.

A linear subspace E of $L^0(\mu)$, equipped with a norm $\|\cdot\|_E$, is called a *Banach function space* if $(E, \|\cdot\|_E)$ is a Banach space and whenever $g \in E$ and $f \in L^0(\mu)$ satisfy $|f| \leqslant |g|$ it follows that $f \in E$ and $\|f\|_E \leqslant \|g\|_E$; so E is an order ideal in $L^0(\mu)$ and $(E, \|\cdot\|_E)$ is a Banach lattice. Without loss of generality it may be assumed that for any $A \in \Sigma$ with $\mu(A) > 0$, there exists $B \in \Sigma$ such that $B \subseteq A, 0 < \mu(B) < \infty$ and $\chi_B \in E$; this can be established by an exhaustion type argument along the lines of §67 of [20]. Examples of Banach functions spaces are of course the classical spaces $L^p(\mu), 1 \leqslant p \leqslant \infty$. Other examples are the spaces $(L^1 \cap L^\infty)(\mu)$ and $(L^1 + L^\infty)(\mu)$ equipped with the norms

$$\|f\|_{L^1 \cap L^\infty} = \max\left(\|f\|_1, \|f\|_\infty\right)$$

and

$$\|f\|_{L^1 + L^\infty} = \inf\{\|g\|_1 + \|h\|_\infty : g \in L^1(\mu),\, h \in L^\infty(\mu),\, f = g + h\},$$

respectively. For computational purposes it is sometimes useful to have available the following alternative description for elements of $(L^1 + L^\infty)(\mu)$. Recall that for $f \in L^0(\mu)$ the decreasing rearrangement $f^* : (0, \infty) \to [0, \infty]$ of $|f|$ is defined by

$$f^*(t) = \inf\{\lambda > 0 : \mu(\{w \in \Omega : |f(w)| > \lambda\}) > t\}, \quad t > 0.$$

Then a function $f \in L^0(\mu)$ belongs to $(L^1 + L^\infty)(\mu)$ if and only if $\int_0^t f^*(s)\mathrm{d}s < \infty$ for all (equivalently, for some) $t > 0$, in which case $\|f\|_{L^1 + L^\infty} = \int_0^1 f^*(s)\mathrm{d}s$. These statements and notions can be found in [2] and [13], for example; for more general measure spaces we refer to [7].

Most of the Banach function spaces E considered in this paper will be *exact* (L^1, L^∞)*-interpolation spaces* and so, in particular, $(L^1 \cap L^\infty)(\mu) \subseteq E \subseteq (L^1 + L^\infty)(\mu)$ for some Maharam measure μ. Such exact (L^1, L^∞)-interpolation spaces are characterized by the property that whenever $f \in L^0(\mu)$ and $g \in E$ satisfy $f \prec\prec g$, then $f \in E$ and $\|f\|_E \leqslant \|g\|_E$; here $f \prec\prec g$ means that f is submajorized by g in the sense of Hardy-Littlewood-Polya, i.e.,

$$\int_0^t f^*(s)\mathrm{d}s \leqslant \int_0^t g^*(s)\mathrm{d}s, \quad t > 0$$

(see e.g. [2], [6], [13]). Function spaces E with this latter property will also be called *fully symmetric Banach function spaces.* Note that Orlicz spaces, Lorentz spaces and Marcinkiewicz spaces are all examples of fully symmetric Banach function spaces.

Assume that E is a Banach function space. The space of all bounded linear operators on E is denoted by $\mathcal{L}(E)$. If $\varphi \in L^\infty(\mu)$, then the multiplication operator M_φ, defined by $M_\varphi f = \varphi f$ for all $f \in E$, satisfies $M_\varphi \in \mathcal{L}(E)$ and $\|M_\varphi\| = \|\varphi\|_\infty$ (if we need to specify the space E on which the multiplication operator is considered to be acting, then we will denote this operator by M_φ^E). For a subalgebra $\mathfrak{U} \subseteq L^\infty(\mu)$ let

$$\mathfrak{M}_E(\mathfrak{U}) = \{M_\varphi^E \in \mathcal{L}(E) : \varphi \in \mathfrak{U}\},$$

which is a commutative subalgebra of $\mathcal{L}(E)$. In most cases $\mathfrak{U}$ is assumed to be unital and norm-closed in which case $\mathfrak{M}_E(\mathfrak{U})$ is also unital and operator norm-closed in $\mathcal{L}(E)$. As usual, the commutant and bicommutant of $\mathfrak{M}_E(\mathfrak{U})$ in $\mathcal{L}(E)$ are denoted by $\mathfrak{M}_E(\mathfrak{U})^{\mathrm{c}}$ and $\mathfrak{M}_E(\mathfrak{U})^{\mathrm{cc}}$, respectively.

The following lemma, which for the case that $E = L^2(\mu)$ is essentially the Fuglede theorem for normal operators in Hilbert space, enables us to restrict our attention to the case of $\mathbb{R}$-valued functions only. The present proof, based on Rosenblum's proof of Fuglede's theorem (see [5], Theorem 7.21), was pointed out to us by Anton Schep.

LEMMA 2.1. *Let $E \subseteq L^0_{\mathbb{C}}(\mu)$ be a Banach function space. If $\varphi \in L^\infty_{\mathbb{C}}(\mu)$ and $T \in \mathcal{L}(E)$ satisfy $M_\varphi T = T M_\varphi$, then $M_{\overline{\varphi}} T = T M_{\overline{\varphi}}$ (where $\overline{\varphi}$ denotes the function defined by $\overline{\varphi}(w) = \overline{\varphi(w)}$, for $w \in \Omega$).*

Proof. Suppose that $\psi \in L^\infty(\mu)$. If p is a complex polynomial, then it is clear that $M_{p\circ\psi} = p(M_\psi)$ and hence $\exp(M_\psi) = M_{\exp(\psi)}$. Now assume that

$\varphi \in L^\infty(\mu)$ and $T \in \mathcal{L}(E)$ are such that $M_\varphi T = TM_\varphi$. Define the analytic function $f : \mathbb{C} \to \mathcal{L}(E)$ by

$$f(z) = \exp(-zM_{\overline{\varphi}})T\exp(zM_{\overline{\varphi}}), \quad z \in \mathbb{C}.$$

Since M_φ and T commute we have $\exp(\overline{z}M_\varphi)T = T\exp(\overline{z}M_\varphi)$, i.e., $T = \exp(\overline{z}M_\varphi)T\exp(-\overline{z}M_\varphi)$ for all $z \in \mathbb{C}$. Hence,

$$f(z) = \exp(\overline{z}M_\varphi - zM_{\overline{\varphi}})T\exp(zM_{\overline{\varphi}} - \overline{z}M_\varphi), \quad z \in \mathbb{C}.$$

By the above observation $\exp(\overline{z}M_\varphi - zM_{\overline{\varphi}})$ is the operator of multiplication by $\exp(\overline{z}\varphi - z\overline{\varphi})$. Since the function $\overline{z}\varphi - z\overline{\varphi}$ assumes only purely imaginary values it follows that

$$\|\exp(\overline{z}M_\varphi - zM_{\overline{\varphi}})\|_{\mathcal{L}(E)} = \|\exp(\overline{z}\varphi - z\overline{\varphi})\|_\infty = 1$$

for all $z \in \mathbb{C}$. Similarly

$$\|\exp(zM_{\overline{\varphi}} - \overline{z}M_\varphi)\|_{\mathcal{L}(E)} = 1$$

for all $z \in \mathbb{C}$. Consequently, $\|f(z)\| \leqslant \|T\|$ for all $z \in \mathbb{C}$ and so f is constant. In particular, $f'(0) = 0$ which yields $TM_{\overline{\varphi}} = M_{\overline{\varphi}}T$. ∎

As already mentioned, in virtue of the above lemma we may restrict ourselves to considering only real function spaces. All results in this paper extend easily, with the appropriate modifications, to spaces E of $\mathbb{C}$-valued functions (see [18] for more details). Therefore, from now on, *all Banach function spaces considered will consist of* $\mathbb{R}$*-valued functions.*

Let $\mathfrak{U}$ be a unital norm-closed subalgebra of $L^\infty(\mu)$, in which case $\mathfrak{U}$ is also a sublattice of $L^\infty(\mu)$. The $\sigma(L^\infty, L^1)$-closure of $\mathfrak{U}$ is also a unital subalgebra and sublattice and is denoted by $\overline{\mathfrak{U}}^*$. We recall the definition of the Dedekind closure (see Section 46 of [11], or [18] for more information). For $\varphi \in L^\infty(\mu)$ define elements $\varphi^\uparrow$ and $\varphi^\downarrow$ of $L^\infty(\mu)$ by

$$\varphi^\uparrow = \sup\{\psi \in \mathfrak{U} : \psi \leqslant \varphi\} \quad \text{and} \quad \varphi^\downarrow = \inf\{\psi \in \mathfrak{U} : \varphi \leqslant \psi\},$$

where the supremum and infimum are taken in the Dedekind complete vector lattice $L^\infty(\mu)$; of course, $\varphi^\uparrow$ and $\varphi^\downarrow$ depend on the algebra $\mathfrak{U}$. It is clear that $\varphi^\uparrow \leqslant \varphi \leqslant \varphi^\downarrow$. The *Dedekind closure* $\mathfrak{R}(\mathfrak{U})$ of $\mathfrak{U}$ in $L^\infty(\mu)$ is defined by

$$\mathfrak{R}(\mathfrak{U}) = \{\varphi \in L^\infty(\mu) : \varphi^\uparrow = \varphi^\downarrow\}.$$

Then $\mathfrak{U} \subseteq \mathfrak{R}(\mathfrak{U})$ and $\mathfrak{R}(\mathfrak{U})$ is a norm-closed unital subalgebra and sublattice of $L^\infty(\mu)$. The weak-star topology $\sigma(L^\infty, L^1)$ is generated by the family of semi-norms $\psi \mapsto p_g(\psi) = \left|\int_\Omega \psi g \mathrm{d}\mu\right|$ as g varies through the non-negative elements of $L^1(\mu)$. These semi-norms have the property that $p_g(\psi_\alpha) \uparrow p_g(\psi)$ for each $0 \leqslant g \in L^1(\mu)$, whenever $\{\psi_\alpha\}$ is an upwards directed net in $L^\infty(\mu)$ satisfying $0 \leqslant \psi_\alpha \uparrow \psi \in L^\infty(\mu)$ (i.e., ψ is the supremum of $\{\psi_\alpha\}$ in the vector lattice $L^\infty(\mu)$). These observations show that $\varphi^\uparrow \in \overline{\mathfrak{U}^*}$ for all $\varphi \in L^\infty(\mu)$, and consequently that $\mathfrak{R}(\mathfrak{U}) \subseteq \overline{\mathfrak{U}^*}$. Hence, for any unital norm-closed subalgebra $\mathfrak{U}$ of $L^\infty(\mu)$ we have the inclusions

$$\mathfrak{U} \subseteq \mathfrak{R}(\mathfrak{U}) \subseteq \overline{\mathfrak{U}^*} \subseteq L^\infty(\mu). \tag{2.1}$$

In particular, if $\mathfrak{U}$ is weak-star closed, then $\mathfrak{R}(\mathfrak{U}) = \mathfrak{U}$.

Next we collect some facts concerning the algebras $\mathfrak{M}_E(\mathfrak{U})$ of multiplication operators acting on a Banach function space E. A basic result which will be used throughout the paper is that $\mathfrak{M}_E(L^\infty(\mu))$ is maximal abelian in $\mathcal{L}(E)$, i.e., $\mathfrak{M}_E(L^\infty(\mu))^{\mathrm{c}} = \mathfrak{M}_E(L^\infty(\mu))$. For special cases this result can be found in the literature. However, for the general setting of this paper we were not able to find an explicit reference. For convenience of the reader we include a proof, actually of a slightly more general result (cf. Proposition 2.2 below) which will be useful in the sequel. It is possible to give a quick proof of this result by an appeal to some rather deep theorems, due to Yu. Abramovich, A.I. Veksler and A.V. Koldunov, concerning band preserving operators on Banach lattices (see e.g. [1], Theorem 15.4 and [17], Theorem 3.1.12), but we prefer to present a proof via more elementary methods.

Proposition 2.2. *Let E be a Banach function space on the Maharam measure space (Ω, Σ, μ). Suppose that $\Sigma_0 \subseteq \Sigma$ is a family of sets with the property that for any $B \in \Sigma$ with $\mu(B) > 0$ there exists $A \in \Sigma_0$ such that $A \subseteq B$ and $\mu(A) > 0$. If $T \in \mathcal{L}(E)$ satisfies $TM_{\chi_A} = M_{\chi_A}T$ for all $A \in \Sigma_0$, then $T \in \mathfrak{M}_E(L^\infty(\mu))$. In particular, $\mathfrak{M}_E(L^\infty(\mu))^{\mathrm{c}} = \mathfrak{M}_E(L^\infty(\mu))$.*

Proof. For $f, g \in E$ we write $f \perp g$ if f and g are disjointly supported (i.e., $|f| \wedge |g| = 0$ in E). First we show that if T is as in the statement of the proposition, then $f \perp g$ in E implies that $Tf \perp g$. Suppose, on the contrary, that $|Tf| \wedge |g| > 0$. Then there exists $B \in \Sigma$ and $\varepsilon > 0$ such that $\mu(B) > 0$ and $|Tf| \wedge |g| \geqslant \varepsilon\chi_B$. Take $A \in \Sigma_0$ such that $A \subseteq B$ and $\mu(A) > 0$. Since $|g| \geqslant |Tf| \wedge |g| \geqslant \varepsilon\chi_B \geqslant \varepsilon\chi_A$ and $|f| \wedge |g| = 0$ it follows that $\chi_A f = 0$, and so $\chi_A Tf = T(\chi_A f) = 0$, which is clearly a contradiction.

We note that the property of T just established is equivalent to $TM_{\chi_A} = M_{\chi_A}T$ for all $A \in \Sigma$. Since the linear span of $\{\chi_A : A \in \Sigma\}$ is norm dense in $L^\infty(\mu)$, it follows that $TM_\varphi = M_\varphi T$ for all $\varphi \in L^\infty(\mu)$, i.e., $T \in \mathfrak{M}_E(L^\infty(\mu))^{\mathrm{c}}$.

Given $h \in L^0(\mu)$ let $C(h) = \{w \in \Omega : h(w) \neq 0\}$. It was just established that T commutes (in particular) with multiplication by $\chi_{C(h)}$ and so $C(Th) \subseteq C(h)$ for all $h \in E$. If $\varphi_h = \chi_{C(h)} \cdot (Th/h)$, then it is clear that $\varphi_h \in L^0(\mu)$ is the unique function satisfying $Th = \varphi_h h$ and $C(\varphi_h) \subseteq C(h)$. For any $f, g \in E$ we claim that $\varphi_f = \varphi_g$ on $C(f) \cap C(g)$. Indeed, let $u = |f| \vee |g|$. Since $|f| \leqslant u$ there exists $\varphi \in L^\infty(\mu)$ such that $f = \varphi u$. Hence $Tf = \varphi Tu = \varphi\varphi_u u = \varphi_u f$ and so $\varphi_f = \varphi_u$ on $C(f)$. Similarly $\varphi_g = \varphi_u$ on $C(g)$ and so $\varphi_f = \varphi_g$ on $C(f) \cap C(g)$. It follows from this property that $|\varphi_f| \leqslant \|T\|\mathbb{1}$ for all $f \in E$, where $\mathbb{1}$ denotes the function constantly equal to 1 on Ω. Using the fact that $L^\infty(\mu)$ is a Dedekind complete vector lattice (as (Ω, Σ, μ) is Maharam), it follows via a standard argument that there exists $\varphi_0 \in L^\infty(\mu)$ such that $\varphi_0 = \varphi_f$ on $C(f)$ for all $f \in E$. Then $Tf = \varphi_0 f$ for all $f \in E$ and we conclude that $T = M_{\varphi_0} \in \mathfrak{M}_E(L^\infty(\mu))$. ∎

Let $\mathfrak{U}$ be a unital norm-closed subalgebra of $L^\infty(\mu)$. Then $\mathfrak{M}_E(L^\infty(\mu)) \subseteq \mathfrak{M}_E(\mathfrak{U})^{\mathrm{c}}$ and so, by the above proposition, $\mathfrak{M}_E(\mathfrak{U})^{\mathrm{cc}} \subseteq \mathfrak{M}_E(L^\infty(\mu))$. Consequently, $\mathfrak{M}_E(\mathfrak{U})^{\mathrm{cc}} = \mathfrak{M}_E(\mathfrak{U}^E_{\mathrm{cc}})$ for some unital norm-closed subalgebra $\mathfrak{U}^E_{\mathrm{cc}}$ of $L^\infty(\mu)$.

For any subset $D \subseteq \mathcal{L}(E)$ we denote the closure of D with respect to the strong (respectively, weak) operator topology τ_{s} (respectively, τ_{w}) by $\overline{D^{\mathrm{s}}}$ (respectively, $\overline{D^{\mathrm{w}}}$). Recall that convex subsets (in particular, linear subspaces) of $\mathcal{L}(E)$ have the same closures for τ_{s} and τ_{w}. Since the commutant of any subset of $\mathcal{L}(E)$ is always τ_{w}-closed, it follows from Proposition 2.2 that $\mathfrak{M}_E(L^\infty(\mu))$ is τ_{w}-closed in $\mathcal{L}(E)$. In particular, for any unital norm-closed subalgebra $\mathfrak{U} \subseteq L^\infty(\mu)$ we have $\overline{\mathfrak{M}_E(\mathfrak{U})^{\mathrm{s}}} = \overline{\mathfrak{M}_E(\mathfrak{U})^{\mathrm{w}}} \subseteq \mathfrak{M}_E(L^\infty(\mu))$ and so we can write $\overline{\mathfrak{M}_E(\mathfrak{U})^{\mathrm{w}}} = \mathfrak{M}_E(\mathfrak{U}^E_{\mathrm{w}})$ for some unital norm-closed subalgebra $\mathfrak{U}^E_{\mathrm{w}} \subseteq L^\infty(\mu)$. Since $\mathfrak{M}_E(\mathfrak{U})^{\mathrm{cc}}$ is τ_{w}-closed, it is clear that $\mathfrak{U}^E_{\mathrm{w}} \subseteq \mathfrak{U}^E_{\mathrm{cc}}$. Hence, for any unital norm-closed subalgebra $\mathfrak{U}$ of $L^\infty(\mu)$ we have the inclusions

$$\mathfrak{U} \subseteq \mathfrak{U}^E_{\mathrm{w}} \subseteq \mathfrak{U}^E_{\mathrm{cc}} \subseteq L^\infty(\mu). \tag{2.2}$$

Via the vector space isomorphism $\varphi \leftrightarrow M_\varphi$ between $L^\infty(\mu)$ and $\mathfrak{M}_E(L^\infty(\mu))$ we can transfer τ_{w} and τ_{s} to locally convex Hausdorff topologies on $L^\infty(\mu)$. We denote these corresponding topologies on $L^\infty(\mu)$ by τ^E_{w} and τ^E_{s}, respectively (note that convex subsets of $L^\infty(\mu)$ have the same closures for τ^E_{w} and τ^E_{s}). It is clear that $\mathfrak{U}^E_{\mathrm{w}}$ is the τ^E_{w}-closure of $\mathfrak{U}$ in $L^\infty(\mu)$. The next lemma gives the relation between the topologies $\sigma(L^\infty, L^1)$ and τ^E_{w} on $L^\infty(\mu)$. First we recall some facts concerning the Banach space dual E^* of a Banach function space E. The *associate space* E' of E is defined by

$$E' = \Big\{ g \in L^0(\mu) : \int_\Omega |fg| \mathrm{d}\mu < \infty \text{ for all } f \in E \Big\}.$$

Any $g \in E'$ defines a linear functional $\xi_g \in E^*$ via

$$\langle f, \xi_g \rangle = \int_\Omega fg \mathrm{d}\mu, \quad f \in E. \tag{2.3}$$

Equipped with the norm $\|\cdot\|_{E'}$ given by $\|g\|_{E'} = \|\xi_g\|_{E^*}$, it turns out that $(E', \|\cdot\|_{E'})$ is also a Banach function space. A linear functional $\xi \in E^*$ can be represented as $\xi = \xi_g$ for some $g \in E'$ if and only if ξ is order continuous, i.e., $\langle f_\alpha, \xi \rangle \xrightarrow{\alpha} 0$ whenever $\{f_\alpha\}$ is a downwards directed net in E satisfying $f_\alpha \downarrow 0$. If the norm on E is order continuous (i.e., $f_\alpha \downarrow 0$ in E implies that $\|f_\alpha\|_E \downarrow 0$), then every $\xi \in E^*$ satisfies this last condition and hence $\xi = \xi_g$ for some $g \in E'$ (so, in this case, we may identify E^* with E'). The above facts can be found in [6]; [7]; [20], for example.

LEMMA 2.3. (i) *For any Banach function space E, the topology τ_{w}^E is stronger than the $\sigma(L^\infty, L^1)$-topology.*

(ii) *If E has order continuous norm, then the $\sigma(L^\infty, L^1)$-topology coincides with τ_{w}^E.*

Proof. The topology τ_{w}^E is generated by the family of seminorms $\{p_{f,\xi} : 0 \leqslant f \in E, 0 \leqslant \xi \in E^*\}$, where $p_{f,\xi}(\varphi) = |\langle \varphi f, \xi \rangle|$ for $\varphi \in L^\infty(\mu)$. The $\sigma(L^\infty, L^1)$-topology is generated by the family of seminorms $\{q_h : 0 \leqslant h \in L^1(\mu)\}$, where $q_h(\varphi) = \Big| \int_\Omega \varphi h \mathrm{d}\mu \Big|$ for $\varphi \in L^\infty(\mu)$. We claim that $\{q_h : 0 \leqslant h \in L^1(\mu)\} = \{p_{f,\xi_g} : 0 \leqslant f \in E, 0 \leqslant g \in E'\}$. Indeed, if $0 \leqslant f \in E$ and $0 \leqslant g \in E'$, then $p_{f,\xi_g} = q_{fg}$ with $0 \leqslant fg \in L^1(\mu)$. Conversely if $0 \leqslant h \in L^1(\mu)$, then by a theorem of Lozanovskii (see [8], [15], [19]) h can be factorized as $h = fg$ with $0 \leqslant f \in E$ and $0 \leqslant g \in E'$. Accordingly, $q_h = p_{f,\xi_g}$. From this claim and the remarks prior to the lemma the statements (i) and (ii) follow. ∎

For later reference we state explicitly the following consequence of the above lemma.

COROLLARY 2.4. (i) *If $\mathfrak{U} \subseteq L^\infty(\mu)$ is a norm-closed unital subalgebra and $E \subseteq L^0(\mu)$ is any Banach function space, then $\overline{\mathfrak{U}^*}$ is τ_{w}^E-closed and $\mathfrak{U}_{\mathrm{w}}^E \subseteq \overline{\mathfrak{U}^*}$.*

(ii) *If E has order continuous norm, then $\mathfrak{U}_{\mathrm{w}}^E = \overline{\mathfrak{U}^*}$.*

Remark 2.5. An alternative description of $\overline{\mathfrak{M}_E(\mathfrak{U})}^{\mathrm{w}}$ is possible. Given a Banach space X and an algebra of operators $\mathfrak{B} \subseteq \mathcal{L}(X)$ we denote by $\mathrm{Lat}(\mathfrak{B})$ the collection of all closed subspaces $Y \subseteq X$ satisfying $BY \subseteq Y$ for every $B \in \mathfrak{B}$. Then $\mathrm{AlgLat}(\mathfrak{B})$ denotes the subalgebra $\{T \in \mathcal{L}(X) : TY \subseteq Y \text{ for all } Y \in \mathrm{Lat}(\mathfrak{B})\}$ of $\mathcal{L}(X)$.

Let E be a Banach function space on the Maharam measure space (Ω, Σ, μ) and $\mathfrak{U} \subseteq L^\infty(\mu)$ be a unital norm-closed subalgebra. The claim is that $\overline{\mathfrak{M}_E(\mathfrak{U})}^{\mathrm{w}} = \mathrm{AlgLat}(\mathfrak{M}_E(\mathfrak{U}))$. Indeed, by Gelfand theory (over $\mathbb{R}$; §17 of [3]) or Kakutani's theorem there is a compact Hausdorff space K and a bounded unital isomorphism m of $C(K)$ into $\mathcal{L}(E)$ such that $m(C(K)) = \mathfrak{M}_E(\mathfrak{U})$. The conclusion then follows from Theorem 7 of [9]. ∎

Finally, we formulate an extension theorem for lattice homomorphisms on vector lattices (sharpening the Luxemburg-Schep extension theorem) which is needed later on. Suppose that L and M are (Archimedean) vector lattices. A linear mapping $T : L \to M$ is called a lattice homomorphism if $|Tx| = T|x|$ for all $x \in L$. As is well known, if T is a linear mapping from $L^\infty(\mu)$ into itself with $T\mathbb{1} = \mathbb{1}$, then T is a lattice homomorphism if and only if T is an algebra homomorphism. Now assume that M is Dedekind complete and $K \subseteq L$ is a vector sublattice which is majorizing (i.e., for every $x \in L$ there exists $y \in K$ such that $x \leqslant y$). The Luxemburg-Schep extension theorem ([16]) states that whenever $T_0 : K \to M$ is a lattice homomorphism there exists a lattice homomorphism $T : L \to M$ such that $T|K = T_0$. For $x \in L$ define

$$\begin{aligned} \theta_\ell(x) &= \sup\{T_0 y : y \in K, y \leqslant x\}, \\ \theta_{\mathrm{u}}(x) &= \inf\{T_0 y : y \in K, x \leqslant y\}. \end{aligned} \tag{2.4}$$

It is clear that any lattice homomorphism $T : L \to M$ which extends T_0 must satisfy $\theta_\ell(x) \leqslant Tx \leqslant \theta_{\mathrm{u}}(x)$ for all $x \in L$. The proof of the following result, as pointed out to us by Anton Schep, follows from a close inspection of the proof (due to Z. Lipecki ([14])) of the Luxemburg-Schep extension theorem in the book [17], Section 1.5. We leave the details to the reader.

Proposition 2.6. *Let L and M be vector lattices with M being Dedekind complete and let $K \subseteq L$ be a majorizing vector sublattice. Let $T_0 : K \to M$ be a lattice homomorphism and define $\theta_\ell : L \to M$ and $\theta_{\mathrm{u}} : L \to M$ by (2.4). Then, given any $x_0 \in L$ and $y_0 \in M$ satisfying $\theta_\ell(x_0) \leqslant y_0 \leqslant \theta_{\mathrm{u}}(x_0)$ there exists a lattice homomorphism $T : L \to M$ such that $T|K = T_0$ and $Tx_0 = y_0$.*

3. SOME GENERAL RESULTS

As before, E denotes a Banach function space on a Maharam measure space (Ω, Σ, μ). Let $\mathfrak{A}$ be a unital norm-closed subalgebra of $L^\infty(\mu)$. In this section we will prove a number of results concerning the relationship between the algebras in (2.1) and (2.2).

Theorem 3.1. *If E is a fully symmetric Banach function space, then* $\mathfrak{A}^E_{\mathrm{cc}} \subseteq \overline{\mathfrak{A}^*}$.

Proof. The statement of the theorem is equivalent to the condition $\mathfrak{A}^E_{\mathrm{cc}} = \mathfrak{A}$ for any $\sigma(L^\infty, L^1)$-closed unital subalgebra $\mathfrak{A}$ of $L^\infty(\mu)$. So assume that $\mathfrak{A} \subseteq L^\infty(\mu)$ is actually $\sigma(L^\infty, L^1)$-closed.

First consider the case when μ is a finite measure. Since $\mathfrak{A}$ is closed for μ-a.e. pointwise convergence of bounded sequences (by the dominated convergence theorem and the fact that $\overline{\mathfrak{A}^*} = \mathfrak{A}$) it is known that there exists a σ-subalgebra $\Sigma_0 \subseteq \Sigma$ such that $\mathfrak{A} = L^\infty(\Sigma_0, \mu)$; see e.g. [10], Chapter II. Let $\mathcal{E}(\,\cdot\,|\Sigma_0)$ be the conditional expectation operator with respect to Σ_0. Since E is fully symmetric (or equivalently, an exact (L^1, L^∞)-interpolation space), $\mathcal{E}(\,\cdot\,|\Sigma_0)$ acts as a bounded linear operator in E. Since each $\varphi \in \mathfrak{A}$ is Σ_0-measurable, it follows that $\mathcal{E}(\varphi f|\Sigma_0) = \varphi\mathcal{E}(f|\Sigma_0)$ for all $f \in E$ and $\varphi \in \mathfrak{A}$ showing that $\mathcal{E}(\,\cdot\,|\Sigma_0) \in \mathfrak{M}_E(\mathfrak{A})^{\mathrm{c}}$. Now take $\psi \in \mathfrak{A}^E_{\mathrm{cc}}$. Then M_ψ commutes, in particular, with $\mathcal{E}(\,\cdot\,|\Sigma_0)$, i.e., $\mathcal{E}(\psi f|\Sigma_0) = \psi\mathcal{E}(f|\Sigma_0)$ for all $f \in E$. Noting that $f = \mathbb{1} \in L^\infty(\mu) \subseteq E$ yields $\mathcal{E}(\psi|\Sigma_0) = \psi$. Hence $\psi \in L^\infty(\Sigma_0, \mu) = \mathfrak{A}$.

The extension to an arbitrary Maharam measure space is obtained by the same method of proof as that of Proposition 3.4 in [18], to which we refer the reader for the details. ∎

In view of Theorem 3.1, if E is a fully symmetric Banach function space and $\mathfrak{A} \subseteq L^\infty(\mu)$ is a unital norm-closed subalgebra, then

$$\mathfrak{A} \subseteq \mathfrak{A}^E_{\mathrm{w}} \subseteq \mathfrak{A}^E_{\mathrm{cc}} \subseteq \overline{\mathfrak{A}^*} \subseteq L^\infty(\mu). \tag{3.1}$$

In combination with Corollary 2.4 this yields immediately the following result.

Corollary 3.2. *If E is a fully symmetric Banach function space with order continuous norm, then $\mathfrak{A}^E_{\mathrm{w}} = \mathfrak{A}^E_{\mathrm{cc}} = \overline{\mathfrak{A}^*}$. In particular, $\overline{\mathfrak{M}_E(\mathfrak{A})^{\mathrm{w}}} = \mathfrak{M}_E(\mathfrak{A})^{\mathrm{cc}}$, that is, the bicommutant theorem holds.*

We note, in particular, that reflexive Banach function spaces have order continuous norm (see e.g. Theorem 2.4.15 of [17] and Theorem 114.8 of [21]) as do the spaces $L^1(\mu)$. In particular, if $E = L^p(\mu), 1 \leqslant p < \infty$, then Corollary 3.2 contains Theorem 1.1 of [18] as a special case.

The assumption that E is fully symmetric cannot be omitted in the above results, as is illustrated by the following example. This example is essentially due to J. Dieudonné ([4]), and was constructed for a different but related purpose.

Example 3.3. ([4]) As the underlying measure space we take the interval $[0,2]$ equipped with Lebesgue measure λ. Let w_1 and w_2 be two decreasing positive functions on $[0,1]$ satisfying $\int_0^1 w_1^2 \mathrm{d}\lambda = \int_0^1 w_2^2 \mathrm{d}\lambda = \infty$ and $\int_0^1 w_1 w_2 \mathrm{d}\lambda < \infty$ (see [4] for an explicit construction of such a pair of functions). For any fixed $1 \leqslant p < \infty$ define the function norm ρ on $L^0([0,2])$ by

$$\rho(f) = \Big(\int_0^1 \{(f\chi_{[0,1]})^*\}^p w_1 \mathrm{d}\lambda \Big)^{1/p} + \Big(\int_0^1 \{(f\chi_{[1,2]})^*\}^p w_2 \mathrm{d}\lambda \Big)^{1/p},$$

and let $E = \{f \in L^0([0,2]) : \rho(f) < \infty\}$. Then E is a Banach function space when equipped with the norm ρ (but not fully symmetric). Clearly ρ is order continuous and if $1 < p < \infty$, then E is even reflexive. Define the unital $\sigma(L^\infty, L^1)$-closed (proper) subalgebra $\mathfrak{U} \subseteq L^\infty([0,2])$ by

$$\mathfrak{U} = \{\varphi \in L^\infty([0,2]) : \varphi(t) = \varphi(t-1) \text{ for } 1 \leqslant t \leqslant 2\}.$$

It is shown in [4] that $\mathfrak{M}_E(\mathfrak{U})^{\mathrm{c}} = \mathfrak{M}_E(L^\infty([0,2]))$ and so $\mathfrak{U}_{\mathrm{cc}}^E = L^\infty([0,2])$. Hence, the conclusion of both Theorem 3.1 and Corollary 3.2 fail in this case. ∎

The next result is more special. As shown in [18], if $E = L^\infty(\mu)$ and $\mathfrak{U} \subseteq L^\infty(\mu)$ is any unital norm-closed subalgebra, then $\mathfrak{U}_{\mathrm{cc}}^E = \mathfrak{R}(\mathfrak{U})$, the Dedekind closure of $\mathfrak{U}$. As noted in the Introduction the choice $E = L^p([0,1]), 1 \leqslant p < \infty$, and $\mathfrak{U} = C([0,1])$ shows that this result does not hold in general. However, for certain spaces E and for algebras $\mathfrak{U}$ with special properties it turns out that the inclusion $\mathfrak{U}_{\mathrm{cc}}^E \subseteq \mathfrak{R}(\mathfrak{U})$ is valid. In the next section we will discuss examples where these conditions are satisfied, and also exhibit examples showing that the assumptions in the theorem below cannot be omitted in general.

Theorem 3.4. *Let E be a Banach function space such that $L^\infty(\mu) \subseteq E \subseteq (L^1 + L^\infty)(\mu)$. Suppose that $\mathfrak{U} \subseteq L^\infty(\mu)$ is a unital norm-closed subalgebra satisfying $\mathfrak{U} \cap L^1(\mu) = \{0\}$.*

(i) *The topology τ_{s}^E restricted to $\mathfrak{U}$ coincides with the norm topology of $L^\infty(\mu)$ restricted to $\mathfrak{U}$. In particular, $\mathfrak{U}_{\mathrm{w}}^E = \mathfrak{U}$.*

(ii) $\mathfrak{R}(\mathfrak{U}) \cap L^1(\mu) = \{0\}$.

(iii) $\mathfrak{U}^E_{\mathrm{cc}} \subseteq \mathfrak{R}(\mathfrak{U})$.

Proof. First observe that the inbeddings $L^\infty(\mu) \subseteq E \subseteq (L^1 + L^\infty)(\mu)$ are necessarily continuous (by the closed graph theorem, for example).

(i) Note that $\mathfrak{U} \cap L^1(\mu) = \{0\}$ implies that $\mu(\{w \in \Omega : |\varphi(w)| > 0\}) = \infty$ for all $0 \neq \varphi \in \mathfrak{U}$. We claim, for each $\varphi \in \mathfrak{U}$, that $\varphi^*(t) = \|\varphi\|_\infty$ for all $t > 0$. Indeed, take $\varphi \in \mathfrak{U} \setminus \{0\}$ and let $0 < \lambda < \|\varphi\|_\infty$. Then $0 < (|\varphi| - \lambda \mathbb{1})^+ \in \mathfrak{U}$ and hence, $\mu(\{w \in \Omega : |\varphi(w)| > \lambda\}) = \infty$. This implies that $\varphi^*(t) \geqslant \lambda$ for all $t > 0$, from which the claim follows. Since $\|\varphi\|_{L^1+L^\infty} = \int_0^1 \varphi^*(t)\mathrm{d}t$, it follows, in particular, that $\|\varphi\|_{L^1+L^\infty} = \|\varphi\|_\infty$ for all $\varphi \in \mathfrak{U}$. Let $\{\varphi_\alpha\}$ be a net in $\mathfrak{U}$ which converges to $\varphi \in L^\infty(\mu)$ with respect to τ^E_{s}, i.e., $M_{\varphi_\alpha} \xrightarrow{\alpha} M_\varphi$ with respect to τ_{s} in $\mathcal{L}(E)$. Then $\varphi_\alpha = M_{\varphi_\alpha}(\mathbb{1}) \xrightarrow{\alpha} M_\varphi(\mathbb{1}) = \varphi$ with respect to $\|\cdot\|_E$ and hence, $\varphi_\alpha \xrightarrow{\alpha} \varphi$ in the norm of $(L^1 + L^\infty)(\mu)$. Since $\|\varphi_\alpha - \varphi\|_{L^1+L^\infty} = \|\varphi_\alpha - \varphi\|_\infty$, this implies that $\varphi_\alpha \xrightarrow{\alpha} \varphi$ for the norm in $L^\infty(\mu)$. This shows that $\mathfrak{U}$ is τ^E_{s}-closed in $L^\infty(\mu)$ and the topologies τ^E_{s} and $\|\cdot\|_\infty$ coincide on $\mathfrak{U}$.

(ii) Fix $0 \leqslant \varphi \in L^\infty(\mu) \cap L^1(\mu)$. If $0 \leqslant \psi \leqslant \varphi$ and $\psi \in \mathfrak{U}$, then $0 \leqslant \psi \in \mathfrak{U} \cap L^1(\mu)$ and so $\psi = 0$. Hence, $\varphi^\uparrow = 0$. This shows that $\mathfrak{R}(\mathfrak{U}) \cap L^1(\mu) = \{0\}$.

(iii) Define the linear subspace K of $L^\infty(\mu)$ by $K = \mathfrak{U} \oplus (L^1 \cap L^\infty)(\mu)$. Each $f \in K$ has a unique decomposition $f = f_1 + f_2$ with $f_1 \in \mathfrak{U}$ and $f_2 \in (L^1 \cap L^\infty)(\mu)$. Define $T_0 : K \to L^\infty(\mu)$ by $T_0 f = f_1$. The claim is that K is a vector sublattice of $L^\infty(\mu)$ and that $T_0 : K \to L^\infty(\mu)$ is a lattice homomorphism. To see this take $f \in K$ and write $f = f_1 + f_2$ as above. Then

$$|f| = |f_1| + (|f| - |f_1|) \tag{3.2}$$

with $|f_1| \in \mathfrak{U}$. Also $\big||f| - |f_1|\big| \leqslant |f - f_1| = |f_2| \in (L^1 \cap L^\infty)(\mu)$ and so $|f| - |f_1| \in (L^1 \cap L^\infty)(\mu)$. Hence, $|f| \in K$ and so (3.2) is the unique decomposition of $|f|$. Accordingly, $T_0|f| = |f_1| = |T_0 f|$, which proves the claim.

For $f \in L^\infty(\mu)$, let

$$\begin{aligned} \theta_\ell(f) &= \sup\{T_0 g : g \in K, g \leqslant f\}, \\ \theta_{\mathrm{u}}(f) &= \inf\{T_0 g : g \in K, f \leqslant g\}. \end{aligned} \tag{3.3}$$

Fix $f_0 \in L^\infty(\mu)$ and suppose that $h_0 \in L^\infty(\mu)$ satisfies $\theta_\ell(f_0) \leqslant h_0 \leqslant \theta_{\mathrm{u}}(f_0)$. By Proposition 2.6 there exists a lattice homomorphism $T_1 : L^\infty(\mu) \to L^\infty(\mu)$ such that $T_1|K = T_0$ and $T_1 f_0 = h_0$.

Now take $f \in E$ and write $f = f_1 + f_\infty$ with $f_1 \in L^1(\mu)$ and $f_\infty \in L^\infty(\mu)$. We define $T : E \to E$ by $Tf = T_1 f_\infty$. Since $T_1 \equiv 0$ on $L^1(\mu) \cap L^\infty(\mu)$, it is easy to see that T is well defined and, via a similar argument as in the beginning of the proof of part (iii), it follows that T is a lattice homomorphism. In particular, as T is positivity preserving it is norm bounded on E. The claim is that $T \in \mathfrak{M}_E(\mathfrak{U})^{\mathrm{c}}$. Indeed, since $T_1 : L^\infty(\mu) \to L^\infty(\mu)$ is a lattice homomorphism with $T_1 \mathbb{1} = \mathbb{1}$, it follows that T_1 is multiplicative. Take $\varphi \in \mathfrak{U}$ and $f \in E$. Writing $f = f_1 + f_\infty$ as above we have that $\varphi f = \varphi f_1 + \varphi f_\infty$ with $\varphi f_1 \in L^1(\mu)$ and $\varphi f_\infty \in L^\infty(\mu)$. Hence, $T(\varphi f) = T_1(\varphi f_\infty) = T_1\varphi \cdot T_1 f_\infty = \varphi T_1 f_\infty = \varphi T f$. This shows that $TM_\varphi = M_\varphi T$ for all $\varphi \in \mathfrak{U}$, i.e., $T \in \mathfrak{M}_E(\mathfrak{U})^{\mathrm{c}}$.

Suppose now that $f_0 \in \mathfrak{U}^E_{\mathrm{cc}}$. Then $TM_{f_0} = M_{f_0} T$ and so, in particular, $TM_{f_0}\mathbb{1} = M_{f_0} T\mathbb{1}$, i.e. $Tf_0 = f_0$. Hence $f_0 = h_0$ which shows that $\theta_\ell(f_0) = \theta_{\mathrm{u}}(f_0) = f_0$. Since $T_0 g \in \mathfrak{U}$ for all $g \in K$, it follows from (3.3) that $\theta_\ell(f_0)$ is a supremum of elements in $\mathfrak{U}$ and hence $\theta_\ell(f_0)^\uparrow = \theta_\ell(f_0)$. Similarly $\theta_{\mathrm{u}}(f_0)^\downarrow = \theta_{\mathrm{u}}(f_0)$. Since f_0 satisfies $f_0 = \theta_\ell(f_0) = \theta_{\mathrm{u}}(f_0)$, we have $f_0 = f_0^\uparrow = f_0^\downarrow$ and so $f_0 \in \mathfrak{R}(\mathfrak{U})$. We have thus shown that $\mathfrak{U}^E_{\mathrm{cc}} \subseteq \mathfrak{R}(\mathfrak{U})$. The proof of the theorem is thereby complete. ∎

Observe that any fully symmetric Banach function space containing $\mathbb{1}$ automatically satisfies the assumptions on E required by Theorem 3.4.

In the setting of Theorem 3.4 above the algebra $\mathfrak{U}^E_{\mathrm{cc}}$ is in some sense “small”. There are also situations in which one can conclude that $\mathfrak{U}^E_{\mathrm{cc}}$ is “large”.

Proposition 3.5. *Let E be a Banach function space on (Ω, Σ, μ) and let $\mathfrak{U} \subseteq L^\infty(\mu)$ be a norm-closed unital subalgebra. Suppose there is a collection $\Sigma_0 \subseteq \Sigma$ such that whenever $B \in \Sigma$ satisfies $\mu(B) > 0$ there exists $A \in \Sigma_0$ such that $A \subseteq B$ and $\mu(A) > 0$. If $\chi_A \in \mathfrak{U}^E_{\mathrm{w}}$ for each $A \in \Sigma_0$, then $\mathfrak{U}^E_{\mathrm{cc}} = L^\infty(\mu)$.*

Proof. Since $\mathfrak{U}^E_{\mathrm{w}} \subseteq \mathfrak{U}^E_{\mathrm{cc}}$, it follows from Proposition 2.2 that $\mathfrak{M}_E(\mathfrak{U})^{\mathrm{c}} = \mathfrak{M}_E(L^\infty(\mu))$. Accordingly, $\mathfrak{U}^E_{\mathrm{cc}} = L^\infty(\mu)$. ∎

Note that the above result applies, in particular, if $L^1(\mu) \cap L^\infty(\mu) \subseteq \mathfrak{U}^E_{\mathrm{w}}$.

4. THE SPACES $(L^1 + L^\infty)(\mu)$ AND $(L^1 \cap L^\infty)(\mu)$

The aim of this final section is to consider the particular fully symmetric Banach function spaces $(L^1 + L^\infty)(\mu)$ and $(L^1 \cap L^\infty)(\mu)$ and specific subalgebras $\mathfrak{U} \subseteq L^\infty(\mu)$. The examples are chosen to be new and non-trivial, to illustrate the variety of the phenomena of significance that can typically occur, and to show the applicability of the general results developed in earlier sections to concrete situations. Along the way we also establish some general results of interest in their own right.

When μ is Lebesgue measure on the Lebesgue measurable subsets of $\mathbb{R}$ we will denote $(L^1+L^\infty)(\mu)$ and $(L^1\cap L^\infty)(\mu)$ simply by $(L^1+L^\infty)(\mathbb{R})$ and $(L^1\cap L^\infty)(\mathbb{R})$, respectively. Lebesgue measure itself will always be denoted by m.

EXAMPLE 4.1. Let $E = (L^1 + L^\infty)(\mathbb{R})$ and $\mathfrak{U} = C_{\mathrm{b}}(\mathbb{R})$ be the algebra of all bounded continuous functions on $\mathbb{R}$. We first determine the algebra $\mathfrak{U}^E_{\mathrm{w}}$, i.e. the closure of $\mathfrak{U}$ with respect to τ^E_{w}. Recall that $\mathfrak{U}^E_{\mathrm{w}}$ is also the closure of $\mathfrak{U}$ with respect to τ^E_{s}, where τ^E_{s} is the topology generated by the family of semi-norms $\varphi \mapsto \|\varphi f\|_E$ as f varies in $(L^1 + L^\infty)(\mathbb{R})$. We will require the following result.

PROPOSITION 4.1.1. *Let (Ω, Σ, μ) be a Maharam measure space, $\mathfrak{U} \subseteq L^\infty(\mu)$ be a unital norm-closed subalgebra and $E = (L^1 + L^\infty)(\mu)$. Then*

$$\mathfrak{U}^E_{\mathrm{w}} = L^\infty(\mu) \cap \mathrm{cl}_E(\mathfrak{U}),$$

where $\mathrm{cl}_E(\mathfrak{U})$ denotes the norm closure of $\mathfrak{U}$ in E.

Proof. Fix $\varphi \in \mathfrak{U}^E_{\mathrm{w}}$ and choose a net $\{\varphi_\alpha\}$ in $\mathfrak{U}$ such that $\varphi_\alpha \xrightarrow{\alpha} \varphi$ with respect to τ^E_{s}. By the definition of τ^E_{s} this implies, in particular, that $\|\varphi_\alpha - \varphi\|_{L^1+L^\infty} \xrightarrow{\alpha} 0$. Hence, $\varphi \in \mathrm{cl}_E(\mathfrak{U})$ which shows that $\mathfrak{U}^E_{\mathrm{w}} \subseteq L^\infty(\mu) \cap \mathrm{cl}_E(\mathfrak{U})$.

Now choose $\varphi \in L^\infty(\mu) \cap \mathrm{cl}_E(\mathfrak{U})$. Then there exists a sequence $\{\varphi_n\}_{n=1}^\infty$ in $\mathfrak{U}$ such that $\|\varphi_n - \varphi\|_{L^1+L^\infty} \to 0$ as $n \to \infty$. By replacing φ_n with $(\varphi_n \wedge \|\varphi\|_\infty \mathbb{1}) \vee (-\|\varphi\|_\infty \mathbb{1})$, if necessary, we may assume that $|\varphi_n| \leqslant \|\varphi\|_\infty \mathbb{1}$ for $n = 1, 2, \ldots$. Now write $\varphi_n - \varphi = \psi'_n + \psi''_n$ where $\psi'_n \in L^1(\mu)$ and $\psi''_n \in L^\infty(\mu)$ satisfy $|\psi'_n| \leqslant |\varphi_n - \varphi|$ and $|\psi''_n| \leqslant |\varphi_n - \varphi|$ and have the property that $\|\psi'_n\|_1 \to 0$ and $\|\psi''_n\|_\infty \to 0$ as $n \to \infty$. By passing to a subsequence we may further assume that $\psi'_n \to 0$ pointwise μ-a.e. as $n \to \infty$. Note that $\psi'_n \in (L^1 \cap L^\infty)(\mu)$ and $|\psi'_n| \leqslant 2\|\varphi\|_\infty \mathbb{1}$ for all $n = 1, 2, \ldots$. Fix $f \in (L^1 + L^\infty)(\mu)$. Then $\psi'_n f \in L^1(\mu)$ and so

$$\begin{aligned}\|(\varphi_n - \varphi) f\|_{L^1+L^\infty} &\leqslant \|\psi'_n f\|_{L^1+L^\infty} + \|\psi''_n f\|_{L^1+L^\infty}\\ &\leqslant \|\psi'_n f\|_1 + \|\psi''_n\|_\infty \|f\|_{L^1+L^\infty}.\end{aligned}$$

Writing $f = g + h$, with $g \in L^1(\mu)$ and $h \in L^\infty(\mu)$, we have $\|\psi'_n f\|_1 \leqslant \|\psi'_n g\|_1 + \|\psi'_n\|_1 \|h\|_\infty$. By the dominated convergence theorem it follows that $\|\psi'_n g\|_1 \to 0$ as $n \to \infty$. This shows that $\|\psi'_n f\|_1 \to 0$ as $n \to \infty$ and hence, that $\|(\varphi_n - \varphi) f\|_1 \to 0$ as $n \to \infty$. Consequently, $\varphi_n \to \varphi$ with respect to τ_{s}^E as $n \to \infty$ and we conclude that $\varphi \in \mathfrak{U}_{\mathrm{w}}^E$. ∎

Returning to the example, where we recall that $\mathfrak{U} = C_{\mathrm{b}}(\mathbb{R})$, let $A \subseteq \mathbb{R}$ be any Lebesgue measurable set. Choose a sequence $\varphi_n \in C_{\mathrm{b}}(\mathbb{R})$, for $n = 1, 2, \ldots$, satisfying $0 \leqslant \varphi_n \leqslant \mathbb{1}$ $(n = 1, 2, \ldots)$ and $\|\varphi_n - \chi_A\|_1 \to 0$ as $n \to \infty$. Then $\|\varphi_n - \chi_A\|_{L^1 + L^\infty} \to 0$ as $n \to \infty$ and so $\chi_A \in \mathrm{cl}_E(\mathfrak{U})$. Since any $\varphi \in L^\infty(\mathbb{R})$ can be approximated uniformly by step functions based on Lebesgue measurable sets, it follows that $L^\infty(\mathbb{R}) \subseteq \mathrm{cl}_E(\mathfrak{U})$. This implies that $\mathrm{cl}_E(\mathfrak{U}) = (L^1 + L^\infty)(\mathbb{R})$. From Proposition 4.1.1 it now follows that $\mathfrak{U}_{\mathrm{w}}^E = L^\infty(\mathbb{R})$. The inclusions (3.1) then yield

$$\mathfrak{U} \subseteq \mathfrak{U}_{\mathrm{w}}^E = \mathfrak{U}_{\mathrm{cc}}^E = \overline{\mathfrak{U}^*} = L^\infty(\mathbb{R}). \quad ∎$$

Example 4.2. Let $E = (L^1 + L^\infty)(\mathbb{R})$ and $\mathfrak{U} = \mathrm{BUC}(\mathbb{R})$ be the algebra of all bounded uniformly continuous functions on $\mathbb{R}$. In this case it is not so straightforward to compute $\mathfrak{U}_{\mathrm{w}}^E$ explicitly. First we show that $\mathfrak{U}_{\mathrm{w}}^E$ contains sufficiently many functions to determine $\mathfrak{U}_{\mathrm{cc}}^E$ (via Proposition 3.5). For this purpose we require the following slightly more general result. Let $C_{\mathrm{c}}^\infty(\mathbb{R})$ denote the space of all compactly supported C^∞-functions on $\mathbb{R}$.

Proposition 4.2.1. *Let $E = (L^1 + L^\infty)(\mathbb{R})$ and suppose that $\mathfrak{U} \subseteq L^\infty(\mathbb{R})$ is any unital norm-closed subalgebra containing $C_{\mathrm{c}}^\infty(\mathbb{R})$. Then $\chi_A \in \mathfrak{U}_{\mathrm{w}}^E$ for all bounded Lebesgue measurable sets $A \subseteq \mathbb{R}$. In particular, $\mathfrak{U}_{\mathrm{cc}}^E = L^\infty(\mathbb{R})$.*

Proof. If A is a bounded Lebesgue measurable subset of $\mathbb{R}$, then there exists a sequence $\{\varphi_n\}_{n=1}^\infty$ in $C_{\mathrm{c}}^\infty(\mathbb{R})$ such that $\|\varphi_n - \chi_A\|_1 \to 0$ as $n \to 0$ and hence also $\|\varphi_n - \chi_A\|_{L^1 + L^\infty} \to 0$ as $n \to \infty$. This shows that $\chi_A \in \mathrm{cl}_E(\mathfrak{U})$ and so, by Proposition 4.1.1, $\chi_A \in \mathfrak{U}_{\mathrm{w}}^E$. The last statement of the lemma follows immediately from Proposition 3.5. ∎

Remark. We note that in $E = (L^1 + L^\infty)(\mathbb{R})$ the condition $C_{\mathrm{c}}^\infty(\mathbb{R}) \subseteq \mathfrak{U}$ actually implies that $(L^1 \cap L^\infty)(\mathbb{R}) \subseteq \mathfrak{U}_{\mathrm{w}}^E$. This follows from Proposition 4.2.1 and a routine approximation argument. ∎

Now we return to the situation of $\mathfrak{U} = \mathrm{BUC}(\mathbb{R})$ in Example 4.2. From Proposition 4.2.1 and (3.1) it follows that

$$\mathfrak{U} \subseteq \mathfrak{U}_{\mathrm{w}}^{E} \subseteq \mathfrak{U}_{\mathrm{cc}}^{E} = \overline{\mathfrak{U}^{*}} = L^{\infty}(\mathbb{R}). \tag{4.1}$$

It is clear that the first inclusion in (4.1) is strict. We now show that the second inclusion is also strict. From Proposition 4.1.1 we know that $\mathfrak{U}_{\mathrm{w}}^{E} = L^{\infty}(\mathbb{R}) \cap \mathrm{cl}_E(\mathfrak{U})$. For any function $\varphi \in (L^1 + L^\infty)(\mathbb{R})$ and $t \in \mathbb{R}$ define the translated function φ_t by $\varphi_t(x) = \varphi(x - t)$. If $\varphi \in \mathfrak{U} = \mathrm{BUC}(\mathbb{R})$, then $\lim_{t\to 0} \|\varphi_t - \varphi\|_\infty = 0$ and hence $\lim_{t\to 0} \|\varphi_t - \varphi\|_{L^1+L^\infty} = 0$. Consequently, $\lim_{t\to 0} \|\varphi_t - \varphi\|_{L^1+L^\infty} = 0$ for all $\varphi \in \mathrm{cl}_E(\mathfrak{U})$. Define the function $\psi \in L^\infty(\mathbb{R})$ by $\psi = \sum_{n=-\infty}^{\infty} \chi_{[2n,2n+1]}$. It is easy to see that $(\psi_t - \psi)^*(s) = 1$ for all $s > 0$ and $|t| < 1$ and hence that $\|\psi_t - \psi\|_{L^1+L^\infty} = 1$ for all $|t| < 1$. Accordingly, $\psi \notin \mathfrak{U}_{\mathrm{w}}^{E}$ and we conclude that the second inclusion in (4.1) is strict.

Although we cannot provide an explicit description of $\mathfrak{U}_{\mathrm{w}}^{E}$ it is possible to give a characterization of $\mathfrak{U}_{\mathrm{w}}^{E}$. In the argument above we observed that $\lim_{t\to 0} \|f_t - f\|_{L^1+L^\infty} = 0$ whenever $f \in \mathrm{cl}_{L^1+L^\infty}(\mathrm{BUC}(\mathbb{R}))$. Actually, this property characterizes functions in $\mathrm{cl}_{L^1+L^\infty}(\mathrm{BUC}(\mathbb{R}))$. Indeed, suppose that $f \in (L^1 + L^\infty)(\mathbb{R})$ satisfies $\|f_t - f\|_{L^1+L^\infty} \to 0$ as $t \to 0$. Fix any non-negative function $\psi \in C_{\mathrm{c}}(\mathbb{R})$ and define $\psi_n : x \mapsto n\psi(nx)$ for each $n = 1, 2, \ldots$. Then the convolutions $f * \psi_n$ belong to $\mathrm{BUC}(\mathbb{R})$ for all $n = 1, 2, \ldots$. Using the assumption on f a routine argument shows that $\|f - (\psi_n * f)\|_{L^1+L^\infty} \to 0$ as $n \to \infty$. Hence $f \in \mathrm{cl}_{L^1+L^\infty}(\mathrm{BUC}(\mathbb{R}))$. In combination with Proposition 4.1.1 this implies that

$$\mathfrak{U}_{\mathrm{w}}^{E} = \{\varphi \in L^\infty(\mathbb{R}) : \lim_{t\to 0} \|\varphi - \varphi_t\|_{L^1+L^\infty} = 0\}$$

for the case of $\mathfrak{U} = \mathrm{BUC}(\mathbb{R})$ and $E = (L^1 + L^\infty)(\mathbb{R})$. ∎

EXAMPLE 4.3. Let $E = (L^1 + L^\infty)(\mathbb{R})$ and $\mathfrak{U} = \mathrm{AP}(\mathbb{R})$ be the real algebra of all almost periodic functions on $\mathbb{R}$, i.e., the norm-closure in $L^\infty_{\mathbb{R}}(\mathbb{R})$ of the real trigonometric polynomials. First observe that $\mathfrak{U} \cap L^1(\mathbb{R}) = \{0\}$; this follows from [12], Lemma VI.5.14, for example. Therefore, the conditions of Theorem 3.4 are satisfied. Moreover, $\overline{\mathfrak{U}^{*}} = L^\infty(\mathbb{R})$ since $\mathfrak{U}$ separates points of $L^1(\mathbb{R})$ by the uniqueness theorem for Fourier transforms of L^1-functions. Hence,

$$\mathfrak{U} = \mathfrak{U}_{\mathrm{w}}^{E} \subseteq \mathfrak{U}_{\mathrm{cc}}^{E} \subseteq \mathfrak{R}(\mathfrak{U}) \subseteq \overline{\mathfrak{U}^{*}} = L^\infty(\mathbb{R}), \tag{4.2}$$

where $\mathfrak{R}(\mathfrak{U})$ is the Dedekind closure of $\mathfrak{U}$ in $L^\infty(\mathbb{R})$. According to Theorem 3.4, also $\mathfrak{R}(\mathfrak{U}) \cap L^1(\mathbb{R}) = \{0\}$ and so the last inclusion in (4.2) is clearly strict. Next we show that the first inclusion in (4.2) is also strict by exhibiting a class of non-continuous functions in $\mathfrak{U}^E_{\mathrm{cc}}$. Take any $\psi \in \mathfrak{U} = \mathrm{AP}(\mathbb{R})$ such that $m(\{x \in \mathbb{R} : \psi(x) = 0\}) = 0$ and let $A = \{x \in \mathbb{R} : \psi(x) > 0\}$. We claim that $\chi_A \in \mathfrak{U}^E_{\mathrm{cc}}$. Indeed, let $T \in \mathfrak{M}_E(\mathfrak{U})^{\mathrm{c}}$. If $f \in (L^1 + L^\infty)(\mathbb{R})$ satisfies $f \perp \chi_A$, then $\psi^+ Tf = T(\psi^+ f) = 0$ and so $Tf \perp \chi_A$. Since $\mathbb{R} \setminus A = \{x \in \mathbb{R} : \psi(x) < 0\} \cup \{x \in \mathbb{R} : \psi(x) = 0\}$ with $m(\{x \in \mathbb{R} : \psi(x) = 0\}) = 0$, a similar argument shows that $f \perp \chi_{\mathbb{R}\setminus A}$ implies $Tf \perp \chi_{\mathbb{R}\setminus A}$. This implies that T is reduced by the projection M_{χ_A}, i.e., $TM_{\chi_A} = M_{\chi_A}T$. The claim is thereby proved.

At the present time we do not know if the second inclusion in (4.2) is strict. ∎

We now wish to concentrate on the space $E = (L^1 \cap L^\infty)(\mu)$. Before turning our attention to specific examples we begin with a general description of $\mathfrak{U}^E_{\mathrm{cc}}$ in this case. First we require some notation and terminology.

Let (Ω, Σ, μ) be a Maharam measure space. For any $A \in \Sigma$ let $L^\infty(A, \mu)$ denote the L^∞-space on A with respect to the restriction of μ to A. Clearly $L^\infty(A, \mu)$ can be identified with the closed subspace of $L^\infty(\mu)$ consisting of all functions which vanish on $\Omega \setminus A$. Let $\Sigma_f = \{A \in \Sigma : \mu(A) < \infty\}$. Given a unital norm-closed subalgebra $\mathfrak{U} \subseteq L^\infty(\mu)$ and $A \in \Sigma_f$ we define the unital norm-closed subalgebra $\mathfrak{U}_A = \{\varphi\chi_A : \varphi \in \mathfrak{U}\}$ of $L^\infty(A, \mu)$. The *local Dedekind closure* $\mathfrak{R}_{\mathrm{loc}}(\mathfrak{U})$ of $\mathfrak{U}$ in $L^\infty(\mu)$ is now defined by

$$\mathfrak{R}_{\mathrm{loc}}(\mathfrak{U}) = \{\varphi \in L^\infty(\mu) : \varphi\chi_A \in \mathfrak{R}(\mathfrak{U}_A) \text{ for all } A \in \Sigma_f\};$$

here $\mathfrak{R}(\mathfrak{U}_A)$ denotes the Dedekind closure of $\mathfrak{U}_A$ in $L^\infty(A, \mu)$. It is easy to check that $\mathfrak{R}(\mathfrak{U}) \subseteq \mathfrak{R}_{\mathrm{loc}}(\mathfrak{U})$. In general this inclusion is strict (see Examples 4.7 and 4.10 below).

Theorem 4.4. *Let (Ω, Σ, μ) be a Maharam measure space, $E = (L^1 \cap L^\infty)(\mu)$ and let $\mathfrak{U} \subseteq L^\infty(\mu)$ be a unital norm-closed subalgebra. Then $\mathfrak{U}^E_{\mathrm{cc}} = \mathfrak{R}_{\mathrm{loc}}(\mathfrak{U})$.*

Proof. First we show that $\mathfrak{R}_{\mathrm{loc}}(\mathfrak{U}) \subseteq \mathfrak{U}^E_{\mathrm{cc}}$. Take $\varphi \in \mathfrak{R}_{\mathrm{loc}}(\mathfrak{U})$ and $T \in \mathfrak{M}_E(\mathfrak{U})^{\mathrm{c}}$. For $A \in \Sigma_f$ define $T_A \in \mathcal{L}(L^\infty(A, \mu))$ by $T_A(f) = \chi_A T(f)$ for all $f \in L^\infty(A, \mu)$. Is is easy to verify that $T_A \in \mathfrak{M}_{L^\infty(A,\mu)}(\mathfrak{U}_A)^{\mathrm{c}}$. By Theorem 1.2 in [18] we have that $\mathfrak{M}_{L^\infty(A,\mu)}(\mathfrak{U}_A)^{\mathrm{cc}} = \mathfrak{M}_{L^\infty(A,\mu)}(\mathfrak{R}(\mathfrak{U}_A))$. By hypothesis $\varphi\chi_A \in \mathfrak{R}(\mathfrak{U}_A)$ and so $\varphi\chi_A T_A(f) = T_A(\varphi\chi_A f)$ for all $f \in L^\infty(A, \mu)$. This shows that $\chi_A \varphi T(\chi_A f) = \chi_A T(\varphi\chi_A f)$ for all $f \in (L^1 \cap L^\infty)(\mu)$ and all $A \in \Sigma_f$. Using the density in $(L^1 \cap L^\infty)(\mu)$ of all step functions based on Σ_f it follows that $M_\varphi T = TM_\varphi$. This shows that $\varphi \in \mathfrak{U}^E_{\mathrm{cc}}$.

To establish that $\mathfrak{U}^E_{\mathrm{cc}} \subseteq \mathfrak{R}_{\mathrm{loc}}(\mathfrak{U})$ let $\varphi \in \mathfrak{U}^E_{\mathrm{cc}}$. By Theorem 1.2 in [18] and the definition of $\mathfrak{R}_{\mathrm{loc}}(\mathfrak{U})$ it suffices to show that $M_{\varphi\chi_A} \in \mathfrak{M}_{L^\infty(A,\mu)}(\mathfrak{U}_A)^{\mathrm{cc}}$ for all

$A \in \Sigma_f$. To this end let $A \in \Sigma_f$ and $S \in \mathfrak{M}_{L^\infty(A,\mu)}(\mathfrak{U}_A)^{\mathrm{c}}$ be given. Define the operator $T \in \mathcal{L}(E)$ by $Tf = S(\chi_A f)$ for all $f \in E$. It is then routine to verify that $T \in \mathfrak{M}_E(\mathfrak{U})^{\mathrm{c}}$. Since $\varphi \in \mathfrak{U}^E_{\mathrm{cc}}$, this implies that $M_\varphi T = TM_\varphi$. From the definition of T it then follows that $M_{\varphi\chi_A} S = SM_{\varphi\chi_A}$. This completes the proof. ∎

EXAMPLE 4.5. Let $E = (L^1 \cap L^\infty)(\mathbb{R})$ and $\mathfrak{U} = C_{\mathrm{b}}(\mathbb{R})$. It is then easy to check that $\mathfrak{R}_{\mathrm{loc}}(\mathfrak{U}) = \mathfrak{R}(\mathfrak{U}) = \mathcal{R}(\mathbb{R})$, where $\mathcal{R}(\mathbb{R})$ is the algebra of all bounded Riemann measurable functions on $\mathbb{R}$ (cf. [18], Example 2.4). Hence, from (3.1) and Theorem 4.4 it follows that

$$\mathfrak{U} \subseteq \mathfrak{U}^E_{\mathrm{w}} \subseteq \mathfrak{U}^E_{\mathrm{cc}} = \mathcal{R}(\mathbb{R}) \subseteq \overline{\mathfrak{U}^*} = L^\infty(\mu). \tag{4.3}$$

It remains to determine the algebra $\mathfrak{U}^E_{\mathrm{w}}$ for which we need the following result.

LEMMA 4.5.1. *Let $E = (L^1 \cap L^\infty)(\mathbb{R})$. Then τ^E_{s} coincides with the $\|\cdot\|_\infty$-norm topology on $C_{\mathrm{b}}(\mathbb{R})$.*

Proof. Recall that τ^E_{s} is generated by the family of semi-norms $\varphi \mapsto \|\varphi f\|_{L^1\cap L^\infty}$ as f varies in $(L^1 \cap L^\infty)(\mathbb{R})$. Clearly τ^E_{s} is weaker than the $\|\cdot\|_\infty$-topology on $L^\infty(\mathbb{R})$. It remains to show that on $C_{\mathrm{b}}(\mathbb{R})$ the $\|\cdot\|_\infty$-topology is weaker than τ^E_{s}. For this purpose it suffices to show that there exists $f \in (L^1 \cap L^\infty)(\mathbb{R})$ such that

$$\{\varphi \in C_{\mathrm{b}}(\mathbb{R}) : \|\varphi f\|_{L^1\cap L^\infty} < 1\} \subseteq \{\varphi \in C_{\mathrm{b}}(\mathbb{R}) : \|\varphi\|_\infty < 1\}.$$

Let A be any open dense subset of $\mathbb{R}$ with $0 < m(A) < \infty$ and suppose that $\varphi \in C_{\mathrm{b}}(\mathbb{R})$ is a non-zero function such that $\|\varphi\chi_A\|_{L^1\cap L^\infty} < 1$. Take $0 < r < \|\varphi\|_\infty$. Then $\{x \in \mathbb{R} : |\varphi(x)| > r\} \cap A$ is open and non-empty and so $r \leqslant \|\varphi\chi_A\|_\infty \leqslant \|\varphi\chi_A\|_{L^1\cap L^\infty}$. Hence $\|\varphi\|_\infty \leqslant \|\varphi\chi_A\|_{L^1\cap L^\infty} < 1$ and the proof is complete. ∎

Since $C_{\mathrm{b}}(\mathbb{R})$ is $\|\cdot\|_\infty$-complete, the above lemma implies, in particular, that $\mathfrak{U} = C_{\mathrm{b}}(\mathbb{R})$ is τ^E_{s}-closed and hence that it is τ^E_{w}-closed, i.e., $\mathfrak{U}^E_{\mathrm{w}} = \mathfrak{U}$. So we conclude in this case that

$$\mathfrak{U} = \mathfrak{U}^E_{\mathrm{w}} \subseteq \mathfrak{U}^E_{\mathrm{cc}} = \mathcal{R}(\mathbb{R}) \subseteq \overline{\mathfrak{U}^*} = L^\infty(\mathbb{R}) \tag{4.4}$$

with both inclusions strict. ∎

EXAMPLE 4.6. Let $E = (L^1 \cap L^\infty)(\mathbb{R})$ and $\mathfrak{U} = \mathrm{BUC}(\mathbb{R})$. From Lemma 4.5.1 it is clear that $\mathfrak{U}^E_{\mathrm{w}} = \mathfrak{U}$. We again have that $\mathfrak{R}(\mathfrak{U}) = \mathfrak{R}_{\mathrm{loc}}(\mathfrak{U}) = \mathcal{R}(\mathbb{R})$. Therefore, the situation is exactly as in (4.4). ∎

Example 4.7. Let $E = (L^1 \cap L^\infty)(\mathbb{R})$ and $\mathfrak{U} = \mathrm{AP}(\mathbb{R})$. Again Lemma 4.5.1 implies that $\mathfrak{U}_{\mathrm{w}}^E = \mathfrak{U}$. Furthermore, since the restriction to a closed bounded interval in $\mathbb{R}$ of the almost periodic functions coincides with the space of all continuous functions on this interval, it is clear that $\mathfrak{R}_{\mathrm{loc}}(\mathfrak{U}) = \mathcal{R}(\mathbb{R})$. Hence, also $\mathfrak{U}_{\mathrm{cc}}^E = \mathcal{R}(\mathbb{R})$ by Theorem 4.4. However, $\mathfrak{R}(\mathfrak{U})$ is strictly contained in $\mathfrak{R}_{\mathrm{loc}}(\mathfrak{U})$. Indeed, as observed in Example 4.3 we have $\mathfrak{R}(\mathfrak{U}) \cap L^1(\mathbb{R}) = \{0\}$. Accordingly, (4.4) is valid for $\mathfrak{U} = \mathrm{AP}(\mathbb{R})$, but now the inclusion $\mathfrak{R}(\mathfrak{U}) \subseteq \mathfrak{R}_{\mathrm{loc}}(\mathfrak{U})$ is strict. ∎

Examples 4.1–4.7 have been chosen to be representative in the sense that they illustrate (non-trivially) the variety of phenomena that occur. Of course, there are other examples of algebras acting in $(L^1 + L^\infty)(\mathbb{R})$ and $(L^1 \cap L^\infty)(\mathbb{R})$ which are of interest for various reasons. For instance, the norm-closed unital subalgebra $C_\ell(\mathbb{R})$ consisting of the continuous functions f for which $\lim_{|x|\to\infty} f(x)$ exists can be identified with (the real part of) the norm-closed unital algebra generated by the subalgebra $\widehat{L}^1(\mathbb{R}) = \{\widehat{g} : g \in L^1(\mathbb{R})\}$ and so is of interest from the viewpoint of harmonic analysis. The same is true of $\widehat{M_{\mathbb{R}}}(\mathbb{R})$, consisting of the real part of the norm closure in $L^\infty(\mathbb{R})$ of the algebra of all Fourier-Stieltjes transforms of finite regular Borel measures on $\mathbb{R}$. Then $C_\ell(\mathbb{R}) \subseteq \widehat{M_{\mathbb{R}}}(\mathbb{R}) \subseteq \mathrm{BUC}(\mathbb{R})$ with all inclusions strict. The algebra $\mathfrak{U}$ of all even functions in $L^\infty(\mathbb{R})$ shows that $\mathfrak{U}$ need not always be translation invariant or satisfy $\overline{\mathfrak{U}^*} = L^\infty(\mathbb{R})$. If A is a Lebesgue measurable subset of $\mathbb{R}$ with $m(A) > 0$, then the algebra consisting of all elements of $L^\infty(\mathbb{R})$ which are constant m-a.e. on A shows that not all algebras need have the property that they contain $x \mapsto f(-x)$ whenever they contain f. And so on. The interested reader may wish to pursue the details of such additional algebras (and many more, of course).

We should also point out that algebras like $C_\ell(\mathbb{R})$ and $C_{\mathrm{b}}(\mathbb{R})$ have analogues in the setting of $(L^1 + L^\infty)(\mu)$ and $(L^1 \cap L^\infty)(\mu)$ for any regular Maharam measure μ on the Borel sets of a σ-compact, locally compact Hausdorff space, for example. Similarly, algebras like $C_\ell(\mathbb{R})$, $\widehat{M_{\mathbb{R}}}(\mathbb{R})$, $\mathrm{AP}(\mathbb{R})$ and $\mathrm{BUC}(\mathbb{R})$ have analogues in $(L^1 + L^\infty)(\mu)$ and $(L^1 \cap L^\infty)(\mu)$ when μ is Haar measure on a σ-compact, locally compact abelian group. By modifying appropriately the arguments used for m and $\mathbb{R}$ the results and examples above can be carried over to these more general settings.

We end this section with some examples relevant to specific kinds of Maharam measure spaces which, nevertheless, illustrate various points. We first collect some elementary facts which are needed; their proofs are routine and so are omitted.

REMARK 4.8. (i) Let (Ω, Σ, μ) be a finite measure space. Then $(L^1 + L^\infty)(\mu) = L^1(\mu)$ with equivalence of norms and $(L^1 \cap L^\infty)(\mu) = L^\infty(\mu)$ with equivalence of norms.

(ii) Let Ω be a non-empty set, $\Sigma = 2^\Omega$ be the σ-algebra of all subsets of Ω and let μ be counting measure. Then $(L^1 + L^\infty)(\mu) = \ell^\infty(\Omega)$ with equality of norms and $(L^1 \cap L^\infty)(\mu) = \ell^1(\Omega)$ with equality of norms.

(iii) Let $E = L^\infty(\mu)$, with (Ω, Σ, μ) an arbitrary Maharam measure space, and $\mathfrak{U} \subseteq L^\infty(\mu)$ be an arbitrary unital norm-closed subalgebra. Then the topology τ_{s}^E restricted to $\mathfrak{U}$ coincides with the $\|\cdot\|_\infty$-norm topology restricted to $\mathfrak{U}$. In particular, $\mathfrak{U}_{\mathrm{w}}^E = \mathfrak{U}$. ∎

EXAMPLE 4.9. Let $\Omega = \mathbb{N}$ and μ be counting measure on $\Sigma = 2^{\mathbb{N}}$. Let $E = (L^1 + L^\infty)(\mu)$, i.e. $E = \ell^\infty$ (cf. Remark 4.8) and consider $\mathfrak{U} = c$, the unital norm-closed subalgebra of $\ell^\infty = L^\infty(\mu)$ consisting of all the convergent sequences. Since $\mathfrak{U}$ separates points of $L^1(\mu) = \ell^1$ it follows that $\overline{\mathfrak{U}^*} = \ell^\infty$. By Theorem 1.2 in [18] we have that $\mathfrak{U}_{\mathrm{cc}}^E = \mathfrak{R}(\mathfrak{U})$. Also $\mathfrak{U} = \mathfrak{U}_{\mathrm{w}}^E$ (cf. Remark 4.8 (iii)). By Theorem 3.1 it follows that $\mathfrak{U}_{\mathrm{cc}}^E \subseteq \overline{\mathfrak{U}^*}$. If $T \in \mathfrak{M}_E(\mathfrak{U})^{\mathrm{c}}$ then, in particular, $TM_{\chi_{\{n\}}} = M_{\chi_{\{n\}}}T$ for each $n \in \mathbb{N}$ (since $\chi_{\{n\}} \in c$). Letting Σ_0 denote the family of all singleton subsets of $\mathbb{N}$ it follows from Proposition 2.2 that $T \in \mathfrak{M}_E(L^\infty(\mu))$. So, we have shown that $\mathfrak{M}_E(\mathfrak{U})^{\mathrm{c}} \subseteq \mathfrak{M}_E(L^\infty(\mu))$ and hence, $\mathfrak{M}_E(\mathfrak{U})^{\mathrm{c}} = \mathfrak{M}_E(L^\infty(\mu))$. This implies that $\mathfrak{U}_{\mathrm{cc}}^E = \ell^\infty$. Combining all of these observations yields

$$c = \mathfrak{U} = \mathfrak{U}_{\mathrm{w}}^E \subseteq \mathfrak{U}_{\mathrm{cc}}^E = \mathfrak{R}(\mathfrak{U}) = \overline{\mathfrak{U}^*} = \ell^\infty,$$

where the indicated inclusion is strict. ∎

EXAMPLE 4.10. Let $\Omega = [0,1]$ and μ be counting measure on $\Sigma = 2^\Omega$. Let $E = (L^1 + L^\infty)(\mu)$, i.e., $E = \ell^\infty(\Omega)$ (cf. Remark 4.8) and consider $\mathfrak{U} = C([0,1])$, the subalgebra of $L^\infty(\mu) = \ell^\infty(\Omega)$ consisting of the continuous functions on $[0,1]$. Theorem 1.2 in [18] shows that $\mathfrak{U}_{\mathrm{cc}}^E = \mathfrak{R}(\mathfrak{U})$. Also $\mathfrak{U} = \mathfrak{U}_{\mathrm{w}}^E$ (cf. Remark 4.8(iii)) and Theorem 3.1 implies that $\mathfrak{U}_{\mathrm{cc}}^E \subseteq \overline{\mathfrak{U}^*}$. It is shown in Example 2.3 of [18] that $\mathfrak{R}(\mathfrak{U}) = \mathfrak{U}$. These facts show that

$$C([0,1]) = \mathfrak{U} = \mathfrak{U}_{\mathrm{w}}^E = \mathfrak{R}(\mathfrak{U}) = \mathfrak{U}_{\mathrm{cc}}^E \subset \overline{\mathfrak{U}^*} = \ell^\infty([0,1]). \tag{4.5}$$

The only point still to be verified is that $\overline{\mathfrak{U}^*} = \ell^\infty([0,1])$, which follows from the observation that $C([0,1])$ separates the points of $L^1(\mu) = \ell^1([0,1])$. To see this, let $\varphi \in \ell^1([0,1])$ and suppose that $\langle \varphi, f \rangle = 0$ for all $f \in C([0,1])$, i.e., $\sum_{w \in A} \varphi(w) f(w) = 0$ for all $f \in C([0,1])$, where the set $A = \{w \in [0,1] : \varphi(w) \neq 0\}$ is countable. But, if ν denotes the finite regular Borel measure given by $\nu(E) = \sum_{w\in A} \varphi(w)\delta_w(E)$, for each Borel set $E \subseteq [0,1]$ (where δ_w is the Dirac point measure at w), then

$$\int_\Omega f \mathrm{d}\nu = \sum_{w\in A} f(w)\nu(\{w\}) = \sum_{w\in A} f(w)\varphi(w) = 0, \quad f \in C([0,1]).$$

Accordingly, ν is the zero measure and so $\varphi(w) = \nu(\{w\}) = 0$, for all $w \in A$, i.e., $\varphi = 0$ in $\ell^1([0,1])$.

It was noted in Section 2 that $\mathfrak{R}(\mathfrak{U}) = \mathfrak{U}$ whenever $\mathfrak{U}$ is weak-star closed in $L^\infty(\mu)$; see (2.1). We see from (4.5) that $\mathfrak{R}(\mathfrak{U}) = \mathfrak{U}$ can also hold without $\mathfrak{U}$ being weak-star closed.

Now let $F = (L^1 \cap L^\infty)(\mu)$, i.e., $F = L^1(\mu) = \ell^1([0,1])$. Then Corollary 3.2 implies that $\mathfrak{U}_{\mathrm{w}}^F = \mathfrak{U}_{\mathrm{cc}}^F = \overline{\mathfrak{U}^*}$. Accordingly,

$$C([0,1]) = \mathfrak{U} = \mathfrak{R}(\mathfrak{U}) \subset \mathfrak{U}_{\mathrm{w}}^F = \mathfrak{U}_{\mathrm{cc}}^F = \overline{\mathfrak{U}^*} = L^\infty(\mu).$$

Theorem 4.4 shows that $\mathfrak{R}_{\mathrm{loc}}(\mathfrak{U}) = \mathfrak{U}_{\mathrm{cc}}^F = L^\infty(\mu)$ and so we have another example where the containment $\mathfrak{R}(\mathfrak{U}) \subseteq \mathfrak{R}_{\mathrm{loc}}(\mathfrak{U})$ is strict. ∎

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