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REDUCED HEAT KERNELS ON HOMOGENEOUS SPACES

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ABSTRACT. If S is the semigroup generated by an n-th order strongly elliptic operator on $L_p(X; \mathrm{d}x)$ associated with the left regular representation of a unimodular Lie group G in the homogeneous space $X = G/M$, where M is a compact subgroup of G, and κ is the reduced heat kernel of S defined by

$$(S_t\varphi)(x) = \int_X \kappa_t(x\,;y)\,\varphi(y)\,\mathrm{d}y$$

then we prove Gaussian upper bounds for κ_t and all its derivatives.

For reduced heat kernels associated with irreducible unitary representations on nilpotent Lie groups we prove similar Gaussian bounds.

KEYWORDS: *Reduced heat kernel, Gaussian bounds, homogeneous space, strongly elliptic operator, left regular representation, nilpotent Lie group, irreducible unitary representation, Kirillov theory.*

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1. INTRODUCTION

Various methods have been developed in the last few years for the derivation of Gaussian bounds on the kernels of strongly elliptic and subelliptic operators on manifolds or Lie groups. These methods are described in the books [5], [12] and [16]. The first method, used by Davies, is a logarithmic Sobolev inequality to obtain semigroup bounds for real second order strongly elliptic operators. Via a perturbation method one then obtains Gaussian type upper bounds. Alternatively, [16] uses Harnack inequalities and that method is also restricted to real second order operators. Robinson, however, first proves Nash inequalities in order to derive semigroup bounds. Then via the Davies perturbation method, Gaussian type upper bounds are established. His method also works for higher order strongly elliptic operators.

In this paper we consider homogeneous spaces $X = G/M$ with G a connected unimodular Lie group and M a connected compact Lie subgroup. Let H be a complex n-th order strongly elliptic operator affiliated to the left regular representation of G in $L_2(X;\, \mathrm{d}x)$, where $\mathrm{d}x$ is the G-invariant measure on X induced by the Haar measures of G and M. Then the semigroup S generated by the closure of H has a smooth kernel K on G such that

$$S_t\varphi = \int_G K_t(g)U(g)\varphi\, \mathrm{d}g$$

for all $t > 0$ and $\varphi \in L_2(X;\, \mathrm{d}x)$, where U denotes the left regular representation of G in $L_2(X;\, \mathrm{d}x)$ (see [12], Theorem III.2.1). We shall show that the semigroup S has a heat kernel on $X \times X$ which can be expressed as an integral of the Lie group kernel K. For this kernel, and all its derivatives, we prove Gaussian type upper bounds in terms of the natural distance on the homogeneous space.

On non-compact symmetric spaces, Anker ([1]), studied the functional calculus of Laplace operators. In [1] heat kernel upper bounds are derived for these Laplace operators. In this paper, the Lie group G need not be semisimple.

We use the notation of [12]. Let G be a connected unimodular d-dimensional Lie group with Haar measure $\mathrm{d}g$ and let M be a $d_{\mathfrak{m}}$-dimensional compact connected Lie subgroup of G with Haar measure $\mathrm{d}m$. Let $a_1, \ldots, a_d$ be a vector space basis for the Lie algebra $\mathfrak{g}$ of G. Define the homogeneous space $X = G/M$. By [14], Satz III.3.2 there exists a G-invariant measure $\mathrm{d}x = \mathrm{d}\dot{g}$ induced by the Haar measures $\mathrm{d}g$ and $\mathrm{d}m$. By $\dot{g}$ we denote the left coset gM for all $g \in G$. If U is a continuous representation of G in a Banach space $\mathcal{X}$ then for all $i \in \{1, \ldots, d\}$ we denote by $A_i = \mathrm{d}U(a_i)$ the infinitesimal generator of the one parameter group $t \mapsto U(\exp(-ta_i))$. We also need multi-index notation. Let $J(d) = \bigoplus_{k=0}^{\infty} \{1, \ldots, d\}^k$ denote the set of all multi-indices over the index set $\{1, \ldots, d\}$. If $\alpha = (i_1, \ldots, i_k) \in J(d)$ then set $A^\alpha = A_{i_1} \circ \cdots \circ A_{i_k}$ and we denote by $|\alpha| = k$ the length of the multi-index α.

For $p \in [1, \infty]$ consider the left regular representation U of G on $L_p(X;\, \mathrm{d}x)$ defined by

$$\big(U(g)\varphi\big)(x) = \varphi(g^{-1}x) \tag{1.1}$$

for all $\varphi \in L_p(X;\, \mathrm{d}x)$ and a.e. $x \in X$.

Let $n \in \mathbb{N}$ be even and for all $\alpha \in J(d)$ with $|\alpha| \leqslant n$ let $c_\alpha \in \mathbb{C}$. We consider the operator H in $L_p(X; \mathrm{d}x)$ associated with the left regular representation U of (1.1)

$$H = \sum_{\alpha:\, |\alpha| \leqslant n} c_\alpha A^\alpha, \tag{1.2}$$

with domain $D(H) = \bigcap_{|\alpha| \leqslant n} D(A^\alpha)$. The operator H is called an n-th order strongly elliptic operator if there exists a $\mu > 0$ such that

$$\mathrm{Re}\,(-1)^{n/2} \sum_{\alpha:\, |\alpha| = n} c_\alpha \xi^\alpha \geqslant \mu |\xi|^n$$

for all $\xi \in \mathbb{R}^d$, where $\xi^\alpha = \xi_{i_1} \cdots \xi_{i_k}$ for all $\alpha = (i_1, \ldots, i_k) \in J(d)$. By [12], Theorem I.5.1, the closure of H generates a continuous semigroup S. Note that the operator H is already closed if $p \in (1, \infty)$ (see [3], Theorem 2.9). Moreover, for all $t > 0$ there exists a smooth, rapidly decreasing, Lie group kernel $K_t \in L_1(G; \mathrm{d}g)$ such that

$$(S_t \varphi)(x) = \int_G K_t(g) (U(g)\varphi)(x) \, \mathrm{d}g = \int_G K_t(g) \varphi(g^{-1}x) \, \mathrm{d}g$$

for all $\varphi \in L_p(X; \mathrm{d}x)$ and a.e. $x \in X$ (see [12], Theorem III.2.1).

Since K_t is continuous and M is compact one can define for all $t > 0$ the continuous function $\kappa_t : X \times X \to \mathbb{C}$ by

$$\kappa_t(\dot{g}; \dot{k}) = \int_M K_t(gmk^{-1}) \, \mathrm{d}m$$

where $g, k \in G$. Note that this definition does not depend on the coset representatives by the unimodularity of M. We call the function κ_t the *reduced heat kernel* because of the following identity.

Proposition 1.1. *If $p \in [1, \infty]$, $\varphi \in L_p(X; \mathrm{d}x)$ and $t > 0$ then*

$$(S_t \varphi)(x) = \int_X \kappa_t(x; y) \varphi(y) \, \mathrm{d}y$$

for a.e. $x \in X$.

Proof. Let $\psi \in C_c^\infty(X)$. Since G and M are unimodular [14], Satz III.3.2 gives

$$\begin{aligned}(\psi, S_t\varphi) &= \int_X \overline{\psi(\dot g)} \int_G K_t(l)\varphi(l^{-1}gM)\,\mathrm{d}l\,\mathrm{d}\dot g \\ &= \int_X \overline{\psi(\dot g)} \int_G K_t(l^{-1})\varphi(lgM)\,\mathrm{d}l\,\mathrm{d}\dot g \\ &= \int_X \overline{\psi(\dot g)} \int_G K_t(gl^{-1})\varphi(lM)\,\mathrm{d}l\,\mathrm{d}\dot g \\ &= \int_X \overline{\psi(\dot g)} \int_X \Big(\int_M K_t(gm^{-1}k^{-1})\,\mathrm{d}m\Big)\varphi(kM)\,\mathrm{d}\dot k\,\mathrm{d}\dot g \\ &= \int_X \overline{\psi(\dot g)} \int_X \Big(\int_M K_t(gmk^{-1})\,\mathrm{d}m\Big)\varphi(\dot k)\,\mathrm{d}\dot k\,\mathrm{d}\dot g\end{aligned}$$

for all $t > 0$ and $\varphi \in L_p(X;\mathrm{d}x)$. ∎

The function $t \mapsto \kappa_t(x;y)$ with $x, y \in X$ fixed extends to a holomorphic function since S is holomorphic (see also [2], Theorem 3.1).

We now discuss the regularity of the reduced heat kernel κ_t.

Proposition 1.2. *For all $t > 0$ one has $\kappa_t \in C^\infty(X \times X)$.*

Proof. Define for $t > 0$ the function $\widetilde{K}_t : G \times G \to \mathbb{C}$ by

$$\widetilde{K}_t(g_1, g_2) = \int_M K_t(g_1 m g_2^{-1})\,\mathrm{d}m.$$

Then $\widetilde{K}_t \in C^\infty(G \times G)$ since $K_t \in C^\infty(G)$ (see [12], Theorem III.4.8). The projection $(g_1, g_2) \mapsto (\dot g_1, \dot g_2)$ from $G \times G$ into $X \times X$ is a C^∞ map. From these observations it follows that $\kappa_t \in C^\infty(X \times X)$ for all $t > 0$. ∎

We denote the (multi-)derivatives of the reduced kernel κ_t with respect to the first variable by A^α and with respect to the second variable by R^α. To avoid confusion we denote the left derivative in the direction a_i on the Lie group G by $\widetilde{A}_i$ and the right derivative by $\widetilde{R}_i$. Derivatives of the reduced heat kernel κ can be expressed in terms of derivatives of the Lie group kernel K.

Corollary 1.3. *If $\alpha, \beta \in J(d)$ then*

$$(A^\alpha R^\beta \kappa_t)(\dot g; \dot k) = \int_M (\widetilde{A}^\alpha \widetilde{R}^\beta K_t)(gmk^{-1})\,\mathrm{d}m$$

for all $t > 0$ and $g, k \in G$.

Proof. The proof of this corollary is similar to the proof of Proposition 1.1. ∎

Introduce the control metric d_1 on X by

$$(1.3)\quad d_1(x;y) = \sup\Big\{|\psi(x)-\psi(y)| \;\Big|\; \psi \in C_{\mathrm{b};\infty}(X) \text{ real and } \sum_{i=1}^{d} |A_i\psi|^2 \leqslant 1\Big\},$$

where $C_{\mathrm{b};\infty}(X)$ denotes the space of all infinitely differentiable functions on X with uniformly bounded derivatives. The main result of this paper is the next theorem.

THEOREM 1.4. *Let $X = G/M$ be a homogeneous space with G a connected unimodular Lie group and M a compact connected subgroup. Let H be an n-th order strongly elliptic operator as in* (1.2) *and κ_t the corresponding reduced heat kernel. Then for all $\alpha, \beta \in J(d)$ there exist $a, b > 0$ and $\omega \geqslant 0$ such that*

$$|(A^\alpha R^\beta \kappa_t)(x;y)| \leqslant a t^{-(|\alpha|+|\beta|+d-d_{\mathrm{m}})/n} e^{\omega t} e^{-b(d_1(x;y)^n t^{-1})^{1/(n-1)}}$$

for all $x, y \in X$ and $t > 0$.

REMARK 1.5. Although this theorem has been formulated for real t, it is also valid in a complex sector. There exists a $\theta_C \in (0, \pi/2]$ such that the operator $e^{i\varphi} H$ is a strongly elliptic operator for all $\varphi \in (-\theta_C, \theta_C)$. Then the reduced heat kernel extends to a holomorphic function on the sector $\Lambda(\theta_C) = \{z \in \mathbb{C}\backslash\{0\} \;\big|\; |\arg z| < \theta_C\}$ and for all $\theta \in (0, \theta_C)$ one has similar kernel bounds for κ_z uniformly for $z \in \Lambda(\theta)$, with t replaced by $|z|$.

EXAMPLE 1.6. If G is a connected semisimple Lie group and M a connected compact subgroup of G then G is unimodular and the conclusions of Theorem 1.4 are valid for the homogeneous space G/M.

In the next example we present an explicit description of a homogeneous space.

EXAMPLE 1.7. Let $X = \mathrm{SL}(r,\mathbb{R})/\mathrm{SO}(r,\mathbb{R})$. Consider the action of $g \in \mathrm{SL}(r,\mathbb{R})$ defined by $A \mapsto gAg^t$ for all strictly positive symmetric matrices A. Then $\mathrm{SO}(r,\mathbb{R})$ is the stabilizer subgroup of the identity matrix I. The $\mathrm{SL}(r,\mathbb{R})$-orbit of I equals the set of all strictly positive symmetric matrices with determinant 1. Indeed, each strictly positive symmetric matrix A with determinant 1 can be written as $A = U\Lambda U^t$ with $U \in \mathrm{SO}(r,\mathbb{R})$ and Λ a diagonal matrix with strictly positive diagonal entries and determinant 1. Let $g = U\Lambda^{1/2} \in \mathrm{SL}(r,\mathbb{R})$. Then $A = gIg^t$. So we can identify X with the set of strictly positive symmetric matrices with determinant 1.

On $\mathrm{SO}(3,\mathbb{R})$ one can use spectral theory to deduce Gaussian bounds for the reduced heat kernel.

Example 1.8. Let $N = (0, 0, 1) \in \mathbb{R}^3$. Let $G = \mathrm{SO}(3, \mathbb{R})$ and $M = G_N = \{g \in \mathrm{SO}(3, \mathbb{R}) \mid gN = N\}$ the stabilisator group of N. Let a_1, a_2, a_3 be the basis for $\mathfrak{g}$ given by

$$a_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

So $d = 3$ and $d_{\mathfrak{m}} = 1$, whence $d - d_{\mathfrak{m}} = 2$. Consider the bijection $\Phi : G/M \to \mathbb{S}^2$ defined by $\Phi(\dot{g}) = gN$ for all $g \in G$. For all $i \in \{1, 2, 3\}$ let $\widehat{A}_i$ be the vector fields on $\mathbb{S}^2$ induced by the bijection Φ and the vector fields A_i on X. The measure $dx = d\dot{g}$ on X induces the surface measure $d\mu$ on $\mathbb{S}^2$. Next, the G-invariant second order strongly elliptic operator $-\widehat{A}_1^2 - \widehat{A}_2^2 - \widehat{A}_3^2$ equals the Laplace-Beltrami operator of $\mathbb{S}^2$. For this particular operator one can derive the Gaussian type upper bounds by a spectral argument. For all $n \in \mathbb{N}_0$ the eigenspace corresponding to the eigenvalue $n(n+1)$ is spanned by $2n+1$ orthonormal eigenvectors $e_{n,j}$ with $j \in \{1, \ldots, 2n+1\}$. Then by the addition theorem, ([11], Theorem 2) there exist $C_1, C_2 > 0$ such that

$$\begin{aligned} |\kappa_t(x; x)| &= \Big| \sum_{n=0}^{\infty} \mathrm{e}^{-t(n^2+n)} \sum_{j=1}^{2n+1} e_{n,j}(x) \overline{e_{n,j}(x)} \Big| \\ &\leqslant C_1 \Big(1 + \sum_{n=0}^{\infty} \mathrm{e}^{-t(n^2+n)} n \Big) \leqslant C_2 (1 + t^{-2/2}) = C_2 (1 + t^{-(d-d_{\mathfrak{m}})/2}) \end{aligned}$$

for all $t > 0$ and $x \in \mathbb{S}^2$. Then off-diagonal Gaussian upper bounds as in Theorem 1.4 can be obtained by an application of [15], Theorem 1. ∎

The techniques used in this paper differ from the usual methods, because we cannot apply the higher order Davies perturbation trick as in [12], Chapter III. First, we derive the appropriate Nash inequalities if $M \cap Z(G)$ is finite, where $Z(G)$ denotes the centre of G. This is inspired by the paper [7], where Gaussian bounds for real second order strongly elliptic operators associated with irreducible unitary representations of nilpotent Lie groups have been established. These Nash inequalities are used to obtain semigroup bounds for second order operators. Then Gaussian bounds for higher order strongly elliptic operators on the homogeneous space are derived via a reduction method from the Gaussian bounds for higher order strongly elliptic operators on the Lie group. In [9] a slightly less delicate version of a transference method was used to deduce large time Gaussian bounds for the kernel associated with a homogeneous operator on a nilpotent Lie group from a similar kernel on a homogeneous group. In the present paper the reduction gives the correct small time singularity in the Gaussian bound of the main theorem. Finally we remove the assumption on $M \cap Z(G)$.

The reduction method also works to extend the results in [7] to higher order operators and Gaussian kernel bounds on the derivatives. In [7] Gaussian kernel bounds have been proved for the reduced heat kernel of the semigroup generated by a real second order strongly elliptic operator associated with an irreducible unitary representation of a nilpotent Lie group. In the last section we show how the other kernel bounds can be proved.

2. VOLUME ESTIMATE

In this section we prove a lower bound on the X-volume of the projection on X of small balls in G if $M \cap Z(G)$ is finite.

Suppose the vector space basis $a_1, \ldots, a_d$ for the Lie algebra $\mathfrak{g}$ is such that $a_1, \ldots, a_{d_\mathfrak{m}}$ is a vector space basis for the Lie algebra $\mathfrak{m}$ of M.

The modulus $|\cdot|$ on the Lie group G is defined by $|g| = d^G(e; g)$, where the metric d^G on G is given by

$$d^G(k; l) = \inf_{\gamma \in \Gamma(k,l)} \int_0^1 \Big(\sum_{i=1}^{d} \gamma_i^2(t) \Big)^{1/2} \, \mathrm{d}t,$$

with

$$\Gamma(k,l) = \{\gamma : [0,1] \to G \mid \gamma \text{ is absolutely continuous on } [0,1], \gamma(0) = k, \gamma(1) = l\}$$

and

$$\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} = \sum_{i=1}^{d} \gamma_i(t) Y_i \Big|_{\gamma(t)}$$

for a.e. $t \in [0,1]$, where the vector fields Y_i are defined by

$$(Y_i \varphi)(g) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \varphi(\exp(-t a_i) g) \tag{2.1}$$

for all $\varphi \in C^\infty(G)$, $g \in G$ and $1 \leqslant i \leqslant d$. The modulus on M, denoted by $|\cdot|_M$, is defined analogously. Let $B_\varepsilon = \{g \in G \mid |g| < \varepsilon\}$ and $B_{\varepsilon,M} = \{g \in M \mid |g|_M < \varepsilon\}$ for all $\varepsilon > 0$.

Frequently we need the following lemma to estimate distances.

Lemma 2.1. *There exist $\varepsilon_0 > 0$ and $C > 0$ such that the restriction of the exponential map to the set $\{t \in \mathbb{R}^d \mid |t_i| < \varepsilon_0$ for all $i \in \{1, \ldots, d\}\}$ is an analytic diffeomorphism onto its image and*

$$C^{-1}\|t\| \leqslant |\exp(t_1 a_1 + \cdots + t_d a_d)| \leqslant C\|t\|$$

uniformly for all $t \in \{t \in \mathbb{R}^d \mid |t_i| < \varepsilon_0$ for all $i \in \{1, \ldots, d\}\}$, where $\|t\| = \max\{|t_i| \mid i \in \{1, \ldots, d\}\}$.

Proof. See [8], Proposition 6.1. ∎

Lemma 2.2. *There exist $\varepsilon' > 0$ and $C > 0$ such that*

$$B_\varepsilon \cap M \subseteq B_{C\varepsilon, M}$$

for all $\varepsilon \in (0, \varepsilon']$.

Proof. Let V be a neighbourhood of 0 in the Lie algebra $\mathfrak{g}$ such that $\exp|_V$ is a diffeomorphism from V onto a neighbourhood of the identity $e \in G$. Write $\mathfrak{g} = \mathfrak{a} + \mathfrak{m}$, where $\mathfrak{m}$ denotes the Lie algebra of M. Then by the proof of [13], Theorem 6.9, there exist neighbourhoods W of 0 in $\mathfrak{a}$ and W' of 0 in $\mathfrak{m}$ such that $W + W' \subseteq V$, the map $w + w' \mapsto \exp(w)\exp(w')$ is a homeomorphism from $W + W'$ onto the neighbourhood $U = \exp(W)\exp(W')$ of the identity $e \in G$ and, moreover, $U \cap M = \exp(W')$. Let $\varepsilon' > 0$ be so small that

$$B_{\varepsilon'} \subseteq U \cap \exp(V).$$

Then $B_{\varepsilon'} \cap M \subseteq U \cap M = \exp(W')$.

Next, let $\varepsilon_0, C > 0$ be as in Lemma 2.1. Now suppose $\varepsilon \in (0, \varepsilon' \wedge \varepsilon_0]$ and $g \in B_\varepsilon \cap M$. Then for all $i \in \{1, \ldots, d\}$ there exist $t_i \in \mathbb{R}$ with $|t_i| \leqslant C\varepsilon$ such that

$$g = \exp(t_1 a_1 + \cdots + t_d a_d).$$

Alternatively, $g \in \exp(W')$ and hence there exists a $w' \in W'$ such that $g = \exp(w')$. Since $\exp|_V$ is injective it follows that $t_{d_{\mathfrak{m}}+1} = \cdots = t_d = 0$. Therefore, by Lemma 2.1 again, there exist $\varepsilon'', C' > 0$ such that $B_\varepsilon \cap M \subseteq B_{C'\varepsilon, M}$ for all $\varepsilon \in (0, \varepsilon'']$. ∎

LEMMA 2.3. *If $\varepsilon > 0$ then*

$$\int\limits_M 1_{B_\varepsilon}(gmk^{-1})\,\mathrm{d}m \leqslant \int\limits_{M\cap g^{-1}B_{2\varepsilon}g} 1\,\mathrm{d}m$$

for all $g,k \in G$.

Proof. We may assume that there exists an $m_1 \in M$ such that $b = gm_1k^{-1} \in B_\varepsilon$. Then

$$\int\limits_M 1_{B_\varepsilon}(gmk^{-1})\,\mathrm{d}m = \int\limits_M 1_{B_\varepsilon}(gmm_1k^{-1})\,\mathrm{d}m = \int\limits_M 1_{B_\varepsilon}(gmg^{-1}b)\,\mathrm{d}m$$

by the unimodularity of M. Since $gmg^{-1}b \in B_\varepsilon$ if, and only if, $m \in g^{-1}B_\varepsilon b^{-1}g$ and $B_\varepsilon b^{-1} \subseteq B_{2\varepsilon}$ one obtains

$$\int\limits_M 1_{B_\varepsilon}(gmk^{-1})\,\mathrm{d}m = \int\limits_{M\cap g^{-1}B_\varepsilon b^{-1}g} 1\,\mathrm{d}m \leqslant \int\limits_{M\cap g^{-1}B_{2\varepsilon}g} 1\,\mathrm{d}m,$$

as required. ∎

We next introduce a technical condition which we remove at the end of Section 4.

LEMMA 2.4. *Suppose there exist $r \in \mathbb{N}$ and a continuous matrix representation ρ' of G in $\mathrm{GL}(r,\mathbb{R})$ such that $M \cap \operatorname{Ker}\rho'$ is finite. Then there exists a matrix representation ρ of G in $\mathrm{GL}(r,\mathbb{R})$ such that the restriction of ρ to M has a finite kernel and $\rho(M)$ is a subgroup of $\mathrm{SO}(r,\mathbb{R})$.*

Proof. Since M is compact and connected also $\rho'(M)$ is a compact and connected subgroup of $\mathrm{GL}(r,\mathbb{R})$ and hence of $\mathrm{SL}(r,\mathbb{R})$. Note that $(\mathrm{SL}(r,\mathbb{R}),\mathrm{SO}(r,\mathbb{R}))$ is a Riemannian symmetric pair of non-compact type. Therefore Theorem VI.2.1 of [10] implies that there exists an $h \in \mathrm{SL}(r,\mathbb{R})$ such that $h\rho'(M)h^{-1} \subseteq \mathrm{SO}(r,\mathbb{R})$. Then the representation $g \mapsto h\rho'(g)h^{-1}$ has the desired properties. ∎

LEMMA 2.5. *Suppose there exist $r \in \mathbb{N}$ and a continuous matrix representation ρ of G in $\mathrm{GL}(r,\mathbb{R})$ such that $M \cap \operatorname{Ker}\rho$ is finite and $\rho(M)$ is a subgroup of $\mathrm{SO}(r,\mathbb{R})$. Then there exists a $C > 0$ such that*

$$\|VEV^{-1} - I\| \leqslant C\varepsilon$$

for all $V \in \mathrm{SO}(r,\mathbb{R})$, $\varepsilon \in [0,2]$ and $E \in \rho(B_\varepsilon)$, where $\|\cdot\|$ denotes the Euclidean matrix norm on $\mathbb{R}^{r\times r}$.

Proof. Define for all $i \in \{1, \ldots, d\}$ the matrices M_i by

$$M_i = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \rho(\exp(ta_i)).$$

By the Campbell-Baker-Hausdorff formula it is obvious that the local map from coordinates of the first kind to coordinates of the second kind is a real analytic diffeomorphism with a Jacobian matrix of determinant 1 in 0. Hence by Lemma 2.1 there exist $\varepsilon_0, C > 0$ such that the map $(t_1, \ldots, t_d) \mapsto \exp(t_1 a_1) \cdots \exp(t_d a_d)$ is an analytic diffeomorphism from $\{t \in \mathbb{R}^d \mid |t_i| < \varepsilon_0 \text{ for all } i \in \{1, \ldots, d\}\}$ onto its image Ω and

$$C^{-1}\|t\| \leqslant |\exp(t_1 a_1) \cdots \exp(t_d a_d)| \leqslant C\|t\|$$

uniformly for all $t \in \{t \in \mathbb{R}^d \mid |t_i| < \varepsilon_0 \text{ for all } i \in \{1, \ldots, d\}\}$, where $\|t\| = \max\{|t_i| \mid i \in \{1, \ldots, d\}\}$. There exists an $\varepsilon' > 0$ such that $B_{\varepsilon'} \subseteq \Omega$. Suppose $\varepsilon \in (0, \varepsilon']$ and $E \in \rho(B_\varepsilon)$. Then for all $i \in \{1, \ldots, d\}$ there exists a $t_i \in \mathbb{R}$ with $|t_i| \leqslant C\varepsilon$ such that

$$\begin{aligned} E &= \rho(\exp(t_1 a_1) \cdots \exp(t_d a_d)) = \rho(\exp(t_1 a_1)) \cdots \rho(\exp(t_d a_d)) \\ &= \exp(t_1 M_1) \cdots \exp(t_d M_d). \end{aligned}$$

Hence

$$\|VEV^{-1} - I\| = \|V(E - I)V^{-1}\| = \|E - I\| \leqslant C'\varepsilon$$

for all $V \in \mathrm{SO}(r, \mathbb{R})$, where $C' > 0$ depends only on ε', C and $M_1, \ldots, M_d$. By compactness there exists an $M > 0$ such that

$$\|VEV^{-1} - I\| \leqslant M$$

for all $V \in \mathrm{SO}(r, \mathbb{R})$ and $E \in \rho(B_2)$. Therefore

$$\|VEV^{-1} - I\| \leqslant \max(C', M(\varepsilon')^{-1})\varepsilon$$

for all $\varepsilon \in [0, 2]$, $E \in \rho(B_\varepsilon)$ and $V \in \mathrm{SO}(r, \mathbb{R})$. ∎

Proposition 2.6. *Suppose there exist $r \in \mathbb{N}$ and a continuous matrix representation ρ of G in $\mathrm{GL}(r, \mathbb{R})$ such that $M \cap \operatorname{Ker} \rho$ is finite. Then there exists a $C > 0$ such that*

$$\mathrm{Vol}_X(B_\varepsilon \dot{g}) \geqslant C\varepsilon^{d - d_{\mathfrak{m}}}$$

for all $\varepsilon \in (0, 1]$ and $g \in G$.

Proof. By Lemma 2.4 we may assume that $\rho(M) \subset \mathrm{SO}(r, \mathbb{R})$.

First,

$$\mathrm{Vol}_G(B_\varepsilon) = \int_G 1_{B_\varepsilon}(gl^{-1})\,\mathrm{d}l = \int_X \int_M 1_{B_\varepsilon}(gmk^{-1})\,\mathrm{d}m\,\mathrm{d}\dot{k}$$
$$\leqslant \int_{B_\varepsilon \dot{g}} \int_{M\cap g^{-1}B_{2\varepsilon}g} 1\,\mathrm{d}m\,\mathrm{d}\dot{k} = \mathrm{Vol}_X(B_\varepsilon\dot{g})\mathrm{Vol}_M(M\cap g^{-1}B_{2\varepsilon}g)$$

for all $g \in G$ and $\varepsilon > 0$ by Lemma 2.3. Next, there exists a $C \geqslant 1$ such that $C^{-1}\varepsilon^d \leqslant \mathrm{Vol}_G(B_\varepsilon) \leqslant C\varepsilon^d$ for all $\varepsilon \in (0,1]$. Therefore it is sufficient to prove that there exists a $C > 0$ such that

$$\mathrm{Vol}_M(M\cap g^{-1}B_{2\varepsilon}g) \leqslant C\varepsilon^{d_{\mathfrak{m}}}$$

for all $g \in G$ and $\varepsilon \in (0,1]$. Since the restriction of ρ to M has a finite kernel and $\rho(M)$ is isomorphic to $M/\operatorname{Ker}(\rho|_M)$ it therefore suffices to show that there exists a $C > 0$ such that

$$\mathrm{Vol}_{\rho(M)}(\rho(M)\cap \rho(g^{-1}B_{2\varepsilon}g)) \leqslant C\varepsilon^{d_{\mathfrak{m}}}$$

for all $g \in G$ and $\varepsilon \in (0,1]$.

Let $\varepsilon \in (0,1]$ and $g \in G$. Let $A \in \rho(M)\cap\rho(g^{-1}B_{2\varepsilon}g)$. Consider the Gaussian decomposition

$$\rho(g^{-1}) = U\Lambda V,$$

where $U, V \in \mathrm{SO}(r,\mathbb{R})$ and $\Lambda = \mathrm{diag}\,(\lambda_1,\ldots,\lambda_r)$. By a suitable permutation of the rows and columns of Λ and the fact that permutation matrices are orthogonal we may assume without loss of generality that $|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_r|$. Then

$$U^{-1}AU = \Lambda V E V^{-1}\Lambda^{-1}$$

for some $E \in \rho(B_{2\varepsilon})$. Hence $U^{-1}AU = \Lambda\widetilde{E}\Lambda^{-1}$, where

$$\widetilde{E} = VEV^{-1} = \begin{pmatrix} 1+\varepsilon_{11} & \cdots & \varepsilon_{1r} \\ \vdots & \ddots & \vdots \\ \varepsilon_{r1} & \cdots & 1+\varepsilon_{rr} \end{pmatrix},$$

with $|\varepsilon_{ij}| \leqslant C\varepsilon$ for some $C > 0$, independent of V, E and ε, by Lemma 2.5. Therefore

$$B = U^{-1}AU = \Lambda\widetilde{E}\Lambda^{-1} = \begin{pmatrix} 1+\varepsilon_{11} & \cdots & \varepsilon_{1r}\lambda_1\lambda_r^{-1} \\ \vdots & \ddots & \vdots \\ \varepsilon_{r1}\lambda_r\lambda_1^{-1} & \cdots & 1+\varepsilon_{rr} \end{pmatrix}.$$

Since $B \in \mathrm{SO}(r, \mathbb{R})$ all columns have length 1 and are mutually orthogonal. As $|\lambda_i \lambda_j^{-1}| \leqslant 1$ for all $i \geqslant j$ there exists a $C > 0$, independent of V, E and ε, such that $|(B - I)_{ij}| \leqslant C\varepsilon$ for all $i \geqslant j$. Evaluating the inner product of the first and second column in B it follows that there is a $C > 0$, independent of V, E, Λ and ε, such that $|B_{12}| \leqslant C\varepsilon$. Repeating this procedure, i.e., evaluating the inner products of column j and the preceding columns $1, \ldots, j-1$, it follows that there exists a $C > 0$, independent of V, E, Λ and ε, such that $|(B - I)_{ij}| \leqslant C\varepsilon$ for all $i, j \in \{1, \ldots, r\}$. Since $U \in \mathrm{SO}(r, \mathbb{R})$ it follows that $|(A - I)_{ij}| = |(U(B - I)U^{-1})_{ij}| \leqslant Cr^2\varepsilon$ for all $1 \leqslant i, j \leqslant r$. Note that Cr^2 is independent of ε and g.

Next there is an $\varepsilon' \in (0, 1]$ such that if $\varepsilon \in (0, \varepsilon']$, then $A = \exp(\log A)$. Hence there exists a $C > 0$, independent of ε and g, such that $|(\log A)_{ij}| \leqslant C\varepsilon$ for all $1 \leqslant i, j \leqslant r$. Therefore, by Lemma 2.1 there is a $C > 0$, independent of g and $\varepsilon \in (0, \varepsilon']$, such that A lies in the $C\varepsilon$-ball in $\mathrm{GL}(r, \mathbb{R})$ induced by the modulus on $\mathrm{GL}(r, \mathbb{R})$. Then by Lemma 2.2 there exists a $C > 0$, independent of g and $\varepsilon \in (0, \varepsilon']$ such that A lies in the $C\varepsilon$-ball in $\rho(M)$ induced by the modulus on $\rho(M)$. Since the dimension of $\rho(M)$ equals $d_{\mathfrak{m}}$ there is a $C > 0$ such that

$$\mathrm{Vol}_{\rho(M)}(\rho(M) \cap \rho(g^{-1} B_{2\varepsilon} g)) \leqslant C\varepsilon^{d_{\mathfrak{m}}}$$

for all $g \in G$ and $\varepsilon \in (0, \varepsilon']$. Finally, the restriction $\varepsilon \in (0, \varepsilon']$ can be weakened to $\varepsilon \in (0, 1]$ by a compactness argument. ∎

Corollary 2.7. *Suppose $M \cap Z(G)$ is finite. Then there exists a $C > 0$ such that*

$$\mathrm{Vol}_X(B_\varepsilon \dot{g}) \geqslant C\varepsilon^{d - d_{\mathfrak{m}}}$$

for all $\varepsilon \in (0, 1]$ and $g \in G$.

Proof. The matrix representation induced by the adjoint representation Ad of G and the basis $a_1, \ldots, a_d$ for $\mathfrak{g}$ has kernel $Z(G)$. ∎

3. NASH INEQUALITIES

The method we use to derive semigroup bounds from $L_1(X;\, \mathrm{d}x)$ into $L_\infty(X;\, \mathrm{d}x)$ is via Nash inequalities, which we prove in this section.

Throughout this section we suppose that $M \cap Z(G)$ is finite. Moreover, we suppose that the vector space basis $a_1, \ldots, a_d$ for the Lie algebra $\mathfrak{g}$ is such that $a_1, \ldots, a_{d_\mathfrak{m}}$ is a vector space basis for the Lie algebra $\mathfrak{m}$ of M.

Let $L_{2;1}(X;\, \mathrm{d}x) = \bigcap_{i=1}^{d} D(A_i) \subset L_2(X;\, \mathrm{d}x)$ with norm

$$\|\varphi\|_{2;1} = \max_{\substack{\alpha \in J(d) \\ |\alpha| \leqslant 1}} \|A^\alpha \varphi\|_2.$$

The bounds of the next proposition are a generalization of the classical Nash inequalities.

Theorem 3.1. *There exists a $C > 0$ such that*

$$\|\varphi\|_2^{2+4/(d-d_\mathfrak{m})} \leqslant C \|\varphi\|_{2;1}^2 \|\varphi\|_1^{4/(d-d_\mathfrak{m})}$$

for all $\varphi \in L_{2;1}(X;\, \mathrm{d}x) \cap L_1(X;\, \mathrm{d}x)$.

In order to prove Theorem 3.1 we first prove Young inequalities on the homogeneous space.

Let $\varphi \in L_2(X;\, \mathrm{d}x) \cap L_1(X;\, \mathrm{d}x)$ and $\psi \in L_1(G;\, \mathrm{d}g)$. Then the *convolution product* $\psi *_U \varphi$ is defined by

$$\psi *_U \varphi = \int_G \psi(g) U(g) \varphi \, \mathrm{d}g.$$

For every integrable function $\psi : G \to \mathbb{C}$ introduce the function $\psi^\flat : X \times X \to \mathbb{C}$ by

$$\psi^\flat(\dot{g}, \dot{k}) = \int_M \psi(g m k^{-1}) \, \mathrm{d}m,$$

where $g, k \in G$.

Next let $\psi : G \to \mathbb{C}$ be integrable and let $q \in [1, \infty)$. Then $|||\psi|||_q$ is defined by

$$|||\psi|||_q = \operatorname*{ess\,sup}_{x \in X} \Big(\int_X |\psi^\flat(x,y)|^q \, \mathrm{d}y \Big)^{1/q}$$

and $|||\psi|||_\infty$ is defined by

$$|||\psi|||_\infty = \operatorname*{ess\,sup}_{x \in X} \operatorname*{ess\,sup}_{y \in X} |\psi^\flat(x,y)|.$$

If $\Psi : X \times X \to \mathbb{C}$ is measurable, then for $q \in [1,\infty)$ set

$$||||\Psi||||_q = \operatorname*{ess\,sup}_{y \in X} \Big(\int\limits_X |\Psi(x,y)|^q \, \mathrm{d}x \Big)^{1/q}$$

and

$$||||\Psi||||_\infty = \operatorname*{ess\,sup}_{y \in X} \operatorname*{ess\,sup}_{x \in X} |\Psi(x,y)|.$$

Note that the integration and essential supremum are taken over different variables.

The next elementary lemma gives a relation between the norms.

Lemma 3.2. *If $\varepsilon > 0$ and $q \in [1,\infty]$ then*

$$||||\psi_\varepsilon^\flat||||_q = |||\psi_\varepsilon|||_q,$$

where $\psi_\varepsilon = 1_{B_\varepsilon}$.

Proof. Let $\varepsilon > 0$. Then by the unimodularity of M one has for all $q \in [1,\infty)$

$$\begin{aligned} ||||\psi_\varepsilon^\flat||||_q &= \operatorname*{ess\,sup}_{\dot{k} \in X} \Big(\int\limits_X |\psi_\varepsilon^\flat(\dot{g},\dot{k})|^q \, \mathrm{d}\dot{g} \Big)^{1/q} \\ &= \operatorname*{ess\,sup}_{\dot{k} \in X} \Big(\int\limits_X \Big(\int\limits_M 1_{B_\varepsilon}(gmk^{-1}) \, \mathrm{d}m \Big)^q \, \mathrm{d}\dot{g} \Big)^{1/q} \\ &= \operatorname*{ess\,sup}_{\dot{k} \in X} \Big(\int\limits_X \Big(\int\limits_M 1_{B_\varepsilon}(km^{-1}g^{-1}) \, \mathrm{d}m \Big)^q \, \mathrm{d}\dot{g} \Big)^{1/q} \\ &= \operatorname*{ess\,sup}_{\dot{k} \in X} \Big(\int\limits_X \Big(\int\limits_M 1_{B_\varepsilon}(kmg^{-1}) \, \mathrm{d}m \Big)^q \, \mathrm{d}\dot{g} \Big)^{1/q} \\ &= \operatorname*{ess\,sup}_{\dot{k} \in X} \Big(\int\limits_X |\psi_\varepsilon^\flat(\dot{k},\dot{g})|^q \, \mathrm{d}\dot{g} \Big)^{1/q} = |||\psi_\varepsilon|||_q. \end{aligned}$$

The equality for $q = \infty$ is proved similarly. ∎

We are now going to prove Young type inequalities. In order to do this we first need some preparation.

Let S denote the set of complex valued integrable simple functions $\Psi : X \times X \to \mathbb{C}$ such that $K = \{x \in X \mid \exists\, y \in X \text{ with } \Psi(x,y) \neq 0\}$ has finite measure.

The following lemma states some very crucial properties for simple functions in the set S.

Lemma 3.3. (i) *If* $\Psi \in S$ *then*

$$|||\Psi|||_\infty = \operatorname*{ess\,sup}_{y\in X} \operatorname*{ess\,sup}_{x\in X} |\Psi(x,y)| = \operatorname*{ess\,sup}_{(x,y)\in X\times X} |\Psi(x,y)| = \operatorname*{ess\,sup}_{x\in X} \operatorname*{ess\,sup}_{y\in X} |\Psi(x,y)|.$$

(ii) *Let* $\Psi \in S$ *and set* $a = |||\Psi|||_\infty$. *Define* $E_{a,y} = \{x \in X \mid |\Psi(x,y)| = a\}$ *for all* $y \in X$. *Then* $\int_X \int_X 1_{E_{a,y}}(x)\,\mathrm{d}x\,\mathrm{d}y \neq 0$.

Proof. If $\Psi = 1_U$ for some measurable $U \subset X \times X$ with measure zero then it is obvious that

$$\operatorname*{ess\,sup}_{x\in X} \operatorname*{ess\,sup}_{y\in X} |\Psi(x,y)| = \operatorname*{ess\,sup}_{(x,y)\in X\times X} |\Psi(x,y)| = \operatorname*{ess\,sup}_{y\in X} \operatorname*{ess\,sup}_{x\in X} |\Psi(x,y)| = 0.$$

Similarly, if $\Psi = 1_U$ for some measurable $U \subseteq X\times X$ with strictly positive measure then it is elementary that

$$\operatorname*{ess\,sup}_{x\in X} \operatorname*{ess\,sup}_{y\in X} |\Psi(x,y)| = \operatorname*{ess\,sup}_{(x,y)\in X\times X} |\Psi(x,y)| = \operatorname*{ess\,sup}_{y\in X} \operatorname*{ess\,sup}_{x\in X} |\Psi(x,y)| = 1.$$

Now let $\Psi \in S$ be arbitrary. Then there exist $k \in \mathbb{N}$, $a_1,\dots,a_k \geqslant 0$ and disjoint measurable sets $U_1,\dots,U_k \subseteq X \times X$ such that $|\Psi| = \sum_{i=1}^k a_i 1_{U_i}$. Since $\operatorname*{ess\,sup}_{(x,y)\in X\times X} ((f\vee g)(x,y)) = \operatorname*{ess\,sup}_{(x,y)\in X\times X} f(x,y) \vee \operatorname*{ess\,sup}_{(x,y)\in X\times X} g(x,y)$ and $|\Psi| = a_1 1_{U_1} \vee \dots \vee a_k 1_{U_k}$, the first part of the lemma follows.

If $\Psi = 1_U$ for some measurable $U \subseteq X\times X$ then it is obvious that the second statement holds. If $|\Psi| = \sum_{i=1}^k a_i 1_{U_i}$ as in the first part and $a_1 = a$ without loss of generality (and the measure of U_1 is strictly positive), then

$$\int_X \int_X 1_{E_{a,y}}(x)\,\mathrm{d}x\,\mathrm{d}y \geqslant \int_X \int_X 1_{U_1}(x,y)\,\mathrm{d}x\,\mathrm{d}y > 0,$$

as required. ∎

Define for $\varphi \in L_2(X;\mathrm{d}x) \cap L_1(X;\mathrm{d}x)$ the linear operator $T_\varphi : S \to \bigcap_{p=1}^\infty L_p(X;\mathrm{d}x)$ by

$$(T_\varphi \Psi)(x) = \int_X \Psi(x,y)\varphi(y)\,\mathrm{d}y$$

for a.e. $x \in X$. The following theorem is a slight variation of the classical Riesz-Thorin theorem.

Theorem 3.4. *Let $\varphi \in L_2(X; \mathrm{d}x) \cap L_1(X; \mathrm{d}x)$, $\alpha_1, \alpha_2 \in (0,1)$, $\beta_1, \beta_2 \in (0,1)$ and $M_1, M_2 \geqslant 0$. Suppose*

$$\|T_\varphi \Psi\|_{1/\beta_1} \leqslant M_1 |||\Psi|||_{1/\alpha_1}, \quad \|T_\varphi \Psi\|_{1/\beta_2} \leqslant M_2 |||\Psi|||_{1/\alpha_2}$$

for all functions $\Psi \in S$. Then

$$\|T_\varphi \Psi\|_{1/\beta} \leqslant M_1^{1-t} M_2^t |||\Psi|||_{1/\alpha}$$

for all $\Psi \in S$ and $t \in (0,1)$, where $\alpha = (1-t)\alpha_1 + t\alpha_2$ and $\beta = (1-t)\beta_1 + t\beta_2$.

Proof. Define the functions $\alpha, \beta : \mathbb{C} \to \mathbb{C}$ by

$$\alpha(z) = (1-z)\alpha_1 + z\alpha_2, \quad \beta(z) = (1-z)\beta_1 + z\beta_2$$

for all $z \in \mathbb{C}$. For $z = 0$, $z = 1$ and $z = t$ the pair $(\alpha(z), \beta(z))$ reduces to (α_1, β_1), (α_2, β_2) and (α, β), respectively. Note that

$$\|T_\varphi \Psi\|_{1/\beta} = \sup_{\substack{\sigma \in S', \\ \|\sigma\|_{1/(1-\beta)} = 1}} \Big| \int_X (T_\varphi \Psi)(x) \sigma(x) \, \mathrm{d}x \Big|$$

for all $\beta \in (0,1)$ and $\Psi \in S$, where S' denotes the set of all simple functions on X. Fix $\Psi \in S$ and $\sigma \in S'$ with $\|\sigma\|_{1/(1-\beta)} = 1$. Define I by

$$I = \int_X (T_\varphi \Psi)(x) \sigma(x) \, \mathrm{d}x.$$

Let $c_1, c_2, \ldots, c_p$ be the different values of Ψ not equal to zero and let $\chi_1, \chi_2, \ldots, \chi_p$ be the corresponding characteristic functions. Write $c_j = |c_j| \mathrm{e}^{\mathrm{i} u_j}$. Define

$$F_z = \sum_{j=1}^p \mathrm{e}^{\mathrm{i} u_j} |c_j|^{\alpha(z)/\alpha} \chi_j$$

for all $z \in \mathbb{C}$. Similarly, let $d_1, d_2, \ldots, d_q$ be the different values of σ not equal to zero and let $\chi'_1, \chi'_2, \ldots, \chi'_q$ be the corresponding characteristic functions. Write $d_j = |d_j| \mathrm{e}^{\mathrm{i} v_j}$ and define

$$\sigma_z = \sum_{j=1}^q \mathrm{e}^{\mathrm{i} v_j} |d_j|^{(1-\beta(z))/(1-\beta)} \chi'_j$$

for all $z \in \mathbb{C}$. Replacing χ_j by $T_\varphi \chi_j$ yields an expression for $T_\varphi F_z$. Define $\Phi : \mathbb{C} \to \mathbb{C}$ by

$$\Phi(z) = \int_X (T_\varphi F_z)(x) \sigma_z(x) \, \mathrm{d}x.$$

Then $\Phi(t) = I$ and it is obvious from these considerations that Φ is a bounded, continuous and holomorphic function on $\{z = x + \mathrm{i}y \in \mathbb{C} \mid 0 < x < 1\}$.

Consider $z \in \mathbb{C}$ with $\mathrm{Re}\, z = 0$. Then $\mathrm{Re}\, \alpha(z) = \alpha_1$ and $\mathrm{Re}\, \beta(z) = \beta_1$. The Hölder inequality gives

$$|\Phi(z)| \leqslant \|T_\varphi F_z\|_{1/\beta_1} \|\sigma_z\|_{1/(1-\beta_1)} \leqslant M_1 ||||F_z||||_{1/\alpha_1} \|\sigma_z\|_{1/(1-\beta_1)}.$$

Moreover, for each $y \in X$ one has

$$\Big(\int_X |F_z(x,y)|^{1/\alpha_1}\, \mathrm{d}x \Big)^{\alpha_1} = \Big(\int_X |\Psi(x,y)|^{1/\alpha}\, \mathrm{d}x \Big)^{\alpha(\alpha_1/\alpha)}.$$

Hence

$$||||F_z||||_{1/\alpha_1} = ||||\Psi||||_{1/\alpha}^{\alpha_1/\alpha}.$$

It follows that

$$|\Phi(z)| \leqslant M_1 ||||\Psi||||_{1/\alpha}^{\alpha_1/\alpha} \|\sigma_z\|_{1/(1-\beta_1)} = M_1 ||||\Psi||||_{1/\alpha}^{\alpha_1/\alpha}$$

for all $z \in \mathbb{C}$ with $\mathrm{Re}\, z = 0$. Similarly,

$$|\Phi(z)| \leqslant M_2 ||||\Psi||||_{1/\alpha}^{\alpha_2/\alpha}$$

for all $z \in \mathbb{C}$ with $\mathrm{Re}\, z = 1$. Then the Phrágmen-Lindelöf lemma (cf. [17], Chapter XII, p. 93) gives

$$|I| = |\Phi(t)| \leqslant M_1^{1-t} M_2^t ||||\Psi||||_{1/\alpha}.$$

Therefore

$$\|T_\varphi \Psi\|_{1/\beta} = \sup_{\substack{\sigma \in S' \\ \|\sigma\|_{1/(1-\beta)}=1}} |\Phi(t)| \leqslant M_1^{1-t} M_2^t ||||\Psi||||_{1/\alpha},$$

as required. ∎

COROLLARY 3.5. *Let* $\varphi \in L_1(X; \mathrm{d}x) \cap L_2(X; \mathrm{d}x)$ *and* $M_1, M_2 \geqslant 0$. *Suppose*

$$\|T_\varphi \Psi\|_\infty \leqslant M_1 ||||\Psi||||_\infty, \quad \|T_\varphi \Psi\|_1 \leqslant M_2 ||||\Psi||||_1$$

for all $\Psi \in S$. *Then*

$$\|T_\varphi \Psi\|_{1/t} \leqslant M_1^{1-t} M_2^t ||||\Psi||||_{1/t}$$

for all $t \in (0,1)$ *and* $\Psi \in S$.

Proof. Let $\Psi \in S$ and $\varepsilon > 0$. From the definition of S it follows that $||||\Psi||||_1 < \infty$. Moreover, if $\Psi(x,y) = 0$ for a.e. $(x,y) \in X \times X$ then the proof is trivial. So let $\Psi \in S$ be such that $a = ||||\Psi||||_\infty > 0$. Set $E_{a,y} = \{x \in X \mid |\Psi(x,y)| = a\}$. Suppose that for all $n \in \mathbb{N}$ the set $\Big\{y \in X \mid \int\limits_X 1_{E_{a,y}}(x)\,\mathrm{d}x \geqslant 1/n\Big\}$ has zero measure. Then the set $\Big\{y \in X \mid \int\limits_X 1_{E_{a,y}}(x)\,\mathrm{d}x > 0\Big\}$ has zero measure, which contradicts Lemma 3.3 (ii). Therefore there exist $b > 0$ and a subset $X_0 \subseteq X$ with strictly positive measure such that $\int\limits_X 1_{E_{a,y}}(x)\,\mathrm{d}x \geqslant b$ uniformly for all $y \in X_0$. Then there exists an $N_1 \in \mathbb{N}$ such that for all $n > N_1$ and $y \in X_0$ the bounds

$$\begin{aligned} ||||\Psi||||_\infty = a &\leqslant (a^n b)^{1/n}(1+\varepsilon) \leqslant \Big(\int\limits_{E_{a,y}} a^n\,\mathrm{d}x\Big)^{1/n}(1+\varepsilon) \\ &= \Big(\int\limits_{E_{a,y}} |\Psi(x,y)|^n\,\mathrm{d}x\Big)^{1/n}(1+\varepsilon) \leqslant \Big(\int\limits_X |\Psi(x,y)|^n\,\mathrm{d}x\Big)^{1/n}(1+\varepsilon) \end{aligned}$$

are valid. Since X_0 has positive measure one deduces that

$$||||\Psi||||_\infty \leqslant ||||\Psi||||_n(1+\varepsilon)$$

for all $n > N_1$. Moreover, with $K = \{x \in X \mid \exists\, y \in X \text{ with } \Psi(x,y) \neq 0\}$ one has

$$\begin{aligned} \|T_\varphi \Psi\|_n &= \Big(\int\limits_X \Big|\int\limits_X \Psi(x,y)\varphi(y)\,\mathrm{d}y\Big|^n\,\mathrm{d}x\Big)^{1/n} = \Big(\int\limits_K \Big|\int\limits_X \Psi(x,y)\varphi(y)\,\mathrm{d}y\Big|^n\,\mathrm{d}x\Big)^{1/n} \\ &\leqslant \operatorname*{ess\,sup}_{x\in X} \Big|\int\limits_X \Psi(x,y)\varphi(y)\,\mathrm{d}y\Big|\Big(\int\limits_K 1\,\mathrm{d}x\Big)^{1/n} = \|T_\varphi\Psi\|_\infty \Big(\int\limits_K 1\,\mathrm{d}x\Big)^{1/n} \end{aligned}$$

for all $n \in \mathbb{N}$. But $\Psi(x,y) = 0$ for a.e. $(x,y) \in X \times X$ if $\int\limits_K 1\,\mathrm{d}x = 0$. Therefore assume that $\int\limits_K 1\,\mathrm{d}x \neq 0$. By definition of S one has $\int\limits_K 1\,\mathrm{d}x < \infty$. Hence there is an $N_2 > 0$ such that for all $n > N_2$

$$\|T_\varphi\Psi\|_n \leqslant \|T_\varphi\Psi\|_\infty(1+\varepsilon).$$

So

$$\|T_\varphi\Psi\|_n \leqslant \|T_\varphi\Psi\|_\infty(1+\varepsilon) \leqslant M_1(1+\varepsilon)^2 ||||\Psi||||_n \tag{3.1}$$

for all $n > N_3 = \max(N_1, N_2)$.

Alternatively, $\lim_{n\to\infty} |\Psi|^{1/(1-1/n)} = |\Psi|$ uniformly on $X \times X$. So there exists an $N \in \mathbb{N}$ such that for all $n > N$ and $y \in X$ the estimates

$$\int_X |\Psi(x,y)|\,\mathrm{d}x \leqslant \int_X |\Psi(x,y)|^{1/(1-1/n)}\,\mathrm{d}x(1+\varepsilon)^{1/2}$$

are valid. Since $0 \leqslant x \leqslant x^{1-1/n}(1+\varepsilon)^{1/2}$ for sufficiently large $n \in \mathbb{N}$, uniformly on $[0, ||||\Psi||||_1]$, there is an $N_4 \in \mathbb{N}$ such that for all $n > N_4$ and a.e. $y \in X$

$$\begin{aligned}\int_X |\Psi(x,y)|\,\mathrm{d}x &\leqslant \Big(\int_X |\Psi(x,y)|\,\mathrm{d}x\Big)^{1-1/n}(1+\varepsilon)^{1/2}\\ &\leqslant \Big(\int_X |\Psi(x,y)|^{1/(1-1/n)}\,\mathrm{d}x\Big)^{1-1/n}(1+\varepsilon).\end{aligned}$$

Therefore,

$$||||\Psi||||_1 \leqslant ||||\Psi||||_{1/(1-1/n)}(1+\varepsilon)$$

for all $n > N_4$. Moreover, the dominated convergence theorem yields

$$\lim_{n\to\infty}\int_X |(T_\varphi\Psi)(x)|^{1/(1-1/n)}\,\mathrm{d}x = \int_X |(T_\varphi\Psi)(x)|\,\mathrm{d}x.$$

Hence there exists an $N_5 \in \mathbb{N}$ such that

$$\|T_\varphi\Psi\|_{1/(1-1/n)} \leqslant \|T_\varphi\Psi\|_1(1+\varepsilon)$$

for all $n > N_5$. It follows that

$$\|T_\varphi\Psi\|_{1/(1-1/n)} \leqslant \|T_\varphi\Psi\|_1(1+\varepsilon) \leqslant M_2(1+\varepsilon)^2||||\Psi||||_{1/(1-1/n)} \tag{3.2}$$

for all $n > N_6 = \max(N_4, N_5)$.

By Theorem 3.4 one can interpolate between the bounds (3.1) and (3.2) and

$$\|T_\varphi\Psi\|_{1/((1-t)/n+t(1-1/n))} \leqslant M_1^{1-t}M_2^t(1+\varepsilon)^2||||\Psi||||_{1/((1-t)/n+t(1-1/n))}$$

for all $t \in (0,1)$ and $n > \max(N_3, N_6)$. Let $\widetilde{t} \in (0,1)$. Then there exists an $N \in \mathbb{N}$ such that

$$\|T_\varphi\Psi\|_{1/\widetilde{t}} \leqslant (1+\varepsilon)^2 M_1^{(1-\widetilde{t}-1/n)/(1-2/n)} M_2^{(\widetilde{t}-1/n)/(1-2/n)}||||\Psi||||_{1/\widetilde{t}}$$

for all $n > N$. Finally, letting $n \to \infty$ and $\varepsilon \to 0$ one obtains

$$\|T_\varphi\Psi\|_{1/\widetilde{t}} \leqslant M_1^{1-\widetilde{t}}M_2^{\widetilde{t}}||||\Psi||||_{1/\widetilde{t}}$$

for all $\widetilde{t} \in (0,1)$. ∎

In order to apply Corollary 3.5 we prove bounds on L_1 and L_∞.

Lemma 3.6. *If* $\varphi \in L_2(X;\, \mathrm{d}x) \cap L_1(X;\, \mathrm{d}x)$ *then*

$$\|T_\varphi \Psi\|_1 \leqslant |||\Psi|||_1 \|\varphi\|_1, \qquad \|T_\varphi \Psi\|_\infty \leqslant |||\Psi|||_\infty \|\varphi\|_1$$

for all $\Psi \in S$.

Proof. Let $\Psi \in S$. Then Fubini's theorem gives

$$\begin{aligned}
\|T_\varphi \Psi\|_1 &= \int_X \Big| \int_X \Psi(x,y)\varphi(y)\,\mathrm{d}y \Big|\,\mathrm{d}x \leqslant \int_X \int_X |\Psi(x,y)|\,|\varphi(y)|\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_X \Big(\int_X |\Psi(x,y)|\,\mathrm{d}x \Big) |\varphi(y)|\,\mathrm{d}y \leqslant \int_X \operatorname*{ess\,sup}_{y' \in X} \Big(\int_X |\Psi(x,y')|\,\mathrm{d}x \Big) |\varphi(y)|\,\mathrm{d}y \\
&= \operatorname*{ess\,sup}_{y \in X} \Big(\int_X |\Psi(x,y)|\,\mathrm{d}x \Big) \|\varphi\|_1 = |||\Psi|||_1 \|\varphi\|_1.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|T_\varphi \Psi\|_\infty &= \operatorname*{ess\,sup}_{x \in X} \Big| \int_X \Psi(x,y)\varphi(y)\,\mathrm{d}y \Big| \leqslant \operatorname*{ess\,sup}_{x \in X} \int_X |\Psi(x,y)||\varphi(y)|\,\mathrm{d}y \\
&\leqslant \operatorname*{ess\,sup}_{x \in X} \int_X \operatorname*{ess\,sup}_{y' \in X} \Big(|\Psi(x,y')| \Big) |\varphi(y)|\,\mathrm{d}y = |||\Psi|||_\infty \|\varphi\|_1,
\end{aligned}$$

by Lemma 3.3 (i), which completes the proof. ∎

We are now able to state Young type inequalities.

Proposition 3.7. *If* $\varphi \in L_2(X;\, \mathrm{d}x) \cap L_1(X;\, \mathrm{d}x)$ *then*

$$\|T_\varphi \Psi\|_p \leqslant |||\Psi|||_p \|\varphi\|_1$$

for all $\Psi \in S$ *and* $p \in [1,\infty]$.

Proof. This is an immediate consequence of Lemma 3.6 and Corollary 3.5. ∎

By approximation we apply the Young inequalities to obtain bounds on $\psi_\varepsilon *_U \varphi$, where we set $\psi_\varepsilon = 1_{B_\varepsilon}$.

Proposition 3.8. *Let $\varphi \in L_2(X;\, \mathrm{d}x) \cap L_1(X;\, \mathrm{d}x)$. Then*

$$\|\psi_\varepsilon *_U \varphi\|_2 \leqslant |||\psi_\varepsilon^\flat|||_2\, \|\varphi\|_1$$

for all $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$. First take $\varphi \in C_c(X)$. Let K be the support of φ. Suppose $\psi_\varepsilon^\flat(x, y) \neq 0$ for some $x = \dot{g} \in X$ and $y = \dot{k} \in X$. Then there exists an $m \in M$ such that $gmk^{-1} \in B_\varepsilon$, and hence, $x \in B_\varepsilon y$. Therefore it is obvious that $x \notin B_\varepsilon K$ implies $\psi_\varepsilon^\flat(x, y) = 0$ for all $y \in K$. Hence the function $(x, y) \mapsto \psi_\varepsilon^\flat(x, y)\varphi(y)$ from $X \times X$ into $\mathbb{C}$ has support in $B_\varepsilon K \times K$. Next

$$\begin{aligned}
\int_X \psi_\varepsilon^\flat(\dot{g}, \dot{k})\varphi(\dot{k})\, \mathrm{d}\dot{k} &= \int_X \Big(\int_M \psi_\varepsilon(gmk^{-1})\, \mathrm{d}m \Big) \varphi(\dot{k})\, \mathrm{d}\dot{k} \\
&= \int_X \Big(\int_M \psi_\varepsilon(gm^{-1}k^{-1})\varphi(kmM)\, \mathrm{d}m \Big)\, \mathrm{d}\dot{k} \\
&= \int_G \psi_\varepsilon(gl^{-1})\varphi(lM)\, \mathrm{d}l = \int_G \psi_\varepsilon(l)\varphi(l^{-1}gM)\, \mathrm{d}l \\
&= \int_G \psi_\varepsilon(l)\varphi(l^{-1}\dot{g})\, \mathrm{d}l = (\psi_\varepsilon *_U \varphi)(\dot{g})
\end{aligned}$$

for all $g \in G$ by the unimodularity of G and M. Now let $\widetilde{\Psi}_n$ be a sequence of non-negative simple functions converging to $\psi_\varepsilon^\flat$ pointwise from below. Set $\Psi_n = \widetilde{\Psi}_n \cdot 1_{B_\varepsilon K \times K}$ for all $n \in \mathbb{N}$. Then

$$\lim_{n \to \infty} \Big(\int_X \Big| \int_X (\psi_\varepsilon^\flat(x, y) - \Psi_n(x, y))\varphi(y)\, \mathrm{d}y \Big|^2 \mathrm{d}x \Big)^{1/2} = 0$$

by the dominated convergence theorem. Hence Proposition 3.7 yields

$$\|\psi_\varepsilon *_U \varphi\|_2 \leqslant \limsup_{n \to \infty} \|T_\varphi \Psi_n\|_2 + \|\psi_\varepsilon *_U \varphi - T_\varphi \Psi_n\|_2 \leqslant |||\psi_\varepsilon^\flat|||_2\, \|\varphi\|_1.$$

Since $\varphi \mapsto \psi_\varepsilon *_U \varphi$ is continuous on $L_2(X;\, \mathrm{d}x)$, the proposition follows immediately from the density of $C_c(X)$ in $L_2(X;\, \mathrm{d}x) \cap L_1(X;\, \mathrm{d}x)$. ∎

Let $\gamma : [0, 1] \to G$ be an absolutely continuous path from the identity e to g with tangents in the space spanned by $a_1, \ldots, a_d$. Then there exist $\gamma_i \in L_\infty([0, 1])$ such that

$$\frac{\mathrm{d}\psi(\gamma(t))}{\mathrm{d}t} = \sum_{i=1}^{d} \gamma_i(t)(Y_i\psi)(\gamma(t))$$

for all $\psi \in C^\infty(G)$ and a.e. $t \in [0,1]$, where Y_i is as in (2.1). Moreover,

$$|g| = \inf_\gamma \int_0^1 \Big(\sum_{i=1}^d \gamma_i(t)^2 \Big)^{1/2} \, \mathrm{d}t,$$

where the infimum is over all absolutely continuous paths from the identity e to $g \in G$. Therefore

$$((I - U(g))\varphi)(x) = \int_0^1 \sum_{i=1}^d \gamma_i(t) (L(\gamma(t)) A_i \varphi)(x) \, \mathrm{d}t$$

for all $\varphi \in C_c^\infty(X)$ and $x \in X$. Consequently,

$$\|(I - U(g))\varphi\|_2 \leqslant \int_0^1 \Big(\sum_{i=1}^d \gamma_i(t)^2 \Big)^{1/2} \Big(\sum_{i=1}^d \|A_i \varphi\|_2^2 \Big)^{1/2} \, \mathrm{d}t$$

by the Schwarz inequality. Then optimization over all possible paths γ gives

$$\|(I - U(g))\varphi\|_2 \leqslant |g| \Big(\sum_{i=1}^d \|A_i \varphi\|_2^2 \Big)^{1/2}.$$

So, if $\psi \in L_1(G; \mathrm{d}g)$ is a positive function with $\|\psi\|_1 = 1$ then

$$\|\varphi - \psi *_U \varphi\|_2 \leqslant \int_G \psi(g) |g| \Big(\sum_{i=1}^d \|A_i \varphi\|_2^2 \Big)^{1/2} \, \mathrm{d}g.$$

In the following proposition we state the Nash inequalities.

Proposition 3.9. *Let $\varphi \in L_{2;1}(X; \mathrm{d}x) \cap L_1(X; \mathrm{d}x)$. Then*

$$\|\varphi\|_2 \leqslant \varepsilon \Big(\sum_{i=1}^d \|A_i \varphi\|_2^2 \Big)^{1/2} + \Big(|||\psi_\varepsilon|||_2 / \|\psi_\varepsilon\|_1 \Big) \|\varphi\|_1$$

for all $\varepsilon > 0$.

Proof. Obviously one has

$$\|\varphi\|_2 \leqslant \|\varphi - (\psi_\varepsilon / \|\psi_\varepsilon\|_1) *_U \varphi\|_2 + \|(\psi_\varepsilon / \|\psi_\varepsilon\|_1) *_U \varphi\|_2.$$

Then the proposition follows from Lemma 3.2, Proposition 3.8 and the preparatory considerations preceding this proposition. ∎

Now we estimate for all $\varepsilon \in (0,1]$ the factor $|||\psi_\varepsilon|||_2/\|\psi_\varepsilon\|_1$ in the Nash inequality stated in Proposition 3.9. First we have $\|\psi_\varepsilon\|_1 = \mathrm{Vol}_G(B_\varepsilon)$. But there exists an $\alpha \geqslant 1$ such that

$$\alpha^{-1}\varepsilon^d \leqslant \mathrm{Vol}_G(B_\varepsilon) \leqslant \alpha\varepsilon^d \tag{3.3}$$

for all $\varepsilon \in (0,4]$. Hence

$$\alpha^{-1}\varepsilon^d \leqslant \|\psi_\varepsilon\|_1 \leqslant \alpha\varepsilon^d$$

for all $\varepsilon \in (0,4]$.

Secondly, we estimate an upper bound for the norm $|||\psi_\varepsilon|||_2$ for all $\varepsilon \in (0,1]$.

Lemma 3.10. *If $g \in G$ then*

$$\mathrm{Vol}_G(B_{2\varepsilon}) \geqslant \mathrm{Vol}_X(B_\varepsilon \dot g)\mathrm{Vol}_M(M \cap g^{-1}B_\varepsilon g)$$

for all $\varepsilon > 0$.

Proof. Let $g \in G$ and $\varepsilon > 0$. Then

$$\int_M 1_{g^{-1}B_\varepsilon g}(m)\,\mathrm{d}m \leqslant \int_M 1_{g^{-1}B_{2\varepsilon} g}(km)\,\mathrm{d}m$$

for all $k \in g^{-1}B_\varepsilon g$. Since the map $k \mapsto \int_M 1_{g^{-1}B_{2\varepsilon} g}(km)\,\mathrm{d}m$ from G into $\mathbb{R}$ is right M-invariant it follows that

$$1_{g^{-1}B_\varepsilon \dot g}(\dot k)\int_M 1_{g^{-1}B_\varepsilon g}(m)\,\mathrm{d}m \leqslant \int_M 1_{g^{-1}B_{2\varepsilon} g}(km)\,\mathrm{d}m$$

for all $k \in G$. Integration over X yields

$$\begin{aligned}\mathrm{Vol}_X(B_\varepsilon \dot g)\mathrm{Vol}_M(M \cap g^{-1}B_\varepsilon g) &= \int_X 1_{g^{-1}B_\varepsilon \dot g}(\dot k)\int_M 1_{g^{-1}B_\varepsilon g}(m)\,\mathrm{d}m\,\mathrm{d}\dot k \\ &\leqslant \int_X \Big(\int_M 1_{g^{-1}B_{2\varepsilon} g}(km)\,\mathrm{d}m\Big)\,\mathrm{d}\dot k \\ &= \int_G 1_{B_{2\varepsilon}}(glg^{-1})\,\mathrm{d}l = \mathrm{Vol}_G(B_{2\varepsilon})\end{aligned}$$

by the unimodularity of G. ∎

Proposition 3.11. *There exists a $C > 0$ such that*

$$|||\psi_\varepsilon|||_2 \leqslant C\varepsilon^{(d+d_{\mathfrak{m}})/2}$$

for all $\varepsilon \in (0,1]$.

Proof. Since we assume that $M \cap Z(G)$ is finite, there exists by Corollary 2.7 a $C > 0$ such that

$$\mathrm{Vol}_X(B_\varepsilon x) \geqslant C\varepsilon^{d-d_{\mathfrak{m}}}$$

for all $x \in X$ and $\varepsilon \in (0,1]$. Next, if $g, k \in G$ and $\int\limits_M 1_{B_\varepsilon}(gmk^{-1})\,\mathrm{d}m \neq 0$ then there exists an $m \in M$ such that $km^{-1} \in B_\varepsilon g$, whence $\dot{k} \in B_\varepsilon \dot{g}$. Then by the Lemmas 2.3 and 3.10 one has

$$\begin{aligned}
|||\psi_\varepsilon|||_2 &\leqslant \operatorname*{ess\,sup}_{\dot{g} \in X} \Big(\int\limits_{B_\varepsilon \dot{g}} \Big(\int\limits_{M \cap g^{-1} B_{2\varepsilon} g} 1\,\mathrm{d}m \Big)^2 \mathrm{d}y \Big)^{1/2} \\
&\leqslant \mathrm{Vol}_G(B_{4\varepsilon}) \operatorname*{ess\,sup}_{x \in X} \mathrm{Vol}_X(B_\varepsilon x)^{1/2} \mathrm{Vol}_X(B_{2\varepsilon} x)^{-1} \\
&\leqslant \mathrm{Vol}_G(B_{4\varepsilon}) \operatorname*{ess\,sup}_{x \in X} \mathrm{Vol}_X(B_\varepsilon x)^{-1/2} \leqslant 4^d \alpha C^{-1/2} \varepsilon^{(d+d_{\mathfrak{m}})/2}
\end{aligned}$$

for all $\varepsilon \in (0,1]$, where we used the estimates (3.3) in the last step. ∎

The following proposition is the key result to prove Theorem 3.1.

Proposition 3.12. *There exists a $C > 0$ such that*

$$\|\varphi\|_2 \leqslant \varepsilon \|\varphi\|_{2;1} + C\varepsilon^{-(d-d_{\mathfrak{m}})/2} \|\varphi\|_1$$

for all $\varepsilon > 0$ and $\varphi \in L_{2;1}(X;\mathrm{d}x) \cap L_1(X;\mathrm{d}x)$.

Proof. By the estimates for $|||\psi_\varepsilon|||_2$ for all $\varepsilon \in (0,1]$ stated in Proposition 3.11 and the Nash inequality stated in Proposition 3.9, there exists a $C > 0$ such that

$$\|\varphi\|_2 \leqslant \varepsilon \|\varphi\|_{2;1} + C\varepsilon^{-(d-d_{\mathfrak{m}})/2} \|\varphi\|_1$$

for all $\varepsilon \in (0,1]$ and $\varphi \in L_{2;1}(X;\mathrm{d}x) \cap L_1(X;\mathrm{d}x)$. But then these estimates are valid for all $\varepsilon > 0$ since $\|\varphi\|_2 \leqslant \|\varphi\|_{2;1}$. ∎

It is now easy to prove Theorem 3.1.

Proof of Theorem 3.1. Optimize the inequalities from Proposition 3.12 over $\varepsilon > 0$. ∎

In the following section the Nash inequalities are used to prove the Gaussian bounds of Theorem 1.4.

4. GAUSSIAN KERNEL BOUNDS

In this section we deduce the kernel bounds stated in Theorem 1.4. For all $r, p \in [1, \infty]$ denote by $\|T\|_{r \to p}$ the operator norm of a linear operator $T : L_r(X; \mathrm{d}x) \to L_p(X; \mathrm{d}x)$.

The next proposition is well known, but for self-consistency we include the proof.

Proposition 4.1. *Suppose $M \cap Z(G)$ is finite. Moreover, suppose that the vector space basis $a_1, \ldots, a_d$ for the Lie algebra $\mathfrak{g}$ is such that $a_1, \ldots, a_{d_\mathfrak{m}}$ is a vector space basis for the Lie algebra $\mathfrak{m}$ of M. Let S be the semigroup generated by the closure of a pure second order strongly elliptic operator H of the form*

$$H = -\sum_{i,j=1}^{d} c_{ij} A_i A_j.$$

Then there exist $a, \omega > 0$ such that

$$|(\tau, S_t \varphi)| \leqslant a t^{-(d - d_\mathfrak{m})/2} \mathrm{e}^{\omega t} \|\varphi\|_1 \|\tau\|_1$$

for all $\varphi, \tau \in C_\mathrm{c}^\infty(X)$ and $t > 0$.

Proof. Let $H_0 = H + \mu I$ and T the semigroup generated by the closure of H_0, where μ denotes the ellipticity constant. Since $\overline{H}_0$ generates a continuous semigroup on $L_1(X; \mathrm{d}x)$, by Theorem I.5.1 of [12], there exist $a, \omega > 0$ such that $\|T_t\|_{1 \to 1} \leqslant a \mathrm{e}^{\omega t}$ for all $t > 0$. Let $C > 0$ be the Nash constant as in Theorem 3.1. Let $\varphi \in L_1(X; \mathrm{d}x) \cap L_2(X; \mathrm{d}x)$. Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \|T_t \varphi\|_2^2 = -2 \,\mathrm{Re}\, (T_t \varphi, H_0 T_t \varphi) \leqslant -2\mu \|T_t \varphi\|_{2;1}^2 &\leqslant -\frac{2\mu}{C} \frac{\|T_t \varphi\|_2^{2 + 4/(d - d_\mathfrak{m})}}{\|T_t \varphi\|_1^{4/(d - d_\mathfrak{m})}} \\
&\leqslant -\frac{2\mu}{C a^{4/(d - d_\mathfrak{m})} \mathrm{e}^{4\omega t/(d - d_\mathfrak{m})}} \frac{(\|T_t \varphi\|_2^2)^{1 + 2/(d - d_\mathfrak{m})}}{\|\varphi\|_1^{4/(d - d_\mathfrak{m})}}
\end{aligned}$$

for all $t > 0$. Therefore,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} (\|T_t \varphi\|_2^2)^{-2/(d - d_\mathfrak{m})} &= -\frac{2}{d - d_\mathfrak{m}} (\|T_t \varphi\|_2^2)^{-1 - 2/(d - d_\mathfrak{m})} \frac{\mathrm{d}}{\mathrm{d}t} \|T_t \varphi\|_2^2 \\
&\geqslant \frac{4\mu}{(d - d_\mathfrak{m}) C a^{4/(d - d_\mathfrak{m})} \mathrm{e}^{4\omega t/(d - d_\mathfrak{m})}} \|\varphi\|_1^{-4/(d - d_\mathfrak{m})}
\end{aligned}$$

and by integration

$$\begin{aligned}
\|T_t \varphi\|_2^{-4/(d - d_\mathfrak{m})} &= (\|T_t \varphi\|_2^2)^{-2/(d - d_\mathfrak{m})} \\
&\geqslant t \frac{4\mu}{(d - d_\mathfrak{m}) C a^{4/(d - d_\mathfrak{m})}} \mathrm{e}^{-4\omega t/(d - d_\mathfrak{m})} \|\varphi\|_1^{-4/(d - d_\mathfrak{m})}.
\end{aligned}$$

So $\|S_t\|_{1\to 2} = \mathrm{e}^{\mu t}\|T_t\|_{1\to 2} \leqslant a' t^{-(d-d_{\mathfrak{m}})/4}\mathrm{e}^{\omega' t}$ for suitable $a', \omega' > 0$. By duality, $\|S_t\|_{2\to\infty} = \|S_t^*\|_{1\to 2} \leqslant a'' t^{-(d-d_{\mathfrak{m}})/4}\mathrm{e}^{\omega'' t}$ for suitable $a'', \omega'' > 0$ and therefore

$$\|S_t\|_{1\to\infty} \leqslant \|S_{t/2}\|_{1\to 2}\|S_{t/2}\|_{2\to\infty} \leqslant a t^{-(d-d_{\mathfrak{m}})/2}\mathrm{e}^{\omega t}$$

for redefined a and ω, uniformly for all $t > 0$. ∎

Together with the reduction formula these bounds are the main ingredient in the proof of the Gaussian bounds for higher order strongly elliptic operators.

Proposition 4.2. *Suppose $M \cap Z(G)$ is finite. Moreover, suppose that the vector space basis $a_1, \ldots, a_d$ for the Lie algebra $\mathfrak{g}$ is such that $a_1, \ldots, a_{d_{\mathfrak{m}}}$ is a vector space basis for the Lie algebra $\mathfrak{m}$ of M. Let H be an n-th order strongly elliptic operator as in* (1.2) *and κ_t the corresponding reduced heat kernel. Then for all $\alpha, \beta \in J(d)$ there exist $a, b > 0$ and $\omega \geqslant 0$ such that*

$$|(A^\alpha R^\beta \kappa_t)(x;y)| \leqslant a t^{-(|\alpha|+|\beta|+d-d_{\mathfrak{m}})/n}\mathrm{e}^{\omega t}\mathrm{e}^{-b(d_1(x;y)^n t^{-1})^{1/(n-1)}}$$

for all $x, y \in X$ and $t > 0$.

Proof. First for all $\varphi : X \to \mathbb{C}$ define the function $\pi^*\varphi : G \to \mathbb{C}$ by $(\pi^*\varphi)(g) = \varphi(\dot g)$. Let $\alpha, \beta \in J(d)$. Then the reduction formula of Corollary 1.3 gives

$$(A^\alpha R^\beta \kappa_t)(\dot g; \dot k) = \int_M (\widetilde{A}^\alpha \widetilde{R}^\beta K_t)(gmk^{-1})\,\mathrm{d}m = \int_M (\widetilde{A}^\alpha \widetilde{R}^\beta K_t)(gm^{-1}k^{-1})\,\mathrm{d}m$$

for all $g, k \in G$, where we use the unimodularity of M in the second equality. Let $\varphi, \tau \in C_c^\infty(X)$, $\rho \in \mathbb{R}$, $\psi \in C_{\mathrm{b};\infty}(X)$ real valued and suppose that $\sum_{i=1}^d |A_i\psi|^2 \leqslant 1$. Then

$$\begin{aligned}
&\int_X \int_X (A^\alpha R^\beta \kappa_t)(\dot g;\dot k)\mathrm{e}^{\rho(\psi(\dot g)-\psi(\dot k))}\varphi(\dot k)\tau(\dot g)\,\mathrm{d}\dot k\,\mathrm{d}\dot g \\
&= \int_X \int_X \int_M (\widetilde{A}^\alpha \widetilde{R}^\beta K_t)(gm^{-1}k^{-1})\mathrm{e}^{\rho((\pi^*\psi)(g)-(\pi^*\psi)(km))}(\pi^*\varphi)(km)(\pi^*\tau)(g)\,\mathrm{d}m\,\mathrm{d}\dot k\,\mathrm{d}\dot g \\
&= \int_X \int_G (\widetilde{A}^\alpha \widetilde{R}^\beta K_t)(gr^{-1})\mathrm{e}^{\rho((\pi^*\psi)(g)-(\pi^*\psi)(r))}(\pi^*\varphi)(r)(\pi^*\tau)(g)\,\mathrm{d}r\,\mathrm{d}\dot g \\
&= \int_X \int_M \int_G (\widetilde{A}^\alpha \widetilde{R}^\beta K_t)(smr^{-1})\mathrm{e}^{\rho((\pi^*\psi)(sm)-(\pi^*\psi)(r))}(\pi^*\varphi)(r)(\pi^*\tau)(sm)\,\mathrm{d}r\,\mathrm{d}m\,\mathrm{d}\dot s \\
&= \int_G \int_G (\widetilde{A}^\alpha \widetilde{R}^\beta K_t)(hr^{-1})\mathrm{e}^{\rho((\pi^*\psi)(h)-(\pi^*\psi)(r))}(\pi^*\varphi)(r)(\pi^*\tau)(h)\,\mathrm{d}r\,\mathrm{d}h
\end{aligned}$$

for all $t > 0$. Secondly,

$$(\widetilde{A}_i \pi^* \psi)(g) = (A_i \psi)(\dot{g}) \tag{4.1}$$

for all $g \in G$ and $i \in \{1, \ldots, d\}$. So $\sum_{i=1}^{d} |\widetilde{A}_i \pi^* \psi|^2 \leqslant 1$. If follows that

$$|(\pi^*\psi)(g) - (\pi^*\psi)(h)| \leqslant |gh^{-1}|$$

for all $g, h \in G$. From [12], Theorem III.4.8 and an elementary transformation to rewrite the right derivatives in terms of left derivatives and an exponential function, one deduces that there exist $a, b > 0$ and $\omega \geqslant 0$ such that

$$|(\widetilde{A}^\alpha \widetilde{R}^\beta K_t)(g)| \leqslant a t^{-(d+|\alpha|+|\beta|)/n} \mathrm{e}^{\omega t} \mathrm{e}^{-2b(|g|^n t^{-1})^{1/(n-1)}}$$

for all $g \in G$ and $t > 0$. Therefore

$$\begin{aligned}
&\Big| \int\limits_X \int\limits_X (A^\alpha R^\beta \kappa_t)(\dot{g}; \dot{k}) \mathrm{e}^{\rho(\psi(\dot{g}) - \psi(\dot{k}))} \varphi(\dot{k}) \tau(\dot{g}) \, \mathrm{d}\dot{k} \, \mathrm{d}\dot{g} \Big| \\
&\quad \leqslant \int\limits_G \int\limits_G a t^{-(d+|\alpha|+|\beta|)/n} \mathrm{e}^{\omega t} \mathrm{e}^{-2b(|hr^{-1}|^n t^{-1})^{1/(n-1)}} \cdot \\
&\qquad\qquad \cdot \mathrm{e}^{\rho((\pi^*\psi)(h) - (\pi^*\psi)(r))} |(\pi^*\varphi)(r)| |(\pi^*\tau)(h)| \, \mathrm{d}h \, \mathrm{d}r \\
&\quad \leqslant a t^{-(d+|\alpha|+|\beta|)/n} \mathrm{e}^{\omega t} \int\limits_G \int\limits_G \mathrm{e}^{-2b(|hr^{-1}|^n t^{-1})^{1/(n-1)}} \mathrm{e}^{|\rho| |hr^{-1}|} |(\pi^*\varphi)(r)| |(\pi^*\tau)(h)| \, \mathrm{d}h \, \mathrm{d}r.
\end{aligned}$$

Using the estimate

$$-b(|hr^{-1}|^n t^{-1})^{1/(n-1)} + |\rho| |hr^{-1}| \leqslant \omega_b |\rho|^n t$$

with $\omega_b = b^{-(n-1)}(n-1)^{n-1} n^{-n}$ one deduces that

$$\begin{aligned}
&\Big| \int\limits_X \int\limits_X (A^\alpha R^\beta \kappa_t)(\dot{g}; \dot{k}) \mathrm{e}^{\rho(\psi(\dot{g}) - \psi(\dot{k}))} \varphi(\dot{k}) \tau(\dot{g}) \, \mathrm{d}\dot{k} \, \mathrm{d}\dot{g} \Big| \\
&\quad \leqslant a t^{-(d+|\alpha|+|\beta|)/n} \mathrm{e}^{\omega t + \omega_b \rho^n t} \cdot \\
&\qquad \cdot \int\limits_G \int\limits_G \mathrm{e}^{-b(|hr^{-1}|^n t^{-1})^{1/(n-1)}} |(\pi^*\varphi)(r)| |(\pi^*\tau)(h)| \, \mathrm{d}h \, \mathrm{d}r.
\end{aligned} \tag{4.2}$$

Therefore it remains to estimate

$$t^{-d/n} \int\limits_G \int\limits_G \mathrm{e}^{-b(|hr^{-1}|^n t^{-1})^{1/(n-1)}} |(\pi^*\varphi)(r)| |(\pi^*\tau)(h)| \, \mathrm{d}h \, \mathrm{d}r$$

for all $t > 0$.

Thirdly, for all $j \in \mathbb{N}_0$ define the annuli Ω_j by

$$\Omega_j = \{(h,r) \in G \times G \mid j \leqslant |hr^{-1}|^n t^{-1} < j+1\}.$$

Let $\mathrm{d}^{(j)}(h,r)$ denote the measure on Ω_j induced by $\mathrm{d}h\,\mathrm{d}r$. Then

$$\begin{aligned}
&\int_G \int_G t^{-d/n} \mathrm{e}^{-b(|hr^{-1}|^n t^{-1})^{1/(n-1)}} |(\pi^*\varphi)(r)||(\pi^*\tau)(h)| \,\mathrm{d}h\,\mathrm{d}r \\
&\qquad = \sum_{j=0}^{\infty} \int_{\Omega_j} t^{-d/n} \mathrm{e}^{-b(|hr^{-1}|^n t^{-1})^{1/(n-1)}} |(\pi^*\varphi)(r)||(\pi^*\tau)(h)| \,\mathrm{d}^{(j)}(h,r) \\
&\qquad \leqslant \sum_{j=0}^{\infty} t^{-d/n} s_j^{d/2} \mathrm{e}^{((j+1)t)^{2/n} s_j^{-1}} \mathrm{e}^{-bj^{1/(n-1)}} \cdot \\
&\qquad\qquad \cdot \int_{\Omega_j} |(\pi^*\varphi)(r)||(\pi^*\tau)(h)| s_j^{-d/2} \mathrm{e}^{-|hr^{-1}|^2 s_j^{-1}} \,\mathrm{d}^{(j)}(h,r) \\
&\qquad \leqslant \sum_{j=0}^{\infty} t^{-d/n} s_j^{d/2} \mathrm{e}^{((j+1)t)^{2/n} s_j^{-1}} \mathrm{e}^{-bj^{1/(n-1)}} \cdot \\
&\qquad\qquad \cdot \int_G \int_G |(\pi^*\varphi)(r)||(\pi^*\tau)(h)| s_j^{-d/2} \mathrm{e}^{-|hr^{-1}|^2 s_j^{-1}} \,\mathrm{d}h\,\mathrm{d}r,
\end{aligned}$$

where $s_j > 0$ for all $j \in \mathbb{N}_0$.

Fourthly, let K^Δ and κ^Δ denote the Lie group kernel and reduced heat kernel of the semigroup S^Δ generated by the Laplacian $\Delta = -A_1^2 - \cdots - A_d^2$. By [12], Theorem III.5.1 there exist $a, c > 0$ and $\omega_1 \geqslant 0$ such that

$$s^{-d/2} \mathrm{e}^{-|g|^2 s^{-1}} \leqslant a \mathrm{e}^{\omega_1 s} K^\Delta_{cs}(g)$$

for all $s > 0$ and $g \in G$. Then by reduction, Corollary 1.3, again, it follows that

$$\begin{aligned}
&\int_G \int_G |(\pi^*\varphi)(r)||(\pi^*\tau)(h)| s_j^{-d/2} \mathrm{e}^{-|hr^{-1}|^2 s_j^{-1}} \,\mathrm{d}h\,\mathrm{d}r \\
&\qquad \leqslant a \mathrm{e}^{\omega_1 s_j} \int_G \int_G |(\pi^*\varphi)(r)||(\pi^*\tau)(h)| K^\Delta_{cs_j}(hr^{-1}) \,\mathrm{d}h\,\mathrm{d}r \\
&\qquad = a \mathrm{e}^{\omega_1 s_j} \int_X \int_X \kappa^\Delta_{cs_j}(x;y) |\varphi(y)||\tau(x)| \,\mathrm{d}x\,\mathrm{d}y \\
&\qquad = a \mathrm{e}^{\omega_1 s_j} (|\tau|, S^\Delta_{cs_j} |\varphi|)
\end{aligned} \tag{4.3}$$

for all $j \in \mathbb{N}_0$. Then by the bounds of Proposition 4.1 there exist $a > 0$ and $\omega \geqslant 0$ such that

$$\int_G \int_G t^{-d/n} e^{-b(|hr^{-1}|^n t^{-1})^{1/(n-1)}} |(\pi^* \varphi)(r)||(\pi^* \tau)(h)| \, dh \, dr$$
$$\leqslant a \sum_{j=0}^{\infty} t^{-d/n} s_j^{d/2} e^{((j+1)t)^{2/n} s_j^{-1}} e^{-bj^{1/(n-1)}} s_j^{-(d-d_{\mathrm{m}})/2} e^{\omega s_j} \|\varphi\|_1 \|\tau\|_1$$

uniformly for all $t, s_1, s_2, \ldots > 0$ and $\varphi, \tau \in C_c^\infty(X)$. Now set $s_j = (j+1)^{1/(n-1)} t^{2/n}$ for all $j \in \mathbb{N}_0$. Then

$$\begin{aligned} t^{-d/n} & s_j^{d/2} e^{((j+1)t)^{2/n} s_j^{-1}} e^{-bj^{1/(n-1)}} s_j^{-(d-d_{\mathrm{m}})/2} e^{\omega s_j} \\ &= (j+1)^{d_{\mathrm{m}}/(2(n-1))} e^{(j+1)^{2/n - 1/(n-1)}} e^{-bj^{1/(n-1)}} e^{\omega (j+1)^{1/(n-1)} t^{2/n}} t^{-(d-d_{\mathrm{m}})/n} \\ &\leqslant (j+1)^{d_{\mathrm{m}}/(2(n-1))} e^{(j+1)^{2/n-1/(n-1)} - 2^{-1} b j^{1/(n-1)}} t^{-(d-d_{\mathrm{m}})/n} \end{aligned}$$

for all $t > 0$ with $2\omega t^{2/n} \leqslant 2^{-1} b$. Since $2/n - 1/(n-1) < 1/(n-1)$ it follows that

$$M = \sum_{j=0}^{\infty} (j+1)^{d_{\mathrm{m}}/(2(n-1))} e^{(j+1)^{2/n-1/(n-1)} - 2^{-1} b j^{1/(n-1)}} < \infty$$

and

$$\begin{aligned} \int_G \int_G & t^{-d/n} e^{-b(|hr^{-1}|^n t^{-1})^{1/(n-1)}} |(\pi^* \varphi)(r)||(\pi^* \tau)(h)| \, dh \, dr \\ &\leqslant a M t^{-(d-d_{\mathrm{m}})/n} \|\varphi\|_1 \|\tau\|_1 \end{aligned} \tag{4.4}$$

for all $t \in (0, ((4\omega)^{-1} b)^{n/2}]$. Alternatively, if $t > ((4\omega)^{-1} b)^{n/2}$ then

$$\begin{aligned} \int_G \int_G & t^{-d/n} e^{-b(|hr^{-1}|^n t^{-1})^{1/(n-1)}} |(\pi^* \varphi)(r)||(\pi^* \tau)(h)| \, dh \, dr \\ &\leqslant t^{-d/n} \int_G \int_G |(\pi^* \varphi)(r)||(\pi^* \tau)(h)| \, dh \, dr \\ &= t^{-d/n} \|\varphi\|_1 \|\tau\|_1 \leqslant ((4\omega)^{-1} b)^{-d_{\mathrm{m}}/2} t^{-(d-d_{\mathrm{m}})/n} \|\varphi\|_1 \|\tau\|_1 . \end{aligned} \tag{4.5}$$

A combination of (4.2), (4.4) and (4.5) yields that for all $\alpha, \beta \in J(d)$ there exist $a, b, \omega > 0$ such that

$$\Big| \int_X \int_X (A^\alpha R^\beta \kappa_t)(\dot g; \dot k) e^{\rho(\psi(\dot g) - \psi(\dot k))} \varphi(\dot k) \tau(\dot g) \, d\dot k \, d\dot g \Big|$$
$$\leqslant a t^{-(|\alpha| + |\beta| + d - d_{\mathrm{m}})/n} e^{\omega t + \omega_b \rho^n t} \|\varphi\|_1 \|\tau\|_1$$

uniformly for all $t > 0$, $\varphi,\tau \in C_{\rm c}^\infty(X)$, $\rho \in \mathbb{R}$ and real valued $\psi \in C_{{\rm b};\infty}(X)$ with $\sum_{i=1}^d |A_i\psi|^2 \leqslant 1$ for all $i \in \{1,\ldots,d\}$. Then

$$|(A^\alpha R^\beta \kappa_t)(\dot{g};\dot{k})| \leqslant a t^{-(|\alpha|+|\beta|+d-d_{\mathfrak{m}})/n} e^{\omega t+\omega_b \rho^n t} e^{-\rho(\psi(\dot{g})-\psi(\dot{k}))}$$

and minimizing first over ψ and finally over ρ gives the bounds

$$|(A^\alpha R^\beta \kappa_t)(\dot{g};\dot{k})| \leqslant a t^{-(|\alpha|+|\beta|+d-d_{\mathfrak{m}})/n} e^{\omega t} e^{-b(d(\dot{g};\dot{k})^n t^{-1})^{1/(n-1)}}$$

and the proof of Proposition 4.2 is complete. ∎

Now we are able to prove the main theorem.

Proof of Theorem 1.4. Obviously the validity of Theorem 1.4 is independent of the choice of the basis $a_1,\ldots,a_d$ for $\mathfrak{g}$, i.e., if Theorem 1.4 is valid for one particular basis then it is valid for any basis. This is because the distance d_1 in (1.3) is independent of the chosen basis, up to equivalence of norms.

Let $D = M \cap Z(G)$. Then D is a closed normal subgroup of G, and also of M, since D is central in G. Therefore G/D is a connected unimodular group and M/D is a compact connected subgroup of G/D. We first show that $(M/D) \cap Z(G/D)$ is finite.

Let $a_1,\ldots,a_{d'},\ldots,a_{d_{\mathfrak{m}}},\ldots,a_d$ be a basis for $\mathfrak{g}$ such that $a_1,\ldots,a_{d'}$ is a basis for $\mathfrak{d}$, the Lie algebra of D and $a_1,\ldots,a_{d_{\mathfrak{m}}}$ is a basis for $\mathfrak{m}$. Then $a_{d'+1}+\mathfrak{d},\ldots,a_d+\mathfrak{d}$ is a basis for the Lie algebra $\mathfrak{g}/\mathfrak{d}$.

Let $a \in \mathfrak{m}$ and suppose $a + \mathfrak{d} \in \mathfrak{z}(\mathfrak{g}/\mathfrak{d})$, the centre of $\mathfrak{g}/\mathfrak{d}$. Then $[a,b] \in \mathfrak{d}$ and hence $[a,b]$ is central for all $b \in \mathfrak{g}$. Therefore,

$$\mathrm{Ad}\,(\exp(ta))b = e^{t\,ad\,a} b = b + t[a,b]$$

for all $t \in \mathbb{R}$ and $b \in \mathfrak{g}$. But $a \in \mathfrak{m}$ and hence $\{\mathrm{Ad}\,(\exp(ta))b \mid t \in \mathbb{R}\}$ is a compact subset of $\mathfrak{g}$. So $[a,b] = 0$ for all $b \in \mathfrak{g}$ and hence $a \in \mathfrak{d}$. Therefore $(\mathfrak{m}/\mathfrak{d}) \cap \mathfrak{z}(\mathfrak{g}/\mathfrak{d}) = \{0\}$. Hence $\dim((M/D) \cap Z(G/D)) = 0$ since the Lie algebra of G/D is naturally isomorphic to the Lie algebra $\mathfrak{g}/\mathfrak{d}$. Then $(M/D) \cap Z(G/D)$ is finite because M/D is compact.

Let $P : G/D \to (G/D)\big/(M/D)$ denote the canonical projection map. Define the map $\Phi : G/M \to (G/D)\big/(M/D)$ by

$$\Phi(gM) = P(gD)$$

for all $g \in G$. It is an elementary exercise to show that the map Φ is well defined, it is a bijection and both Φ and Φ^{-1} are C^∞ maps. Moreover, if one normalizes the Haar measure on D to have total measure one then

$$\int_{(G/D)\big/(M/D)} \varphi = \int_{G/M} \varphi \circ \Phi$$

for all $\varphi \in C_{\mathrm{c}}((G/D)\big/(M/D))$. Let again $a_1, \ldots, a_{d'}, \ldots, a_{d_{\mathfrak{m}}}, \ldots, a_d$ be a basis for $\mathfrak{g}$ such that $a_1, \ldots, a_{d'}$ is a basis for $\mathfrak{d}$ and $a_1, \ldots, a_{d_{\mathfrak{m}}}$ is a basis for $\mathfrak{m}$. Then it is obvious that $A_i = 0$ for all $i \in \{1, \ldots, d'\}$. The Lie algebra of G/D is isomorphic to $\mathfrak{g}/\mathfrak{d}$ in the natural manner, and as vector space the latter is naturally isomorphic to span $\{a_{d'+1}, \ldots, a_d\}$. Let $\widetilde{a}_i$ be the element in the Lie algebra of G/D assigned to a_i for all $i \in \{d'+1, \ldots, d\}$ and let $\widetilde{A}_i$ be the associated infinitesimal generator on $(G/D)\big/(M/D)$. Then $A_i(\varphi \circ \Phi) = (\widetilde{A}_i \varphi) \circ \Phi$ for all $\varphi \in C^\infty((G/D)\big/(M/D))$ and $i \in \{d'+1, \ldots, d\}$.

Let $\widetilde{d}_1$ be the distance on $(G/D)\big/(M/D)$ as in (1.3). Then it is easy to see that

$$\widetilde{d}_1(\Phi(gM); \Phi(hM)) = d_1(gM; hM)$$

for all $g, h \in G$.

Next let

$$\widetilde{H} = \sum_{\substack{\alpha \in J(d) \\ |\alpha| \leqslant n}} c_\alpha \widetilde{A}^\alpha$$

be the strongly elliptic operator of order n on $L_2((G/D)\big/(M/D))$, where we set $\widetilde{A}_1 = \cdots = \widetilde{A}_{d'} = 0$ (see [6] Lemma 3.9). Then $(\widetilde{H}\varphi) \circ \Phi = H(\varphi \circ \Phi)$ for all $\varphi \in C_{\mathrm{c}}^\infty((G/D)\big/(M/D))$ and hence $(\widetilde{S}_t \varphi) \circ \Phi = S_t(\varphi \circ \Phi)$ for all $t > 0$, where $\widetilde{S}_t$ is the semigroup generated by $\widetilde{H}$. Let $\widetilde{\kappa}$ denote the reduced heat kernel corresponding to $\widetilde{H}$. Then

$$\kappa_t(gM; hM) = \widetilde{\kappa}_t(\Phi(gM); \Phi(hM))$$

for all $g, h \in G$. By Proposition 4.2 applied to the group G/D, the compact subgroup M/D and the strongly elliptic operator $\widetilde{H}$, the kernel $\widetilde{\kappa}$ has the required Gaussian estimates. But then also κ satisfies the desired Gaussian bounds. ∎

REMARK 4.3. In general $M \cap Z(G)$ is not finite. An example is the Heisenberg group $G = A(\mathbb{R})$, topologically isomorphic to $\mathbb{R} \times \mathbb{R} \times \mathbb{T}$, together with the compact subgroup $M \sim \{0\} \times \{0\} \times \mathbb{T}$.

5. Reduced heat kernels on nilpotent groups

In the previous section we used the semigroup estimates for the Laplacian on the homogeneous space (Proposition 4.1) and the general kernel bounds for n-th order operators on the Lie group to deduce Gaussian bounds for the reduced heat kernel associated with an n-th order strongly elliptic operator on the homogeneous space. But it turns out that this method also works for reduced heat kernels associated with n-th order strongly elliptic operators associated with an irreducible unitary representation on a nilpotent Lie group. In this section we prove Gaussian bounds for these kinds of reduced heat kernels on nilpotent Lie groups, together with all its derivatives. In [7] Gaussian bounds were proved for second order strongly elliptic operators with real symmetric principal coefficients.

We first recall some notation from [7], Section 2. Note that the representations in [7] are defined with respect to right cosets, but the difference will cause no problem. From now on let G be a connected, simply connected, d-dimensional, nilpotent Lie group with Lie algebra $\mathfrak{g}$ and fix $l \in \mathfrak{g}^*$. Let $\mathfrak{m}$ denote a polarizing subalgebra for l of dimension $d_\mathfrak{m}$ and let $M = \exp(\mathfrak{m})$ denote the corresponding subgroup of G. Further let $a_1, \ldots, a_{d_\mathfrak{m}}, \ldots, a_{d_\mathfrak{m}+k}$ be a weak Malcev basis of $\mathfrak{g}$ passing through $\mathfrak{m}$, i.e., span $\{a_1, \ldots, a_j\}$ is a subalgebra of $\mathfrak{g}$ for all $j \leqslant d = d_\mathfrak{m} + k$ and $\mathfrak{m} = \text{span}\,\{a_1, \ldots, a_{d_\mathfrak{m}}\}$. Define the one dimensional representation $\chi : M \to \mathbb{C}$ by $\chi(\exp a) = \mathrm{e}^{2\pi \mathrm{i} l(a)}$ for all $a \in \mathfrak{m}$. Further define $\gamma : \mathbb{R}^k \to G$ by

$$\gamma(x) = \gamma(x_1, \ldots, x_k) = \exp(x_1 a_{d_m+1}) \cdots \exp(x_k a_{d_m+k}).$$

The map $(m, x) \mapsto m \cdot \gamma(x)$ is a diffeomorphism from $M \times \mathbb{R}^k$ into G which preserves measures. Let $E = (E_1, E_2) : G \to M \times \mathbb{R}^k$ be the inverse of this map. For $g \in G$ define $U(g) : L_2(\mathbb{R}^k) \to L_2(\mathbb{R}^k)$ by

$$(U(g)\varphi)(x) = \chi(E_1(\gamma(x)g))\varphi(E_2(\gamma(x)g))$$

for all $\varphi \in L_2(\mathbb{R}^k)$ and a.e. $x \in \mathbb{R}^k$. Then U is the basis realization of the irreducible representation $\mathrm{ind}(M \uparrow G, \chi)$ with respect to the weak Malcev basis (see [4], p. 125). In the sequel we also need the representation U° of G on $L_2(\mathbb{R}^k)$ given by

$$(U^\circ(g)\varphi)(x) = \varphi(E_2(\gamma(x)g)).$$

The representations U and U° extend to continuous isometric representations on all the $L_p(\mathbb{R}^k)$-spaces with $p \in [1, \infty]$ (see [7], Lemma 2.1).

Let $b_1, \ldots, b_d$ be a vector space basis for $\mathfrak{g}$ and for all $i \in \{1, \ldots, d\}$ let $B_i = \mathrm{d}U(b_i)$ and $B_i^\circ = \mathrm{d}U^\circ(b_i)$ be the associated infinitesimal generators. The B_i° can be used to define a distance on $\mathbb{R}^k$ by

$$d(x;y) = \sup\Big\{|\psi(x) - \psi(y)| \;\Big|\; \psi \in C_c^\infty(\mathbb{R}^k) \text{ real and } \sum_{i=1}^d |B_i^\circ \psi|^2 \leqslant 1\Big\}.$$

(Cf. [7] p. 495.) Next let $n \in \mathbb{N}$ and

$$H = \sum_{\alpha\colon |\alpha| \leqslant n} c_\alpha B^\alpha$$

be a strongly elliptic operator of order n with complex (constant) coefficients on $L_p(\mathbb{R}^k)$. Then the closure $\overline{H}$ of H generates a continuous semigroup S on $L_p(\mathbb{R}^k)$ and for all $t > 0$ the operator S_t has a smooth reduced heat kernel $\kappa_t \in \mathcal{S}(\mathbb{R}^k \times \mathbb{R}^k)$ such that

$$(S_t\varphi)(x) = \int_{\mathbb{R}^k} \kappa_t(x;y)\varphi(y)\,\mathrm{d}y$$

for all $\varphi \in L_p(\mathbb{R}^k)$ and $x \in \mathbb{R}^k$. Let B^α denote the (multi-)derivative of the reduced kernel κ_t with respect to the first variable and R^β the derivative with respect to the second variable and the basis $b_1, \ldots, b_d$.

Now we are able to state the Gaussian bounds for the reduced heat kernels on nilpotent Lie groups.

THEOREM 5.1. *For all $\alpha, \beta \in J(d)$ there exist $a, b > 0$ and $\omega \geqslant 0$ such that*

$$|(A^\alpha R^\beta \kappa_t)(x;y)| \leqslant a t^{-(k+|\alpha|+|\beta|)/n} \mathrm{e}^{\omega t} \mathrm{e}^{-b(d(x;y)^n t^{-1})^{1/(n-1)}}$$

for all $x, y \in \mathbb{R}^k$ and $t > 0$.

Proof. For $t > 0$ let $K_t \in L_1(G;\mathrm{d}g)$ be the Lie group kernel of the operator S_t. As in Section 1 we denote the left derivative in the direction b_i on the Lie group G by $\widetilde{B}_i$ and the right derivative in the direction b_i by $\widetilde{R}_i$. Again the reduced heat kernel is obtained from K by a reduction formula.

LEMMA 5.2. *If $\alpha, \beta \in J(d)$ and $t > 0$ then*

$$(B^\alpha R^\beta \kappa_t)(x;y) = \int_M \chi(m)(\widetilde{B}^\alpha \widetilde{R}^\beta K_t)(\gamma(x)^{-1} m \gamma(y))\,\mathrm{d}m$$

for all $x, y \in \mathbb{R}^k$.

Proof. By [4], Proposition 4.3.2, the right hand side is the kernel of the operator $U(\widetilde{B}^\alpha \widetilde{R}^\beta K_t) = U(\widetilde{B}^\alpha K_{t/2}) U(\widetilde{R}^\beta K_{t/2})$. But $U(\widetilde{B}^\alpha K_{t/2}) = \widetilde{B}^\alpha U(K_{t/2})$ and has as kernel $B^\alpha \kappa_{t/2}$. By duality, $U(\widetilde{R}^\beta K_{t/2})$ has the kernel $R^\beta \kappa_{t/2}$. Then the lemma follows by taking the convolution. ∎

Similar results are also valid with respect to the representation U°. Let $H^{\circ\Delta} = -\sum_{i=1}^{d} (B_i^\circ)^2$ be the Laplacian and $S^{\circ\Delta}$ be the semigroup generated by $\overline{H^{\circ\Delta}}$. Then $S_t^{\circ\Delta}$ has a fast decaying reduced heat kernel $\kappa_t^{\circ\Delta}$ satisfying

$$\kappa_t^{\circ\Delta}(x;y) = \int_M K_t^\Delta(\gamma(x)^{-1} m\gamma(y))\, \mathrm{d}m \tag{5.1}$$

for all $x, y \in \mathbb{R}^k$, where $K_t^\Delta \in L_1(G; \mathrm{d}g)$ is the kernel of the semigroup generated by $-\sum_{i=1}^{d} \widetilde{B}_i^2$. Proposition 4.1 has the following form in the present context.

Proposition 5.3. *There exist $a, \omega > 0$ such that*

$$|(\tau, S_t^{\circ\Delta}\varphi)| \leqslant a t^{-k/2} \mathrm{e}^{\omega t} \|\varphi\|_1 \|\tau\|_1 \tag{5.2}$$

for all $\varphi, \tau \in C_{\mathrm{c}}^\infty(\mathbb{R}^k)$ and $t > 0$.

Proof. If the weak Malcev basis $a_1, \ldots, a_d$ has the ideal property (1.3) of [7], i.e., if

$$[a, a_{d_{\mathrm{m}}+j}] \in \operatorname{span}\{a_1, \ldots, a_{d_{\mathrm{m}}+j-1}\} \text{ for all } a \in \mathfrak{g} \text{ and } j \in \{1, \ldots, k\},$$

then the bounds (5.2) follow from the Nash inequalities Corollary 3.10 of [7] for U°, similarly as in the proof of Proposition 4.1. But by Lemma 2.3 of [7] one can then remove the restriction that the weak Malcev basis has the ideal property. ∎

The modulus on G is defined with respect to the right invariant vector fields (2.1). But if one uses the left invariant vector fields instead of the right invariant vector fields then one obtains the same modulus.

For all $\varphi : \mathbb{R}^k \to \mathbb{C}$ define the function $\pi^*\varphi : G \to \mathbb{C}$ by $(\pi^*\varphi)(m\gamma(x)) = \varphi(x)$. Then

$$(\widetilde{R}_i(\pi^*\varphi))(m\gamma(x)) = (B_i^\circ\varphi)(x)$$

for all $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^k)$, $m \in M$, $x \in \mathbb{R}^k$ and $i \in \{1, \ldots, d\}$. This is the present substitute for (4.1).

Now we are able to prove Theorem 5.1. Since the proof is very similar, we indicate the differences. Let $\alpha, \beta \in J(d)$, $t > 0$, $\varphi, \tau \in C_{\mathrm{c}}^\infty(\mathbb{R}^k)$, $\rho \in \mathbb{R}$,

$\psi \in C_{\mathrm{b};\infty}(\mathbb{R}^k)$ real valued and suppose that $\sum_{i=1}^{d} |B_i^\circ \psi|^2 \leqslant 1$. Then by [4], Lemma 1.2.13 and Theorem 1.2.10, one has

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} (B^\alpha R^\beta \kappa_t)(x;y) \mathrm{e}^{\rho(\psi(x)-\psi(y))} \varphi(y)\tau(x)\,\mathrm{d}y\,\mathrm{d}x \Big| \\
&\leqslant \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \int_{M} |(\widetilde{B}^\alpha \widetilde{R}^\beta K_t)(\gamma(x)^{-1} m \gamma(y))| \mathrm{e}^{\rho(\psi(x)-\psi(y))} |\varphi(y)|\,|\tau(x)|\,\mathrm{d}m\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\mathbb{R}^k} \int_{G} |(\widetilde{B}^\alpha \widetilde{R}^\beta K_t)(\gamma(x)^{-1} g)| \mathrm{e}^{\rho((\pi^*\psi)(\gamma(x))-(\pi^*\psi)(g))} |(\pi^*\varphi)(g)|\,|\tau(x)|\,\mathrm{d}g\,\mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^k} \int_{G} |(\widetilde{B}^\alpha \widetilde{R}^\beta K_t)(\gamma(x)^{-1} g)| \mathrm{e}^{|\rho||\gamma(x)^{-1}g|} |(\pi^*\varphi)(g)|\,|\tau(x)|\,\mathrm{d}g\,\mathrm{d}x.
\end{aligned}$$

If one defines and uses the annuli

$$\Omega_j = \{(x,g) \in \mathbb{R}^k \times G \mid j \leqslant |\gamma(x)^{-1} g|^n t^{-1} < j+1\}$$

then one can argue as in the proof of Proposition 4.2 up to equality (4.3). But now it follows from (5.1) that

$$\begin{aligned}
&\int_{\mathbb{R}^k} \int_{G} |(\pi^*\varphi)(g)|\,|\tau(x)| K^{\Delta}_{cs_j}(\gamma(x)^{-1} g)\,\mathrm{d}g\,\mathrm{d}x \\
&= \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} \kappa^{\circ\Delta}_{cs_j}(x;y) |\varphi(y)|\,|\tau(x)|\,\mathrm{d}y\,\mathrm{d}x = (|\tau|, S^{\circ\Delta}_{cs_j}|\varphi|) \leqslant a s_j^{-k/2} \mathrm{e}^{\omega s_j} \|\varphi\|_1\,\|\tau\|_1
\end{aligned}$$

for suitable $a, \omega > 0$, by Proposition 5.3. The rest of the proof is as before. ∎

Although the proof in the nilpotent case is very similar to the proof in the homogeneous case, the nilpotent case is not a special case of the homogeneous case since M is not compact in general.

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